Chapter – 4

JOINT AND CONDITIONAL PROBABILITY DISTRIBUTION

- In most cases the outcomes of an experiment may be characterized by more than one random variables.
- And hence, if a given experiment characterized by two random variables it is called bivariate distributions.
- Where as three or more random variables, for instance in the case where X may be income and Y the total expenditures of a house hold and Z is a family size,.....

.....it is possible to observe a different combinations of variables (X,Y,Z) and this type of distribution is called multivariate distribution.

Definition: If X and Y are discrete random variables, the function given by f(x, y) = P(X = x, Y = y) for each pair of values (x, y) within the range of x and y is called the joint probability distribution of x and y.

Example 1: Assuming that if the variable x denotes the sex of the person and the variable y denotes whether the person is smoker or not. Based on the above understanding if the probability of male who is smoker is 50%, probability of female who smoker is 20%, probability of male who is non-smoker is 20% and probability of female who is non – smoker is 10%. Then the joint probability distribution of x

Joint distribution of x and y

Values of y	Value of x	
	X_1 = male	X_2 – Female
Y_1 smoker	0.50	0.20
Y_2= non – smoker	0.20	0.10

Theorem: A bivariate function can serve on the joint probability distribution of a pair of discrete random variable, x and y $\Leftrightarrow$ its values f(x, y), satisfy the conditions:

1) $f(x, y) \geq 0$ for each pair of values (x, y), within its domain,

2) $\sum_x \sum_y f(x, y) = 1$,

Where the double summation extends over all possible pairs (x,y) within its domain.

Examples:

1) Check theorem based on problem one from the above example one.

Solution:

i) $f(x, y) \geq 0 \Rightarrow f(x_1, y_1) = 0.5, f(x_1 y_2) = 0.2, f(x_2, y_1) = 0.2, f(x_2, y_2) = 0.1$

ii) $\sum_x \sum_y f(x, y) = 1 \Rightarrow f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(\chi_2 y_2)$

$$= 0.5 + 0.2 + 0.2 + 0.1 = 1$$

Example 2: Determine the value of k for which the function given by f(x, y) = k[y(y-x)], for **x= 0, 1, 2 & y =0, 1, 2** can serve as a joint probability distribution.

Solution:

- Substituting the various value of x, and y, we get

f(0,0) = 0;f(0,1) = k; f(0,2) = 4k, f(1,0) = 0;f(1,1) = 0; f(1,2) = 2K ; f(2,2) =0 , f(2.0)= 0;f(2,1)=-k is not a probability distribution since it violets the first condition of theorem

3) Determine the value of K for each the function given by f(x, y) =k(x + y), for x=1, 2; y=1, 2.

Solution: f(1, 1) =2k; f(1, 2) =3k; f (2, 1) =3k;f(2,2)=4k

$$\sum_x \sum_y f(x,y) = 1 = 2k + 3k + 3k + 4k = k(3+3+3+4) = 12k, \therefore k = \frac{1}{12}$$

Definition 4.2 If X and Y are discrete random variables the function given by:

$$F(x, y) = P(X \le x, Y \le y) = \sum_{s \le x} \sum_{t \le y} f(s,t) \text{ for } -\infty < x < \infty, -\infty < y < \infty,$$

Where f(s, t) is the value of the joint probability distribution of x and y at (s, t), and is

Called the joint distribution function or the joint commutative distribution of x and y.

Examples:

1) If the values of the joint probability distribution of x and y are as shown in the table:

X / Y	0	1	2
0	$6/36$	$12/36$	$3/36$
1	$8/36$	$6/36$	0
2	$1/36$	0	0

Solution

i) $F(2,0) = f(0,0) + f(1,0) + f(2,0) = P(\leq 2, y \leq 0)$

$$= \frac{1}{6} + \frac{1}{3} + \frac{1}{12} = \frac{7}{12}$$

ii) $F(1,1) = f(0,0) + f(0,1)\ f(1,0) + f(1,1) = P(\leq 1, y \leq 1)$

$$\frac{6}{36} + \frac{8}{36} + \frac{12}{36} + \frac{6}{36} = \frac{32}{36} = \frac{8}{9}$$

iii) $F(3,4) = f(0,0) + f(0,1) + f(0,2) + f(0,3) + f(0,4) +$

$+f(1,0) + f(1,1) + f(1,2) + f(1,3) + f(1,4) +$

$+f(2,0) + f(2,1) + f(2,2) + f(2,3) + (2,4) +$

$+f(3,0) + f(3,1) + f(3,2) + f(3,3) + (3,4) =$

$$= \frac{6}{36} + \frac{8}{36} + \frac{1}{36} + 0 + 0 + \frac{12}{36} + \frac{6}{36} + 0 + 0 + 0 + \frac{3}{36} +$$

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 =$$

$$= \frac{6}{36} + \frac{8}{36} + \frac{1}{36} + \frac{12}{36} + \frac{6}{36} + \frac{3}{36} = \frac{36}{36} = 1$$

IV) $F(-2,1) = 0$, there is no value of $x<0$, there fore $P[x<0;\ y \leq (0;1)] = 0$

Definition: A bivariate function with values f(x, y), defined over the x y- plane, is called a joint probability density function of the continuous random variables (CRV⇔ $p[(x,y)\varepsilon A] = \int_A \int f(x,y)dxdy$; egion A in the x y= plane.

