
UNIT 7: TRANSPORTATION AND ASSIGNMENT PROBLEMS

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7.0 AIMS AND OBJECTIVES

After completing this unit, you should be able to:

- describe the nature of transportation and assignment problems
- use the transportation method to solve problems manually
- deal with special cases of transportation problems
- use the assignment method to solve problems manually
- deal with special cases of assignment problems

7.1 INTRODUCTION

Certain types of linear programming problems can be solved using special purpose algorithms instead of using the simplex method. Prior to the wide spread use of computers to solve LP problems, special purpose algorithms were particularly useful because they enabled practitioners to obtain solutions to these special cases with much less computational burden than simplex would have required. Now, these special purpose algorithms provide further insight into LP problems and their solution.

This unit describes two special purpose algorithms: the transportation method and the assignment method. Model formulation, and manual solution of models.

7.2 TRANSPORTATION PROBLEMS

The transportation method is usually applied to distribution type problems, in which supplies of goods that are held at various locations are to be distributed to other receiving locations. For example, a company may have 10 warehouses that are used to supply 50 retail stores. Obviously, there are many different combinations of warehouse store supply lines that could be used. Generally, some of these combinations will involve transportation costs that are higher than others. The purpose of using an LP model would be to identify a distribution plan that would minimize the cost of transporting the goods from the warehouses to the retail stores, taking into account warehouse supplies and store demands as well as transportation costs. Other examples of transportation problems include shipments from factories to warehouses, shipments between departments within a company, and production scheduling. Moreover, some companies use the transportation method to compare location alternatives (i.e., to decide where to locate factories and warehouse in order to achieve the minimum cost distribution configuration).

Formulating the Model

A transportation problem typically involves a set of sending locations, which are referred to as origins, and a set of receiving locations, which are referred to as *destinations*. In order to develop a model of a transportation problem, it is necessary to have the following information:

1. Supply quantity (capacity) of each origin.
2. Demand quantity of each destination.

3. Unit transportation cost for each origin destination route.

The transportation algorithm requires the assumption that all goods be homogeneous, so that any origin is capable of supplying any destination, and the assumption that transportation costs are a direct linear function of the quantity shipped over any route. We shall add one additional requirement that will simplify the problem: the total quantity available for shipment is equal to the total quantity demanded. Later in the unit we shall see how the problem can be modified to handle cases where this assumption is not met.

Let's consider an example. ABC's Sand and Gravel Pit has contracted to provide topsoil for three residential housing developments. Topsoil can be supplied from three different "farms" as follows:

<u>Farm</u>	Weekly Capacity (Cubic yards)
A	100
B	200
C	200

Demand for the topsoil generated by the construction project is:

<u>Project</u>	Weekly Capacity (Cubic yards)
1	50
2	150
3	300

The manager of the sand and gravel pit has estimated the cost per cubic yard to ship over each of the possible routes:

<u>From</u>	Cost per Cubic Yard to		
	Project #1	Project #2	Project #3
Farm A	\$4	\$2	\$8

		1	
Farm B	5	6	9
Farm C	7		3

This constitutes the information needed to solve the problem. The next step is to arrange the information into a transportation table. This is shown in Table 7-1. The origins (farms) are listed down the left side of the table and their respective supply quantities are listed down the right side of the table. The destinations (projects) are listed across the top of the table, and their respective demands are listed across the bottom of the table. The unit shipping costs are shown in the upper right hand corner of each cell, which represents a shipping route. Hence, the cost per cubic yard to ship topsoil from Farm A to Project # 1 is \$4. (For convenience, dollar signs are not shown.)

To From:				Supply
	Project #1	Project #2	Project #3	
Farm A	4	2	8	100
Farm B	5	1	9	200
Farm C	7	6	3	200
Demand	50	150	300	

Table 7-1 Transportation Table for ABC's Sand and Gravel

Overview of the Transportation Method

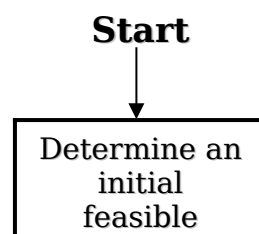
The transportation method is similar in certain respects to the simplex technique because both involve an initial feasible solution that is

evaluated to determine if it can be improved. Moreover, both involve displaying initial and improved solutions in a series of tableaus or tables. However, as noted earlier, the transportation method requires considerably less computational effort. Moreover, it is not unusual to discover that the initial feasible solution in a transportation problem is the optimum. Figure 7-1 gives an overview of the transportation method.

A solution to a transportation problem consists of quantities that are assigned to the various routes (i.e., cells in the table). These values can range from zero, which implies that no units will be shipped over that route, to a maximum that equals the smaller of two quantities: the row (supply) and column (demand) totals. The logic of the maximum quantity is simple: The quantity shipped cannot exceed the available supply in a row, and it should not exceed the amount demanded (column total). Such a solution is a feasible solution. The starting point of the transportation method is a feasible solution.

Finding an Initial Feasible Solution

A feasible solution is one in which assignments are made in such a way that all supply and demand requirements are satisfied. In general, the number of nonzero (occupied) cells should equal one less than the sum of the number of rows and the number of columns in a transportation table. In the case of a table with 3 rows and 3 columns, the number of occupied cells should be $3 + 3 - 1 = 5$ in order to be able to use the transportation algorithm. Sometimes, fewer occupied or completed cells appear in a solution. When that happens, the solution is referred to as a degenerate solution; such a solution requires modification in order to be able to determine if it is optimal. This topic is covered later in the unit.



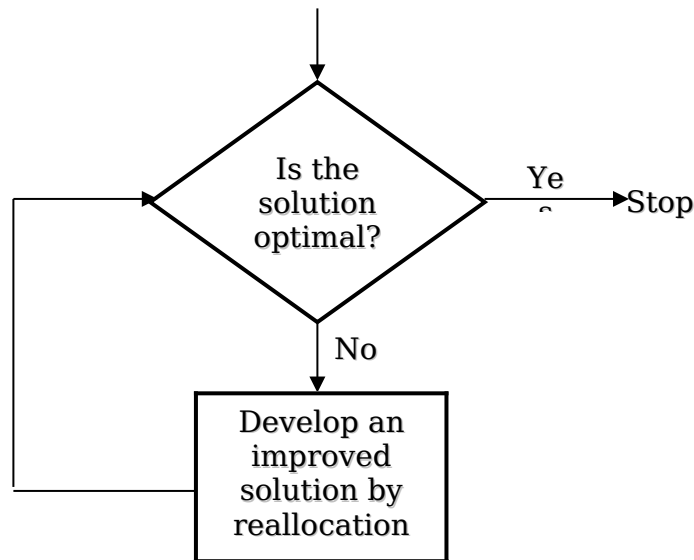


Figure 7-1 Overview of the Transportation Method

Aside from a degenerate case, the transportation method will generate solutions that have the number of completed cells equal to the number of rows plus number of columns minus one. Other feasible solutions that use more of the cells are imaginable, but not really desirable. For instance, it may be possible to concoct a solution that uses all of the cells. However, the cell costs are marginal costs to transport single units. Often, there are fixed costs associated with the number of cells (routes) used. For example, in the ABC problem, consider that, undoubtedly, each route requires a separate truck, which represents, perhaps, a high fixed cost. Therefore, it would be desirable to devise a solution that used as few routes as possible. In general, such a solution will require the above mentioned number of completed cells.

