

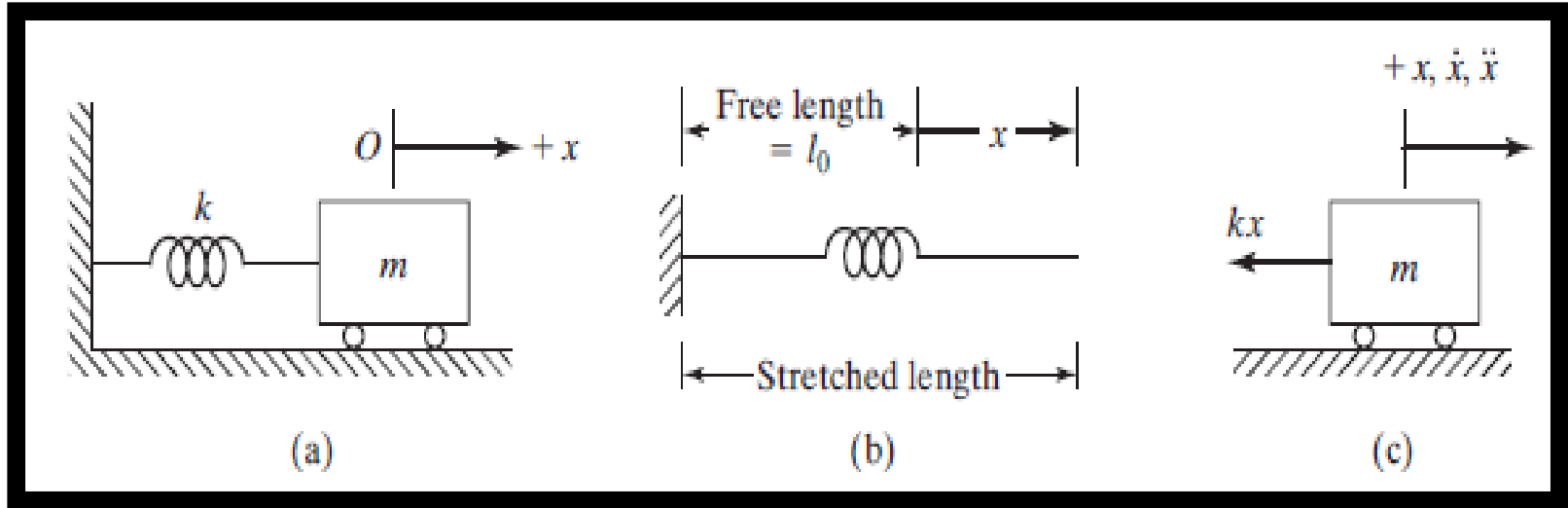
CHAPTER - III

SINGLE DEGREES OF FREEDOM

Free vibration (single dof)

- A system is said to undergo **free vibration** when it oscillates only under an initial disturbance with no external forces acting afterward.
- Some examples are the **oscillations of the pendulum** of a grandfather clock, the vertical oscillatory motion felt by a bicyclist after hitting a road bump, and the motion of a child on a swing after an initial push.
- Figure shows a spring-mass system that represents the simplest possible vibratory system. It is called a **single-degree-of-freedom system**,
- since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be free vibration.

A spring-mass system in horizontal position.



- $Kx = \text{Reactive force}$ $m \ddot{x} = \text{Inertia force}$

From newtons 2nd law of motion

- $\sum F = ma$ $-k x = m \ddot{x}$

- $m \ddot{x} + k x = 0$ Equation of

Contd.,

- **D Alemberts principle:**

The sum of all active and reactive forces minus the inertia force gives the virtual state of equilibrium known as dynamic equilibrium state.

$$\sum F - m\ddot{X} = 0$$

$$-kx - m\ddot{X} = 0$$

$$m\ddot{X} + kx = 0$$

Principle of conservation of energy

- A system is said to be conservative if no energy is lost due to friction or energy-dissipating non elastic members.
- If no work is done on a conservative system by external forces (other than gravity or other potential forces)
- Then the total energy of the system remains constant. Since the energy of a vibrating system is partly **potential and partly kinetic**, the **sum of these two energies remains constant**.
- The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic

Contd.,

- Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant}$$

$$\frac{d}{dt}(T + U) = 0$$

The kinetic and potential energies are given by

$$T = \frac{1}{2}m\dot{x}^2$$

and

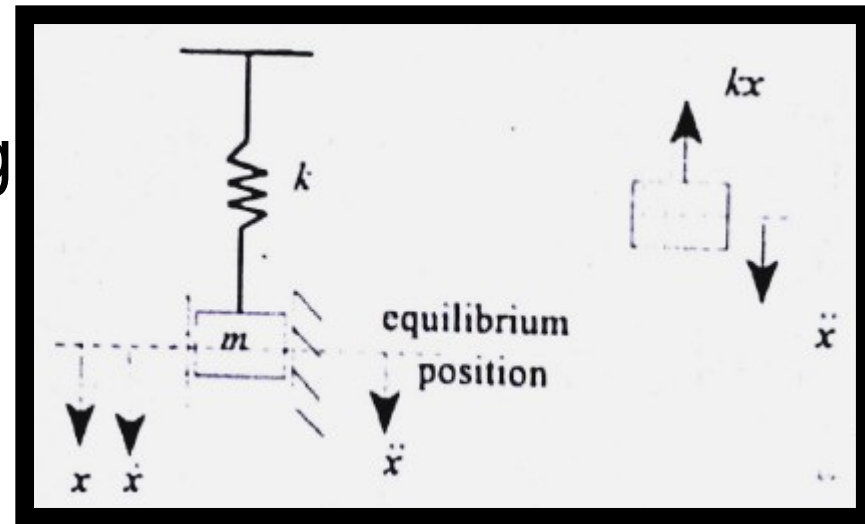
$$U = \frac{1}{2}kx^2$$

$$m\ddot{x} + kx = 0$$

Response of un damped free vibration

- Force acting on the mass due to the spring (the motion resisting force)

$$m\ddot{x} + kx = 0;$$
$$\ddot{x} + \frac{k}{m}x = 0.$$



- For the above differential equation we have three possible solutions.

$$X=Ae^{st}, \quad X=A \sin \omega t, \quad X=B \cos \omega t$$

- Also any combination of these solution can be a solution, the most general form of solution for this equation will be

Contd.,

- The most general form of solution for this equation will be
- $X = A \sin \omega t + B \cos \omega t$
- Substitute the differential equation

$$\ddot{x} + \frac{k}{m}x = 0.$$

$$-A\omega^2 \sin \omega t - B\omega^2 \cos \omega t + \frac{k}{m}(A \sin \omega t + B \cos \omega t) = 0;$$

$$A\left[\frac{k}{m} - \omega^2\right] \sin \omega t + B\left[\frac{k}{m} - \omega^2\right] \cos \omega t = 0;$$

$$\left[\frac{k}{m} - \omega^2\right][A \sin \omega t + B \cos \omega t] = 0;$$

$$\left[\frac{k}{m} - \omega^2\right] x = 0.$$

Contd.,

- This equation must hold true for all values of x . the displacement x cannot be zero to satisfy the equation, the term $(k/m - \omega^2)$ must be equal to zero

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

- The frequency given by the above equation is said to be the un damped natural frequency.

DAMPED FREE VIBRATION

- In many practical systems, the vibrational energy is gradually converted to heat or sound.
 - Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases.
 - The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping.
 - Damping is modelled as one or more of the following types.
-
- **Viscous Damping**
 - **Coulomb or Dry-Friction Damping**
 - **Material or Solid or Hysteretic Damping**

Damping Elements

□ **Viscous** Damping:

Damping force is proportional to the velocity of the vibrating body in a fluid medium such as air, water, gas, and oil.