Theorem 4.3 A bivariate function can serve as a joint probability density function of a pair of continuous random variables x and y if its values, f(x, y), s

1) $f(x,y) \geq 0, for -\infty < x < \infty, -\infty < y < \infty$

2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = 1$

Examples 1) If the joint probability density of x and y is given by f(x, y)

$$\begin{cases} \frac{x}{5}(x+y), for, o < x < 2, 0 < y < 1, \\ 0, elsewhere \end{cases}$$

Find P [(x, y) EA], where A is the region $\left\{(x, y) \middle| 0 < x < \frac{3}{2}, 0 < y < 1\right\}$

Solution

$$P[(x, y)\ EA] = P[0 < x < \frac{3}{2}, 0 < y < 1] = \int_0^1 \int_0^{3/2} \frac{x}{5}(x+y)dxdy =$$

$$= \int_0^1 \int_0^{3/2} \frac{1}{5}(X^2 + xy)dxdy = \frac{1}{5}\int_0^1 [\frac{x^3}{3} + y\frac{x^2}{2}]_0^{\frac{3}{2}} dy =$$

$$= \frac{1}{5}\int_0^1 (\frac{9}{8} + \frac{9}{8}y)dy = \frac{1}{5} \times \frac{8}{9}\int_0^1 (1+y)dy = \frac{1}{5} \times \frac{8}{9}[y + \frac{y^2}{2}]_0^1 =$$

$$\frac{1}{5} \times \frac{8}{9}(1 + \frac{1}{2}) = \frac{1}{5} \times \frac{8}{9} \times \frac{3}{2} = \frac{27}{80}$$

Example 2) Determine R, so that $f(x,y)=\begin{cases} Rx(x-y), for, 0<x<1, -1<y<1, \\ 0, elsewhere \end{cases}$

Solution:

$$1=\int_{-1}^{1}\int_{0}^{1} Rx(x-y)dxdy = R\int_{-1}^{1}\int_{0}^{1}(x^2-yx)dxdy =$$

$$= R\int_{-1}^{1}\left(\frac{x^3}{3}-y\frac{x^2}{2}\right)\Big|_0^1 dy = R\int_{-1}^{1}\left(\frac{1}{3}-\frac{y}{2}\right)dy =$$

$$= R\left(\frac{y}{3}-\frac{y^2}{4}\right)\Big|_{-1}^{1} = R\left[\left(\frac{1}{3}-\frac{1}{4}\right)-\left(-\frac{1}{3}-\frac{1}{4}\right)\right] = \frac{2}{3}R$$

$$\Rightarrow R=\frac{3}{2}$$

Definition 4.4 If x and y are continuous random variables (CRVs), the function given by:

$F(x, y) = P(x \le x, y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)dsdt$, for $-\infty < x < \infty; -\infty < y < \infty$, where f(s, t) is the

Value of the joint probability distribution at (s, t), is called the joint distribution function of x and y.

Example 1) Find the joint probability density of the two random variables x and y whose joint distribution function is given by:

$F(x, y) = \begin{cases} \left(1 - e^{-x^3}\right)\left(1 - e^{-y^3}\right), for, x > 0, and, y > 0, 0, elsewhere \end{cases}$, also use the joint probability density to determine $P(1 < x \le 2, 1 \le 2, 1 < y \le 2)$.

Solution

1st we have to find the primitive (joint distribution function of x and y) function. Hence

$$\frac{dd}{dxdy}[F(x,y)] = \frac{dd}{dxdy}\left[\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)dxdy\right] = f(s,t),\ \text{So}\ \frac{dd}{dxdy}[(1-e^{-x3})(1-e^{-y3})] = 3x^2e^{-x3}\frac{d}{dy}(1-e^{-y3}) =$$

$= 9x^2y^2\ e^{-x3}\ e^{-y3} = 9\ x^2y^2e^{-(x3+y3)}$, But joint probability density P $(1<x\leq 2, 1<y\leq 2)$

$$\Rightarrow \int_1^2\int_1^2 9x^2y^2e^{-(x3+y3)}dxdy = \int_1^2 3y^2e^{-y^3}[-e^{-x3}]_1^2dy =$$

$$\Rightarrow (e^{-(1)3}-e^{-(2)3})\int_1^2 3y^2e^{-y3}dy = (e^{-1}-e^{-8})[-e^{-y3}]_1^2 =$$

$$= (e^{-1}-e^{-8})(e^{-1}-e^{-8}) = \left(e^{-1}-e^{-8}\right)^2 = e^{-2}-2e^{-9}+e^{-16}$$

Marginal probability distribution

Definition: If x and y are discrete random variables and f(x, y) is the value of their joint probability distribution at (x, y), the function given by g(x) = $\sum_y f(x, y)$ for each x within the range of x is called the marginal distribution of x.

