UNIT 3: INTRODUCTION TO LINEAR PROGRAMMING

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3.0 OBJECTIVES AND AIMS

At the end of this chapter students will be able to apply the knowledge of linear programming to solve managerial problems, they will be able to use linear programming models to solve managerial problems having quantitative application.

3.1 INTRODUCTION

Linear programming- is an optimization method which shows how to allocate scarce resources in the best possible way subject to more than one limiting condition expressed in the form of inequalities and /or equations. It

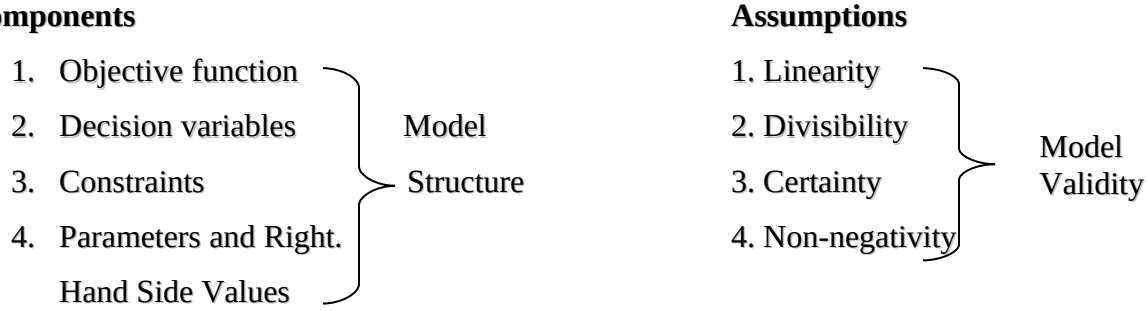
- enables users to find optional solution to certain problems in which the solution must satisfy a given set of requirements or constraints.
- Optimization in linear programming implies either maximization (max) Profit, revenue, sales, market share or minimization (min) Cost, time, distance, or a certain objective function.
- Involves linearly related multi-variety functions i.e. functions with more than one independent variables.
- The goal in linear programming is to find the best solution given the constraints imposed by the problem, hence the term constrained optimization.

We can't max/min two quantities in one model.

3.2 LINEAR PROGRAMMING MODELS

LP models are mathematical representation of LP problems. Some models have a specialized format where as others have a more generalized format. Despite this, LPMs

have certain characteristics in common knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories: Components and assumptions. The components relate to the structure of a model, where as the assumptions describe the conditions under which the model is valid.



3.2.1 Components of LP Model

a) The Objective function: is the mathematical/ quantitative expression of the objective of the company/ model. The objective in problem solving is the criterion by which all decisions are evaluated. In LPMs a single quantifiable objective must be specified by the decision maker. For example, the objective might relate to profits, or costs or market share, best to only one of these. Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time

b) The Decision Variables: represent unknown quantities to be resolved for. These decision variables may represent such things as the:
   - number of units of different products to be sold
   - the # of dollars to invest in various projects
   - the # of ads to place with different media

Since the decision maker has freedom of choice among actions, these decision variables are controllable variables.

c) The constraints: are restrictions which define or limit the attainability (achievability) feasibility of a proposed course of action. They limit the degree to which the objective can be pursued.

A typical restriction embodies scarce resources (such as labor supply, RMs, production capacity, machine time, storage space), legal or contractual requirements (leg. Product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.

d) Parameters- are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as to the numerical value of each constraint.

The components are the building blocks of an LP model. We can better understand their meaning by examining a simple LP model as follows.

Example:

Maximize:

Maximize: $4X_1 + 7X_2 + 5X_3$ (profit)… objective function subject to

$2X_1 + 3X_2 + 6X_3 \leq 300$ labor hrs
$5X_1 + 4X_3 \leq 200$ raw mata. } System Constraints
$3X_1 + 5X_2 + 2X_3 \leq 360$

$X_1 = 30$
$X_2 \geq 40$ } Individual Constraints

Decision Variables

X_1 – qty of product 1
X_2 qty of product 2
X_3 qty of product 3

$X_1, X_2, X_3 \geq 0$ → Non negativity constructs

System constraints- involve more than one decision variables

Individual constraint- involve only one decision variable.

None-negativity constrains- specify that no variable will be allowed to take on a negative value. The non negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

3.2.2 Assumption of LP models

a) Linearity

The linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. Taking the above example,

producing one more unit of products add br 4 to the total profit. This is true over the entire range of possible values of x_1. The same applies (true) to each of the constraints.

b) Divisibility: The divisibility requirement pertains to potential values of decision variables. If is assumed that non-integer values are acceptable. For example: 3.5 TV sets/ hr would be acceptable →7TV sets/ 2hr.

c) Certainty: The parameters are known and constant. The certainty requirement involves two aspects of LP models. The constraint equations do not change.

(1) With respect to model parameters (i.e. the numerical values) –If is assumed that these values are known and constant. Eg. In the above example each unit of product 1 requires 2 labor hours is known and remain constant, and also the 300 labor available is deemed to be known and constant.

(2) All the relevant constraints identified and represented in the model are as they are.

d) Non-negativity- The non-negativity constraint is that negative values of variables are unrealistic and, therefore, will not be considered in any potential solutions, only positive values and zero will be allowed.

3.3 FORMULATING LP MODELS

Once a problem has been defined, the attention of the analyst shifts to formulating a model. Just as it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lend to poor decisions.

Formulating linear programming models involves the following steps:

1) Define the problem/ problems definition: to determine the no. of type 1 and type 2 products to be produced per month so as to maximize the monetary profit given the restriction.
2) Identity the decision variables or represent unknown quantities.
   * Let X_1 and X_2 be the monthly quantities of type 1 and type 2 products.
3) Determine the objective function: Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is a

maximization or a minimization problem and the coefficients of each decision variable.

4) Identify the constraints
   - system constraints- more than one variable
   - individual constraints- one variable
   - non-negativity constraints.

Q(1) Check Your Progress Question

1. A firm that assembles computers computer equipment is about to start production of two new micro computers. Each type of microcomputer will require assembly time, inspection time, and storage space. The amount of each of there resources that can be devoted to the production of micro computers is limited. The mgnager. Of the firm would like to determine the quantity of each micro computer to produce in order to maximize the profit generated by sales of these micro computers.