□ **Coulomb** or **Dry Friction** Damping:

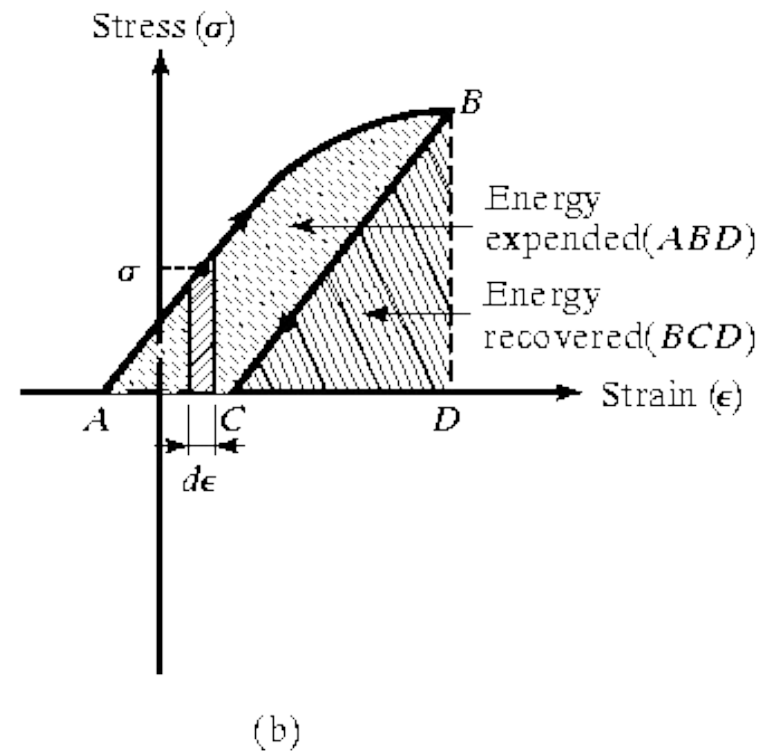
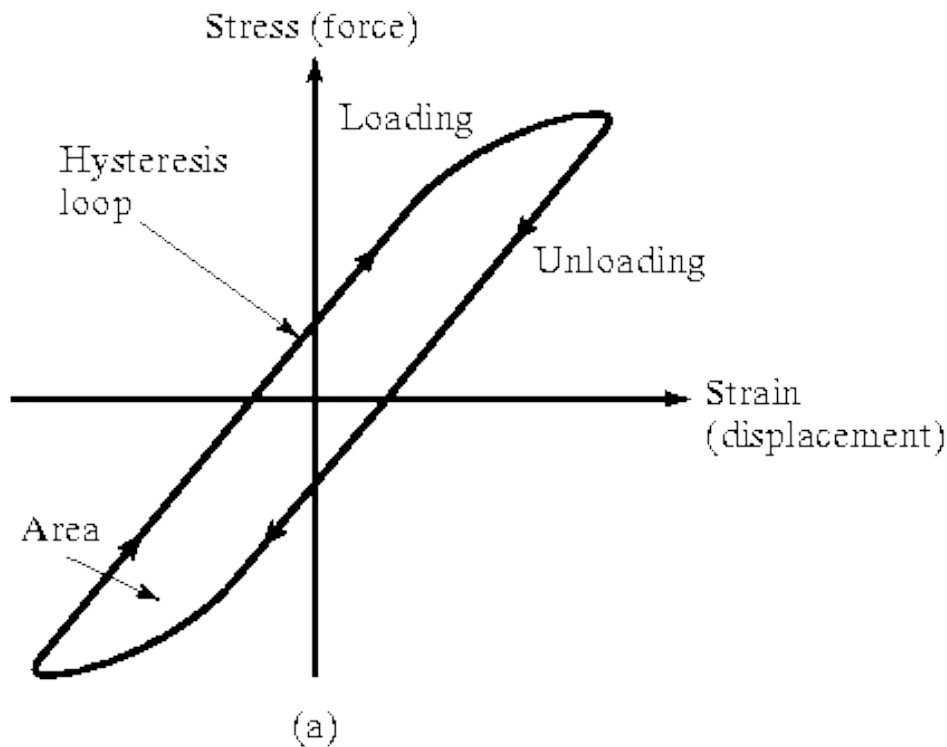
Damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body between dry surfaces.