Correspondingly, the function given by h(y) = $\sum_x f(x, y)$ for each y within the range of y is called the marginal distribution of y.

Definition: If x and y are continuous random variables (CRVs) and f(x, y) is the value of their joint probability density at (x, y), th$\int_{-\infty}^{\infty} f(x,y)dy, for -\infty < x < \infty$ given by

g(x)= is called the marginal den$\int_{-\infty}^{\infty} f(x,y)dx, for -\infty < y < \infty$

Correspondingly, the function given by

h(y)= $f(x,y)=\frac{1}{30}(x+y), for, x=0,1,2,3; y=0,1,2, find:$

marginal density of y.

Examples 1: The joint probability distribution of X and Y is given by:

I. The marginal distribution of x.

Solution

i) The marginal distribution of x.

$$g(x) = \sum_{y=0}^{2} f(x, y), \text{ for } x = 0,1,2,3$$

$$g(0) = f(0,0) + f(0,1) + f(0,2) = 0 + \frac{1}{30} + \frac{2}{30} = \frac{3}{30}$$

$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{1}{30} + \frac{2}{30} + \frac{3}{30} = \frac{6}{30}$$

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{2}{30} + \frac{3}{30} + \frac{4}{30} = \frac{9}{30}$$

$$g(3) = f(3,0) + f(3,1) + f(3,2) = \frac{3}{30} + \frac{4}{30} + \frac{5}{30} = \frac{12}{30}$$

ii) The marginal distribution of y

$$h(y) = \sum_{x=2}^{3} f(x, y); \text{ for } y = 0,1,2$$

$$h(0) = f(0,0) + f(1,0) + f(2,0) + f(3,0) = 0 + \frac{1}{30} + \frac{2}{30} + \frac{3}{30} = \frac{6}{30}$$

$$h(1) = f(0,1) + f(1,1) + f(2,1) + f(3,1) = \frac{1}{30} + \frac{2}{30} + \frac{3}{30} + \frac{4}{30} = \frac{10}{30}$$

$$h(2) = f(0,2) + f(1,2 + f(2,2) + f(3,2) = \frac{2}{30} + \frac{3}{30} + \frac{4}{30} + \frac{5}{30} = \frac{14}{30}$$

Here we can also illustrate in diagram (tabular) form as follows:

x / y	0	1	2	3	Marginal distribution of Y
0	0	$1/30$	$2/30$	$3/30$	$6/30$
1	$1/30$	$2/30$	$3/30$	$4/30$	$10/30$
2	$2/30$	$3/30$	$4/30$	$5/30$	$14/30$
Marginal distribution of X	$3/30$	$6/30$	$9/30$	$12/30$	1

Example 2) Given the joint probability density f(x, y)

$$\begin{cases} \frac{2}{3}(2x+y), for, 0<x<1; 0<y<1, and \\ 0, elsewhere \end{cases}$$

Solution

i) The marginal density of x.

$$g(x) = \int_{-\infty}^{\infty}(x,y)dy = \int_0^1 \frac{(4x+2y)}{3}dy = \frac{2}{3}\int_0^1 (2x+y)dy =$$

$$= \frac{2}{3}[2xy + \frac{y^2}{2}]_0^1 = \frac{2}{3}[(2x(1) + \frac{(1)^2}{2} - (2x(0) + \frac{1}{2}(0)^2]$$

$$= \frac{2}{3}(2x + \frac{1}{2}) = \frac{1}{3}(4x+1)$$

ii. The marginal density of y.

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{2}{3}(2x+y)dx = \frac{2}{3}\int_0^1 (2x+y)dx =$$

$$= \frac{2}{3}[\frac{2x^2}{2} + yx]_0^1 = \frac{2}{3}[(x^2 + yx]_0^1 = \frac{2}{3}[]^2 + y(1)) - (0^2 + y(0))] = \frac{2}{3}(1+y)$$

Conditional Distribution and Independence

Definition: If f(x, y) is the value of the joint probability distribution of the discrete random variables x and y at (x, y) and h(y) is the value of the marginal distribution of y at y,

the function given by $f(x/y) = \frac{f(x,y)}{h(y)}, h(y) \neq 0$ for each of x within the range of x, is called the conditional distribution of x given Y=x

Correspondingly, if g(x) is the value of the marginal distribution f(y, x) at x, the function given by

$W(y/x) = \frac{f(x,y)}{g(x)}; g(x) \neq 0,$ for each y within the range of y, is called the conditional distribution of y given x=x