Additional information

In order to develop a suitable model of the problem, the manager has met with the design and manufacturing personnel. As a result of these meetings, the manager has obtained the following information:

	Type 1	Type 2
Profit per unit	$ 60	$ 50
Assembly time per unit	4hrs	10hrs
Inspection time per unit	2hrs	1hr
Storage space per unit	3 cubic ft	3 cubic ft

The manager also has acquired information on the available company resources. These (weekly) amounts are:

Resource	Resource available
Assembly time	100hrs
Inspection time	22hrs
Storage space	39cubic feet

The manager has also met with the firms marketing manager and learned that demand for the micro computers was such that what ever combination of these two types of micro computers is produced, all of the out put can be sold.

Answer for the above check your progress question

Step 1: Problem definition

- To determine the no. of two types of microcomputers to be produced (and sold) per week so as to maximize the weekly profit given the restrictions.

Step 2: Variable representation

- Let X_1 and X_2 be the weekly quantities of type 1 and type 2 microcomputers respectively.

Step 3: Develop the objective function

Maximize or Z max = $60X_1 + 50X_2$

Step 4: Constraint identification

- System constraints: $4X_1 + 10X_2 \leq 100$hrs Assembly
  $2X_1 + X_2 \leq 22$hrs Inspective
  $3X_1 + 3X_2 \leq 39$ cub Feet Storage
- Individual constraints ….No
- Non-negativity constraint ….$X_1, X_2 \geq 0$

In summary, the mathematical model for the microcomputer problem is:

$$Z \text{ max} = 60X_1 + 50X_2$$

Subject to

$$4X_1 + 10X_2 \leq 100$$
$$2X_1 + X_2 \leq 22$$
$$3X_1 + 3X_2 \leq 39$$
$$X_1, X_2 \geq 0$$

Q(2) Check your progress

2. An electronics firm produces three types of switching devices. Each type involves a two-step assembly operation. The assembly times are shown in the following table:

Assembly time per unit (minutes)

	Station 1	Station 2
Model A	2.5	3.0
Model B	1.8	1.6
Model C	2.0	2.2
	7.5hr	7.5hr

Each workstation has a daily working time of 7.5 hrs. The manager want to obtain the greatest possible profit during the next five working days. Model A yields a profit of br. 8.25 per unit, Model B a profit of br 7.5 per unit and model C a profit of Br 7.8 per unit. Assume that the firm can sell all it produces. During this time, but it must fill outstanding orders for 20 units of each model type.

Required. Formulate the linear programming model of this problem.

Solution for check your progress question number 2.

Step 1: Problem definition: to determine the number of three types of searching devices to be produced to be produced and sold for the next 5 days (working) so as to maximize the 5 days profit.

2. Variable representation

Let X_1, X_2, and X_3 be the number of model A, B and C sketching devices to be produced and sold.

3. Develop objective function

$$Z \max = 8.25X_1 + 7.50X_2 + 7.80X_3$$

4.Constraint identification

$2.5X_1 + 1.8X_2 + 2.0X_3 \leq 450$minutes…Assembly time station 1

$3.0X_1 + 1.6X_2 + 2.2X_3 \leq 450$minutes…. Assembly time station2

} System Constraints

$X_1 \geq 20$ …. Model A

$X_2 \geq 20$ ….Model B

$X_3 \geq 20$….Model C

} Individual Constraints

$X_1, X_2, X_3 \geq 0$ ….no negativity

Summary

$$Z \max = 8.25X_1 + 7.50X_2 + 7.8X_3$$

Subject to: $2.5X_1 + 1.8X_2 + 2X_3 \leq 450$

$$3X_1 + 1.6X_2 + 2.2X_3 \leq 450$$
$$X_1 \geq 20$$
$$X_2 \geq 20$$
$$X_3 \geq 20$$
$$X_1\ X_2\ X_3 \geq 0$$

3.4 SOLUTION APPROACHES TO LINEAR PROGRAMMING PROBLEMS

There are too approaches to solve linear programming problems.

1. The graphic solution method
2. The algebraic solution/ simplex algorithm

3.4.1 The Graphic solution Method

It's a relatively straight forward method for determining the optional solution to certain linear programming problems.

It gives us a clear picture.

This method can be used only to solve problems that involve two decision variables. However, most linear programming applications involve situations that have more than two decision variables, so the graphic approach is not used to solve these.

Example 1. Solving the micro-computer problem with graphic approach.

Z max: $60X_1 + 50X_2$

S.t: $4X_1 + 10X_2 \leq 100$

$2X_1 + X_2 \leq 22$

$3X_1 + 3X_2 \leq 39$

$X_1\ X_2 \geq 0$

Steps

1. Plot each of the constraints and identify its region.
2. Identify the common region, which is all area that contains all of the points that satisfy the entire set of constraints.
3. Determine the optional solution-identify the point which lead to maximum benefit or minimum cost.

$4X_1 + 10X_2 = 100$

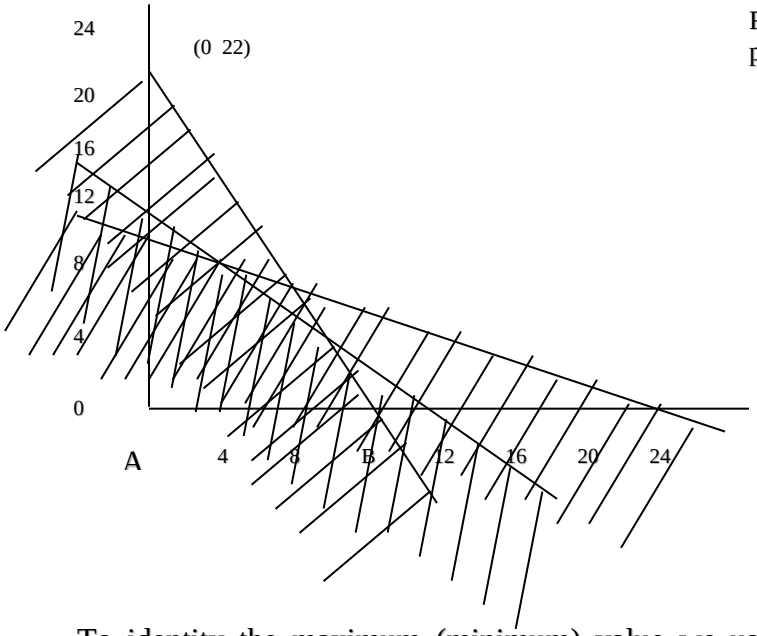
X_1	0	25
X_2	10	0

$2X_1 + X_2 = 22$

X_1	0	11
X_2	22	0

$3X_1 + 3X_2 = 39$

X_1	0	13
X_2	13	0



Region ABCDE is called feasible region

point C = ? $2x_1 + x_2 = 22$ x-3

$3x_1 + 3x_2 = 39$

$-6x_1 + 3x_2 = -66$

$3x_1 + 3x_2 = 39$

$-3x_1 = -27$

$x_1 = -27/-3 = 9$

$2(9) + x2 = 22$

$x_2 = 22 – 18 = 4$

Point D = $3x_1 + 3x2 = 39$ x - 4

$4x1 + 10x2 = 100$x3

$= -12x1 + 12x2 = -156$

$12x1 + 30x2 = 300$

$18x2 = 144$

$x2 = 8$

$3x1 + 3(8) = 39$

$3x1 = 39 – 24$

$x1 = 15/3 = 5$

To identity the maximum (minimum) value we use the corner point approach or the extreme point approach. The corner point/ extreme point approach has one theorem. It states that:

