

Chapter -2

Random variable and probability distribution

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2.1 Discrete random variable and their probability distribution

- **Probably distribution:** This depicts the possible out comes of an experiment and the probability of each of these outcomes.
- **Random variable:** If S is a sample space with a probability measure and X is a valued function defined over the elements of S , and then X is called random variable.
- It is a quantity resulting from an experiment that, by chance, can assume different values.
- Random variable may have two forms which are discrete random variable and continuous random variable.

Discrete random variable (DRV) : A variable that can assume only certain clearly separated values resulting from **a count** of some item of interest or if a random variable X can assume only **a particular finite** or **countable infinite** set of values, it is said to be discrete random variable.

- ✓ Toss a coin n -times and count the number of heads
- ✓ Number of children in a family
- ✓ Number of defective item in given company

Example

1.The number of students earning a grade of A in 1st year economic class. Here, there can be 25A's, but can not be 25.15 A grades.

2.Suppose, when a coin is tossed 5 times, then the number of heads from the given experiment will be:

$X = 0, 1, 2, 3, 4, 5$ (i.e, if we let X be the number of heads, then the values for x are those.)

3. For an experiment in which we roll a pair of dice say die A and die B, observing that the random variable X

X	2	3	4	5	6	7	8	9	10	11	12
$P(X) = X$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1//36

Each of the possible out comes has the probability 1/36.

$S = \{(1, 1), (1,2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1,), (2, 3), (3, 2), (1, 5),$
 $(5, 1), (2, 4), (4, 2), (3, 3), (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), (2, 6),$
 $(6, 2), (3, 5), (5, 3), (4, 4), (3, 6), (6, 3), (4, 5), (5, 4), (5, 5), (4, 6), (6, 4),$
 $(5, 6), (6, 5), (6, 6)\}$

Definition 2.2 If X is a discrete random variable, the function given by $f(x) = P(X=x)$ for each x with in the range of X is called the probability distribution of X.

Theorem 2.1 A function can serve as a probability distribution of a discrete random variable its value $f(x)$, satisfy the conditions:

1. $f(x) \geq 0$ for each value within its domain
2. $\sum_x f(x) = 1$ where the summation extends over all the values within its domain

Example: Consider the experiment of

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2. For the probability of the total number of heads obtained in 4 tosses of a coin. Describe the probability distribution for the given experiment.

- **Solution:** Let the possible outcomes of heads be noted by X the random variable X and its corresponding probabilities by

X	0	1	2	3	4
$P(X)$	1/16	4/16	6/16	4/16	1/16

$$S = \left\{ \begin{array}{l} \text{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,} \\ \text{TTHH, TTHT, TTTH, TTTT} \end{array} \right\}$$

Definition 2.3 If a discrete random variable, the function given $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$, for $-\infty < x < \infty$, where $f(t)$ is the value of the probability distribution of X at t is called the **distribution function or the cumulative distribution of X .**

Theorem 2.2 the values $F(X)$ of the distribution function of a discrete random

- i) $F(-\infty) = 0$ and $F(\infty) = 1$
- ii) If $a < b$, then $F(a) \leq F(b)$, for any real numbers a & b .
- iii) $0 \leq F(x) \leq 1$

- **Example:** Find the distribution function of the total number of heads obtained in five tosses of a coin.

- **Solution:**

- $S = \{ \text{HHHHH, HHHHT, HHHTH, } \dots, \text{TTTTH, TTTT} \}$

we get $f(0) = 1/32$; $f(1) = 5/32$; $f(2) = 10/32$; $f(3) = 10/32$; $f(4) = 5/32$; $f(5) = 1/32$

