UNIT 2 MATRIX ALGEBRA AND ITS APPLICATION

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2.0 AIMS AND OBJECTIVES

After reading this chapter students will be able to:

- explain what a matrix is
- define the different types of matrices
- perform matrix operations
- find inverse of a square matrix
- explain how to solve a systems of linear equations
- solve word problems applying matrices
- understand the concept of Markov chain

2.1 INTRODUCTION

Brevity in mathematical statements is achieved through the use of symbols. The price paid for brevity, of course, is the effort spent in learning the meaning of the symbol.

In this unit we shall learn the symbols for matrices, and apply them in the statement and solution of input-output problems and other problem involving linear systems

2.2 MATRIX ALGEBRA

Algebra is a part of mathematics, which deals with operations (+, -, x, ÷).

A matrix is a rectangular array of real numbers arranged in m rows & n columns. It is symbolized by a bold face capital letter enclosed by a bracket or parentheses.

eg. $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} - - - - - - a_{1n} \\ a_{21} & a_{22} - - - - - - a_{2n} \\ a_{m1} & a_{m2} - - - - - - a_{mn} \end{bmatrix}$ in which a_{jj} are real numbers

Each number appearing in the array is said to be an element or component of the matrix. Element of a matrix are designated using a lower case form of the same letter used to symbolize the matrix itself. These letters are subscripted as a_{ij}, to give the row & column location of the element with in the array. The first subscript always refers to the raw location of the element; the second subscript always refers to its column location. Thus, component a_{ij} is the component located at the intersection of the i^{th} raw and j^{th} column.

The number of rows (m) & the number of columns (n) of the array give its order or its dimension, M x n (reads "M" by "n")

Eg. The following are examples of matrices

$a_{12} = 7$

$$A = \begin{pmatrix} 1 & 7 \\ 5 & 3 \end{pmatrix}$$

element

this is 3 x 2 matrix

$a_{21} = 5$

4 2

$a_{32} = 2$

$$X = \begin{pmatrix} 1 & 5 & 9 & 15 \\ 2 & 6 & 10 & 20 \\ 3 & 7 & 11 & 30 \\ 4 & 8 & 12 & 45 \end{pmatrix}$$

This is a 4 x 4 matrix

Element $X_{44} = 45$

$X_{34} = 30$

$X_{42} = 8$

$X_{32} = 7$

2.2.1 Types of Matrices

There are deferent types of matrices. These are

1. Vector matrix is a matrix, which consists of just one row or just one column. It is an

m x 1 or 1 x n matrix.

1.1 Row vector is a 1 x n matrix i.e. a matrix with 1 row

eg. $W = \begin{bmatrix} -1 & 0 & 6 \end{bmatrix}$ 1 x 3

1.2. Column vector: is an m x 1 matrix i.e. a matrix with one column only

eg. $Z = \begin{bmatrix} 0 \\ 20 \\ 5 \end{bmatrix}$ 3 x 1

2. Square matrix: - a matrix that has the same number of rows & columns. It is also called n-th order matrix

eg. 2 x 2 , 3 x 3, n x n $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 2 x 2

3. Null or zero matrix: - is a matrix that has zero for every entry.

$\begin{bmatrix} & \\ & \end{bmatrix}$

It's generally denoted by Om x n eg. Y = 0 0

0 0

4. Identity (unit) matrix: - a square matrix in which all of the primary diagonal entries are

ones & all of the off diagonal entries are zeros. Its denoted by I.

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2 x 2 $I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3 x 3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 4 x 4

N.B. Each identity matrix is a square matrix

* Primary diagonal represents: a_{11}, a_{22}, a_{33}, a_{44}---------a_{nn} entries element

A x I = A & I x A = A that is, the product of any given matrix & the identity matrix is the given matrix itself. Thus, the identity matrix behaves in a matrix multiplication like number 1 in an ordinary arithmetic.

5. Scalar matrix: - is a square matrix where elements on the primary diagonal are the same.

 An identity matrix is a scalar matrix but a scalar matrix may not be an identity matrix.

2.2.2 Matrix operations (Addition, Subtraction, Multiplication)

Matrix Addition/ Subtraction

Two matrices of the same dimension are said to be CONFORMABLE FOR ADDITION. Adding corresponding elements from the two matrices & entering the result in the same raw-column position of a new matrix perform the addition.

If A & B are two matrices, each of site m x n, then the sum of A & B is the m x n matrix C whose elements are:

$C_{ij} = a_{ij} + b_{ij}$ for i = 1, 2 --------m

j = 1, 2 ---------n

$C_{11} = a_{11} + b_{11}$

$C_{22} = a_{22} + b_{22}$

$C_{12} = a_{12} + b_{12}$ etc

eg. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{4}^{3} + \begin{bmatrix} 7 & 9 \\ 8 & -10 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 10 & -6 \end{bmatrix}$

eg. $\begin{bmatrix} -2 & 7 \\ 4 & 6 \end{bmatrix}_{9}^{2} + \begin{bmatrix} 8 & 7 \\ 6 & 4 \end{bmatrix}$ = These two matrices aren't conformable for addition because they aren't of the same dimension.

Laws of matrix addition

The operation of adding two matrices that are conformable for addition has these two basic properties.

1. A + B = B + A → The commutative law of matrix addition

2. (A + B) + C = A + (B + C) → the associative law of matrix addition

→ The laws of matrix addition are applicable to laws of matrix subtraction, given

that the two matrices are conformable for subtraction A − B = A + (-B)

$$\text{eg. } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Matrix Multiplication

a) <u>By a constant (scalar multiplication)</u>

A matrix can be multiplied by a constant by multiplying each component in the

matrix by a constant. The result is a new matrix of the same dimension as the

original matrix.

If K is any real number & A is an M x n matrix, then the product KA is defined to

be the matrix whose components are given by K times the corresponding

component of A; i.e.

$$KA = [K a_{ij}] \quad (m \times n)$$

eg. If $X = \begin{bmatrix} 6 & 5 & 7 \end{bmatrix}$, then $2X = [\quad]_{(2 \times 6)}$ (2 x 5) (2 x 7)

$$2X = \begin{bmatrix} 12 & 10 \\ 14 & \end{bmatrix}$$

Laws of scalar multiplication

The operation of multiplying a matrix by a constant (a scalar) has the following basic properties. If X & Y are real numbers & A & B are m x n matrices, conformable for addition, then

1. XA = AX
2. (X + Y) A = XA + YA
3. X (A + B) = XA + XB
4. X (YA) = XY (A)

Laws of scalar multiplication

eg. Given matrices A & B and two real numbers X & Y

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 5 \end{bmatrix}$$

X = 2 Y = 4

1) XA = AX

Proof: $XA = 2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $AX = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} 2$

$$XA = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} \qquad AX = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Therefore XA = AX

2) (X + Y) A = (XA + YA)

Proof:

→ (X + Y) A means first add X with Y and then multiply the result by matrix A.