Examples 1: Given the joint probability distribu $\frac{xy}{36}$ f(x, y) = for x = 1,2,3 and y = 1,2,3 , find the conditional distribution of x give

y \ x	1	2	3	$h(y_i)$
1	$1/36$	$2/36$	$3/36$	$h(1)=\frac{6}{36}$
2	$2/36$	$4/36$	$6/36$	$h(2)=\frac{12}{36}$
3	$3/36$	$6/36$	$9/36$	$h(3)=\frac{18}{36}$

$$f(1,1)=\frac{f(1,1)}{h(1)}=\frac{f(1,1)}{f(1,1)+f(2,1)+f(3,1)}=\frac{\left[\frac{1\times1}{36}\right]}{\frac{1\times1}{36}+\frac{2\times1}{36}+\frac{3\times1}{36}}=\frac{\left[\frac{1}{36}\right]}{\left[\frac{6}{36}\right]}=\frac{1}{6}$$

$$f(2/1)=\frac{f(2,1)}{h(1)}=\frac{\left[\frac{2\times1}{36}\right]}{\frac{1}{36}+\frac{2}{36}+\frac{3}{36}}=\frac{2}{6}=\frac{1}{3}$$

$$f(3/1)=\frac{f(3,1)}{h(1)}=\frac{\left[\frac{3\times1}{36}\right]}{\frac{1}{36}+\frac{2}{36}+\frac{3}{36}}=\frac{3}{6}=\frac{1}{2}$$

Example 2) Suppose if a textile factory is interested to know the relation ship between the work experience of employees (x) and their grade (work level) in the factory (y) and if these two bivariate random variables are constructed as a joint probability distribution denoted by f(x, y)

for x = 0, 1, 2 and y= 0, 1,2 when the individual employee is selected at random there may be 9 possible outcomes. These are shown in the table below as follows which depicts the relation ship between work experience of employees and their level of work in a textile factory.

x / y	0	1	2	h(y)
0	0.44	0.22	0.03	0.69
1	0.22	0.06	0	0.28
2	0.03	0	0	0.03
g(x)	0.69	0.28	0.03	1=F(x,y)

x= Number of work experience of employees

y= employees work level (grade)

g(x) = marginal distribution of x.

h(y) = marginal distribution of y.

F(x, y) = joint probability distribution of x and y

i) Calculate the marginal distribution of x

ii) Calculate the marginal distribution of y

iii) What is the probability that x = o given that y=1?

iv) What is probability that y =1 given tat x = 1?

(Find the conditional distribution of x = 0 given y=1?)

Solution:

i) $g(x) = \sum_y f(x,y)$, for x = 0,1,2

$g(0) = f(0,0) + f(0,1,) + f(0,2) = 0.44 + 0.22 + 0.03 = 0.69$

$g(1) = f(1,0) + f(1,1) + f(1,2) = 0.22 + 0.06 + 0 = 0.28$

$g(2) = f(2,0) + f(2,1) + f(2,2) = 0.03 + 0 + 0 = 0.03$

$\therefore g(x) = 0\ ,\ 69,\ 0.28,\ 0.03$

iii) $h(y) = \sum_x f(x,y)$, for y = 0,1,2.

$h(0) = (0,0) + f(1,0) + f(2,0) = 0.44 + 0.22 + 0.03 = 0.69$

$h(1) = (0,1) + f(1,1) + f(2,1) = 0.22 + 0.06 + 0 = 0.28$

$h(2) = f(0,2) + f(1,2) + f(2,2) = 0.03 + 0 + 0 = 0.03$

$\therefore h(y) = 0.69,\ 0.28,\ 0.03$

And the joint probability distribution function or

Joint cumulative distribution of x and y = F(x, y) = 1

iii) $f(x=0/y=1) = p(x=0/y=1) = \frac{f(0,1)}{h(1)} = \frac{p(x=0,y=1)}{p(y=1)} = \frac{0.22}{0.22+0.06+0} = \frac{0.22}{0.28}$

iv) $w(y=1/1x=1) = p(y=1/x=1) = \frac{f(x=1,y=1)}{g(x=1)} = \frac{f(1,1)}{g(1)} = \frac{p(x=1,y=1)}{p(x=1)} =$

$$= \frac{0.06}{0.22+0.06+0} = \frac{0.06}{0.28} = 0.21$$

Definition: If f(x, y) is the value of the joint density of the continuous random variables (CRVs) X and Y at (x, y) and h(y) is the value of the marginal density of y at y, the function given by

$$f(x|y) = \frac{f(x,y)}{h(y)}; h(y) \neq 0, for -\infty < x < \infty,$$

is called the conditional density of x given y=y,

- Correspondingly, if f(x) is the value of the marginal density of x at x, the function given by

$$W(y|x) = \frac{f(x,y)}{g(x)}, g(x) \neq 0, for -\infty < y < \infty,$$

is called the conditional density of y given x=x.