$$F(0) = f(0) = 1/32$$

$$F(1) = f(0) + f(1) = 1/32 + 5/32 = 6/32 = F(0) + f(1) = \sum_0^1 f(x)$$

$$F(2) = f(0) + f(1) + f(2) = 1/32 + 5/32 + 10/32 = 16/32 = F(1) + f(2) = \sum_0^2 f(x)$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = 1/32 + 5/32 + 10/32 + 10/32 = 26/32 = F(2) + f(3) = \sum_0^3 f(x)$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1/32 + 5/32 + 10/32 + 10/32 + 5/32$$

$$= 31/32 = F(3) + f(4) = \sum_0^4 f(x)$$

$$F(5) = f(0) + f(1) + f(2) + f(3) + f(4) + f(5) = 1/32 + 5/32 + 10/32 + 10/32$$

$$+ 5/32 + 1/32 = 1 = F(4) + f(5) = \sum_0^5 f(x)$$

Hence, the distribution function is:

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{32}, & \text{for } 0 \leq x < 1 \\ \frac{6}{32}, & \text{for } 1 \leq x < 2 \\ \frac{16}{32}, & \text{for } 2 \leq x < 3 \\ \frac{26}{32}, & \text{for } 3 \leq x < 4 \\ \frac{31}{32}, & \text{for } 4 \leq x < 5 \\ 1, & \text{for } x \geq 5 \end{cases}$$

2) Find the distribution function (X) of the random variable of the probabilities given:

$$f(0) = 3/28, f(1) = 15/28; f(2) = 10/28$$

Solution: $F(0) = f(0) = 3/28$; $F(1) = f(0) + f(1) = 3/28 + 15/28 = 18/28$,

$$F(2) = f(0) + f(1) + f(2) = 3/28 + 15/28 + 10/28 = 1$$

$$F(X) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{3}{28}, & \text{for } 0 \leq x < 1 \\ \frac{18}{28}, & \text{for } 1 \leq x < 2 \\ 1, & \text{for } x \geq 2 \end{cases}$$

Theorem 2.3. If the range of a random variable X consists of the values $x_1 < x_2 < x_3 < \dots < x_n$, then $f(x_1) = F(x_1)$, and $f(x_i) = F(x_i) - F(x_{i-1})$, for $i = 2, 3, \dots, n$.

Example:- It is given by

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{32}, & \text{for } 0 \leq x < 1 \\ \frac{6}{32}, & \text{for } 1 \leq x < 2 \\ \frac{16}{32}, & \text{for } 2 \leq x < 3 \\ \frac{26}{32}, & \text{for } 3 \leq x < 4 \\ \frac{31}{32}, & \text{for } 4 \leq x < 5 \\ 1, & \text{for } x \geq 5 \end{cases}$$

is the cumulative distribution function of X

Solution – Using theorem 2.3.

$$f(0) = F(0) = 1/32$$

$$f(1) = F(1) - F(0) = 6/32 - 1/32 = 5/32$$

$$f(2) = F(2) - F(1) = 16/32 - 6/32 = 10/32$$

$$f(3) = F(3) - F(2) = 26/32 - 16/32 = 10/32$$

$$f(4) = F(4) - F(3) = 31/32 - 26/32 = 5/32$$

$$f(5) = F(5) - F(4) = 32/32 - 31/32 = 1/32$$

This is the total number of heads obtained in five tosses of a coin.

2.2. Expected value and variance of discrete random variable

- In order to summarize the behavior of the probability distribution, some of the most common measures include measuring **central tendency and measures of dispersion**.
- And these terminologies, the so called average and dispersion are known as the **expected value** and the **variance** of the probability distribution respectively.
- The mean of a population is the expected value of the population and it is denoted by $E(X)$ or
- Therefore; the mean:
 - Reports the central location of the data
 - Is the long run average value of the random variable in probability distribution
 - Is a weighted average

Defn : If X is discrete random variable and $f(x)$ is the value of its probability distribution at x , the $\mu = E(x) = \sum_x x \cdot f(x)$ value of x is:

Example: A lot of 12 TV sets includes 2 with white cords. If 3 of the sets are chosen at random for shipment to a hotel, how many sets of white cords