For problems that have optional solutions, a solution will occur at an extreme, or corner point. Thus if a problem has a single optional solution, it will occur at a corner point. If it has multiple optional solutions, at least one will occur at a corner point consequently, in searching for an optional solution to a problem, we need any consider the extreme points because one of those must be optional. Further, determining the value of the objective function at each corner point, we could identify the optional solution by selecting the corner point that has the best value (i.e. maximum or minimum, depending on the optimization case) of the objective function. Extreme points represent interactions of constraints.

Determine the values of the decision variables at each corner point. Some times, this can be done by impaction (observation) and sometimes by simultaneous equation.

Substitute the value of the decision variables at each corner point into the objective function to obtain its value at each corner point.

After all corner points have been so evaluated, select the one with the highest or lowest value depending on the optimization case.

Corner Points	Coordinates X_1	X_2	How determined?	Value of the obi function $Z = 60X_1 + 50X_2$
A	0	0	observation	0 br
B	11	0	observation	660 br
C	9	4	Simultaneous equation	740 br
D	5	8	Simultaneous equation	700 br
E	0	10	Observation	500 br

Basic Solution

$X_1 = 9$ $X_2 = 4$ $Z = 740$ Br.

After we have got the optimal solution, we have to substitute the value of the decision variables into the constraints and check whether all the resources available are used or not. If there is any unused resources we can use it for any other purpose. The amount of unused resource is known as **slack**- the amount of a scarce resource that is unused by a given solution. The slack can range from zero, for a case in which all of a particular resource is used, to the original amount of the resource that was available (i.e. none of it is used.)

Computing the amount of slack

	Amount used	Originally available	unused Amount of slack (Available-used)
Constraint	$X_1 = 9X_2 = 4$		
Assembly	4(9) + 10(4) = 76	100	100 – 76 = 24 hrs
Impective	2(9) 9+ 1(4) = 22	22	22 – 22 = 0 hr
Storage	3(9) + 3(4) = 39	39	39 – 39 = 0 hr

Constraints that have no slack are sometimes referred to as binding constraints since they limit or bind the solution. In the above cases, inspection time and storage space are binding constraints, while assembly time has slack.

Knowledge of unused capacity can be useful for planning. A manager may be able to use the remaining assembly time for other products, or, perhaps to schedule equipment maintenance, safety seminars, training sermons or other activities

Interpretation: The company is advised to produce 9 units of type 1 micro computer and 4 units of type 2 micro computers per week to maximize its early profit to Br. 740, and in doing so the company would be left with unused resource of 24 assembly hrs which can be used for other purposes.

Example 2: Solving the diet problem with graphic approach.

C min = $5X_1 + 8X_2$

$10X_1 + 30X_2 \geq 140$

$20X_1 + 15X_2 \geq 145$

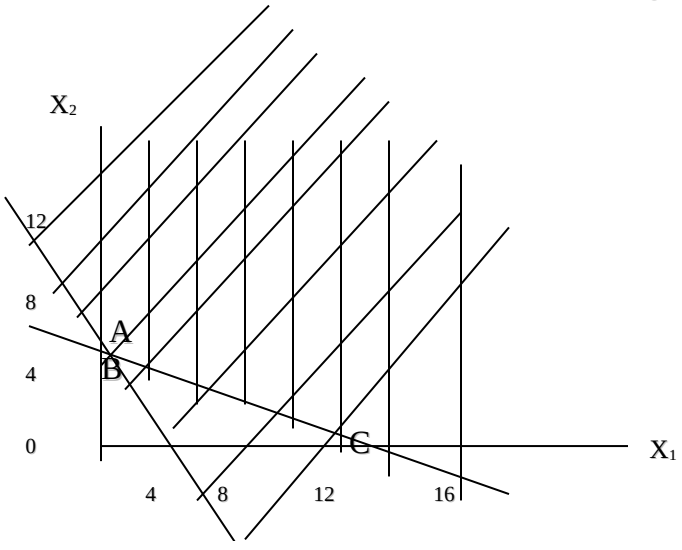
$X_1 , X_2 \geq 0$

$10X_1 + 30X_2 = 140$

X_1	0	14
X_2	14/3	0

$20X_1 + 15X_2 = 145$

X_1	0	+-25
X_2	9.67	0



$20X_1 + 15X_2 = 145$

Points	coordinates X_1	coordinates X_2	How determined	Value of obj Function cmin = $5X + 8X_2$

A	0	9.67	Observation	77.3 br.
B	5	3	Simul. equ^{n}.	49 br.
C	14	0	Observation	70 br.

Basic solution $X_1 = 5$ pounds $X_2 = 3$ pounds C = 49 br.

Interpretation to make the diet the minimum cost of br 49 we have to purchase 5 pounds of type 1 food and 3 pounds type 2 food.

If there is a difference between the minimum required amount and the optimal solution, we call the difference surplus; that is: surplus is the amount by which the optimal solution causes a $\geq$ constraint to exceed the required minimum amount. It can be determined in the same way that slack can: substitute the optimum values of the decision variables into the left side of the constraint and solve. The difference between the resulting value and the original right-hand side amount is the amount of surplus.

Surplus can potentially occur in a $\geq$ constraint.

3.4.2 The Simplex Algorithm/ Algebraic Solution Method

The simplex method is an iterative technique that begins with a feasible solution that is not optimal, but serves as a starting point. Through algebraic manipulation, the solution is improved until no further improvement is possible (i.e. until the optional solution has been identified.) Each iteration moves one step closer to the optional solution.

The optimal solution to a linear programming model will occur at an extreme point of the feasible solution space. This is true even if a model involves more than two variables; optsmal solutions will occur at these point of the feasible solution space; some will be outside of the feasible solution space. Hence, not every solution will be a feasible solution. Solutions which represent infasseetwim of constraints are called basic solutions; those which also satisfy all of the constraints, including the non-negativity constraints, are called basic feasible solutions. The simplex method is an algebraic procedure for systematically examining basic feasible solutions. If an optimal solution exists, the simplex method will identify it. # of basic solution n + mCm $\Rightarrow$ not all basic solutions are feasible.