The result of X + y is (2 + 4) = 6 then 6 will be multiplied by matrix A

$$\begin{bmatrix} 6 & 1 & 2 \\ & 4 & 5 \end{bmatrix} \begin{matrix} 3 \\ 6 \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} 6 & & \\ & 24 & \end{bmatrix} \begin{matrix} 12 & 18 \\ 30 & 36 \end{matrix}$$

Therefore $(X + Y)\,A = \begin{matrix} 6 & 12 & 18 \\ 24 & 30 & 36 \end{matrix}$

→ XA + YA means multiply the constant numbers X and Y with matrix A

independently, then add the two results together

$$XA = \begin{bmatrix} 2 & 1 & 2 \\ & 4 & 5 \end{bmatrix} \begin{matrix} 3 \\ 6 \end{matrix} \qquad YA = \begin{bmatrix} 4 & 1 & 2 \\ & & 4 \end{bmatrix} \begin{matrix} 3 \\ 5 \quad 6 \end{matrix}$$

$$XA = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} \begin{matrix} 6 \\ 12 \end{matrix} \qquad \begin{bmatrix} YA = & 4 & \\ & & 16 \end{bmatrix} \begin{matrix} 8 & 12 \\ 20 & 24 \end{matrix}$$

Then add the result XA with YA

$$XA + YA = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} \begin{matrix} 6 \\ 12 \end{matrix} + \begin{bmatrix} 4 & \\ & 16 \end{bmatrix} \begin{matrix} 8 & 12 \\ 20 & 24 \end{matrix}$$

$$= \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \end{bmatrix}$$

Therefore it is true that (X + Y) A is equal with XA + YA

3) X (A + B) = XA + XB

Proof:

→ X (A + B) means add matrix A and B first and then multiply the result by a constant X

Given constant number X = 2 matrices A and B then the result of X (A + B) will be

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 5 \end{bmatrix} \quad A + B = \begin{bmatrix} 5 & 4 & 4 \\ 7 & 5 & 11 \end{bmatrix}$$

$$X(A + B) = \begin{bmatrix} 2 & 5 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 & 8 \\ 14 & 10 & 22 \end{bmatrix}$$

→ XA + XB means multiply matrices A and B by a constant number X independently

then add the results

$$XA = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix} 3 \qquad XB = 2 \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 5 \end{bmatrix}$$

$$XA = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} \begin{matrix} 6 \\ 12 \end{matrix} \qquad XB = \begin{bmatrix} 8 & 4 & 2 \\ 6 & 0 & 10 \end{bmatrix}$$

Then Add XA with XB

$$\text{i.e. } XA + XB = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} \begin{matrix} 6 \\ 12 \end{matrix} + \begin{bmatrix} 8 & 4 & 2 \\ 6 & 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 & 8 \\ 14 & 10 & 22 \end{bmatrix}$$

Therefore it is true that X (A + B) is equivalent with XA + XB

4) X (YA) = XY (A)

Proof:

→ X (YA) means multiply the second constant number Y with matrix A first and then

multiply the first constant number X with the result.

$$YA = \begin{bmatrix} 4 & 1 \\ 4 & 5 \end{bmatrix} \begin{matrix} 2 & 3 \\ 6 & \end{matrix} \qquad X(YA) = 2 \begin{bmatrix} 4 & 8 \\ 16 & 20 \end{bmatrix} \begin{matrix} 12 \\ 24 \end{matrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix} \qquad = \begin{bmatrix} 8 & 16 \\ 32 & 40 \end{bmatrix} \begin{matrix} 24 \\ 48 \end{matrix}$$

→ XY (A) means multiply the two constant real numbers X and Y first and multiply

the result by matrix A.

XY = 2 X 4

= 8

$$XY(A) = \begin{bmatrix} 8 & 1 \\ 4 & 5 \end{bmatrix} \begin{matrix} 2 & 3 \\ 6 & \end{matrix}$$

$$= \begin{bmatrix} 8 & 16 \\ 32 & 40 \end{bmatrix} \begin{matrix} 24 \\ 48 \end{matrix}$$

Therefore it is also true that X (YA) = XY (A) of columns in B

b) <u>Matrix by matrix multiplication</u>

If A & B are two matrices, the product AB is defined if and only if the number of

Columns in A is equal to the number of rows in B, i.e. if A is an m x n matrix, B

should be an n x b.

If this requirement is met., A is said to be conformable to B for multiplication. The

matrix resulting from the multiplication has dimension equivalent to the number of

rows in A & the number columns in B

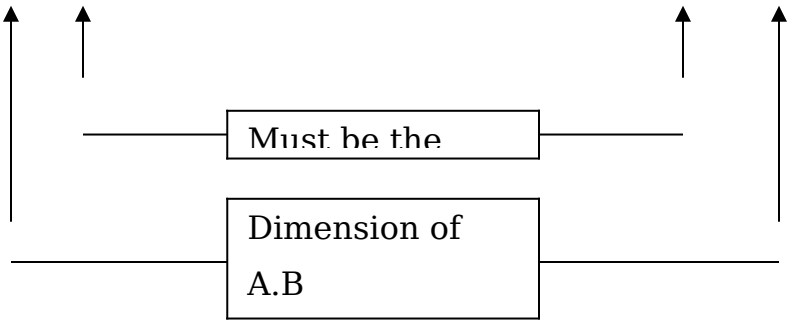
If A is a matrix of dimension n x m (which has m columns) & B is a matrix of dimension p x q (which has p rows) and if m and p aren't the same product A.B is not defined. That is, multiplication of matrices is possible only if the number of columns of the first equals the number of rows of the second.

If A is of dimension n x m & if B is of dimension m x p, then the product A.B is of dimension n x p

A B

Dimension Dimension

n x m m x p



eg. $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ 4, 7 — 2 x 3 $B = \begin{pmatrix} -1 & 7 \\ 0 & 8 \\ 5 & 1 \end{pmatrix}$ 3 x 2

AB = (2x □ 1) + (3 x 0) + (4 x 5) (2 x 7) + (3 x 8) + (4 x 1)

= 18 = 42

(6x □ 1) + (9 x 0) + (7 x 5) (6 x 7) + (9 x 8) + (7 x 1)

= 29 = 121

$$\therefore \quad AB = \begin{bmatrix} 18 & 42 \\ 29 & 121 \end{bmatrix}$$

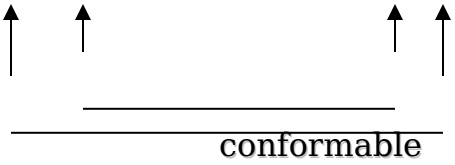
Find BA =

$$B = \begin{bmatrix} -1 & 7 \\ 0 & 8 \\ 5 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 9 & 7 \end{bmatrix} \quad 2 \times 3$$