Solution - X of the two sets with white cords and $3-x$ of the 10 other sets can be chosen in

ways, 3 of the 12 sets can be chosen in $\binom{12}{3}$ ways and, then $\binom{12}{3}$ possibilities are

equi-probable, we find that the probability distribution of X , the number of sets with white cords shipped to the hotel is given by:

$$f(x) = \frac{\binom{2}{x} \binom{10}{3-x}}{\binom{12}{3}} =, \text{ for } x=0, 1, 2.$$

⊕

x	0	1	2
f(x)	6/11	9/12	1/22

□

$$\text{And then, } E(x) = \sum_{i=0}^2 x \cdot f(x) = 0 \times (6/11) + 1 \times (9/22) + 2 \times (1/22) = 1/2$$

Theorem 2.4. If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x , the expected value of $g(x)$ is given by $E[g(x)] = \sum_x g(x) f(x)$

Example: - (2) If X is the number of points rolled with a balanced die, find the expected value of $g(x) = 2X^2 + 1$.

Solution: Since each possible outcome has the probability $1/6$, we get:

$$E[g(x)] = \sum_{i=1}^6 (2X^2 + 1) \cdot 1/6 = (2 \times 1^2 + 1) \cdot 1/6 + (2 \times 2^2 + 1) \cdot 1/6 + (2 \times 3^2 + 1) \cdot 1/6 + (2 \times 4^2 + 1) \cdot 1/6$$

$$+ (2 \times 5^2 + 1) \cdot 1/6 + (2 \times 6^2 + 1) \cdot 1/6 = 1/6 [3 + 9 + 19 + 33 + 51 + 73] = 188/6 = 94/3$$

1. If a and b are constants then the expected value of $ax+b$ is given by: $E(ax+b) = a E(x) + b$

2. if we set $b = 0$ or $a = 0$, it follows from the above that; If a is constant, then

3. The expectation of the sum of two functions $g(x)$ and $h(x)$ is the sum of the expectations.

$$\text{i.e., } E[g(x) + h(x)] = E[g(x)] + E[h(x)]$$

4. If x and y are two random variables, then $E(x + y) = E(x) + E(y)$

5. If x and y are two independent random variables, then $E(xy) = E(x) \cdot E(y)$

Example

- a. If X is the total number of heads obtained in two tosses of a coin, find the expected value of $g(x) = 3x + 5$.

Solution: $E(3x + 5) = (3 \times 0 + 5) \frac{1}{4} + (3 \times 1 + 5) \frac{2}{4} + (3 \times 2 + 5) \frac{1}{4}$
 $= 3 [0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4}] + 5 = 3 \times \frac{4}{4} + 5 = \frac{32}{4}$

b. Given

X	0	1	2
$f(x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

Find the expected value of $g(x) = 4x^2$

Solution: $E[g(x)] = E(4x^2) = 4 [0^2 \times \frac{3}{28} + 1^2 \times \frac{15}{28} + 2^2 \times \frac{10}{28}]$
 $= 4 [0 + \frac{15}{28} + \frac{5}{7}] = \frac{4 \times 35}{14} = \frac{35}{14}$

c. Given

X	0	1	2	4
$F(x)$	0.11	0.53	0.16	0.20

Solution:

$E(3) = 3 \times 0.11 + 3 \times 0.53 + 3 \times 0.16 + 3 \times 0.20 = 3 \times 0.20 =$
 $3[0.11 + 0.53 + 0.16 + 0.20] = 3$

- **The variance** is the amount of spread/variation of distribution of a random variable.