Example 1) given the joint probability density $f(x,y)=\begin{cases}\frac{2}{3}(2x+y), for, 0<x<1, 0<y<1,\\ 0, elsewhere\end{cases}$

i. Find the conditional density of x given y=y and use it evaluate $p\left[x\le\frac{1}{2}, y=\frac{1}{2}\right]$

ii. Find the conditional density of y given X=x and use to evaluate $P\left[x=\frac{1}{2}; y\le\frac{1}{2}\right]$

Solution

i. 1st find the marginal density of y at y

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{2}{3}(2x+y)dx = \frac{2}{3}[2\frac{x2}{2} + yx]_0^1 = \frac{2}{3}(y+1),$$

then $f(x/y) = \frac{f(x,y)}{h(y)} = \frac{\frac{2}{3}(2x+y)}{\frac{2}{3}(y+1)}$, for o<x<1 and $f(x/y) = 0$ elsewhere $= \frac{2x+y}{y+1}$

Hence, $f\left(x/\frac{1}{2}\right) = \frac{2x+\frac{1}{2}}{\frac{1}{2}+1} = \frac{4x+1}{3}$

There fore, $P[x \le \frac{1}{2} / y = \frac{1}{2}] = \int_0^{\frac{1}{2}} \frac{1}{3}(4x+1)dx = \frac{1}{3}[\frac{4x^2}{2} + x]_0^{\frac{1}{2}} =$

$$= \frac{1}{3}[(2(\frac{1}{2})^2 + \frac{1}{2}) - (2(0)^2 + (0)^2] = \frac{1}{3}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{3}$$

ii. 1st find the marginal density of x at x

$$g(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_0^1 \frac{2}{3}(2x+y)dy=\frac{2}{3}[2xy+\frac{y^2}{2}]_0^1=\frac{2}{3}(2x+\frac{1}{2}),$$

then $w(y/x)=\dfrac{f(x,y)}{g(x)}=\dfrac{\frac{2}{3}(2x+y)}{\frac{2}{3}(2x+\frac{1}{2})}$, for $0<y<1$ $w(y/x)=0$ elsewhere

$$=\frac{[2(\frac{1}{2})+y]}{[2(\frac{1}{2})+\frac{1}{2}]}=\frac{2}{3}(y+1),$$ there fore, $P[y\le\frac{1}{2}/x=\frac{1}{2}]=\int_0^{1/2}\frac{2}{3}(y+1)dy$

$$=\frac{2}{3}[\frac{y^2}{2}+y]_0^{1/2}=\frac{2}{3}[\frac{(1/2)^2}{2}+\frac{1}{2}]=\frac{2}{3}\times\frac{5}{8}=\frac{5}{12}$$

Independent random variables

Two or more random variables are independent when all the marginal probability and the joint probability of any group of them are unaffected by values of any others.

Definition: If $f(x_1, x_2 \ldots x_n)$ is the value of the joint probability distribution of the n discrete random variables $x_1, x_2, \ldots, x_n$ at $(x_1, x_2, \ldots, x_n)$, and

$f_i(x_i)$ is the value of the marginal distribution of x_i at x_i for $i = 1, 2, \ldots, n$, then the n random variables are independent if and only if

$f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_2). \ldots f_n(x_n)$, for all $(x_1, x_2, \ldots, x_n)$ within their range. i.e., $f(x_1, x_2, \ldots = \prod_{i=1}^{n} fi(xi)$

Definition: If $f(x_1, x_2, \ldots, x_n)$ is the value of the joint probability density of the n continuous random variables $x_1, x_2, \ldots, x_n$ at $(x_1, x_2, \ldots, x_n)$, and $f_i(x_i)$ is the value of the marginal distribution of xi at x_i for i = 1, 2, …, n, then the n random variables $\prod_{i=1}^{n} fi(xi)$ dependent if and only if $f(x_1, x_2, \ldots, x_n) = f_1$ (x_2). …. $f_n(x_n) =$, for all $(x_1, x_2, \ldots, x_n)$ with in their range.

Examples:10 independent tosses of a balanced coin, let xi be the number of Heads (0 or 1) obtained in the 10^{th} toss for i = 1, 2… 10.

Find the joint probability distribution of these 10 random variables.

Solution: Since each of the random variables xi, for i = 1, 2,....., 10 has the probability distribution $f_i(x_i) = ½$, for $x_i =$ 0, 1 {H, T} i.e., head and/or tail and the 10 random variables are independent, their joint probability distribut $= \prod_{i=1}^{10} fi(xi) = \prod_{i=1}^{10} (\frac{1}{2}) = (\frac{1}{2})^{10}$ by

$= \frac{1}{1024} \approx 9.8 \times 10^{-4}$

$f(x_1, x_2, ..., x_{10}) = f_1(x_1).f_2(x_2).....f_{10}(x_{10})$

where $x_i = 0$ or 1, for i = 1, 2, ..., 10

Exercise !!