The simplex procedure for a maximization problem with all $\leq$ constraints consists of the following steps.

1. Write the LPM in a Standard form: When all of the constraints are written as equalities, the LP program is said to be in a standard form. We convert the LPM in to a standard form by applying the slack variables, s, which carries a subscript that denotes which constraint it applies to. For example, s_1 refers to the amount of slack in the first constraint, S_2 to the amount of slack in the second constraint, and so on. When slack variables are introduced to the constraints, they are no longer inequalities b/c the slack variable accounts, they become equalities. Further more, every variable in a model must be represented in the objective function. However, since slack does not provide any real contribution to the objective, each slack variable is a assigned a coefficient of zero in the objective function.

   Slack = Requirement – Production, scruples – Production – Requirement

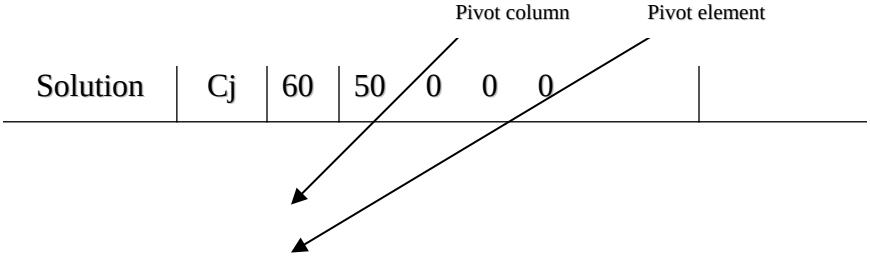
   Taking the microcomputer problem, its standard form is as follows

   $Z \max = 60X_1 + 50X_2$
   $4X_1 + 10X_2 \leq 100$
   $2X_1 + X_2 \leq 22 \Rightarrow$
   $3X_1 + 3X_2 \leq 39$
   $X_1, X_2 \geq 0$

   $Z \max = 60X_1 + 50X_2 + 0S_1 + 0S_2 + 0S_3$
   $4X_1 + 10X_2 + S_1 = 100$
   $2X_1 + X_2 + S_2 = 22$
   $3X_1 + 3X_2 + S_3 = 39$
   All variables ≥ 0
   $(X_1, X_2, S_1, S_2, S_3 \geq 0)$

2. Develop the initial tableau
   a. List the variables across the top of the table and write the objective function coefficient of each variables just above it.
   b. There should be one row in the body of the table for each constraint. List the slack variables in the basis column, one per row.
   c. In the Cj column, enter the objective function coefficient of zero for each slack variable.
   d. Compute values for row Zj.
   e. Computer values for Cj – Zj.

Cj = Coeff of variable J in the obj function.
bj = RHSV of constraint i.
Aij – coefficient of variable j in constraint i



Basis		X_1	X2	S1	S2	S3	RHSV
S1	0	4	10	1	0	0	100
S2	0	2	1	0	1	0	22
S3	0	3	3	0	0	1	39
Zj		0	0	0	0	0	0
Cj - Zj		60	50	0	0	0	

100/4 = 25
leaving
22/2 = 11
39/3 = 13

Entering variable

Initial feasible solution
S_1 = 100 Obtained by
S_2 = 22 equating tow
S_3 =39 Variables to
X_1 = 0 Zero
X_2 = 0 Decision
Z = 0 Variable

3. Develop subsequent tables

   3.1 Identify the entry variable –variable that has a largest positive value in the Cj – Zj row.

   3.2 Identify the leaving variable –using the constraint coefficient or substitution rates in the entering variable column divide each one into the corresponding quantity value. However do not divide by a zero or negative value. The <u>smalls non negative</u> ratio that results indicate which variable will leave the solution

4. Find unique vectors for the new basic variable using row operations on the pivot element.

   -1/2 $R2_1$ –4R_2new + R_1old, -3R_2new + R_3old

Solution Cj		60	50	0	0	0	
Basics		X_1	X_2	S_1	S_2	S_3	RHSV
S1	0	0	8	1	-2	0	56
X1	60	1	½	0	½	0	11
S3	0	0	3/2	0	-3/2	1	6
Zs		60	30	0	30	0	660
Cj – Zj		0	20	0	-30	0	

Cj = bi/xj (aij)
56/8 = 7
11/1/2 = 22
6/3/2 = <u>4</u>

Incoming Variable

Solution basis	Cj	60 X_1	50 X_2	0 S_1	0 S_2	0 S_3	RHSV

S_1	0	0	0	1	6	-16/3	24
X_1	60	1	0	0	1	-1/3	9
X_2	50	0	1	0	-1	2/3	4
Zj		60	50	0	10	40/3	740 → Opportunity cost
Cj - Zj		0	0	0	-10	-40/3	

5. Compute Cj – Zj row
6. If all Cj – Zj Values are zeros and negatives, you have reached optimality
7. If this is not the case (step 6), repeat 2 to 5 until you get optional solution.

   "A simplex solution in a maximization problem in optional if the Cj – Zj row consists entirely of zeros and negative numbers (i.e. there are no positive values in the bottom row.)"

   Note: The variables in solution all have unit vectors in their respective columns for the constraint equations. Further, note that a zero appears in row C – Z in every column whose variable is in solution, in row C – Z in every column whose variable is in solution, indicating that its maximum contribution to the objective function has been realized.

Example 2

A manufacture of lawn and garden equipment makes two basic types of lawn mowers; a push type and a self propelled model. The push type require 9 minutes to assemble and 2 minutes to package; the self-propelled mover requires 12 minute to assemble and 6 minutes to package. Each type has an engine. The company has 12hrs of assembly time available, 75 engines, and 5hrs of packing time profits are Birr 70 for the self propelled model and br 45 for the push type mower per unit.