3 x 2

B A

3 x 2 2 x 3 result 3 x 3 matrix



conformable

BA = (-1 x 2) + (7 x 6) (-1 x 3) + (7 x 9) (-1 x 4) + (7 x 7)

=40 60 45

(0 x 2) + (8 x 6) (0 x 3) + (8 + 9) (0 x 4) + (8 x 7)

= 48 72 56

(5 x 2) + (1 x 6) (5 x 3) + (1 x 9) (5 x 4) + (1 x 7)

16 24 27

$$BA = \begin{bmatrix} 40 & 60 & 45 \\ 48 & 72 & 56 \\ 16 & 24 & 27 \end{bmatrix}$$

Special properties of matrix application

1 The associative & distributive laws of ordinary algebra apply to matrix multiplication.

Given three matrices A, B & C which are conformable for multiplication,

i. A (BC) = AB (C) → Associative law, (not C (AB)
ii. A (B+C) = AB + AC → Distributive property
iii. (A + B) C = AC + BC → Distributive property

2 on the other hand, the commutative law of multiplication doesn't apply to matrix multiplication. For any two real numbers X & Y, the product XY is always identical to the product YX. But for two matrices A & B, it is not generally true that AB equals BA. (in the product AB, we say that B is pre multiplied by A & that A is post multiplied by B.)

3 In many instances for two matrices, A & B, the product AB may be defined while the

product BA is not defined or vice versa.

In some special cases, AB does equal BA. In such special cases A & B are said to be Commute. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 2 x 2 $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ 2 x 2 $AB = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ $BA = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

4 Another un usual property of matrix multiplication is that the product of two matrices can be zero even though neither of the two matrices themselves is zero: we can't conclude from the result AB = 0 that at least one of the matrices A or B is a zero matrix

$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 0 0 7 −10 4
0 0 0

1 0 0 8 3 2
0 0 0

5 Also we can't, in matrix algebra, necessarily conclude from the result aB = AC that B= C even if A ≠ 0. Thus the cancellation law doesn't hold, in general, in matrix multiplication

eg. $A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ $C = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$

$$AB = AC = \begin{bmatrix} 10 & 14 \\ -20 & -28 \end{bmatrix} \quad \text{but } B \neq C$$

2.2.3 The multiplicative inverse of a matrix

If A is a square matrix of order n, then a square matrix of its inverse (A^{-1}) of the same order n is said to be the inverse of A, if and only if A x $A^{-1} = I = A^{-1}$ x A

Two square matrices are inverse of each other, if their product is the identity matrix.

$AA^{-1} = A^{-1} A = I$

Not all matrices have an inverse. In order for a matrix to have an inverse, the matrix must, first of all, be a square matrix.

Still not all square matrices have inverse. If a matrix has an inverse, it is said to be INVERTIBLE OR NON-SINGULAR. A matrix that doesn't have an inverse is said to be singular. An invertible matrix will have

only one inverse; that is, if a matrix does have an inverse, that inverse will be unique.

Note: i. Inverse of a matrix is defined only for square matrices

ii. If B is an inverse of A, then A is also an inverse of B

iii. Inverse of a matrix is unique

iv. If matrix A has an inverse, A is said to be invertible & not all. Square matrices

are invertible.

Finding the inverse of a matrix

Lets begin by considering a tabular format where the square matrix A is AUGMENTED with an identity matrix of the same order as [A / I] i.e. the two matrices separated by a vertical line

Now if the inverse matrix A^{-1} were known, we could multiply the matrices on each side of the vertical line by A^{-1} as

$[AA^{-1} / A^{-1} I]$

Then because $AA^{-1} = I$ & $A^{-1}I = A^{-1}$, we would have $[I / A^{-1}]$. We don't follow this procedure, because the inverse is not known at this juncture, we are trying to determine the inverse. We instead employ a set of permissible row operations on the augmented matrix [A / I] to transform A on the left of the vertical line in to an identity matrix (I). As the identity matrix is formed on the left of the vertical line, the inverse of A is formed on the right side. The allowable manipulations are called Elementary raw operations. ELEMENTARY ROW OPERATIONS: are operations permitted on the rows of a matrix.
In a matrix Algebra there are three types of row operations

Type 1: Any pair of rows in a matrix may be interchanged / Exchange operations

Type 2: a row can be multiplied by any non-zero real number / Multiple operation

Type 3: a multiple of any row can be added to any other row. / Add A-multiple operation

In short the operation can be expressed as

1. Interchanging rows
2. The multiplication of any row by a non-zero number.
3. The addition / subtraction of (a multiple of) one row to /from another row

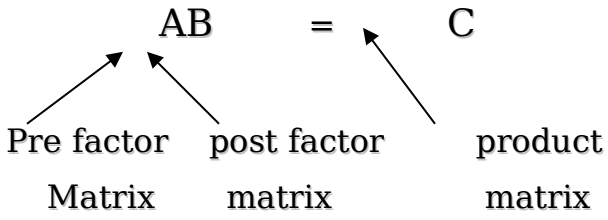
eg.1. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ ⇒ interchanging rows

2. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 6 & 4 \\ -2 & 6 & 7 \end{bmatrix}$ Multiplying the first rows by 2

3. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 & 2 \\ 6 & 12 & 11 \end{bmatrix}$ multiplying the 1st row by 2 & add to the 2nd row. This case there is no charge to the first row.

Theorem on row operations

A row operation performed on product of two matrices is equivalent to row operation performed on the pre factor matrix.



eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{2 \times 3}$ $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}_{3 \times 2}$ $C = \begin{matrix} 9 & 13 \\ 13 & 19 \end{matrix}_{2 \times 2}$

$2 \times 3 \qquad 3 \times 2$

Interchange row1 (R_1) with row-2 (R_2)

$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$ $C = \begin{matrix} 13 & 19 \\ 9 & 13 \end{matrix}$

Basic procedures to find the inverse of a square matrix

1. To set ones first in a column & next zeros (with in a given column)
2. To set zeros first in a matrix & next ones.

<u>Ones first method</u>

Find the inverse of the following matrix

$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

augment A $\quad 3 \quad 2 \quad 1 \quad 0$

with the same dimension $\Rightarrow \quad 1 \quad 1 \quad 0 \quad 1$

identity matrix first

Interchange rows (row 1 with row 2)

$$\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}$$

Multiply R_1 by -3 & add to R_2

$(-3R_1 + R_2)$ i.e. No change to R_1

$$\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}$$

Multiply R_2 by $-1 = (-R_2)$

$$\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}$$

Multiply R_2 by -1 & add to R_1

$$\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 \end{array}$$

Therefore inverse of A is

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

Ones first: try to set ones first in a column and then zeros of the same column. Goes from left to right

Zeros first method

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

Find inverse of A

Augmentation $\rightarrow$

$$\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}$$

$-2R_2 + R_1$

1	0	1	-2
1	1	0	1

$-1R_1 + R_2$

1	0	1	-2
0	1	-1	+3

Therefore, inverse of A

i.e. $A^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

Exercise: Find the inverse for the following matrices (if exist)

1. $A = \begin{pmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ $A^{-1} = 1/3 \begin{pmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{pmatrix}$

2. $B = \begin{pmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}$

3. $C = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}$

4. What do you conclude from question 2 and 3?

5. $D = \begin{pmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{pmatrix}$

Answers for exercises

1) Finding inverse of matrix A next change the remaining number

Uses ones first method. With in the same column into zero.