2) Suppose x_1, x_2 and x_3 are three independent random variables with the probability densities:

$$f_1(x_1) = \begin{cases} e^{-3x_1}, for\ x_1>0 \\ 0, elsewhere \end{cases}$$

$$f_2(x_2) = \begin{cases} e^{-x_2}, for x2>0 \\ 0, elsewhere \end{cases}$$

$$f_3(x_3) = \begin{cases} e-x_3, for x_3>0 \\ 0, elsewhere \end{cases},$$

Find their joint probability density, and use it to evaluate the probability $P[x_3 + x_2 \leq 1, x_1>1)$

Expectations

- If X is discrete random variable and f(x) is the value of its probability distribution at x the expected value of x is E(x) = $\sum_x x.f(x)$
- If X is a continuous random variable and f(x) is the value of its probability density at x, the expected value of x is E(x) = $\int_{-\infty}^{\infty} x.f(x)dx$
- If X and Y are discrete random variables and f(x, y) is the value of their joint probability distribution at (x, y) the expected value of g(x, y) is $\sum_x \sum_y g(x,y).f(x,y)$

  E[g(x, y)] =
- If X and Y are continuous random variables and f(x, y) is the value of their joint probability density at (x, y), the expected value of g(x, y) is $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y).f(x,y)dxdy$

  E[g(x, y)] =

Example 1: Suppose two balls are selected at random form a sack of containing three red balls, two black balls and fouryellow balls. If X and Y are, respectively, the numbers of red balls and black balls included among the two balls drawn from the sack, then the probabilities associated with all possible pairs of values of X and Y are (0,0), (0,1), (1,0), (1,1),(0,2), and (2,0),and the number of ways in which two of the nine balls can be selected is =36.Then calculate the expected value of g(x, y)= x+y.

Solution: f(X=x, Y=y)

Solution

1st find the probability of f(x,y), i.e, $f(0,0)=\frac{1}{6}; f(0,1)=\frac{2}{9}; f(1,0)=\frac{1}{3};$

$$f(1,1)=\frac{1}{6}; f(0,2)=\frac{1}{36}; f(2,0)=\frac{1}{12}$$

$$E(x+y)=\sum_{x=0}^{2}\sum_{y=0}^{2}(x+y).f(x,y)=(0,0)\times\frac{1}{6}+(0+1)\times\frac{2}{9}+(0+2)\times\frac{1}{36}+$$

$$(1+0)\times\frac{1}{3}+(1+1)\times\frac{1}{6}+(2+0)\times\frac{1}{12}=1.11$$

Example: 2) If the joint probability density of X and Y is given by $f(x, y) = \begin{cases} \frac{1}{5}(x+y), for\ 0<x<1, 1<y<2 \\ 0, elsewhere \end{cases}$

Find the expected value of $g(x, y) = \frac{x}{y^4}$

Solution: $E(x/y^4) = \int_1^2 \int_0^1 \frac{x(x+y)}{5y4} dxdy$

$$= \frac{1}{5}\int_1^2 \left[\frac{(\frac{x^3}{3})}{y^4} + \frac{x^2 y}{2y4} \right]_0^1 dy = \frac{1}{5}\int_1^2 \frac{1}{y4}(\frac{1}{3} + \frac{y}{2}) dy$$

$$= \frac{1}{5}\left[\int_1^2 \frac{1dy}{3y^4} + \int_1^2 \frac{1}{2y^3} dy \right] = \frac{1}{15}\int_1^2 \frac{dy}{y^4} + \frac{1}{10}\int_1^2 \frac{dy}{y^3}$$

$$= \frac{1}{15}\left[-\frac{1}{3}.\frac{1}{y^3} \right]_1^2 + (-\frac{1}{20})\frac{1}{y^2}\Big|_1^2$$

$$= -\frac{1}{45}(\frac{1}{8} - 1) - \frac{1}{20}(\frac{1}{4} - 1) = \frac{7}{45 \times 8} + \frac{1}{20} \times \frac{3}{4} = \frac{668}{28{,}800} = 0.023$$

Covariance and correlation

Let X and Y are two random variables with mean E(X) and E(Y).

Their relation ship is measured by covariance and correlation between them. Covariance and variance are interrelated,

- Covariance of X and Y measures how these variables change together,
- While correlation measures the direction and closeness (association) of the linear relationship (association) between them.