□ **Material** or **Solid** or **Hysteretic** Damping:

Energy is absorbed or dissipated by material during deformation due to friction between internal planes.

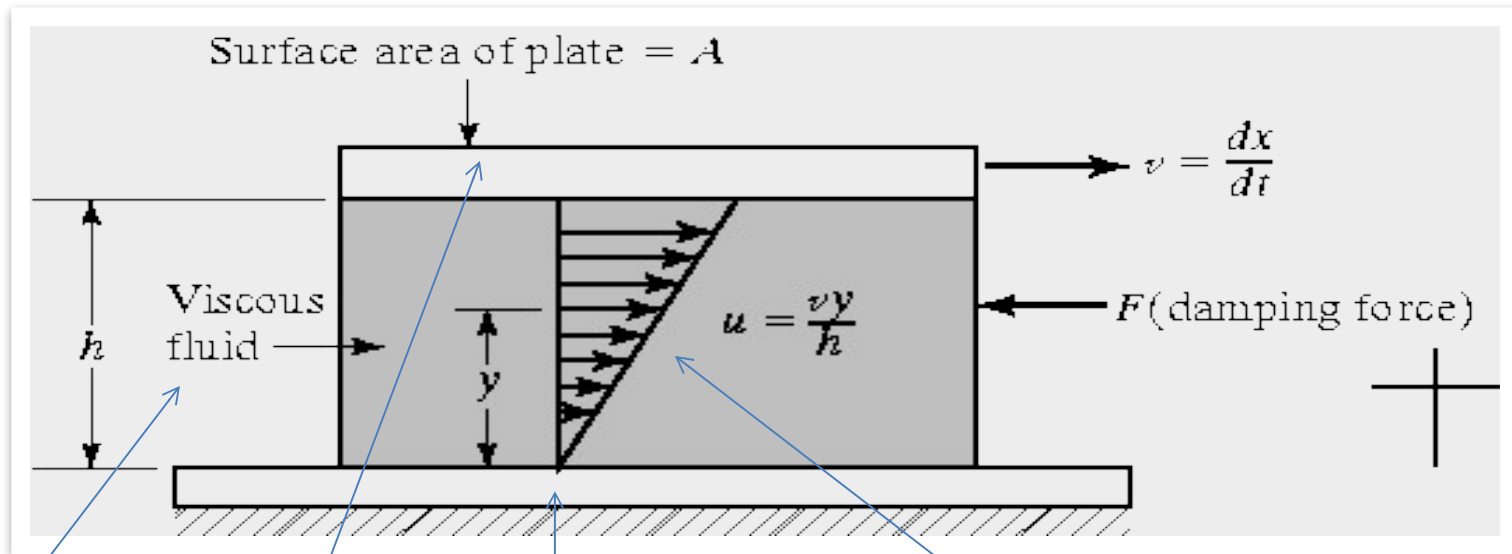
Damping Elements

Hysteresis loop for elastic materials:



Damping Elements

Construction of Viscous Dampers



μ

Fixed plane

Velocity of intermediate fluid layers are assumed to vary linearly

Plate be moved with a velocity v in its own plane

Damping Elements

- *Shear Stress (τ)* developed in the fluid layer at a distance y from the fixed plate is:

$$\tau = \mu \frac{du}{dy} \quad (1.26)$$

where $du/dy = v/h$ is the velocity gradient.

- *Shear or Resisting Force (F)* developed at the bottom surface of the moving plate is:

$$F = \tau A = \mu \frac{Av}{h} = cv \quad (1.27)$$

where A is the surface area of the moving plate and $c = \frac{\mu A}{h}$ is the damping constant.