A number of different approaches can be used to find an initial feasible solution. Two of these are described here:

1. The northwest corner method
2. An intuitive approach.

Finding an Initial Feasible Solution: The Northwest-Corner Method

The northwest-corner method is a systematic approach for developing an initial feasible solution. Its chief advantages are that it is simple to use and easy to understand. Its chief drawback is that it does not take transportation costs into account. Consequently, such a solution may require much additional effort to obtain the optimal solution.

The northwest corner method gets its name because the starting point for the allocation process is the upper left hand (northwest) corner of the transportation table. For the ABC problem, this would be the cell that represents the route from Farm A to Project #1. The following set of principles guides the allocation:

1. Begin with the upper left hand cell, and allocate as many units as possible to that cell. This will be the smaller of the row supply and the column demand. Adjust the row and column quantities to reflect the allocation.
2. Remain in a row or column until its supply or demand is completely exhausted or satisfied, allocating the maximum number of units to each cell in turn, until all supply has been allocated (and all demand has been satisfied because we assume total supply and demand are equal).

For the ABC problem, the sequence would be as follows (see Table 7-2)

1. Beginning in Cell A-1, allocate 50 units, exhausting demand in column 1 and leaving 50 units of supply in row A.
2. Staying in row A, move to cell A-2, where supply is now 50 and demand is 150 units. Allocate 50 units to this cell, exhausting the supply of row A and leaving 100 units in column 2.
3. Staying in column 2, move down to cell B-2, where supply is 200 units. Allocate 100 units to this cell, exhausting demand in column 2 and leaving 100 units of supply in row B.
4. Staying in row B, move to cell B-3 and allocate 100 units, exhausting that row's supply and leaving 200 units in column 3.

5. Staying in column 3, move down to cell C-3 and allocate 200 units, exhausting both the row and column quantities.

In terms of minimizing total transportation cost, this solution may or may not be optimal. We shall make that determination very shortly. At this point, let's simply compute the total cost this solution would generate if it was implemented.

From:\nTo	Project #1	Project #2	Project #3	Supply
Farm A	50 (first) 4	50 (Second) 2	8 8	100
Farm B	5	100 (thrid) 1	100 (fourth) 9	200
Farm C	7	6	200 (last) 3	200
Demand	50	150	300	500

Table 7-2 Initial Feasible Solution for ABC Using Northwest Corner Method

The total cost is found by multiplying the quantities in “completed” (i.e., nonempty) cells by the cell's unit cost and, then, summing those amounts. Thus:

$$\begin{aligned}
 \text{Total cost} &= 50(4) = \$ 200 \\
 &50(2) = 100 \\
 &100(1) = 100
 \end{aligned}$$

$$\begin{array}{r}
 100(9) = \quad \quad 900 \\
 200(3) = \quad \underline{600} \\
 \hline
 \$1,900
 \end{array}$$

As noted earlier, the main drawback of the northwest corner method is that it does not consider cell (route) costs in making the allocation. Consequently, if this allocation is optimal, that can be attributed to chance rather than the method used.

Finding an Initial Feasible Solution: The Intuitive Approach

This approach, also known as the *minimum cost method*, uses lowest cell cost as the basis for selecting routes. The procedure is as follows:

1. Identify the cell that has the lowest unit cost. If there is a tie, select one arbitrarily. Allocate a quantity to this cell that is equal to the lower of the available supply for the row and the demand for the column.

Table 7-3a The ABC Transportation Table's Sand and Gravel

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4	2	8	100
Farm B	5	1	9	200
Farm C	7	6	3	200

Demand	50	150	300	500
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Lowest cell
cost

Table 7-3b The ABC Transportation Table

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4	2	8	100
Farm B	5	1 150	9	200 50
Farm C	7	6	3	200
Demand	50	150	300	500

2. Cross out the cells in the row or column that has been exhausted (or both, if both have been exhausted and adjust the remaining row or column total accordingly).

Table 7-4 200 Units Are Assigned to Cell C-3

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4	2	8	100

Farm B		5	1	9	~~200~~ 50
Farm C		7	6	3	~~200~~
Demand	50	~~150~~	~~300~~ 100		500

- Identify the cell with the lowest cost from the remaining cells. Allocate a quantity to this cell that is equal to the lower of the available supply of the row and the demand for the column.
- Repeat steps 2 and 3 until all supply and demand has been allocated. Let's see how this method can be applied to the ABC problem. The cell with the lowest cost is B-2, in which the cost is \$1 (See Table 7-3a). Farm B has a supply of 200 tones of gravel, whereas Project #2 has a demand for only 150 tones. Consequently, we allocate 150 tons (the lesser of the two quantities) to sell B-2. Because this exhausts the demand of Project #2, we cross out the cells in that column and the demand, and we change the row total to 50 (See Table 7-3b).

Of the cells that remain, C-3 has the lowest cost (A-2 cannot be used because it has been crossed out). The supply of Farm C is 200 tons, the demand of Project #3 is 300 tons. The lesser of these is 200 tons, so that quantity is placed in cell C-3. Since the supply of Farm C has been completely used, the cells in row 3 are crossed out along with the row total. The remaining demand for Project #3 is 100 tons, so that column total must be adjusted accordingly (See Table 7-4).

Of the remaining cells, A-1 has the lowest cell cost. The column total is 50 and the row total is 100; hence, the quantity 50 is

assigned to that cell. Because the demand of Project #1 has been satisfied, the cells in the column must be crossed out, and 50 units must be subtracted from the supply of Farm A (See Table 7-5).

At this point, only two remaining cells have not been crossed out: cells A-3 and B-3. Cell A-3 has the lower cost, so it is net in line for allocation. The remaining supply is 50 units, and the remaining demand is 100 units; consequently, the quantity 50 is placed in A-3. This completes the use of supply for Farm A, and it leaves a demand of 50 units for Project #3. The last remaining cell, then, receives a quantity of 50 units, canceling the remaining supply and demand for *both* its row and column (See Table 7-6).

Table 7-5 50 Units Are Assigned to Cell A-1

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4 50	2	8	100 50
Farm B	5	1 150	9	200

				50
	~~7~~	~~6~~	~~3~~	
Farm C			200	~~200~~
Demand	~~50~~	~~150~~	~~300~~ 100	500

We can easily verify that this is a feasible solution by checking to see that the row and column totals of the assigned cell quantities equal the supply and demand totals for the rows and columns. For example, 50 + 50, or 100 units have been allocated in the first row, which equals the supply of Farm A. Likewise, 50 units have been assigned in the first column, which equals the demand of Project #1.

This solution may or may not be optimal. In the next section, the procedure for testing for optimality will be described. For now, let us simply compute the total cost of this solution and compare it to that of the northwest corner solution. here we have:

<u>Cell</u>	<u>Cost per Unit</u>	<u>Number of Units</u>	<u>Transporta tion Cost</u>
A - 1	\$4	50	\$200
A - 3	8	50	400
B - 2	1	150	150
B - 3	9	50	450
C - 3	3	<u>200</u>	<u>600</u>
		500	\$1800

Compared to the plan generated using the northwest-corner method, this one has a total cost that is \$100 less. The fact that this plan is less costly than the previous one was expected: The previous one did not involve the use of cost information in allocating units. Whether this plan is optimal, or can be improved upon, remains to be seen.