First augment the matrix with the appropriate operation is multiply

The same dimension identity matrix

i.e. $\begin{array}{cc|cccc} -2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}$

Then our objective is trying to change the given matrix into identify format by applying elementary row operations exchange row −1 with row −2

$\begin{array}{cc|cccc} 1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}$

Next multiply Row 1 by 2 and add the result to Row −2 i.e. $2R_1 + R_2$

$\begin{array}{cc|cccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}$

row 2 by 1 and add the result to row number 1.

i.e. $1R_2 + R_1$

$\begin{array}{cc|cccc} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 0 \end{array}$

now proceed to the third column and change the column into its required form. First change the primary diagonal entries into one. By multiplying the third row by 1/3

i.e. $1/3\ R_3$

$\begin{array}{cc|cccc} 1 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 1/3 \end{array}$

0 1 4 0 0 1 then change the remaining numbers

Now proceed to the 2nd column and change the primary diagonal entry into positive one by applying elementary row operation. The best operation is is exchanging row 2 with row 3

1 −1 0 0 1 0 2/3 0

0 1 4 0 0 1

0 0 3 1 2 0

into zero.

multiply Row 3 by −4 and add the result to raw 2

1 0 4 0

0 1 0 −4/3

0 0 1 1/3

now multiply the third row by −4 add the result to row 1 i.e. $-4R_3 + R_1$

1 0 0 −4/3 −5/3 1

0 1 0 −4/3 −8/3 1

0 0 1 1/3 2/3 0

Therefore the resulting matrix, that is a matrix consisting of the elements at the right side is assumed to be inverse of matrix A i.e. A^{-1}

$$A^{-1} = \begin{pmatrix} -4/3 & -5/3 & 1 \\ -4/3 & -8/3 & 1 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

2) Inverse of matrix B is

$$\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}$$

$$B^{-1} = \begin{matrix} 2 & 4 & -3 \\ 3 & 6 & -5 \end{matrix}$$

3) Inverse of matrix C is

$$C^{-1} = \begin{pmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{pmatrix}$$

4) We can conclude (observe) that matrix B and C are inverse to each other.

5) Matrix d doesn't have an inverse because it is not a square matrix.

2.3 MATRIX APPLICATIONS

2.3.1 Solving Systems of Linear Equations

I. n by n systems

Systems of linear equations can be solved using different methods. Some are:

i. Estimation method for two (2) variable problems (equation)

ii. Matrix method

- Inverse method
- Gaussian method

Inverse method:

Steps 1. Change the system of linear equation into matrix form. The result will be 3

different matrices constructed using coefficient of the variables, unknown values

and right hand side (constant) values

2. Find the inverse of the coefficient matrix

3. Multiply the inverse of coefficient matrix with the vector of constant, and the

resulting values are the values of the unknown matrix.

eg. $2X + 3Y = 4$ Given this system of linear equation applying

$X + 2Y = 2$ inverse method we can find the unknown values.

Step 1. Change it into matrix form

- Using coefficient construct one matrix i.e. coefficient matrix

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$ = Coefficient matrix

- Using the unknown variables construct unknown matrix & it is a column vector (a matrix which has one column)

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$ = vector of unknown

-Using the constant values again construct vector of constant

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$ = vector of constant

Step 2. Find inverse of the coefficient matrix

Now we are familiar how to find an inverse for any square matrix. Assuming once first method find the inverse for matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

Its inverse become $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Step 3. Multiply the coefficient inverse with the vector of constant

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \\ \end{bmatrix}$$

Therefore the resulting matrix that is $\begin{matrix} 2 \\ 0 \end{matrix}$ is

the value for the unknown variables i.e. $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{matrix} 2 \\ 0 \end{matrix}$

Then X = 2 and Y = 0 that is unique solution

* The logic is this given three matrices, coefficient matrix, unknown matrix and vector of constant in the following order.

AX = B

Given this we can apply different Operations, say multiply both sides Of the expression by A^{-1}

A = coefficient matrix

X = vector of unknown

B = vector of constant

$\underline{A^{-1}AX} = A^{-1}B$

$\underline{IX} = A^{-1}B$

$X = A^{-1}B$ this implies that multiplying inverse of the coefficient matrix will gives us the value of the unknown matrix

Limitations of inverse method

- It is only used whenever the coefficient matrix is square matrix
- In addition to apply the method the coefficient matrix needs to have an inverse
- It doesn't differentiate between no solution and infinite solution cases.

Gausian method

It is developed by a mathematician Karl F. Gauss (1777-1855). It helps to solve systems of linear equations with different solution approaches i.e. unique solution, No solution and infinite solution cases.

"n" by "n" systems

Step: 1. Change the system of linear equation into a matrix form

2. Augument the coefficient matrix with the vector of constant.

3. Change the coefficient matrix into identity form by applying elementary row operation and apply the same on the vector of constant.

4. The resulting values of the vector of constant will be the solution or the value of the unknown

Example: $\begin{cases} 2X + 3Y = 4 \\ X + 2Y = 2 \end{cases}$

Step 1. Change it into matrix form

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Coefficient Matrix | unknown matrix | vector of constant

Step: 2. Augumentation

$$\left[\begin{array}{cc|cc} 2 & 3 & 4 \\ 1 & 2 & 2 \end{array}\right]$$

Step: 3. Change the coefficient matrix into identity form by applying elementary row operation (use ones first method)

$$\left[\begin{array}{cc|cc} \bigcirc\, 2 & 3 & 4 \\ 1 & 2 & 2 \end{array}\right]$$

Change first the primary diagonal entry from the first row into positive one. Possible operation is exchange row one with row two.