Required:

1. a. To determine how many units of each type of mower to produce so as to maximize profit.

   b. Let X_1 be push type mower

      X_2 be self propelled mower

   c. Determine the objective function

      Z max = $45X_1 + 70\ X_2$

d. Identify constraints

$9X_1 + 12X_2 \leq 720$ minutes ……assembly time

$2X_1 + 6X_2 \leq 300$ minutes ……..packing time

$X_1 + X_2 \leq 75$ engine ……Engines

$X_1, X_2 \geq 0$

In summary

$Z \max = 45X_1 + 70X_2$

S.t. $9X_1 + 12X_2 \leq 720$

$2X_1 + 6X_2 \leq 300$

$X_1 + 2 \leq 75$

$X_1, X_2 \geq 0$

2. a. Write the Lpm in a standard form

$Z \max = 45X_1 + 70X_2 + 0S_1 + 0S_2 + 0S$

S.T. $9X_1 + 12X_2 + S_1 = 720$

$2X_1 + 6X_2 + S_2 = 300$

$X_1 + X_2 + S_3 = 75$

$X_1 X_2, S_1, S_2, S_3 \geq 0$

b. Develop the initial tableau

Solution basis	C	45 X1	70 X2	0 S1	0 S2	0 S3	RHSV
S1	0	9	12	1	0	0	720
S2	0	2	6	0	1	0	300
S3	0	1	1	0	0	1	75
	Z	0	0	0	0	0	0
	C - Z	45	70	0	0	0	

Qij = Bi/aij

720/12 = 60

300/6 = 50 – Leaving

75/1 = 75

entering

c. Develop the subsequent tableaus

1/6R2, - 1R2new + R3, -12R2new + R1

Solution Basis	C	45 X1	70 X2	0 S1	0 S2	0 S3	RHSV

S1	0	5	0	1	-2	0	120
X2	70	1/3	1	0	1/6	0	50
S3	0	2/3	0	0	-1/6	1	25
	Z	70/30	70	0	70/6	0	3500
	C - Z	65/3	0	0	-70/6	0	

Entering

$1/5R_1$, $-1/3\ R_1$new + R_2old, $-2/3$R, New + R_3 old

Solution Basis	C	45 X_1	70 X_2	0 S_1	0 S_2	0 S_3	RHSV
X_1	45	1	0	1/5	-2/5	0	24
X_2	70	0	1	-1/15	3/10	0	42
S_3	0	0	0	-2/15	1/10	1	9
	Z	45	70	13/3	3	0	4020
	C - Z	0	0	-13/3	-3	0	

Optimal solution

X1 = 24units

X2 = 42units

S1 = 0

S2 = 0

S3 = 9 engine

Z = 4020

$9X_1 + 12X_2 = 720$

X_1	0	80
X_2	60	0

$2X_1 + 6X_2 = 300$

X_1	0	150
X_2	50	0

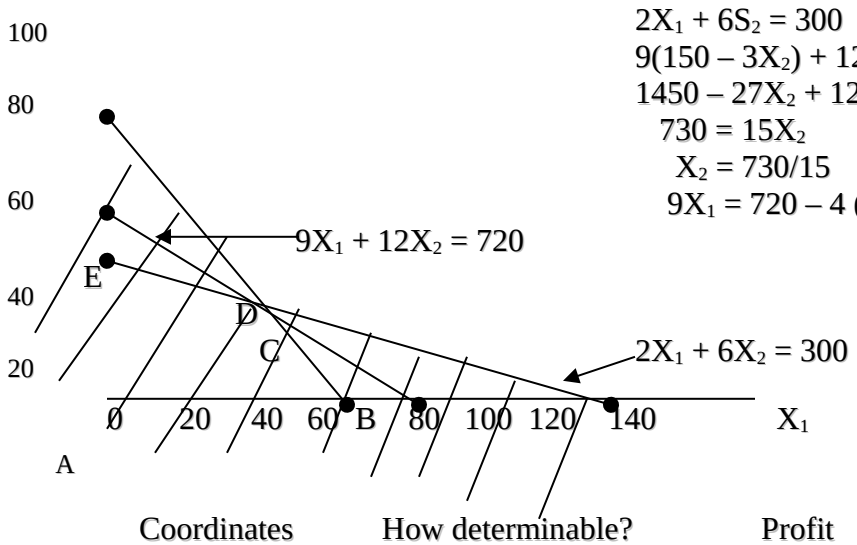
$X_1 + X_2 = 75$

X_1	0	75
X_2	75	0

X_2

120

$9X_1 + 12X_2 = 720$



Points	Coordinates X_1	X_2	How determinable?	Profit $45X_1 + 70X_2$
A	0	0	observation	45(0) + 70(0) = 0
B	75	0	observation	45(75) + 70(0) = 3375
C	60	15	simultaneous	45(60) + 70 (15) = 3750
D	24	42	simultaneous	45 (24) + 70 (42) = 4020
E	0	50	observation	45(0) + 70 (50) = 3500

$X_1 = 24$

$X_2 = 42$

Z = br. 4020

b) 2 (24) + 6 (42) =300

c) 24 + 42 = 75

66

Constraints

a) (24) + 12(42) = 720 75 – 66 = 9

$S_3 = 9$

Interpretation. The company is advised to produce 24 units of push type mower and 40 units of self-propelled mowers so as to realize a profit of Br. 4020. in doing so, the company would be left with unused resource of engine which can be used for other purposes.

Exercises

1. firm manufactures three products which must be processed through some of or all four departments. The table below indicates the number of hours a unit of each product requires in the different departments and the number of pounds of raw materials required. Also listed are the cost per unit, selling price, and weekly capacities of both work-hours and raw materials. If the objective is to maximize total weekly profit, formulate the linear programming model. (Only the model)

Department 1	Product A 3	Product B 4	Product C 2	Weekly Availability 120 hours
Department 2		2	2	100 hours
Department 3	4	1		80 hours
Department 4	2	3	6	150 hours
Pounds of raw Material per unit	5	4	3	250 pounds
Selling price	$ 50	$ 60	$ 65	
Cost / unit	41	40	43	

2. Solve the following linear programming problem

(a) (Use simplex method)

$$Z \text{ max. } 40X_1 + 32X_2$$

Subject to

$$40X_1 + 20X_2 \leq 600$$

$$4X_1 + 10X_2 \leq 100$$

$$2X_1 + 3X_2 \leq 38$$

$$X_1, X_2 \geq 0$$

(b) C min. : 50X1 + 80X2

Subject to

$20X_1 + 30X_2 \geq 1400$

$10X_1 + 40X_2 \geq 1200$

$X_1 , X_2 \geq 0$

(Use the graphic method)

(c) Z max: $X_1 + 1.2X_2 + 2X_3$

Subject to

$X_1 + 2X_2 \leq 150$

$X_1 + 2X_3 \leq 150$

$2X_1 + X_2 \leq 80$

$2X_1 + 3X_2 + X_3 \leq 225$

$X_1X_2X_3 \geq 0$

(Use the simplex method)

Solutions

(1) * Problem: Determine the number of three products (A, B, & C) that must be produced and sold in order to maximize the total weekly profit of the firm, given the different limiting factors.