*Table 7-6 Completion of the Initial Feasible Solution for the ABC Problem
Using the Intuitive Approach*

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4 50	2 150	8 50	100
Farm B	5	1 150	9 50	200
Farm C	7	6	3 200	200
Demand	50	150	300	500

Evaluating a Solution for Optimality

The test for optimality for a feasible solution involves a cost evaluation of empty cells (i.e., routes to which no units have been allocated) to see if an improved solution is possible. We shall consider two methods for cell evaluation:

1. The stepping stone method
2. The MODI method.

The *stepping stone method* involves a good deal more effort than the MODI method, as you will note. However, it provides an intuitive understanding of the evaluation process. Moreover, when a solution is not optimal, the distribution plan must be revised by reallocating units into and out of various cells, and only the stepping stone method can be used for the reallocation. It is, therefore, necessary to be able to use the stepping stone approach, although the preferred choice is first to use the MODI method and, then, the stepping stone method, if necessary.

Evaluation Using The Stepping Stone Method

The stepping stone method involves tracing a series of closed paths in the transportation table, using one such path for each empty cell. The path represents a shift of one unit into an empty cell, and it enables the manager or analyst to answer a “what if” question: What impact on total cost would there be if one unit were shifted into an unused route? The result is a cost change per unit shifted into a cell. If the shift would result in a cost savings, the stepping stone path also can be used to determine the maximum number of units that can be shifted into the empty cell, as well as modifications to other completed cells needed to compensate for the shift into the previously unused cell.

Table 7-7 Initial Feasible Solution Obtained Using the Northwest Corner Method

From: To:	Project #1	Project #2	Project #3	Supply
Farm A	50	50		100
Farm B		100	100	200
Farm C			200	200
Demand	50	150	300	500

The initial feasible solution we found using the northwest corner method is reproduced in Table 7-7. We know it is not optimal because the intuitive method generated an initial solution that has a lower total cost. However, it will be instructive to analyze the northwest corner solution in order to see how the stepping stone method works and to gain some insight into the process of developing an improved solution.

Only the unoccupied cells need to be evaluated because the question at this point is not how many units to allocate to a particular route but only if converting a cell from zero units to non zero (a positive integer) would decrease or increase total costs. The unoccupied cells are A-3, B-1, C-1, and C-2. They must be evaluated one at a time, but in no particular order.

Let's begin with cell B-1. We start by placing a + in the cell being evaluated, which stands for the addition of one unit (e.g., a cubic yard of topsoil) to the cell.

Table 7-8 Evaluation Path for Cell B - 1

To From:	Project #1	Project #2	Project #3	Supply
Farm A	- 4 50	2 50	8	100
Farm B	5	1 100	9 100	200
Farm C	7	6	3 200	200
Demand	50	150	300	500

In order to maintain the column total of 50, we must subtract one unit from an occupied cell; cell A-1 is the only option. This is designated by placing a - in cell A-1 (see Table 7-8). Because we subtracted one unit from row A, we must compensate for this, which we can do by adding a unit (i.e., placing a + sign) in cell A-2. Similarly, we compensate for the addition of one unit to column 2 by subtracting a unit from cell B-2, and place a - in that cell to reflect this. Because we initially added one unit to row B in cell B-1, this last subtraction also compensates for that, and we

have traced a completed path, which we can use to evaluate B-1. Before doing that, let's consider rules that will guide tracing these paths.

Rules for tracing stepping stone paths:

1. All unoccupied cells must be evaluated. Evaluate cells one at a time.
2. Except for the cell being evaluated, only add or subtract in occupied cells. (It is permissible to skip over occupied cells to find an occupied cell from which the path can continue).
3. A path will consist of only horizontal and vertical moves, starting and ending with the empty cell that is being evaluated.
4. Alternate + and - signs, beginning with a + sign in the cell being evaluated.

Note that it is not necessary to actually alter the quantities in the various cells to reflect the one unit change, the + and - signs suffice.

The general implication of the plus and minus signs is that cells with + signs mean one unit would be added, cells with a - sign indicate one unit would be subtracted. The net impact of such a one unit shift can be determined by adding the cell costs with signs attached and noting the resulting value. Thus, for cell B-1, we have a net change of +2:

B-1	
+	-
5	4
2	1
+7	-5
$+7 - 5 = +2$	

This means that for each unit shifted into cell B-2 in this way (which is the *only* way a shift could be made), the total cost would *increase* by \$2. Consequently, such a shift would not be desirable.

Turning our attention to cell C-1, we begin its evaluation by placing a + sign in that cell (See Table 7-9). We can move horizontally or vertically to an occupied cell. Suppose we move to cell C-3 and place a - sign there. Next, we move vertically to cell B-3, place a + sign in it, then horizontally to B-2, place a - sign there, move up to cell A-2, place a + sign there, and, then, move horizontally to A-1 and place a - sign there. Consequently, some of the cells have + or - signs that are the same as in the previous evaluation path. However, this is not necessary; each evaluation path is independent of others in terms of assigning + and - signs. The evaluation of this path is + 10.

C-1	
+	-
7	3
9	1
<u>2</u>	<u>4</u>
+18	-8
+10	

Hence, using this path for reallocation would increase total cost by \$10 per unit. Again, this would be undesirable.

Evaluation paths for empty cells A-3 and C-2 are shown in Table 7-10. Their net changes are:

A-3		C-2	
+	-	+	-
8	2	6	1
<u>1</u>	<u>9</u>	<u>9</u>	<u>3</u>
+9	-11	+15	-4
-2			

Table 7-9 Evaluation Path for Cell B - 1

To	Project #1	Project #2	Project #3	Supply
		+1		

From:						
Farm A	50	4	50	2	8	100
Farm B		5	100	1	9	200
Farm C		7		6	3	200
Demand	50		150		300	500

The negative value for cell A-3 indicates an improved solution is possible: For each unit can shift into that cell, the total cost will decrease by \$2. The next question is, how many units can be reallocated into that cell? Before addressing that question, let's consider an alternative method for cell evaluation that avoids having to trace all of the evaluation paths.

Evaluation Using the MODI Method

The MODI (Modified Distribution) method of evaluating a transportation solution for optimality involves the use of index numbers that are established for the rows and columns. These are based on the unit costs of the occupied cells. The index numbers can be used to obtain the cell evaluations for empty cells without the use of stepping stone paths.

There is one index number for each column and one for each row. These can be conveniently displayed along the left and upper edges of a matrix. The index numbers are determined in such a way that for any occupied cell, the sum of the row index and the column index equal the cell's unit transportation costs:

$$\begin{array}{rcccl} \text{Row index} & + & \text{Column index} & & = \text{Cell cost} & \textbf{(7-1)} \\ r_i & + & k_j & = & c_{ij} \end{array}$$

The index numbers are determined sequentially in a manner dictated by the position of occupied cells. The process always begins by assigning a value of zero as the index number of row 1.

Table 7-10 Evaluation Paths for Cells A-3 and C-2

From: \ To	Project #1	Project #2	Project #3	Supply
Farm A	50 4	50 2	8	100
Farm B	5	100 1	9	200
Farm C	7	6	200 3	200
Demand	50	150	300	500

The method will be illustrated by developing index numbers for the initial feasible solution for the ABC problem generated by the northwest corner method, which is repeated in Table 7-11.