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 \\ \bigcirc\, 2 & 3 & 4 \end{array}\right]$$

Next change the remaining numbers in the first column into zero, this case number 2

Now multiply the 1st row by −2 & add the result to row −2

$$\begin{array}{cc|cc} 1 & & 2 & 2 \\ 0 & & -1 & 0 \end{array}$$

Then proceed to column 2 and change the primary diagonal entry i.e. −1 into 1

Multiply the 2nd row by −1 ($-1R_2$)

$$\begin{array}{cc|cc} 1 & & 2 & 2 \\ 0 & & 1 & 0 \end{array}$$

Now change the remaining number with in the same column (column −2) into zero i.e. number 2

Multiply 2nd row by −2 and add the result to the 1st row

$$\begin{array}{cc|cc} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}$$

Therefore X = 2 and Y = 0

Example 2. $\begin{cases} X + Y = 2 \\ 2X + 2Y = 4 \end{cases}$

Step-1

$$\begin{bmatrix} 1 & \\ 2 & \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{matrix} 2 \\ 4 \end{matrix}$$

Step-2

$$\begin{array}{cc|cc} 1 & & 1 & 2 \\ 2 & & 2 & 4 \end{array}$$

Multiply Row-1 by −2 & add the result to raw-1 ($-2R_1 + R_2$)

$$\begin{array}{cc|cc} 1 & & 1 & 2 \\ 0 & & 0 & 0 \end{array}$$

The next step is changing the primary diagonal entry in the 2nd row to 1. But there is no possible operation that can enable you to change it in to number 1

Therefore the implication is that you can't go further but we can observe something from the result. And it is implying an infinite solution case

Example 3. $\begin{cases} X + Y = 5 \\ X + Y = 9 \end{cases}$

Step 1. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{matrix} 5 \\ 9 \end{matrix}$

Step 2 $\left[\begin{array}{cc|cc} 1 & & 1 & 5 \\ (1) & & 1 & 9 \end{array}\right]$

Change the encircled number above in to zero

Multiply the first row by −1 & add the result to the 2nd row.

$\left[\begin{array}{cc|cc} 1 & & 1 & 5 \\ 0 & & 0 & 4 \end{array}\right]$

0 = 4 no solution

There is no possible operation that we can apply in order to change the primary diagonal entry in the 2nd column without affecting the first column structure. Therefore stop there, but here we can observe something i.e. it is no solution case.

Therefore, Gaussian method makes a distinction between No solution & infinite solution. Unlike the inverse method.

* Summarizing our results for solving an "n" by "n" system, we start with the matrix.

(A/B), & attempt to transform it into the matrix (I/C) one of the three things will result.

1. an □n□ by □n□ matrix with the unique solution.

eg.

1	0	0	10
0	1	0	-5
0	0	1	3

2. A row that is all zeros except in the constant column, indicating that there are no solutions,

eg.

1	0	0	3
0	1	0	-5
0	0	0	7

3. A matrix in a form different from (1) & (2), indicating that there are an unlimited number of solutions. Note that for an n by n system, this case occurs when there is a row with all zeros, including the constant column.

Eg.

1	0	2	5
0	1	3	-3
0	0	0	0

Reference Exercise

1. $\begin{cases} X + 2Y - 3Z = 11 \\ 3X + 2Y + Z = 1 \end{cases}$ 2. $\begin{cases} X + Y + Z = 4 \\ 5X - Y + 7Z = 25 \end{cases}$ 3. $\begin{cases} X + Y + Z = 4 \\ 5X - Y + 7Z = 20 \\ 2X + Y - 5Z = 11 \end{cases}$ $\begin{cases} 2X - Y + 3Z = 8 \\ X - Y + 3Z = 8 \end{cases}$

Unique solution case * No solution case *

Many solution case

i.e X = -1

Y = 3

Z = 2

4. $\begin{cases} 2X + 6Y - Z = 18 \\ Y + 3Z = 9 \\ 3X - 5Y + 8Z = 4 \end{cases}$ X = 1, Y = 3, Z = 2

II M by n linear systems

The m x n linear systems are those systems where the number of rows (m) and number of columns (n) are unequal or it is the case where the number of equations (m) & the number of variables (n) are unequal. And it may appear as m > n or m < n.

Linear equation where m > n

To solve an m by n system of equations with m > n, we start with the matrix (A/B) and attempt to transform it into the matrix (I/C).

One of the three things will result:

1. An m by n identifying matrix above m − n bottom rows that are all zeros, giving the

unique solution:

1	0	0	3	$3X_1 + 2X_2 + X_3 = 23$
0	1	0	-5	$X_1 + 3X_2 + 2X_3 = 26$
0	0	1	4	$2X_1 + X_2 + 2X_3 = 10$
3, 5, 4				
0	0	0	0	$4X_1 + 5X_2 + 3X_3 = 49$

2. A row that is m − n bottom raw is all zeros except in the constant column, indicating that there are no solutions

eg.	1	0	0	3	$2X_1 + X_2 = 30$
	0	1	0	-5	$X_1 + 2X_2 = 24$
	0	0	1	7	$4X_1 + 5X_2 = 72$
	0	0	0	1	

3. A matrix in a form different from (1) & (2), indicating that there are an unlimited

number of solutions

eg.	1	0	2	-4	$3X_1 + 2X_2 + X_3 = 6$
	0	1	3	8	$6X_1 + 4X_2 + 3X_3 = 12$
	0	0	0	0	$9X_1 + 6X_2 + 3X_3 = 18$
	0	0	0	0	$15X_1 + 10X_2 + 5X_3 = 30$

Linear Equations where m < n

Our attempts to transform (A/B) into (I/C) in the case where m < n will result in:

1. A raw which is all zeros except in the constant columns, indicating that there are no solutions, or

2. A matrix in a form different from number one above indicating that there are an unlimited number of solutions.

Every system of linear equations has either No solution, Exactly one solution or infinitely many solutions.

Example Solve the following systems of linear equations

1) $4X_1 + 6X_2 - 3X_3 = 12$

$6X_1 + 9X_2 - 9/2X_3 = 20$

No solution

$X_1 + 3X_2 + X_3 = 6$

$-X_1 + X_2 + X_3 = 2$

Unlimited solution

$2X_1 + 3X_2 + 4X_3 + X_4 = 37$

$X_1 + 2X_2 + 3X_3 + 2X_4 = 24$

$3X_1 + X_2 + X_3 + 3X_3 = 33$

Unlimited solution

Solution for an □n□ by □n□ system

1.

1	2	-3	11	
3	2	1	1	25
2	1	-5	11	

$-3R_1+R_2$ / $-2R_1+R_3$

1	2	-3	11	
0	-4	10	-32	
0	-5	4	-22	

$-1/4\ R_2$ / $-2\ R_2 + R_1$ / $3\ R_2 + R_3$ + R_3

1	0	2	-5	
0	1	-5/2	8	13/2
0	0	-13/2	13	□5/6

$-2/13\ R_3$ / $5/2\ R_3 + R_2$ / $-2\ R_3 + R_1$

1	0	0	-1
0	1	0	3
0	0	1	-2

2.