Covariance

Let X and Y are two random variables with means E(X) and E(Y). Then the covariance between the two variables is defined as.

$$\begin{aligned}\text{Cov}(X,Y) &= E[(X-E(X))\,(Y-E(Y))\\ &= E[xy - yE(x) - x\,E(y) + E(x)\,E(y)]\\ &= E(xy)-E(y)E(x)-E(x)E(y)+E(x)E(y)\\ &= E(xy) - 2E(y)E(x)+E(y)E(x)\\ &= E(xy)- E(x)E(y)\end{aligned}$$

Properties of covariance

1. $\text{var}(x) = \text{cov}(x, x)$
2. $\text{cov}(x, y) = E(x)\,E(y) - E(x)\,E(y) = 0$, *if x and y are independent*
3. $\text{cov}(a_0 + a_1 x, b_o + b_1 y) = a_1 b_1\,\text{cov}(x, y)$

If X and Y are discrete, the covariance is given by:

$$\text{Cov}(x,y) = \sum_j \sum_i (x_i - E(x))(y_j - E(y))$$

$$= \sum_j \sum_i (x_j, y_i) f(x_i, y_j) - E(x)E(y) = E(xy) - E(x)E(y), \text{ where, } i \text{ and } j$$

Are all Possible value of x and y subscripts respectively.

For a continuous case, we have:

$$\text{Cov}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - E(x))(y - E(y)) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(xy) dx dy - E(x)E(y)$$

Example 1: For a discrete case, consider the following joint probability distribution function.

X / Y	0	1	2	h(y)
0	0.46	0.20	0.04	0.70
1	0.22	0.06	0	0.28
2	0.02	0	0	0.02
g(x)	0.70	0.26	0.04	

Find cov (x,y)? Where g(x) and h(y) are the marginal probability distribution functions.

Solution

$\mathrm{Cov}(x,y) = E(xy) - E(x)E(y)$

Let us 1st compute E(x), E(y) and E (xy).

$$E(x) = \sum_{i=0}^{2} x.f(x) = 0\times 0.70 + 1\times 0.26 + 2\times 0.04 = 0.34$$

$$E(y) = \sum_{i=0}^{2} y.f(y) = 0\times 0.70 + 70 + 1\times 0.28 + 2\times 0.02 = 0.32$$

$$E(xy) = \sum_{i=0}^{2}\sum_{j=0}^{2} (x\times y) f(x,y) = 0\times 0\times 0.46 + 0\times 1\times 0.22 + 0\times 2\times 0.02 +$$

$$+1\times 0\times 0.20 + 1\times 1\times 0.06 + 1\times 2\times 0 +$$

$$+2\times 0\times 0.04 + 2\times 1\times 0 + 2\times 2\times 0 = 0.06$$

There fore, $\mathrm{cov}(x,y) = E(xy) - E(x)E(y) = 0.06 - (0.34)(0.32) = -0.0488$

The correlation between two variables may be defined as the degree of relationship between two variables.

If the number of variables is more than two then we have multiple correlations.

The degree of relation ship between two variables in simple correlation, and it is expres

$$\rho_{xy} = \frac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y}$$

,where ρ- read as rho, stands for the population parameter, if the data for variables are obtained form the sample, then the symbol for correlation is “r” and $\boldsymbol{\sigma_x}$ **and** $\boldsymbol{\sigma_y}$ are standard deviation of x and y respectively; - $\boldsymbol{\rho_{xy}}$ is the quantity of coefficient correlation between the two variables.

- The range of coefficient correlation is form negative one up to positive one (i.e,$-1 \leq \rho_{xy} \leq 1$) if the coefficient of correlation is close to 1 there is a strong positive correlation and if it is close to -1, there is a strong negative (inverse) correlation between variables.
- If cov (x, y) = 0, then cov (y, x)=0 in this case the two variables are said to be uncorrelated, which is that variables are independent and their correlation is zero.
- But zero correlation does not mean independence because uncorrelated variables may be statistically dependent in non- linear relationship.
- There fore, in general of the correlation coefficient is a measure of linear association between variables.

Conditional expectation

- If X and Y are discrete random variable

✓ The conditional expectation (mean) of X given Y=y is given by: E(X/Y=y) = $\sum_x x.f(x/y)$

✓ The conditional expectation (mean) of Y given X = x is given by: E(Y/X=x) = E(Y/X) = $\sum_y y.f(y/x)$

- If X and Y are a continuous random variable

✓ The conditional expectation (mean) of X given Y=y is given by: $\int_{-\infty}^{\infty} x.f(x/y)dx$)=

✓ The conditional expectation (mean) of Y given X = x is given $\int_{-\infty}^{\infty} y.f(y/x)dy$. Y/X=x)=

Example: 1) Consider the following probabilities given in the table bellow:

y \ x	0	1	2
0	0.46	0.20	0.04
1	0.22	0.06	0
2	0.02	0	0

Compute:

a. E(x/y=2)

b. E(x/y) =0)

c. E(y/x = 2)