Damping Elements

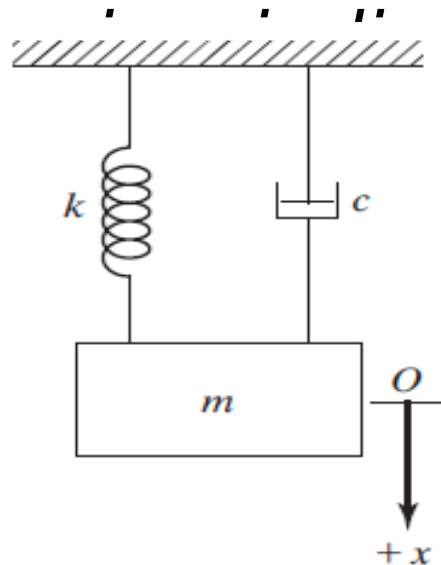
- If a damper is **nonlinear**, a linearization process is used about the operating velocity (v^*) and the equivalent damping constant is:

$$c = \left. \frac{dF}{dv} \right|_{v^*}$$

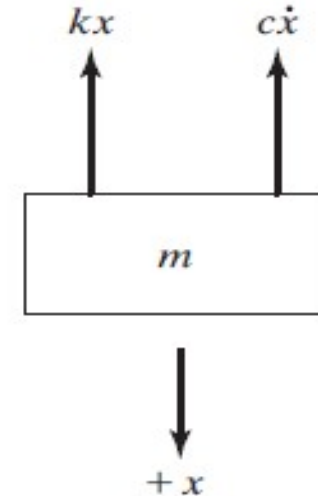
Free Vibration with Viscous Damping

- The viscous damping force F is proportional to the velocity $\dot{x}$ or v can be expressed by $F = -c\dot{x}$

- where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity



System



Free-body diagram

Contd.,

- The equation of motion is then
- Newton's second law $\Sigma F = ma$

$$m\ddot{x} = -c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

- We assume that solution in the form of

$$x(t) = Ce^{st}$$

- Insert this equation into the previous equation $ms^2 + cs + k = 0$

- Tr

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Contd.,

- The roots gives the two solutions
- The most general solution in the combination of these solutions

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

•

$$= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t}$$

- C1 & C2 are arbitrary constants to be find from initial conditions of the system

Critical damping constant and damping ratio :

- The critical damping is defined as the value of the damping constant C_c *for which the radical in Eq. becomes zero:*

Contd.,

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

- For any damped system the damping ratio ζ is defined as the ratio of damping constant to critical damping constant $\zeta = c/c_c$
- We can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n$$

- And hence

$$s_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

Contd.,

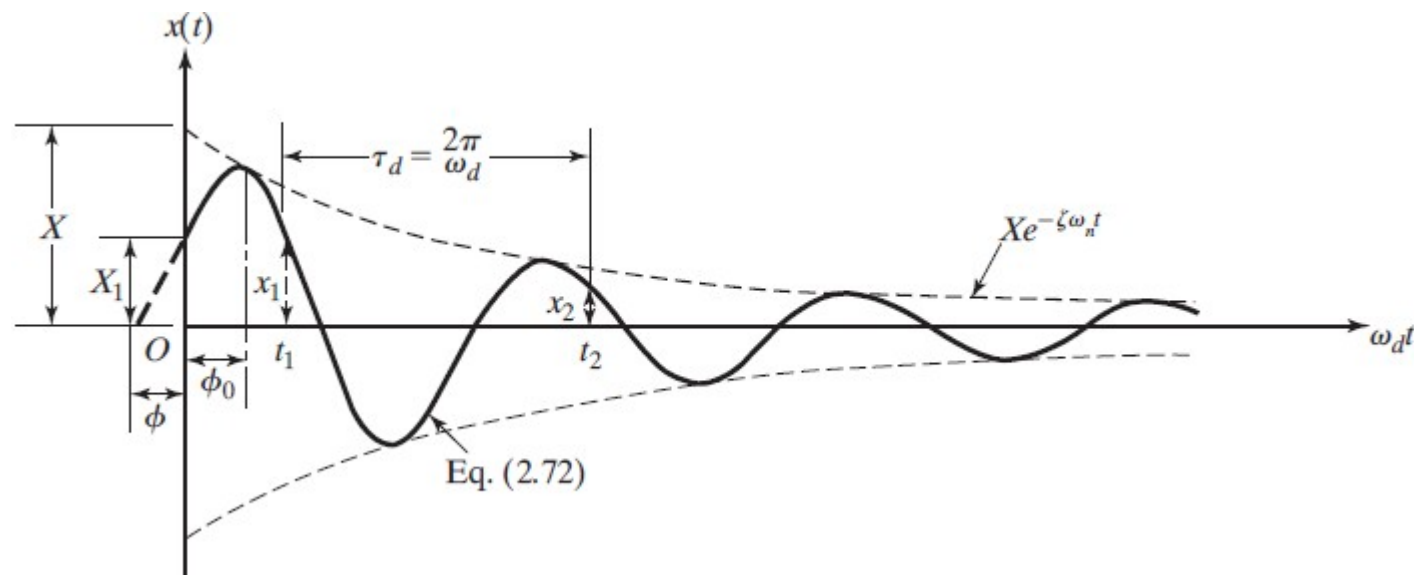
Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = (-\zeta + i\sqrt{1 - \zeta^2})\omega_n$$