1	1	1	4	
5	-1	7		
2	-1	3	8	

□$5R_1 + R_2$ / $-2R_1 + R_3$

1	1	1	4	
0	-6	2	5	
0	-3	1	0	

$-1/6\ R_2$ / $-R_2+R_1$ / $3\ R_2$ + R_3

1	0	4/3		13/2
0	1	-1/3		□5/6
0	0	0		-5/2

∴this implies No solution case

3) 1 1 1

4

1 0 0 X -1

5 -1 7

20

0 1 0 Y = 3

2 -1 3

8

0 0 1 Z -2

−5R₁ + R₂ / -2R₁ + R₂

X -1

= 3

4

Y

Z −2

1 1 1

0

0 -6 2

∴X = -1 Y = 3 Z = -2

0 -3 1

0

-1/6 R₂ / 3 R₂ + R₃ / -R₂ + R₁

1 0 4/3

4

0 1 -

1/3 0

0 0 0

0

This implies that there are so many or infinite solution

"m" by "n" systems

1) 3 2 1 23

1 3 2 26

2 1 2 19

2) 2 1 30

1 2 24

4 5 72

4 5 3 49

$R_1 \Leftrightarrow R_2 / -2R_1 +$ $R_2 / -4R_1 + R_3$

$R_1 \Leftrightarrow R_2 / -3R_1 + R_2 / -2R_1 + R_3 / -4R_1 + R_4$

1 3 | 2 26

0 -7 | -5 -55

0 -5 | -2 -33

0 -7 | -5 -55

1 2 24

0 -3 -18

0 -3 -24

$-1/7\ R_2 / -3R_2 + R_1 / 5R_2 + R_3 / 7R_2 + R_4$ $-1/3\ R_2 / -2R_2 + R_1$ $/ 3R_2 + R_3$

1 0 -1/7 17/7 | 1 0 14

0 1 5/7 55/7 | 0 1 6

0 0 11/7 44/7 | 0 0 -6

$7/11R_3 / -5/7R_3 + R_2 / 1/7R_3 + R_1$ this implies that there is no solution which uniquely satisfy the system

1 0 | 0 3

0 1 | 0 5

0 0 | 1 4

0 0 0 0

3) 3 2 | 1 6

6 4 | 3 12

9 6 | 3 18

15 10 | 5 30

Unique solution case

$1/ 3R_1 / -6R_1 + R_2 / -9R_1 + R_3 /$

$-15R_1 + R_4$

$\therefore \underline{X_1 = 3} \quad \underline{X_2 = 5} \quad \underline{X_3 = 4}$

$$\left.\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{0} \end{array}\right.$$

$\therefore$This implies that there are unlimited numbers of solutions

m by n system where m <n

i.e. number of equations are less than # of variables

1) $4X_1 + 6X_2 - 3X_3 = 12$

$6X_1 + 9X_2 - 9/2X_3 = 20$

$$\begin{pmatrix} 4 & 6 & -3 \\ 6 & 9 & -9/2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$$

$$\left.\begin{array}{ccc|c} 4 & 6 & -3 & 12 \\ 6 & 9 & -9/2 & 20 \end{array}\right.$$

$1/4R_1 \;/\; -6R_1 + R_2$

$$\left.\begin{array}{ccc|c} 1 & 3/2 & -3/4 & 3 \\ 0 & 0 & 0 & \underline{2} \end{array}\right.$$

No solution

2) $X_1 + 3X_2 + X_3 = 6$

$-X + X_2 + X_3 = 2$

$$\begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

1	3	1	6
-1	1	1	2

$1xR_1 + R_2$

1	3	1	6
0	4	2	8

$1/4R_2$

1	3	1	6
0	1	½	1/2

Infinite solution

2.3.2 Word problems

Steps

1. Represent one of the unknown quantities by a letter usually X & express other

unknown quantities if there is any in terms of the same letter like X_1 X_2 etc

2. Translate the quantities from the statement of the problem in to algebraic form & set

up an equation

3. Solve the equation (s) for the unknown that is represented by the letter & find other

unknowns from the solution

4.check the findings according to the statement in the problem

Example:

1) A Manufacturing firm which manufactures office furniture finds that it has the

following variable costs per unit in dollar/unit

	Desks	Chairs	Tables	Cabinet
Material	50	20	15	25
Labor	30	15	12	15
Overhead	30	15	8	20

Assume that an order of 5 desks, 6 chairs, &4 tables & 12 cabinets has just been received. What is the total material, labor & overhead costs associated with the production of ordered items?

Answer:

Material cost = $ 750

Labor cost = $ 918

Overhead cost = $ 512

2. Kebede carpet co. has an inventory of 1,500 square yards of wool & 1,800 square yards of nylon to manufacture carpeting. Two grades of carpeting are produced. Each roll of superior grade carpeting requires 20 sq. yards of wool & 40sq. yards of nylon. Each roll of quality-grade carpeting requires 30 square yards of wool & 30 square yard of nylon. If Kebede would like to use all the material in inventory, how many rolls of superior & how may rolls of quality carpeting should be manufactured?

15 & 40

3. Getahun invested a total of br. 10000 in three different saving accounts. The accounts paid simple interest at an annual rate of 8%, 9% & 7.5% respectively. Total interest earned for the year was br. 845. The amount in the 9% account was twice the amount invested in the 7.5% account. How much did Getahun invest in each account?

1000, 6000, 3000

4. A certain manufacturer produces two product P & q. Each unit of product P requires (in its production) 20 units of row material A & 10

units of row material B. each unit of product of requires 30 units of raw material A & 50 units of raw maternal B. there is a limited supply of 1200 units of raw material A & 950 units of raw material B. How many units of P & Q can be produced if we want to exhaust the supply of raw materials?

Answer: 45 units of P and
10 units of Q

5. Attendance records indicate that 80,000 South Koreans attended the 2002 world cup at its opening ceremony. Total ticket receipts were Birr 3,500,000. Admission prices were Birr 37.5 for the second-class and Birr 62.50 for the first class. Determine the number of South Koreans who attended the football game at first class and second class.

Solutions / word problems

1)

	D	Ch	T	Cb		
Mt	50	20	15	25		5
Desk						
Lab.	30	15	12	15		6
Chair						
FOH	30	15	8	20		4
Tables						
						12
Cabinet						

Material cost = (50 x 5) + (20 x 6) + (5 x 4) + (25 x 12)
= 730 br.

Labor cost = (30 x 5) + (15 x 6) + (12 x 4) + (15 x 12)
= $ 468.00

FOH cost = (30 x 5) + (15 x 6) + (8 x 4) + (20 x 12)
= $ 1710.00

2) Given: wool inventory 1500 yards

nylon inventory 1800 yards

	Grade one carpet	The other grade carpet
Wool requirement per roll	20 Sq.	30 Sq.
Nylon requirement per roll	40 Sq.	30 Sq.