We begin by assigning a value of zero as the index for row 1. Once a row index has been established, it will enable us to compute column index numbers for all occupied cells in that row. Similarly, once a column index number has been determined, index numbers for all rows corresponding to occupied cells in that column can be determined.

The index number for column 1 is based on the fact that the sum of its value and the row index number must equal the cell cost of \$4 for cell A-1. Thus, $0 + k_1 = 4$, so $k_1 = +4$. Similarly, using occupied cell A-2, the index number for row B using the unit cost for occupied cell B-2: $r_2 + 2 = 1$, so $r_2 = -1$. The value of r_2 then enables us to compute the index number for column 3: $-1 + k_3 = 9$ so $k_3 = +10$. The remaining index number, that of row 3, can be determined using the unit cost of occupied cell C-3 and the column 3 index number, $k_3: r_3 + 10 = 3$, so $r_3 = -7$.

The complete set of row and column index numbers is shown in Table 7-12. Generally, it is advisable to do a quick check of the values by confirming that for all occupied cells, the sum of the row and the column index number equals the unit cell cost.

We can now readily determine the cell evaluations (improvement potentials) for each of the unoccupied cells using the relationship:

$$\begin{array}{rcll} \text{Cell evaluation} & = & \text{Cell cost} - \text{Row index} - \text{Column index} & \mathbf{(7-2)} \\ e_{ij} & = & c_{ij} & - \quad r_i \quad - \quad k_j \end{array}$$

Table 7-11 Initial Feasible Solution Obtained Using the Northwest-Corner Method

Index numbers to

To From:	Project #1	Project #2	Project #3	Supply
0 Farm A	4 50	2 50	8	100
r_2 Farm B	5	1 100	9 100	200
r_3 Farm C	7	6	3 200	200
Demand	50	150	300	500

Table 7-12 Index Numbers for Initial Northwest-Corner Solution to the ABC Problem

These determinations can be made in any order. For example, the cell evaluation for A-3 is $8 - 0 - 10 = -2$. This implies an improvement (decrease in total cost) of \$2 per unit for units that can be shifted into cell A-3. Similarly, for empty cell B-1, the improvement potential is $5 - (-1) - 4 = +2$, which indicates any units shifted into this cell would *increase* total cost by \$2 each. For unoccupied cell C-1, the evaluation is $7 - (-7) - 4 = +10$, and for cell C-2, the cell evaluation is $6 - (-7) - 2 = +11$. These values are summarized in Table 7-13. Note that they agree with the values we computed earlier using the stepping stone method.

When cell evaluations are positive or zero, an optimal solution has been found. If one or more is negative, the cell with the *largest* negative should be brought into solution because that route has the largest potential for improvement *per unit*. In this case, we found that cell A-3 had an evaluation of -2, which represented an improvement potential of \$2 per unit. Hence, an improved solution is possible.

Developing an Improved Solution

Developing an improved solution to a transportation problem requires focusing on the unoccupied cell that has the largest negative cell evaluation. In the ABC problem, the only negative evaluation was for cell A-3.

Improving the solution involves reallocating quantities in the transportation table. More specifically, we want to take advantage of the improvement potential of cell A-3 by transferring as many units as possible into the cell. The stepping stone path for that cell is necessary

for determining *how many* units can be reallocated while retaining the balance of supply and demand for the table. The stepping stone path also reveals which cells must have quantity changes and both the *magnitude* and the *direction* of the changes. The stepping stone path for cell A-3 is reproduced in Table 7-14.

Table 7-14 Stepping Stone Path for Cell A-3

	To				Supply
	Project #1	Project #2	Project #3		
Farm A	4 50	2 50 - → + ↑	8		100
Farm B	5	1 100	9 100		200
Farm C	7	6	3 200		200
Demand	50	150	300		500

The + path units the - units

signs in the indicate to be added, signs indicate to be subtracted.

The limit on subtraction is the *smallest quantity in a negative position* along the cell path. There are two quantities in negative positions, 50 and 100. Because 50 is the smaller quantity, that amount will be shifted in the following manner: Subtract 50 units from each cell on the path with a - sign, and add 50 units to the quantity of each cell with a + sign in it. The result is shown in Table 7-15. A quick check reveals that the sums of

quantities in each row and in each column are equal to original row and column totals.

With each iteration (new solution), it is necessary to evaluate the empty cells to see if further improvement is possible. This requires use of either the MODI or the stepping stone method. Both will yield the same values. Suppose we use the MODI method.

We begin by setting the index number for row 1 equal to zero. The column 1 index number is found using the equality $0 + k_1 = 4$. Solving, we find $k_1 = +4$. Similarly, for column 3, $0 + k_3 = 8$, so $k_3 = +8$. Using cell B-3 as a "pivot," the index number for row 2 can be found from the equality $r_2 + k_3 = 9$. Because k_3 was found to equal +8, this means that $r_2 = +1$. This value now allows us to compute the index number for column 2 because $r_2 + k_2 = 1$: $1 + k_2 = 1$, so $k_2 = 0$. Lastly, the index number for row 3 can be determined on the basis of the unit cost of cell C-3 and our finding that $k_3 + 8 = 3$, so $r_3 = -5$. These values provide the basis for computing evaluations for the empty cells using the relationship $e_{ij} = c_{ij} - r_i - k_j$:

Cell A-2:	$2 - 0 - 0 =$	+2
Cell B-1:	$5 - 1 - 4 =$	0
Cell C-1:	$7 - (-5) - 4 =$	+8
Cell C-2:	$6 - (-5) - 0 =$	+11

Because none of these numbers is negative, this is an optimal solution.

Table 7-15 Distribution Plan after Reallocation of 50 Units

From: \ To	Project #1	Project #2	Project #3	Supply
Farm A	4 50	2	8 50	100
Farm B	5	1 100	9 50	200
Farm C	7	6	3 200	200
Demand	50	150	300	500

The
the
are
in
You
that
same

index
number and
cell
evaluations
summarized
Table 7-16.
may recall
this was the
solution

obtained using the intuitive method for the initial feasible solution (see Table 7-6). At that point, it was determined that the total cost for the distribution plan was \$1,800.

Table 7-16 Index Numbers and Cell Evaluations

$$\begin{matrix} + & 0 & + \\ 1 & & 0 \end{matrix}$$

To From:		Project #1	Project #2	Project #3	Supply
0 + 1 - =	Farm A	4 50	2 + 2	8 50	100
	Farm B	5 0	1 150	9 50	200
	Farm C	7 + 8	6 + 1	3 200	200
	Demand	50	150	300	500

A
issues
this
order
your

**Special
Issues**
number of
special
are
discussed in
section in
to round out

understanding of the transportation model. They are:

1. Determining if there are alternative optimal solutions.
2. Defining and handling degeneracy (too few occupied cells to permit evaluation of a solution).
3. Avoiding unacceptable or prohibited route assignments.
4. Dealing with problems in which supply and demand are not equal.
5. Solving maximization problems.

Alternate Optimal Solutions

Sometimes transportation problems have multiple optimal solutions. In such instances, it can be useful for a manager to be aware of alternate solutions, because this gives the manager an option of bringing non-quantitative considerations into the decision.

In the case of the transportation problem, the existence of an alternate solution is signaled by an empty cell's evaluation equal to zero. In fact, you may have noted that cell B-1 had an evaluation equal to zero in the final solution of the ABC problem (see Table 7-16). We can find out what that alternate solution is by reallocating the maximum number of units possible around the stepping stone path for that cell. That path is shown in Table 7-17a. The smallest quantity in a negative position on that path is 50 units. Shifting those 50 units results in the distribution plan shown in Table 7-17b.