* Let, X_1 = the number of product A, that must be produced & sold

X_2 = the number of product B, that must be produced & sold

X_3 = the number of product C, that must be produced & sold

* Objective function: profit maximization

$$Z \text{ max} = 9X_1 + 20X_2 + 22X_3$$

* Constraints:

→ Production time constraints: $3X_1 + 4X_2 + 2X_3 \leq 120$

→ Production time constraints $2X_2 + 2X_3 \leq 100$

→ Production time constraints $4X_1 + X_2 \leq 80$

→ Production time constraints $2X_1 + 3X_2 + 6X_3 \leq 150$

→ Raw material constraints: $5X_1 + 4X_2 + 3X_3 \leq 250$

* Non-negative restrictions: $X_1, X_2, X_3 \geq 0$

⇒ Therefore, Z max: $9X_1 + 20X_2 + 22X_3$

Subject to:

$3X_1 + 4X_2 + 2X_3 \leq 120$

$2X_2 + 2X_3 \leq 100$

$4X_1 + X_2 \leq 80$

$2X_1 + 3X_2 + 6X_3 \leq 150$

$5X_1 + 4X_3 + 3X3 \leq 250$

$X_1, X_2, X_3 \geq 0$

(2) a. * Problem: determining the number of bentwood rocking chairs and bentwood coffee tables that must be manufactured in order to maximize the total profit, given the different limiting factors.

* Let, X_1 = the number of bentwood rocking chairs that must be manufactured & sold

X_2 = the number of bentwood coffee tables that must be manufactured & sold

* Objective function: profit maximization

$Z \text{ max} = 40X_1 + 32X_2$

* Constraints

→ Raw material const. (Rosewood): $40X_1 + 20X_2 \leq 600$

→ Wood working hour const. $4X_1 + 10X_2 \leq 100$

→ Finishing hour constraint $2X_1 + 3X_2 \leq 38$

* Non-negative restrictions: $X_1, \quad X_2 \geq 0$

⇒ Therefore, $Z \text{ max} = 40X_1 + 32X_2$

Subject to:

$40X_1 + 20X_2 \leq 600$

$4X_1 + 10X_2 \leq 100$

$2X_1 + 3X_2 \leq 38$

$X_1, X_2 \geq 0$

Standard form

$Z \max = 40X_1 + 32X_2 + OS_1 + OS_2 + OS_3$

Subject to:

$40X_1 + 20X_2 + S_1 = 600$

$4X_1 + 10X_2 + S_2 = 100$

$2X_1 + 3X_2 + S_3 = 38$

$X_1, X_2, S_1, S_2, S_3, \geq 0$

Initial Table

Soln. bases	Cj	40 X_1	32 X_2	0 S_1	0 S_2	0 S_3	Quantity (RHSV)	Q/X1
S_1	0	40	20	1	0	0	600	15
S_2	0	4	10	0	1	0	100	25
S_3	0	2	3	0	0	1	38	19
	Zj	0	0	0	0	0	0	
$\Delta j = Cj - Zj$		40	32	0	0	0		

1/40 R1 / -4R1 new + R2 old / / -2R1 new + R3 old

Soln. bases	Cj	40 X_1	32 X_2	0 S_1	0 S_2	0 S_3	quantity (RHSV)	Q/X2
X_1	40	1	½	1/40	0	0	15	30
S_2	0	0	8	–1/10	1	0	40	5
S_3	0	0	2	–1/20	0	1	8	4
	Zj	40	20	1	0	0	600	
$\Delta j = Cj - Zj$		0	12	–1	0	0		

1/2 R_3 / -1/2R_3 new + R1 old / -8R_3 new + R_3 old

Soln. bases	Cj	40 X_1	32 X_2	0 S_1	0 S_2	0 S_3	(Quantiy) (RHSV)

X_1	40	1	0	3/80	0	–1/4	13
S_2	0	0	0	1/10	1	–4	8
X_2	32	0	1	–1/40	0	½	4
	Zj	40	32	7/10	0	6	648
Δj = cj –zj		0	0	–7/10	0	–6	

⇒ Since, the numbers in Δj = cj – zj row are all 0 & Negative it shows that this solution is optimal solution.