Solution:

a) $E(x/y=2) = \sum_{i=0}^{2} x.f(x/y=2)$

$= 0 \times f(x=0/y=2) + 1 \times f(x=1/y=2) + 2 \times f(x=2/y=2)$

$= \frac{0 \times 0.02}{0.02} + \frac{1 \times 0}{0.02} + \frac{2 \times 0}{0.02} = 0$

b) $E(x/y=0) = 0 \times f(0/0) + 1 \times f(1/0) + 2 \times f(2/0)$

$= \frac{0 \times 0.46}{0.70} + \frac{1 \times 20}{0.70} + \frac{2 \times 0.04}{0.70} = \frac{0.74}{0.70} = 1.057$

c) $E(y/x=2) = 0 \times f(2/0) + 1 \times f(2/1) + 2 \times f(2/2)$

$= \frac{0 \times 0.04}{0.04} + \frac{1 \times 0}{0.04} + \frac{2 \times 0}{0.04} = 0$

2) If the joint probability density of x and y is given by:

$$f(x,y)=\begin{cases}\frac{2}{3}(x+2y), for 0<x<1, 0<y<1\\ 0, elsewhere\end{cases}$$

Find the conditional mean of x given $y=\frac{1}{2}$

Solution

i) First calculate the marginal density of y.

$$\text{i.e } h(y)=h(y)=\int_{-\infty}^{\infty} f(x,y)dx=\int_0^1 \frac{2}{3}(x+2y)dx=\frac{1}{3}(1+4y), \quad \text{for } 0<y<1.$$

ii) Find the conditional density of x given Y=y

$$f(x/y)=\frac{f(x,y)}{h(y)}=\frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)}=\frac{2x+4y}{1+4y}, \text{ for } 0<x<1$$

iii. find $f(x/\frac{1}{2})=\frac{2x+4\times\frac{1}{2}}{1+4\times\frac{1}{2}}=\frac{2x+2}{3}$

iv. finally, $E(x/\frac{1}{2})=\int_0^1 \frac{2}{3}x(x+1)dx=\frac{2}{3}\left[\int_0^1 x^2dx+\int_0^1 xdx\right]$

$$=\frac{2}{3}[\frac{x^3}{3}\Big|_0^1+\frac{x^2}{2}\Big|_0^1]=\frac{2}{3}\left[\frac{1}{3}+\frac{1}{2}\right]=\frac{2}{3}\times\frac{5}{6}=\frac{5}{9}$$

Definition: If X is a discrete random variable and f(x/y) is the value of the conditional probability distribution of X given Y = y at x, the conditional expectation of g(x) given Y = y is:

$$E[g(X)] = \sum_x g(x).f(x/y)$$

Correspondingly, if X is a continuous random variable and f(x/y) is the value of the conditional probability density of X given Y = y at x, the conditional expectation of g(X) given Y = y is:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x).f(x/y)dx$$

- Similar expressions based on the conditional probability distribution or density of Y given X = x defined the conditional expectation of h(y) given X = x.

Variance

i. For the discrete random variables X and Y, the conditional variance of X given Y = y is defined as: var $(X/Y\text{-}y) = E\{[X - E(X/Y=y)]^2 / Y = y\}$

$$= \sum [X - E(X/Y=y)]^2 f(x/y),$$

Similarly, the conditional variance of Y given X = x is defined as:

$Var(Y/X) = E\{[Y\text{-}E(y/x)]^2 / X = x\}$

$$= \sum [Y - E(y/x)]^2 f(y/x)$$

ii. For continuous variables, the conditional variances are:

var $(X/Y) = \int_{-\infty}^{\infty} [x - E(x/y)]^2 . f(x/y) dx$, and

var $(Y/X) = \int_{-\infty}^{\infty} [y - E(y/x)]^2 f(y/x) dy$,

Independence and Expectation

if X and Y are independent, their covariance is zero. i.e., if X and Y are independent, then E(XY)=E(X).E(Y), and

Example 1) If the joint probability distribution of X and Y is given by:

$$f(-1,-1) = \frac{1}{6}, f(-1,0) = 0, f(-1,1) = \frac{1}{6}$$

$$f(0,-1) = \frac{1}{3}, f(0,0) = 0, f(0,1) = 0$$

$$f(1,-1) = \frac{1}{6}, f(1,0) = 0, f(1,1) = \frac{1}{6}$$

Find the covariance of the two variables.

Solution: $E(x)=(-1)\times\frac{1}{3}+0\times\frac{1}{3}+1\times\frac{1}{3}=0$

$$E(y)=(-1)\times\frac{2}{3}+0\times 0+1\times\frac{1}{3}=-\frac{1}{3}$$

$$E(xy)=(-1)(-1)\times\frac{1}{6}+0\times(-1)\times\frac{1}{3}+1(-1)\times\frac{1}{6}+(-1)\times 0$$

$$\times 0+0\times 0\times 0+1\times 0\times 0+$$

$$+(-1)\ (1)\times\frac{1}{6}+0\times(1)\times 0+(1)(1)\times\frac{1}{6}=0$$

$$\sigma_{xy}=E(xy)-E(x)E(y)=0-0\times(-\frac{1}{3})=0$$