As a
that
cost is

50(5)

Table

and

From: \ To:	Project #1	Project #2	Project #3	Supply
Farm A	4 50	2	8 50	100
Farm B	5	1 150	9 50	200
Farm C	7	6	3 200	200
Demand	50	150	300	500

check, note
the total
the same as
before,

\$1,800:
 $100(8) +$
 $+ 150(1) +$
 $200(3) =$
 \$1,800.

7-17a Index
Numbers
Cell
Evaluations

Degeneracy

A transportation problem is degenerate when there are too few occupied cells to enable all the empty cells to be evaluated. In the case of the stepping stone method, this means that there will be at least one empty cell for which an evaluation path cannot be constructed. For the MODI method, it means that it will be impossible to determine all of the row and column index numbers.

Table 7-17b Alternate Optimal Solution

It is a A if the	To From:				
		Project #1	Project #2	Project #3	Supply
	Farm A	4	2	8	100
	Farm B	5	1	9	200
	Farm C	7	6	3	200
	Demand	50	150	300	500

relatively simple to determine if solution is degenerate: solution is degenerate number of occupied cells is less than the number of rows plus the number of columns minus one.

*Test of degeneracy:
Number of occupied cells must equal*

$$R + C - 1$$

Where

R = number of rows

A quick check of the alternate solution to the ABC problem developed in the preceding section will reveal that the solution was degenerate: The

number of completed cells is 4, while the required number of completed cells is $3 + 3 - 1 = 5$. This presented no difficulty because we were not concerned with evaluating the empty cells; we simply were interested in comparing the total cost of that solution with the total cost of the original optimal solution to verify that the two yielded the same total cost.

However, try to trace a stepping stone path for any of the empty cells and you will understand the nature of the problem. It should be mentioned that this particular case is somewhat atypical in that usually some paths can be traced, but not all of them. Similarly, if you attempt to compute index numbers for the rows and columns, you will be unable to compute them for row B, column 1, or column 2.

Obviously, some modification has to be made in order to determine if a given solution is optimal. The modification is to treat some of the empty cells as occupied cells. This is accomplished by placing a delta (Δ) in one of the empty cells. The delta represents extremely small quantity (e.g., .001 unit); it is so small that supply and demand for the row and column involved will be unaffected even without modifying other quantities in the row or column, and so small that total cost will not change.

The purpose of the delta is to enable evaluation of the remaining empty cells. The choice of location for the delta can be somewhat tricky: some empty cells may be unsuitable if they do not enable evaluations of the remaining empty cells. Moreover, the delta cannot be placed in a cell which later turns out to be in a negative position of a cell path involved in reallocation because delta will be the “smallest quantity in a negative position” and shifting that minute quantity around the cell path will leave the solution virtually unchanged. Consequently, a certain amount of trial and error may be necessary before a satisfactory location can be identified for delta.

The technique can be demonstrated for the degenerate alternate solution of the ABC problem. Suppose that after some experimentation, cell A-1 has been selected for the location of delta. (Note all choices would be acceptable. For example, try placing the delta in cell C-2 and compute the improvement potentials for the empty cells. Remember that delta cannot be in a negative position of a negative cell). The resulting index numbers generated using MODI and the improvement potential for empty cells based on delta in cell A-1 are shown in Table 7-18. This confirms that the solution is optimal.

Table

From: \ To	Project #1	Project #2	Project #3	Supply
Farm A	4 Δ	2 $\begin{pmatrix} + \\ 2 \end{pmatrix}$	8 100	100
Farm B	5 $\begin{pmatrix} + \\ 5 \end{pmatrix}$	1 150	9 $\begin{pmatrix} + \\ 0 \end{pmatrix}$	200
Farm C	7 $\begin{pmatrix} + \\ 8 \end{pmatrix}$	6 $\begin{pmatrix} + \\ 1 \end{pmatrix}$	3 200	200
Demand	50	150	300	500

7-18 ABC
Alternate
Solution
Modified for
Degeneracy

0

+1

-5

Unacceptable Routs

In some cases, certain origin-destination combinations may be unacceptable. This may be due to weather factors, equipment breakdowns, labor problems, or skill requirements that either prohibit, or make undesirable, certain combinations (routes).

Suppose that in the ABC problem route A-3 was suddenly unavailable because of recent flooding. In order to prevent that route from appearing in the final solution (as it originally did), the manager could assign a unit cost to that cell that was large enough to make that route uneconomical and, hence, prohibit its occurrence. One rule of thumb would be to assign a cost that is 10 times the largest cost in the table. Thus because the largest cost is \$9, a unit cost of \$90 could be assigned instead of the original cost of \$8 per unit. Then this revised problem could be solved using northwest-corner, intuitive, or VAM for an initial solution and either stepping stone or MODI to evaluate the initial solution and any possible reallocations.

The optimal solution is shown in Table 7-19. The prohibited route may appear in a non-optimal solution, but it will be eliminated by the time the optimal solution is reached.

Table 7-19 Solution to ABC Problem with a Prohibitive Route

Diagram illustrating a prohibited route. An arrow points from the text "Prohibited route" to the node labeled $+1$.

To From:		Project #1	Project #2	Project #3	Supply
0	Farm A	4 50	2 50	90	100
-1				8	
-7	Farm B	5 100	1 100	9 100	200
		2			
Up to	Farm C	7 200	6 200	3 200	200
have					
cases					
were					
you					
there	Demand	50	150	300	500

Unequal Supply and Demand

this point, examples involved in which supply and demand equal. As might guess, are situations in

which the two are not equal. When such a situation is encountered, it is necessary to modify the original problem so that supply and demand are equal. This is accomplished by adding either a dummy column or a dummy row; a dummy row is added if supply is less than demand and dummy column is added if demand is less than supply. The dummy is assigned unit costs of zero for each cell, and it is given a supply (if a row) or a demand (if a column) equal to the difference between supply and

demand. Quantities in dummy routes in the optimal solution are not shipped. Rather, they serve to indicate which supplier will hold the excess supply, and how much, or which destination will not receive its total demand, and how much it will be short.

Let's consider an example. Suppose that Farm C in the ABC problem has experienced an equipment breakdown, and it will be able to supply only 120 cubic yards of topsoil for a period of time. Therefore, total supply will be 80 units less than total demand. This will require adding a dummy origin with a supply of 80 units. The modified problem is shown in Table 7-21. We interpret the solution indicating that Project #3 will be short 80 units per week until the equipment is repaired. Note, though, that this analysis has considered only transportation costs, and that other factors, such as shortage costs or schedules of the projects, may dictate some other course of action.

If the intuitive approach is used to obtain the initial feasible solution when a dummy is involved, make assignments to the dummy last. Hence, begin by assigning units to the cell with the lowest nonzero cost, then the next lowest nonzero cost, and so on. For the ABC problem, this would mean that units would be assigned first to cell B-2 because its cost of \$1 is the lowest nonzero cell cost.