Definition 2.5. μ_2 is called the variance of the distribution of X , or simply the variance of X ,

and it is denoted by σ^2 , $\text{var}(x)$; σ is the positive square root of the variance, and is called the

standard deviation. Therefore $\mu_2 = E[(x - \mu)^2] = \sum (x - \mu)^2 \cdot f(x)$

Variance of X in terms of expectation

$$\sigma^2 = \mu_2^1 - \mu_2^2$$

Proof $\sigma^2 = E[(X - \mu)^2]$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu E[X] + E[\mu^2] \quad \because E(x^2) = \sum x^2 f(x)$$

$$\because \mu_2^1 = 2\mu \cdot \mu - \mu^2$$

$$= \mu_2^1 - \mu^2$$

Some properties on variances of a discrete random variable

1. If X is a random variable and b is a constant, then the variance of the sum/ difference/ of the random variable X and the constant b is given by :

$$\sigma^2 = \text{Var}(x \pm b) = \text{Var}(x), \text{ where } b \text{ is constant.}$$

Proof:- $E(X \pm b) = E(x) \pm E(b) = E(x) \pm b$

$$\begin{aligned}\text{Var}(x \pm b) &= E[(x \pm b) - (E(x) \pm b)]^2 = E[x \pm b - E(x) \mp b]^2 \\ &= E(x - E(x))^2 = E(x^2) - 2E(x)E(x) + [E(x)]^2 \\ &= E(x^2) - [E(x)]^2 = \text{Var}(x)\end{aligned}$$

$\therefore$ Variance is independent of change of origin.

2. $\text{Var}(ax) = a^2 \text{var}(x)$, where a is constant

Proof:- $\text{var}(ax) = E[ax - E(ax)]^2 = E[ax - aE(x)]^2$

$$= E[a\{x - E(x)\}]^2 = a^2 E[x - E(x)]^2 = a^2 \text{var}(x)$$

$\therefore$ Variance is not independent of change of scale

3. If X is variance σ^2 , then $\text{var}(ax+b) = a^2 \sigma^2$

$$\text{Or } \text{var}(ax \pm b) = \text{var}(ax) = a^2 \text{var}(x)$$

i.e, $\text{var}(ax \pm b) = a^2 \text{var}(x)$,,,,,, by property number 1.

4. $\text{var}(b) = 0$, where b is constant.

$$\textbf{Proof:} - \text{Var}(b) = E[b - E(b)]^2 = E[b - b]^2 = E(0) = 0$$

5. If x and y are independent random variables, then the variance of the sum of X and Y is given by:

$$V(X \pm Y) = V(X) \pm V(Y)$$

6. If a and b are constants, then the variance of $ax+by$ is given by: $\text{Var}(ax+by) = a^2 \text{var}(x) + b^2 \text{var}(y)$ in general, if x and y are two random variables, then the variance of their sum is given by: $\text{var}(x+y) = \text{Var}(x) + \text{Var}(y) + 2 \text{cov}(x,y)$ and $\text{var}(x-y) = \text{Var}(x) + \text{Var}(y) - 2 \text{cov}(x,y)$. Note that $\text{cov}(x,y) = 0$, if x and y are independent.

- **Examples(1)** Calculate the variance of X , representing the number of points rolled with a balanced die.

• **Solution:** $\mu = E(x) = \sum_{x=1}^6 x f(x) = 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6 = 7/2$

$$\mu_2^1 = E(x^2) = 1^2 \times 1/6 + 2^2 \times 1/6 + 3^2 \times 1/6 + 4^2 \times 1/6 + 5^2 \times 1/6 + 6^2 \times 1/6 = 91/6$$

Hence, $\sigma^2 = \mu_2^1 - (\mu)^2 = 91/6 - (7/2)^2 = 35/12$ Or $f(x) = 1/6; X=1, 2, 3, 4, 5, 6$

$$\mu_2 = \sigma^2 = \sum (x - \mu)^2 \cdot f(x)$$

$$= (1 - 7/2)^2 \times 1/6 + (2 - 7/2)^2 \times 1/6 + (3 - 7/2)^2 \times 1/6 + (4 - 7/2)^2 \times 1/6 + (5 - 7/2)^2 \times 1/6 + (6 - 7/2)^2 \times$$

$$1/6 = 1/6 [25/4 + 9/4 + 1/4 + 1/4 + 9/4 + 25/4] = 1/6 \times 70/4 = 35/12.$$

- 2. Calculate the variance of $g(x) = x^2 + 1$, representing the number of points rolled with a balanced die.