$$s_2 = (-\zeta - i\sqrt{1 - \zeta^2})\omega_n$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1 - \zeta^2} \omega_n$, but because of the factor $e^{-\zeta \omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n$$



Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. (2.68) are equal:

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6]¹

$$x(t) = (C_1 + C_2 t) e^{-\omega_n t}$$

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$C_1 = x_0$$

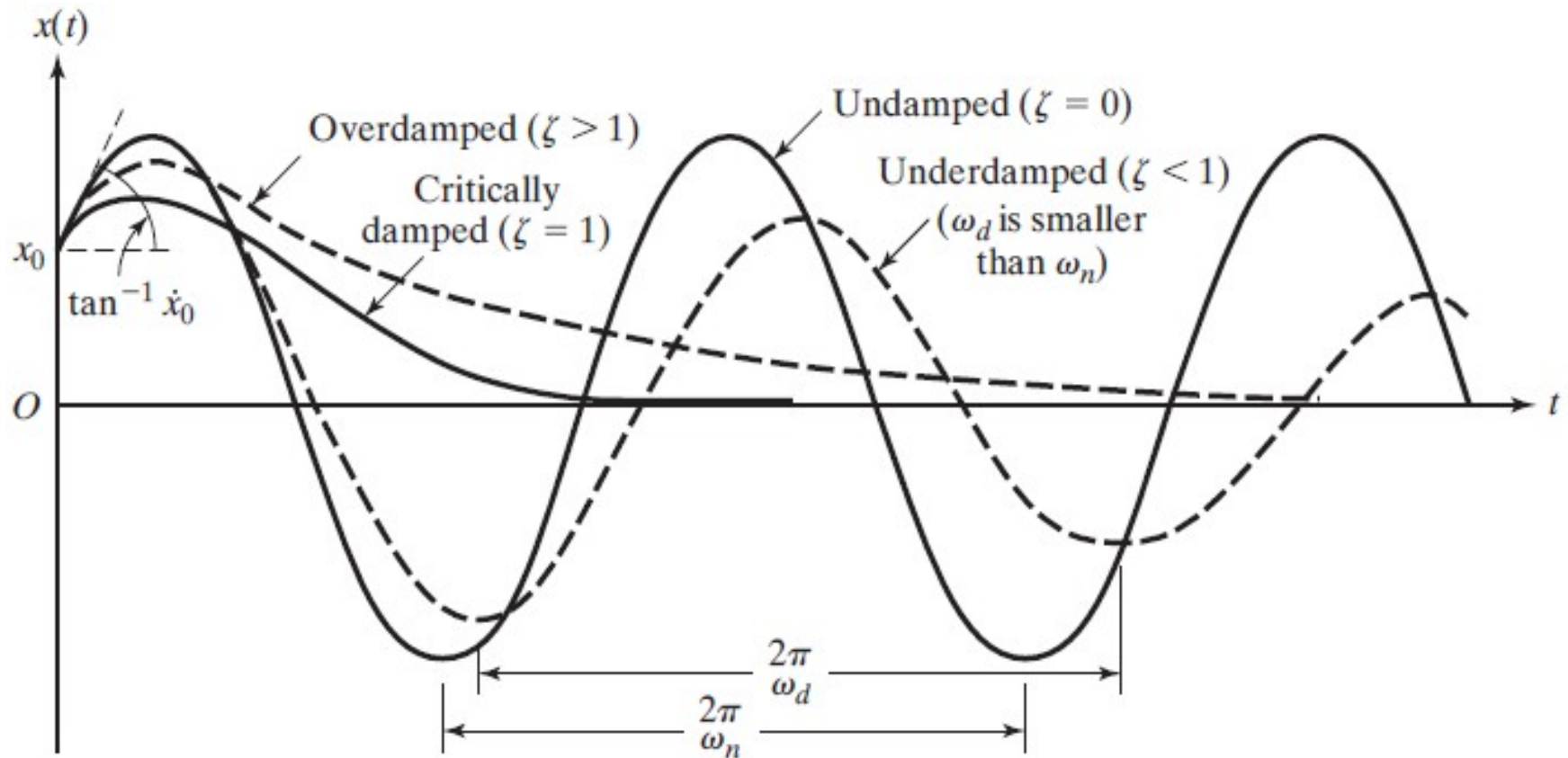
$$C_2 = \dot{x}_0 + \omega_n x_0$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t] e^{-\omega_n t}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \rightarrow 0$ as $t \rightarrow \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