Table 7-20 A Dummy Origin Is Added to Make Up 80 Units

To From:	Project #1	Project #2	Project #3	Supply
Farm A	4	2	8	100
To From:	Project #1	Project #2	Project #3	Supply
Farm B	5	1	9	200
Farm C	7	6	3	
Farm A	4	2	8	120
Farm B	5	1	9	200
Farm C	7	6	3	
Dummy	0	0	0	80
	50	150	300	500
	50	150	300	500

Table
Using
Origin

7-21
Solution
the Dummy

Maximization

Some transportation type problems concern profits or revenues rather than costs. In such cases, the objective is to maximize rather than to minimize. Such problems can be handled by adding one additional step at the start: identify the cell with the largest profit and subtract all the other cell profits from that value. Then replace the cell profits with the resulting values. These values reflect the *opportunity costs* that would be incurred by using routes with unit profits that are less than the largest unit profit. Replace the original unit profits with these opportunity costs and solve in the usual way for the *minimum* opportunity cost solution. This will be identical to maximizing the total profit. For example, suppose in the ABC problem, the cell values had been unit profits instead of unit costs. Cell B-3 had the largest dollar value: \$9. Hence, each cell's dollar amount would be subtracted from 9. For cell A-1, the resulting opportunity cost would have been $9 - 4 = 5$, and so on. Cell B-3 would have an opportunity cost of 0, making it the most desirable route.

The remainder of the steps for developing an initial feasible solution, evaluation of empty cells, and reallocation are identical to those used for cost minimization. When the optimal distribution plan has been identified, use the original *cell values* (i.e., profits) to compute the total profit for that plan.

7.3 THE ASSIGNMENT PROBLEMS

Assigning problems are a special case of linear programming problems. They tend themselves to manual solution using a special purpose algorithm that is described in this section. Assignment problems are characterized by a need to pair items in one group with items in another group in a one-for-one matching. For example, a manager may be faced with the task of assigning four jobs to four machines, one job to a machine. Another manager may be faced with the task of assigning five projects to five staff members, with each staff member responsible for a single project.

Typically, the time or cost required to complete a job or a project will differ, depending on the machine used for the job or the staff member doing the project. The manager's goal in such cases is to develop a set of assignments that will minimize the total time or cost of doing the work. In other cases, the goal may be to minimize distance traveled, scrap, or some other measure of effectiveness. Moreover, some problems involve profit or revenue, and in those cases, the obvious goal would be to develop a set of assignments that would lead to the maximum total profit or revenue. Because minimization problems are more common, we begin with a minimization problem.

Example 1

A manager has prepared a table that shows the cost of performing each of four jobs by each of four employees (see Table 7-22). According to this table, Job 1 will cost \$15 if done by Employee A, \$20 if it is done by Employee B, and so on. The manager has stated that his goal is to develop a set of job assignments that will minimize the total cost of getting all four jobs done. It is further required that the jobs be

performed simultaneously, thus requiring one job being assigned to each employee.

Although the manager recognizes that this problem can be solved using the simplex routine, he also knows that he can solve the problem by hand using the assignment method.

Table 7-22 Job Costs for Each Possible Pairing

		<i>Employees</i>			
		A	B	C	D
Job	1	\$15	20	18	24
	2	12	17	16	15
	3	14	15	19	17
	4	11	14	12	13

The Assignment Method

The assignment method is based on minimization of opportunity costs that would result from potential pairings. There are additional cost that would be incurred if the lowest cost assignment is not made, either in terms of jobs (i.e., rows) or employees (i.e., columns). For example, we can see in Table 7-22 that the lowest processing cost for Job 1 is \$15 when done by Employee A. Therefore, if the job were assigned for some reason to Employee B, the additional (i.e., opportunity) cost would be $\$20 - \$15 = \$5$. Similarly if Job 1 were assigned to Employee C, the opportunity cost would be $\$18 - \$15 = \$3$, and if it were assigned to Employee D, the opportunity cost would be $\$24 - \$15 = \$9$. We can perform similar calculations for the other rows by identifying the lowest cost for each row and, then, subtracting that value from each of the other costs in that row to obtain job opportunity costs for all job assignments. This is usually referred to as a *row reduction*. The results of the row reduction for the costs are shown in Table 7-23.

7-23 Row Reduction

Original Costs
reduction

		<i>Employee</i>				Row Minimu m
		A	B	C	D	
Job	1	15	20	18	24	15
	2	12	17	16	15	12
	3	14	15	19	17	14
	4	11	14	12	13	11

Row
Reducti
on

Costs after the row

		<i>Employee</i>			
		A	B	C	D
Job	1	0	5	3	9
	2	0	5	4	3
	3	0	1	5	3
	4	0	3	1	2

The procedure for a row reduction is summarized in the following box.

Procedure for row reduction:

1. Identify the minimum value in each row.
2. Subtract the minimum value in each row from all the values in that row.
3. Use the resulting values to develop a new

Now, the same logic can be applied from the perspective of the employees. That is, because each employee will have a job, and because there are cost differences among employees, there can be opportunity costs in that respect. These will be in addition to the job opportunity costs because the minimum cost (column) assignments will not necessarily be the same as the minimum cost row assignments. For instance, the lowest-cost assignment for Job 1 was Employee A. However, the lowest cost assignment for Employee A would be Job 4, which has a cost of \$11. The opportunity costs for employees can be determined using the values obtained from the row reductions because they are in addition to those opportunity costs.

The procedure is similar to that for a row reduction:

Procedure for column reduction:

1. Identify the lowest-cost value in each column.
2. Subtract the lowest cost value in each column from each of the values in that column.

Revised (row reduction costs)

<i>Column</i>	0	1	1	2
<i>minimum</i>	0	1	1	2

Once both the row and column opportunity costs have been determined, we can attempt to make the minimum cost assignments. Recognizing that assignment with opportunity costs of zero reflect minimum costs, it would be desirable to attempt to make assignments only with matches that have costs of zero. For instance, assigning any job to employee A would have a zero cost. However, once a job is assigned to A, no other job can be

assigned to A (i.e., the other zero costs in that column become irrelevant). Consequently, even though it may appear on the surface that there are enough zeros to make zero cost assignments, the fact that each assignment eliminates an entire row and column from further consideration (i.e., assignments) means that a complete set of zero cost assignments might not be possible at this juncture.

A quick method of determining if a set of zero opportunity cost assignments can be made is to find the minimum number of lines needed to “cover” all the zero costs. That is, if we draw a line through the zero costs in column A, this will account for four of the seven zeros. How many such lines will be needed to cross out all zeros? The answer is two more (a line through row 3 and one through row 4), for a total of three (see Table 7-25). Now if the minimum number of covering lines equals the number of rows (or columns, because this is a square table), an optimal assignment is possible. In this case, apparently, an optimal assignment is not possible because only three lines were necessary. (Note that there is another way of covering the second zero in row 3: a vertical line could have been drawn through column b. The point is that only three lines, however drawn, would be needed).

Table 7-25 Determine the Minimum Number of Lines Needed to Cover the Zeros

	A	B	C	D		A	B	C	D
1	0	4	2	7	1	0	3	1	6
2	0	4	3	1	2	0	3	2	0
3	0	0	4	1	3	1	0	4	1
4	0	2	0	0	4	1	2	0	0

Table 7-26 Further Revision of the Cost Table

Revised costs

	A	B	C	D
1	0	4	2	7
2	0	4	3	①
3	0	0	4	1
4	0	2	0	0

Smallest
uncovered
←

We must, therefore, make further reductions. Further row or column reductions are possible because there is a zero in every row and column. To get around this, we do the following: subtract the smallest uncovered cost (\$1 in this case), from every other uncovered cost and adding that same amount to costs that lie at an intersection of two covering lines. This is done in Table 7-26. The rationale for this is: Subtracting the smallest *uncovered* cost reveals the next smallest increment in opportunity costs, whereas adding that amount to intersections removes those assignments from consideration. Because both are at an intersection means that *another assignment* already exists in both that row and that column: hence, we can ignore those intersection possibilities.