Solution: $\text{Var} [g(x)] = E [(x^2+1) - E(x^2+1)]^2 = E [x^2 - E(x^2)]^2$

$$E(x^2+1) = (1^2+1) \frac{1}{6} + (2^2+1) \times \frac{1}{6} + (3^2+1) \frac{1}{6} + (4^2+1) \frac{1}{6} + (5^2+1) \frac{1}{6} + (6^2+1) \frac{1}{6} = \frac{1}{6} [2 + 5 + 10 + 17 + 26 + 37] = \frac{97}{6} = \sum_{i=1}^6 (x^2 + 1) f(x)$$

$$\begin{aligned} E(x^2 - 97/6)^2 &= \sum_{i=1}^6 (x^2 - 97/6)^2 \cdot \frac{1}{6} \\ &= \frac{1}{6} [(1^2 - 97/6)^2 + (2^2 - 97/6)^2 + (3^2 - 97/6)^2 + (4^2 - 97/6)^2 + (5^2 - 97/6)^2 + (6^2 - 97/6)^2] \\ &= \frac{1}{6} [1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4] - (97/6)^2 \\ &= \frac{1}{6} [1 + 16 + 81 + 256 + 625 + 1296] - (9409/36) \\ &= (2275/6) - (9409/36) = (13650 - 9409)/36 = (4241/36) = \text{var}(x^2) \end{aligned}$$

3. Calculate the variance of $g(x) = 2x^2$ for the number of points rolled with a balanced die.

$$\text{Var}(2x^2) = 4 \text{ var}(x^2)$$

Solution: from example 2 we obtained that $\text{var}(x^2) =$

$$= (4241/36), \text{ hence } \text{var}(2x^2) = 4 \text{ var}(x^2) = 4 \times (4241/36) = (4241/9)$$

2.3. Continuous random variables (CRV)

Continuous random variable is a variable that can assume one of an infinitely large number of values

Example:-4 Height of a student

4 Weight of a student

Continuous probability distribution for a continuous variable X is called **the probability density function** and is represented by a curve bounded by two values from the abscissa $X = a$ and $X = b$.

The region bounded by the two values and the curve gives the probability that x lies between the values

a and b

$$\text{I.e. } P(a \leq X \leq b) = \int_a^b f(x) dx$$

- Therefore a function of the above expression can serve as a continuous probability density function of a continuous random variable its value

$$\text{i)} \quad f(x) \geq 0$$

$$\text{ii)} \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{iii)} \quad \int_a^b f(x) dx = P(a \leq X \leq b)$$

As in the discrete case, $F(x)$ is called the cumulative probability of X , you should note the following here too.

$$\text{i.} \quad F(-\infty) = 0, F(+\infty) = 1$$

$$\text{ii.} \quad F(b) - F(a) = \int_a^b f(x) dx = P(a < X < b); \text{ for } a < b.$$

2.4 Expected value and variance of a continuous random variable

- **Definition 2** If X is a continuous random variable and $f(x)$ is the value of its probability density $\mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) dx$ expected

1. A certain coded measurements of an objects diameter of threads of a fitting have the probability density.

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & \text{for } 0 < x < 1, \text{ and} \\ 0, & \text{elsewhere} \end{cases}, \text{ find the expected value of this random variable.}$$

Solution: Using the definition of 2.6

$$\mu = E(x) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)} dx = \frac{4}{\pi} \int_0^1 \frac{x}{1+x^2} dx = \frac{4}{\pi} \cdot \frac{1}{2} \cdot \ln(x^2 + 1) \Big|_0^1 = \frac{\ln 4}{\pi} = 0.44$$