Let X be rolls of superior grade carpeting &

Y be rolls of quality grade carpeting

$$\begin{bmatrix} 20 & 30 \\ 40 & 30 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1500 \\ 1800 \end{bmatrix} \qquad \left| \begin{array}{cc} 20 & 30 \\ 40 & 30 \end{array} \right| \begin{array}{c} 1500 \\ 1800 \end{array}$$

$$\left| \begin{array}{cc} 2 & 3 \\ 4 & 3 \end{array} \right| \begin{array}{c} 150 \\ 180 \end{array}$$

$\therefore 20X + 30Y = 1500$

$\underline{40X + 30Y = 1800}$ $\quad 1/2\ R_1 \,/\, -4R_1 + R_2$

$-20X = -300$

$\underline{\underline{X = 15}}$

$$\left| \begin{array}{cc} 1 & 3/2 \\ 0 & -3 \end{array} \right| \begin{array}{c} 75 \\ -120 \end{array}$$

20X + 30Y = 1500 □1/3 R_2/ -3/2 R_2 + R_1

20 (15) + 30Y = 1500 | 1 0 15

30Y = 1500 □ 300 0 1 40

Y = 1200
30

Y = 40 ∴X = 15 & Y = 40

3) Total investment 10000.00

interest earned from the three accounts 8%, 9% and 7.5%

total interest earned for the year was 845.00

amount in 9% = 2 (amount in 7.5%)

Let amount invested in 8% account be X_1

amount invested in 9% account be X_2

amount invested in 7.5% account be X_3

$X_1 + X_2 + X_3 = 10{,}000$

$0.08X_1 + 0.09X_2 + 0.075X_3 = 845$

$X_2 = 2(X_3)$

or

$X_1 + X_2 + X_3 = 10{,}000$

$0.08X_1 + 0.09X_2 + 0.075X_3 = 845$

$X_2 - 2X_3 = 0$

1	1	1	10,000
80	90	75	845,000
0	1	-2	0

$-80R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10{,}000 \\ 0 & 10 & -5 & 45{,}000 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$1/10R_2 \quad -R_2 + R_1 \quad -R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 5500 \\ 0 & 1 & -1/2 & 4500 \\ 0 & 0 & -3/2 & -4500 \end{array}\right]$$

$-2/3\ R_3 \quad ½\ R_3 \quad ½\ R_3 + R_2 \quad -3/2\ R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1000 \\ 0 & 1 & 0 & 6000 \\ 0 & 0 & 1 & 3000 \end{array}\right]$$

$\therefore \underline{X_1 = 1000}, \quad \underline{X_2 = 6000}, \quad \underline{X_3 = 3000}$

2.3.3 Markov Chains

This model is a forecasting model. It is probabilistic (stochastic) model. A Russian Mathematician called Andrew Markov around 1907 develops this model.

Markov chains are models, which are useful in studying the evolution of certain systems over repeated trials. These repeated trials are often successive time periods where the state (outcome condition) of the systems in any particular time period can□t be determined with certainty. Therefore, a set of transition probabilities is used to describe the manner in which the system makes transition from one period to the next. Hence,. We can predict the probabilities of the system being in a particular state at a given time period. We can also talk about the long run or equilibrium or steady state.

The necessary assumptions of the chain:

1.The system condition (outcome) state in any given period depends on its state in the

Preceding period & on the transition probabilities

2. The transition probabilities are constant overtime

3. Change in the system will occur once & only once each period

eg. If it's a week, its only once in a week

4. The transition period occurs with regularities

* if we start with days, we use the day until we reach our end.

Information flow in the analysis

The Markov model is based on two sets of input data

- The set of transition probabilities
- The existing or initial or current conditions or states

The Markov process, therefore, describes the movement of a system from a certain state in the current state/time period to one of n possible states in the next stage. The system makes in an uncertain environment, all that is known is the probability associated with any possible move or transition. This probability is known as transition probability, symbolized by P_{ij}. It is the likelihood that the system which is currently in state i will move to state j in the next period. From these inputs the model makes two predictions usually expressed as vectors.

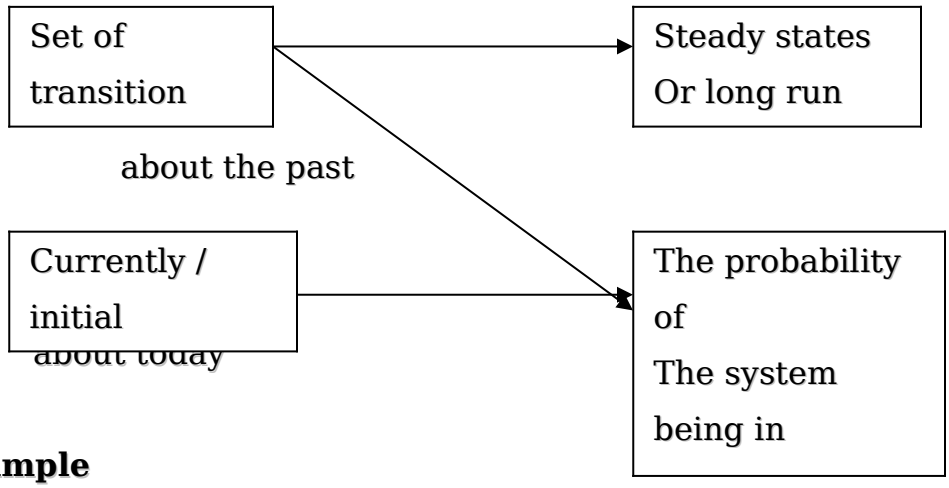
1. The probabilities of the system being in any state at any given future time

period

2. The long run (equilibrium) or steady state probabilities.

The set of transition probabilities are necessary for both prediction (time period n, & steady state), but the initial state is needed for only the first prediction.

Input data **prediction (outcome)**



Example

Currently its known that 80% of customers shop at store 1 & 20% shop at store 2. In reviewing a past data suppose we find that out of all customer who shopped at store 1 in a given week 90% remain loyal for the next week (store one again), 10% switch to store 2. On the other hand, out of all customers who shopped at store 2, in a given week 80% remains loyal for the next week (store 2 again), 20% switch to store 1. What will be the proportion of customers shopping at store 1 & 2 in each of the next two weeks.

Solution

Let S_1 be the proportion of store 1

S_2 be the proportion of store 2

- Initial state/current state probability matrix for store one and two will be:

  V_{12} = (0.8 0.2)

To the next weekly shopping period

from one S_1 S_2 Transition probability matrix

week S_1 0.9 0.1 is a square matrix such that each

shopping S_2 0.2 0.8 entry indicates the prob. Of the

period system moving from a given state to

another state.