We know can repeat the process of finding the minimum number of covering lines. With the revised table, a minimum of four lines is needed (see Table 27). Hence an optimal assignment is now possible.

To avoid confusion in making the assignments, begin by identifying a row or column that has only a single zero. Candidates are columns B and C and rows 1 and 3 because each has only one zero. Suppose we take row 1. We can denote the assignment of Job 1 to Employee A by boxing in the 0 at that intersection, and because this assignment eliminates Job 1 and Employee A from further consideration, we can cross out both column A and row 1. Next, we might select row 3 because it has a single zero, and box that one. This eliminates row 3 and Employee B, and they can be

crossed out. Similarly, we can box the remaining zero in column D, row 2 to complete our assignments. These assignments are shown in Table 7-28.

Table 7-27 A Minimum of Four Lines Are Needed to Cover All Zero Costs

	A	B	C	D
1	0	3	1	6
2	0	3	2	0
3	~~1~~	0	~~4~~	~~1~~
4	~~1~~	2	0	0

Table 7-28 Optimal Assignments

	A	B	C	D
1	0	3	1	6
2	0	3	2	0
3	1	0	4	1
4	1	2	0	0

The total cost of the assignments can be determined by referring to the original cost table (Table 7-22). The total cost is:

1 - A	\$15
2 - D	15
3 - B	15
4 - C	<u>12</u>
	\$57

Summary of Procedure

1. Perform a *row reduction* on the cost table by subtracting the least cost in each row from all costs in that row.

2. Perform a *column reduction* on the cost table that results from the row reduction by subtracting the least cost in each column from all costs in that column.
3. Determine if an optimal assignment can be made by drawing the minimum number of horizontal and/or vertical lines necessary to cover all zero costs. If the number of lines equals the number of rows, go to step 5.
4. If the minimum number of lines is less than the number of rows, identify the smallest uncovered opportunity cost. Subtract that amount from all uncovered costs and add that amount to the covered costs that lie at line intersections. Repeat Step 3.
5. Make the assignments. Begin with a row or column that has a single zero. Box that zero to indicate the assignment and eliminate that row and column from further consideration by drawing a line through the row and another line through the column. Continue assigning rows or columns with single zero cost elements and, then, choose arbitrarily for assignments where multiple zero cost elements exist.

Requirements for Use of the Assignment Method

Situations in which the assignment method of linear programming can be used are characterized by the following:

1. There will be a one-for-one matching of two sets of items.
2. The goal is to minimize costs (or to maximize profits) or a similar objective (e.g., time, distance, etc.)
3. The costs or profits (etc.) are known or can be closely estimated.

Special Situations

Certain situations can arise in which the model deviates slightly from that previously described. Among those situations are the following:

1. The number of rows does not equal the number of columns.
2. The problem involves maximization rather than minimization.

3. Certain matches are undesirable or not allowed.
4. Multiple optimal solutions exist.

The procedure outlined previously for assignment problems requires an equal number of rows and columns. However, certain problems may not satisfy that requirement. For example, a situation might involve four jobs that need processing but there are only three machines available for processing. Consequently, one job will not be processed immediately. In order to perform the analysis, and to learn which job will not be processed, an extra “machine” must be added to the table. In analyzing the problem, one job will be assigned to the nonexistent machine. Hence, that will be the job that is not immediately processed.

Example 2

Prepare this assignment table so that the optimal set of assignments can be made using the previously described procedure.

		Job			
		1	2	3	4
Machine	A	15	19	12	16
	B	23	21	18	17
	C	20	16	11	19

Solution

Compensate for too few machines by adding another machine (row) to the table. Because no such machine exists, there will be no cost for the assignment. Hence, use costs of 0 for this dummy row:

		Job			
		1	2	3	4

	A	15	19	12	16
Machi	B	23	21	18	17
ne	C	20	16	11	19
	D	0	0	0	0

Then, proceed as before. Note, though, that the column reduction step will have no effect because each column already has a zero. Therefore, simply skip that step and go on to the next step. If the goal is to *maximize* rather than to minimize, one extra step must be added to the start of the process: Identify the largest value in each column and, then, subtract all numbers in each column from the column maximum. Having done that, perform the same steps that would be required if the problem were minimization because the modified values represent *opportunity* costs. The set of assignments that minimizes the opportunity costs will also maximize the original values.

Example 3

The following table contains profits that would be realized from various possible pairings. Prepare the table so that the optimal solution can be obtained using the assignment method for *minimization*.

	1	2	3
A	14	22	30
B	20	18	40
C	11	12	50

Solution

Identify the maximum value in each column and then subtract every value in a given column from the column maximum, as shown below:

Original Values			
	1	2	3
A	14	22	30
B	20	18	40
C	11	12	50
<i>Column maximum</i>	20	22	50

Opportunity costs			
	1	2	3
A	6	0	20
B	0	4	10
C	9	10	0

In certain instances, a particular match or pairing may be either understandable or otherwise unacceptable. For example, an employee may not have the skills necessary to perform a particular job or a machine may not be equipped to handle a particular operation. When such a restriction is present, a letter (e.g., a capital M) is often placed in the table in the position that would represent a pairing. Analysis is performed as usual except that the M is ignored throughout the analysis. That is, the M is not used in any reductions, nor is any value added to it or subtracted from it during the course of the analysis.

Example 4

Determine the optimal set of pairings given the following cost table. Note that assignments B-3 is undesirable, as denoted by the M in that position.

	1	2	3
A	8	7	2
B	1	4	M

C	7	9	3
---	---	---	---

Solution

Notice how the M does not change throughout the analysis:

	Original values		
	1	2	3
A	8	7	2
B	1	4	M
C	7	9	3

	After row reduction		
	1	2	3
A	6	5	0
B	0	3	M
C	4	6	0

	After column Reduction		
	1	2	3
A	6	②	0
B	0	0	M
C	4	3	0

further reduction, optimal

	1	2	3
A	4	0	0
B	0	0	M
C	2	1	0

After
solution

In some cases, there are multiple optimal solutions to a problem. This conduction can be easily recognized when making the optimal assignments: No *unique 0* will exist at some point, resulting in more than one choice for assignment and, hence, more than one optimal solution. It should be noted that *all* optimal solutions will yield the *same value* of the objective function (e.g., the same minimum cost).

Example 5

Give this final assignment table, identify two solutions.

	1	2	3
A	4	0	0
B	0	3	2
C	1	0	0

Solution

The first assignment must be B-1, because B-1 is the only 0 that appears in a single row or column. Having made that assignment, there are two choices for the remaining two rows, and two choices for the remaining two columns. This results in two possible solutions, as shown:

	1	2	3
A	4		0
B		0	2
C	0	0	0

	1	2	3
A	4	0	
B		3	0
C	0	0	0

Check Your Progress Exercise

1. Solve this transportation problem for the optimal solution. if there is an alternate optimal solution, identify it.

To From:	1	2	3	Supply
A	2	3	9	61
B	5	6	4	61
Demand	50	55	17	

2. The following table contains processing costs for two jobs on various machines. Each job must be processed on one machine. Determine a set of pairing that will minimize total processing costs.