Comparison of motions with different types of damping



Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0$$

$$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0$$

with $s_2 \ll s_1$. In this case, the solution, Eq. (2.69), can be expressed as

logarithmic decrement

- The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases.
- It is defined as the natural logarithm of the ratio of any two successive amplitudes.
- Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes

Contd.,

But $t_2 = t_1 + \tau_d$, where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence
 $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}$$

The logarithmic decrement δ can be obtained from Eq.

$$\delta = \ln \frac{x_1}{x_2} = \zeta\omega_n \tau_d = \zeta\omega_n \frac{2\pi}{\sqrt{1 - \zeta^2}\omega_n} = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} = \frac{2\pi}{\omega_d} \cdot \frac{c}{2m}$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi\zeta \quad \text{if} \quad \zeta \ll 1$$

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ . Once δ is known, ζ can be found by solving Eq. (2.85):

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ . Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2 . By taking the natural logarithm of the ratio of x_1 and x_2 , we obtain δ . By using Eq. (2.87), we can compute the damping ratio ζ . In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$, where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2} \frac{x_2}{x_3} \frac{x_3}{x_4} \cdots \frac{x_m}{x_{m+1}}$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\begin{aligned} \frac{x_j}{x_{j+1}} &= e^{\zeta\omega_n\tau_d} \\ \frac{x_1}{x_{m+1}} &= (e^{\zeta\omega_n\tau_d})^m = e^{m\zeta\omega_n\tau_d} \end{aligned}$$