$X_1 = 13$, $X_2 = 4$, $S_2 = 8$ (unused) hours ∴Z max = $ 648

⇒ Proof for Question No. 2 by Graphic Method

* $40X_1 + 20X_2 = 600$ ⟶

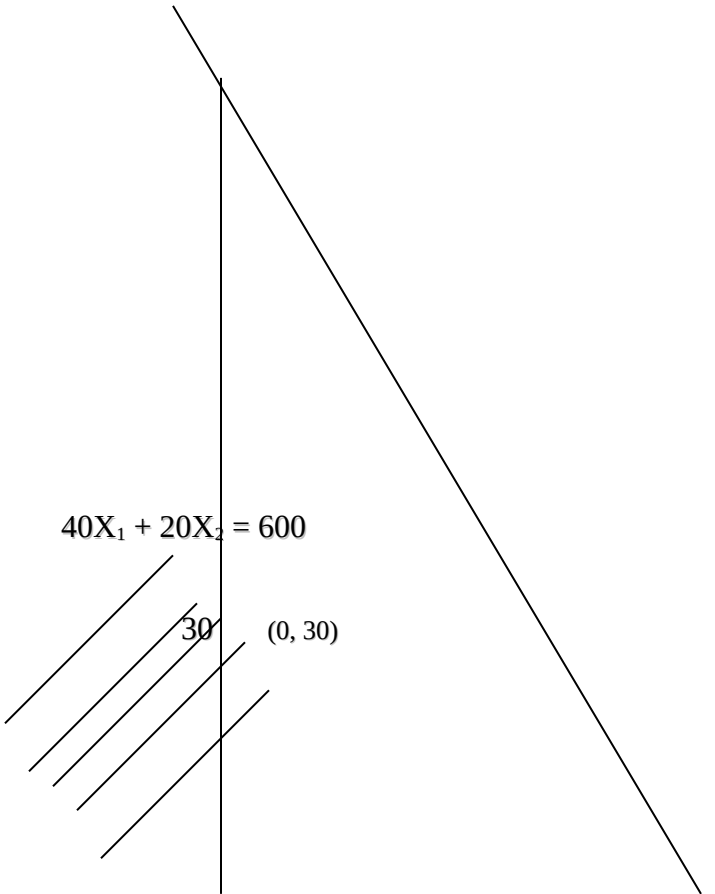
X_1	0	15
X_2	30	0

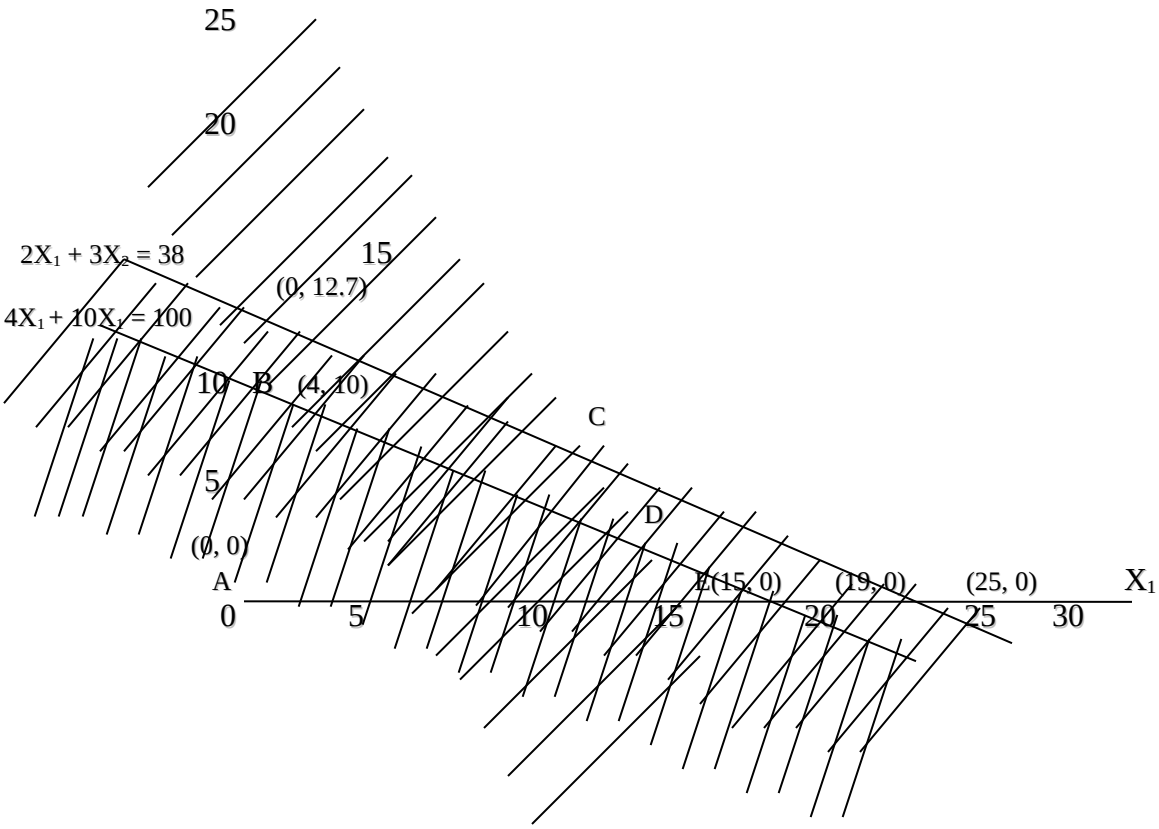
* $4X_1 + 10X_2 = 100$ ⟶

X_1	0	25
X_2	10	0

* $2X_1 + 3X_2 = 38$ ⟶

X_1	0	19
X_2	12.7	0





C.P	X_1	X_2		Value of O.F $40X_1 + 32X_2$	
A	0	0	Observation	0	
B	6	10	Observation	320	
C	10	6	Elimination	592	
D	13	4	Elimination	648	*
E	15	0	Observation	600	

* Point C

$$\begin{cases} 4X_1 + 10X_2 = 100 \\ 2X_1 + 3X_2 = 38 \end{cases} \times -2$$

$4X_1 + 10X_2 = 100$

$-4X_1 - 6X_2 = -76$

$\frac{4X_2}{4} = \frac{24}{4}$

$X_2 = 6$

$4X_1 + 10(6) = 100$

$4X_1 + 60 = 100$

$4X_1 = 100 - 60$

$\frac{4X_1}{4} - \frac{40}{4}$

$X_1 = 10$

Point D

$40X_1 + 20X_2 = 600$

$-20 \quad 2X_1 + 3X_2 = 38$

$40X_1 + 20X_2 = 600$

$-40X_1 - 60X_2 = -760$

$-40X_2 = -160$

$\frac{-40X_2}{-40} = \frac{-160}{-40}$

$X_2 = 4$

$40X_1 + 20(4) = 600$

$40X_1 + 80 = 600$

$40X_1 = 600 - 80$

$\frac{40X_1}{40} = \frac{520}{40}$

$X_1 = 13$

Interpretation

It is advisable for the company to produce 13 units of bentwood rocking chairs and 4 units coffee tables and maximize its profit to birr 648, by doing so the company may be left with 8 unused wood working hours, therefore, the company can use these unused hours for other purposes.

(b) * Problem: determine the number of two types of that must be produced by the two machines (M I & M II) in order to minimize costs of operating the machines.

* Let X_1 = the number of tiers that must be produced by M I

  X_2 = the number of tiers that must be produced by M II

* Objective function : cost Minimization

  $Z \min = 50X_1 + 80X_2$

* Constraints:

→ Grade A tiers constraints: $20X_1 + 30X_2 \geq 1{,}400$

→ Grade B tiers constraints :$10X_1 + 40X_2 \geq 1{,}200$

* Non-negative restrictions $X_1 , X_2 \geq 0$

⇒ Therefore, $Z \min = 50X_1 + 80X_2$

Subject to:

$20X_1 + 30X_2 \geq 1{,}400$

$10X_1 + 40X_2 \geq 1{,}200$

$X_1, X_2 \geq 0$

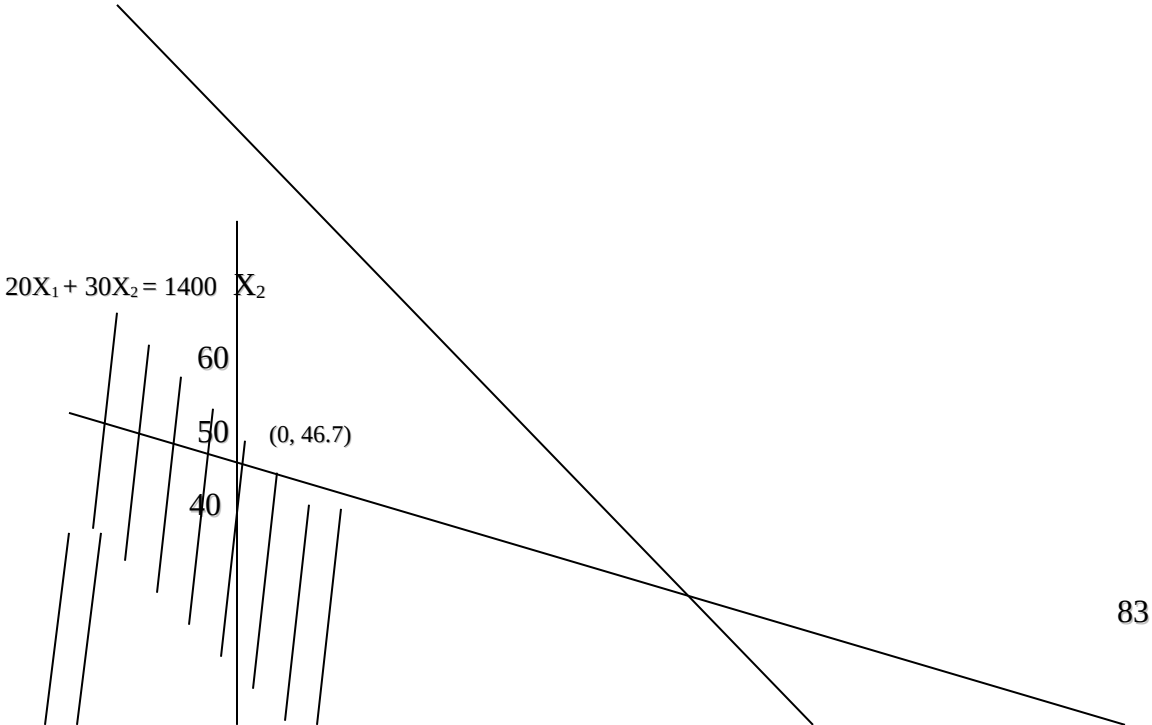
⇒ Solution for Question No. 3 Graphic Method

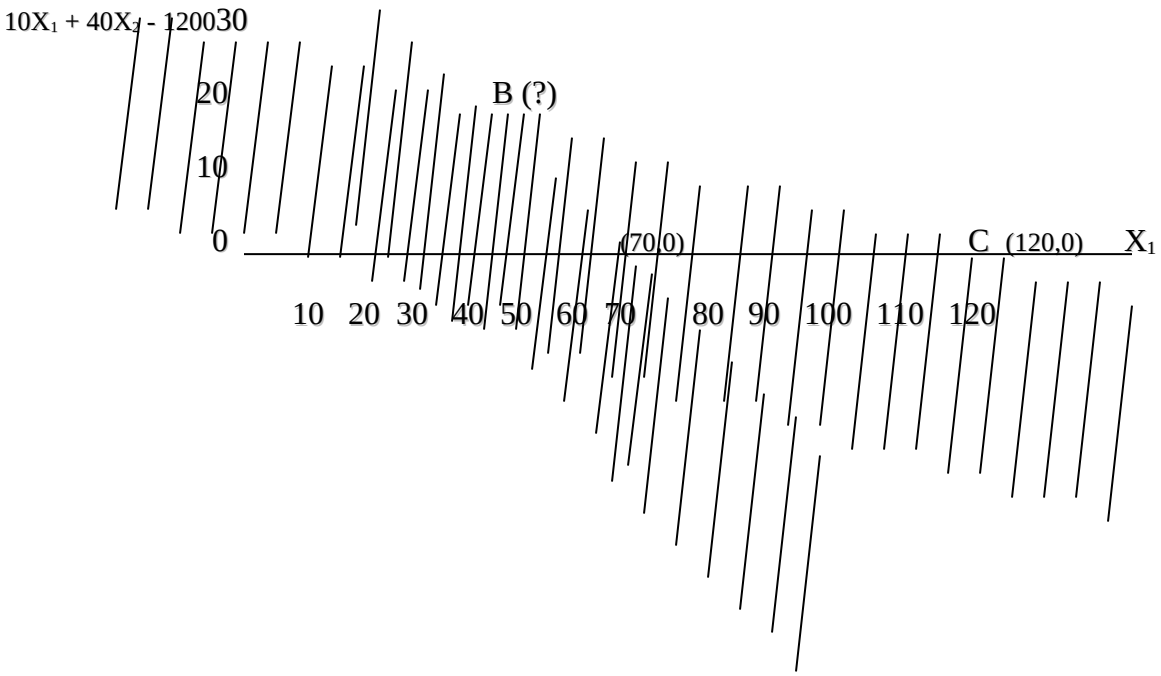
* $20X_1 + 30X_2 = 1{,}400$

X_1	0	70
X_2	46 – 7	0

* $10X_1 + 40X_2 = 1{,}200$

X_1	0	120
X_2	30	0





Corner points	X_1	X_2	How	Value of objective F: $50X_1 + 80X_2$	
A	0	46.7	Obser.	3736	
B	40	20	Elimin.	3600	*
C	120	0	Obser.	6000	

* <u>Point B</u>

$20X_1 + 30X_2 = 1400$

$-2 \quad 10X_1 + 40X_2 = 1200$

$20X_1 + 30X_2 = 1400$

$-20X_1 + 80X_2 = -2400$

⇒ in order to Minimize costs of operating the Machines

M I → Has to produce 40 tires

M II → has to produce 20 tires

$-50X_2 = -1000$ Z min = 3600

50 50

$X_2 = 20$

$10X_1 + 40(20 = 1200$

$10X_1 + 800 = 1200$

$10X_1 = 1200 - 800$

10 ½ R2 / -R2new + R4 old

Soln bases	Cj	1 X_1	1.2 X_2	2 X_3	0 S_1	0 S_2	0 S_3	0 S_4	Quantity (RHSV)	Q/x_2
S_1	0	1	2	0	1	0	0	0	150	∞ 75
X_3	2	½	0	1	0	½	0	0	75	75 ∞
S_3	0	2	1	0	0	0	1	0	80	∞ 80
S_4	0	3/2	3	0	0	-1/2	0	1	150	225 50 →
	Zj	1	0	2	0	1	0	0	150	
$\Delta j = Cj - Zj$		0	1.2	0	0	-1	0	0		

1/3 R4 / -R4 New + R3 old / -2R4 new + R1 old

Soln bases	Cj	1 X1	1.2 X2	2 X3	0 S1	0 S2	0 S3	0 S4	Quantity (RHSV)	
S1	0	0	0	0	1	1/3	0	-2/3	150	
X3	2	½	0	1	0	½	0	0	75	
S3	0	1	0	0	0	1/6	1	-1/3	30	
X2	1.2	1/2	1	0	0	-1/6	0	1/3	50	

Zj	1.6	1.2	2	0	0.8	0	0.4	210	
Δ j = Cj - Zj	-0.6	0	0	0	-0.8	0	-0.4		

⇒ since there is no positive number in Aj = Cj – Z raw we can suggest that we reach the optimum solution therefore, X2 = 50

X3 = 75

S1 = 50

S3 = 30

and Zj = 210