		Job	
		1	2
Machine	A	8	7
	B	5	4
	C	3	1

7.4 SUMMARY

This unit describes two types of problems that lend themselves to solution using linear programming techniques: the transportation problem and the assignment problem. Transportation type problems often involve the distribution of goods, whereas assignment type problems involve the matching or pairing of two items. Usually, such problems have different costs for different distribution alternatives or different pairings, and the objective is to identify the plan which minimizes total cost.

For each type of problem, a procedure for formulating the problem in a way that lends itself to solution using a simple algorithm is described. Solving such problems manually provides additional insight into LP models and their solutions.

7.5 ANSWER TO CHECK YOUR PROGRESS EXERCISE

Answer for Question 1

First, check to see if supply and demand are equal. Demand is $50 + 55 + 17 = 122$; supply is $61 + 61 = 122$. Hence, they are equal. Next, determine an initial feasible solution. This can be done using either the northwest-corner method or the intuitive method. Because the question

did not specify which one, either can be used. Suppose we use the intuitive method.

row	From:\nTo	1	2	3	Supply	Because the and column quantities allocated
	A	2 50	3	9	61	
	B	5	6 44	4 17	61	
	Demand	50	55	17	122	

add to the row and column totals, this is a feasible solution.

Next we must ensure that the minimum number of occupied cells exists, or else we will have to adjust for degeneracy. That number is one less than the sum of the number of rows and the number of columns. With two rows and three columns, the desired number of occupied cells is: $2 + 3 - 1 = 4$. Since this agrees with the number we have, we can proceed to the next step, which is evaluation for optimality.

We can use either the stepping-stone method or MODI because no method was specified. Since MODI is not as messy as stepping-stone, let's use MODI. The following table shows these index numbers.

2 3 1

		To	1	2	3	Supply
		From:				
0	A	2	50	11	9	61
3	B	5		44	17	61
	Demand		50	55	17	122

Now, we can compute the evaluations for the empty cells, using the relationship: Cell evaluation = Cell cost - Row index - Column index.

		To	2	3	1	Supply
		From:	1	2	3	
0	A	2	50	11	9	61
3	B	5		44	17	61
	Demand		50	55	17	122

Because no cell has a negative evaluation, the current solution is optimal.

evaluations indicates that this is the optimal solution. Thus, to minimize transportation cost, ship 50 units from A to 1, 11 units from A to 2, 44 units from B to 2, and 17 units from B to 3. The total cost will be:

<u>Route</u>	<u>Cost Per Unit</u>	<u>X</u>	<u>Route Cost</u>
A-1	\$2	50	\$100
A-2	3	11	33
B-2	6	44	264
B-3	4	17	68
			\$465

The appearance of a cell evaluation of zero indicates that an *alternate solution* with the same total cost exists. To determine what that other solution is, we must trace the stepping-stone path for the cell that has the zero evaluation. It is:

To From:	1	2	3	Supply
A	2 50	3 11	9	61
B	5	6 44	4 17	61
Demand	50	55	17	122

The
the
cells
The

negative positions in path are A-1 and B-2. smallest

quantity in a negative position is the 44 units in cell B-2. It is this amount that can be reallocated around the path, subtracting it from the quantity at each negative position and adding it at each positive position. After doing this, the alternate solution becomes apparent. It is:

To From:	1	2	3	Supply
	2	3	9	
A	6	55		61
B	44		17	61
Demand	50	55	17	122

If you
total
will
the
total

compute the
cost, you
see that it is
same as the
cost of the

original optimal solution. Hence, the two solutions are equivalent in that regard.

Answer for Question 2

The problem requires a pairing, or matching, of jobs and machines. Thus, it is an assignment problem. Assignment techniques require square tables (i.e., in this case, an equal number of jobs and machines). Because there are three machines and only two jobs, a third (dummy) job must be added. Processing costs of zero are assigned to each of the possible pairing of the dummy job and the machines, resulting in this table:

	Job		
	1	2	3
A	8	7	0

Machine	B	5	4	0
	C	3	1	0

Ordinarily, the next step would be to perform a row reduction. Note, however, that the smallest value in each row is a zero. Consequently, a row reduction (i.e., subtracting the smallest value in each row from the other values in the row) will result in the same values. Therefore, we move to the next step, which is a column reduction. The smallest value in the first column is 3, the smallest in the second column is 1, and the smallest in the third column is 0. The column reduction is illustrated in the next set of tables.

		Job			(Original
		1	2	3	
Machine	A	8	7	0	
	B	5	4	0	
	C	3	1	0	
Smallest value:		3	1	0	

		Job			(Values after the column reduction)
		1	2	3	
Machine	A	5	6	0	
	B	②	3	0	
	C	0	0	0	

Now, we test the optimal found at this point. This involves crossing out the zeros in the table with as few (horizontal or

vertical) lines as possible. As shown in the preceding table, only two lines are required.

Identify the smallest value that is not crossed out. It is the 2 in location B-1. Circle it, and then subtract it from any value that has been crossed out. Also, add it to the intersection of the crossing lines. The results are shown in the next table.

		Job		
		1	2	3
Machine	A	8	4	0
	B	0	1	0
	C	0	0	0

At this point, three lines (not shown) would be needed to cross out all of the zero values. Therefore, we can identify the optimal solution. To do this, begin with a row or column that has only one zero. In this case, the first row has only one zero; so does the second column. Hence, either of those could be chosen for the initial assignment.

The (minimum) cost associated with this set of assignments can be determined by referring to the original table of values and noting the cost of each assignment. Thus:

Assignment	Cost
A-3	0
B-1	5
C-2	<u>1</u>
	6

Because Job 3 (the dummy) has been assigned to Machine A, this tells us that, in reality, Machine A will not have a job assigned to it.

7.6 MODEL EXAMINATION QUESTIONS

1. Consider the following transportation problem:

	From:\nTo:	To:			Supply
		Store1	Store 2	Store 3	
a.	Warehouse A	12	20	15	50
b.	Warehouse B	9	11	4	15
c.	Warehouse C	20	14	8	55
	Demand	25	50	45	120

an
Compute the
total cost for
this solution.
the
optimal?
Explain.
the
those
obtained

using the stepping-stone method.

- Obtain an improved solution and evaluate it using MODI. Is it optimal?
- What is the total for your optimal solution?

2. Consider the following transportation problem:

	To:				Supply
	1	2	3	4	
From:					
	12	5	7	2	98
a. B	11	3	6	1	80
Demand	35	50	25	68	

Develop an intuitive solution to this problem; evaluate it, and revise if necessary to obtain the minimum cost solution. Then, compute the total cost.

b. Solve the preceding problem using the northwest-corner and stepping-stone methods. Are there alternate optimal solutions to this problem? How do you know?

3. Solve the following transportation problem for the minimum-cost solution using the intuitive method for the initial solution and MODI for evaluation. Is there an alternate optimal solution? What is it?

From:\nTo:	Diredawa	Mekele	Ambo	Supply
Addis Ababa	14	8	3	40
Awassa	9	5	7	80
Bale	6	12	4	60
Demand	70	60	50	