Equations (2.91) and (2.85) yield

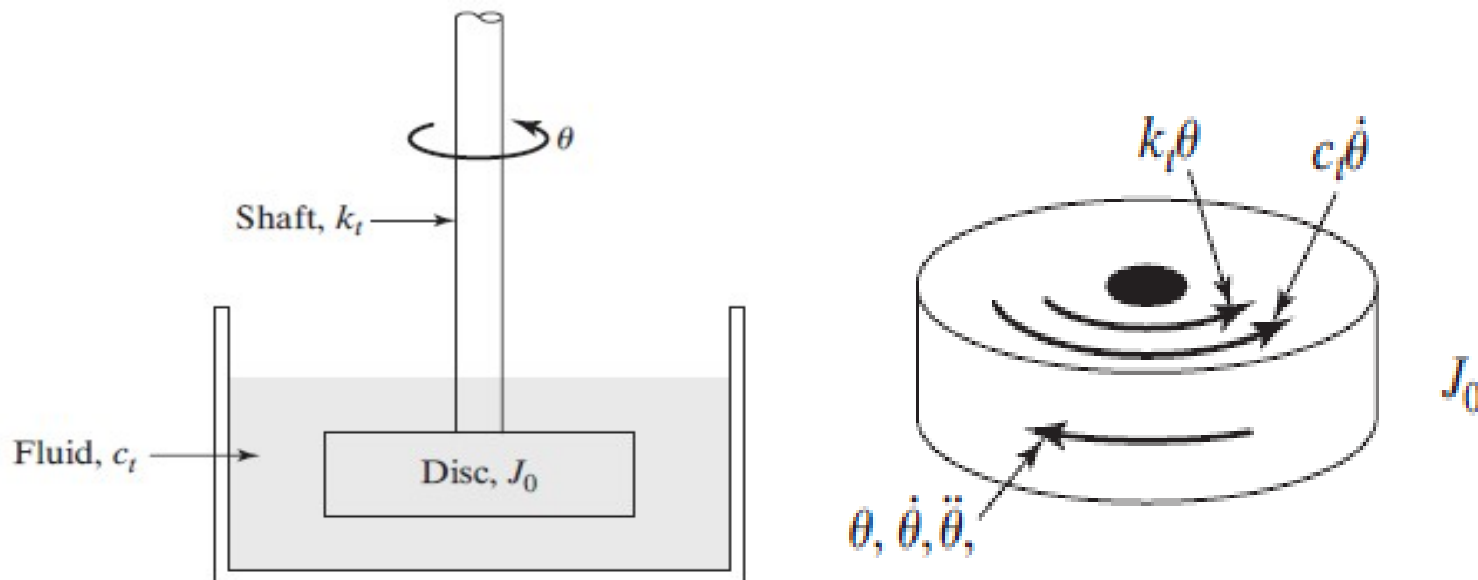
$$\delta = \frac{1}{m} \ln \left(\frac{x_1}{x_{m+1}} \right)$$

Torsion system with viscous damping

- The viscous damping torque for a single dof torsional system $T = -c_t \dot{\theta}$
- Where c_t is the torsional viscous damping
- The equation of motion is

$$J_0 \ddot{\theta} + c_t \dot{\theta} + k_t \theta = 0$$

where J_0 = mass moment of inertia of the disc, k_t = spring constant of the system (ring torque per unit angular displacement), and θ = angular displacement of the disc



$$\omega_d = \sqrt{1 - \zeta^2} \omega_n$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}}$$

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}}$$

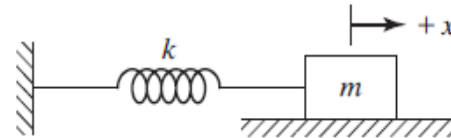
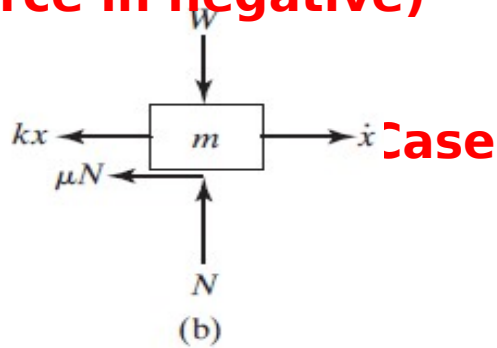
where c_{tc} is the critical torsional damping constant.

Free vibration with coulomb damping

- In many mechanical systems, Coulomb or dry-friction dampers are used because of their mechanical simplicity and convenience.
- Coulomb's law of dry friction states that, when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact.
$$F = \mu N = \mu W = \mu mg$$
- Thus the friction force F is given by

- The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called constant damping.
- since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.
- Consider a single-degree-of-freedom system with dry friction as shown in Fig.
- Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figs.

Case (1): SPRING MASS WITH COULOMB DAMPING (When Frictional force in negative)



$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N$$

Case (2): (When Frictional force in positive)

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N$$