Theorem 2.5 If X is continuous random variable and $f(x)$ is the value of its probability density

at x , the expected value of $g(x)$ is given by: $E[g(x)] = \int_{-\infty}^{\infty} g(x).f(x)dx.$

Example:- If X has the probability density $f(x) = \begin{cases} e^{-x}, & \text{for } x > 0, \text{ and} \\ 0, & \text{elsewhere} \end{cases}$

Find the expected value of $g(x) = e^{4x/5}$

Solution: According to theorem 2.5 we have:

$$E\left[e^{\frac{4x}{5}}\right] = \int_0^{\infty} e^{\frac{4x}{5}} \cdot e^{-x} dx = \int_0^{\infty} e^{\frac{-x}{5}} dx = \frac{1}{\left(\frac{-1}{5}\right)} e^{\frac{-x}{5}} \Bigg|_0^{\infty} = -5 \left[e^{\frac{-\infty}{5}} - e^{\frac{-0}{5}} \right] = -5[0 - 1] = 5$$

Properties of expectation for continuous random variable

- The properties of expectation for discrete random variables work here too.

For instance, if a and b are constant and X is

Proof:- Let $g(x) = ax+b$, then

$$\begin{aligned} E(ax + b) &= \int_{-\infty}^{\infty} (ax + b) \cdot f(x) dx \\ &= a \int_{-\infty}^{\infty} x \cdot f(x) dx + b \int_{-\infty}^{\infty} f(x) dx \\ &= a E(x) + b \end{aligned}$$

Variance of continuous random variable

Definition 2.7. μ_2 - is called the variance of the density of x or it is denoted by

$$\sigma^2_{\text{var}(x)} \quad \therefore \mu = E[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \rightarrow \text{for continuous RV.}$$

Example: If $f(x) = \begin{cases} \frac{1}{x \ln 4}, & \text{for } 1 < x < 3 \\ 0, & \text{elsewhere} \end{cases}$, find the variance

Solution: 1st find $E(x) = \int_1^3 x \cdot \frac{1}{x \ln 4} dx = \frac{1}{\ln 4} \int_1^3 dx = \frac{1}{\ln 4} [x]_1^3 = \frac{2}{\ln 4} = 1.44$

$$\sigma^2 = \frac{1}{\ln 4} \int_1^3 (x - 1.44)^2 \cdot \frac{1}{x} dx = \frac{1}{\ln 4} \left[\int_1^3 x dx - 2.88 \int_1^3 dx + 2.07 \int_1^3 \frac{1}{x} dx \right]$$

Then,

$$= \frac{1}{\ln 4} \left[\frac{x^2}{2} - 2.88x + 2.07 \ln x \right]_1^3 =$$

$$= \frac{1}{\ln 4} \left[\frac{9}{2} - \frac{1}{2} - 2.88x2 + 2.07 \ln 3 \right]$$

$$= 0.72 [4 - 5.76 + 2.27] = 0.72 \times 0.51 = 0.37$$

Moment generating function:

Definition: Let X be a random variable, the moment generating function is defined as be a random variable.

Its moment generating function is defined as $M(t) = \mathbf{E}(e^{tX})$

In the continuous case this becomes

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

In the continuous case this becomes

$$M(t) = \sum_x e^{tx} p(x)$$

Properties of the moment generating function:

Let X be a random variable with probability generating function $M(t)$.

1. M is a positive function with $M(0)=1$

2. If $M(t)$ is finite in an interval $(-\varepsilon, \varepsilon)$ then

- $\mathbf{E}(X) = M'(0)$
- $\mathbf{E}(X^r) = M^{(r)}(0)$ (where $M^{(r)}$ is the r -th derivative).
- $\mathbf{Var}(X) = M^{(2)}(0) - (M'(0))^2$ (where $M^{(2)}$ is the second derivative).