- The sum of rows in the transition matrix should be 1
- We have to be consistent in writing the elements

Markov Chain Formula

a) $V_{ij}(n) = [V_{ij}(n - 1)] \times P$ where P = Transition matrix

$V_{ij}(n)$ = Vector for period n

$V_{ij}(n - 1)$ = Vector for period n − 1

$V_{12}(0) = (0.8\ 0.2)$ ← current share

$V_{12}(1) = V_{12}(1 - 1) \times P$

$= V_{12}(0) \times P$

$$(0.8\ 0.2) \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$$

$= (0.8 \times 0.9) + (0.2 \times 0.2)$ $(0.8 \times 0.1) + (0.2 \times 0.8)$

$= 0.72 + 0.04$ $0.08 + 0.16$

$= \underline{0.76}$ $\underline{0.24}$

$V_{12}(1) = (0.76\ \ 0.24)$

$V_{12}(2) = V_{12}(2 - 1) \times P$

$= V_{12}(1) \times P$

$$(0.76\ 0.24)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$$

(0.76x 0.9) + (0.24 x 0.2) 0.76 (0.1) + (0.24 x 0.8)

(0.732 0.268)

b) Long run market share

Assumption

In the log run the share of the systems is assumed to be constant.
Let - the share of store 1 in the long run be V_1
- the share of store 2 in the long run be V_2

$$\underset{(V_1\ V_2)}{n}\ \underset{\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}}{p} = \underset{(V_1\ V_2)}{n+1}$$

$0.9V_1 + 0.2V_2 = V_1$

$0.1V_1 + 0.8V_2 = V_2$

$-V_1 + 0.9V1 + 0.2V_2 = 0$

$0.1V_1 + 0.8V2 - V_2 = 0$

$-0.1V_1 + 0.2V_2 = 0$

$0.1V_1 + (0.2V_2) = 0$

$0.9V_1 + 0.2 (V_2 - V_1) = V_1$

$0.9V_1 + 0.2 - 0.2V_1 = V_1$

$0.7V_1 + 0.2 = V_1$

$0.2 = 0.3V_1$

$V_1 = 2/3$

$V_2 = 1 - V_1$

$= 1 - 2/3$

$V_2 = 1/3$

In the long run 67% of the customers will shop in store 1 & 33% in shop 2.

Prediction:

Long run: only the transition matrix

= At specified time:- the transition matrix & state vector. Hence unless the transition matrix is affected, the long run state will not be affected. Moreover, we can't know the number of years, weeks to attain the long run state / point but we can know the share

Exercises

1. A division of the ministry of public health has conducted a simple survey on the public
attitude to wards smoking . From the results of the survey the department concluded that currently only 20% of the population smokes cigarette & every month 10% of non-smokers become smokers where as 5% of smokers discontinue smoking.

Required:

1. Write the current & transition matrices
2. What will be the proportion of the non-users (non-smokers) & users (smokers) in the
long run

Solution

1) Let U stands for Smokers
N stands for non-Smokers

Initial state $V_{UN}^{(0)}$ = (0.2 0.8)

To the next month

		Smokers	Non smokers
From	Smokers	0.95	0.05

$$P = \begin{matrix} \\ .95 \\ .05 \end{matrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

one month Non-smokers 0.10 0.90

.10 .90

2. $V_{UN}^{(1)} = V_{UN}^{(0)} \times P$ $V_{UN}^{(2)} = V_{UN}^{(1)} \times P$

$= (0.2\ 0.8) \begin{pmatrix} .95 & .05 \\ .10 & .90 \end{pmatrix}$ $= (.27\ .73) \begin{pmatrix} .95 & .05 \\ .10 & .90 \end{pmatrix}$

$= (\ 0.27 \quad 0.73)$ $\equiv (0.3295 \quad 0.6705)$

2. A population of 100,000 consumers make the following purchases during a particular week: 20000 purchase Brand A, 35,000 Brand B & 45000 purchase neither Brand. From a market study, it is estimated that of those who purchase Brand A, 80% will purchase it again next week, 15% will purchase brand B next week, & 5% will purchase neither brand. Of those who purchase B, 85% will purchase it again next week, 12% will purchase brand A next week, & 3% will purchase neither brand. Of those who purchase neither brand, 20% will purchase A next week, 15% will purchase Brand B next week, & 65% will purchase neither brand next week. If this purchasing pattern continues, will the market stabilize? What will the stable distribution be?

Yes. The share of A, B and C is = (0.4 0.5 0.1) respectively

3. A vigorous television advertising campaign is conducted during the football reason to promote a well-known brand X shaving cream. For each of several weeks, a survey is made & it is found that each week 80% of those using brand X continue to use it & 20% switch. It is also

found that those not using brand X, 20% switch to brand X while the other 80% continue using another brad.

a) Write the transition matrix, assuming the transition percentage continue to hold for

succeeding weeks.

b) If 20% of the people are using brand X at the start of the advertising campaign,

what percentage will be brand X 1week later? Two weeks later?

Solutions for question number 2, 3 & 4

2) Given: 20,000 purchase brand A

35,000 purchase brand B

45,000 purchase neither brand

Total consumers = 100,000 (20,000 + 35,000 + 45,000)

Let V_A represents the share of brand A purchasers

V_B represents the share of Brand B purchasers

V_N represent the share of neither brand purchasers

The system is arranged in a weekly basis

		To the next week shopping period			
From		A	B	N	
One week	A	0.80	0.15	0.05	
Shopping	B	0.12	0.85	0.03	= P
Period	N	0.20	0.15	0.65	

The stable market means the long run or steady state market because it is noted that in the long run the share will be stable.

And in the long run we have said that the share at n period is equal with the share at n + 1 period. Therefore

(The share at n period) x (the transition probabilities) = (the share at n + 1 period)

Let the share of brand A purchasers be V_1 in the long run

the share of brand B purchasers be V_2 in the long run

the share of neither purchasers be V_3 in the long run

Then $(V_1 \quad V_2 \quad V_3)$ $\begin{pmatrix} 0.80 & 0.15 & 0.05 \\ 0.12 & 0.85 & 0.03 \\ 0.20 & 0.15 & 0.65 \end{pmatrix}$ $= (V_1 \quad V_2 \quad V_3)$

$0.8V_1 + 0.12V_2 + 0.2V_3 = V_1$

$0.15V_1 + 0.85V_2 + 0.15V_3 = V_2$

$0.05V_1 + 0.03V_2 + 0.65V_3 = V_3$

$V_1 + V_2 + V_3 = 1$

Therefore $V_1 = \underline{1 - V_2 - V_3}$

Substitute $1 - V_2 - V_3$ in place of V_1 to reduce your system in to two variable case

Take the first equation

$0.8V_1 + 0.12V_2 + 0.2V_3 = V_1$

$0.8 (1- V_2 - V_3) + 0.12V_2 + 0.2V_3 = 1 - V_2 - V_3$

$0.8 - 0.8V_2 - 0.8V_3 + 0.12V_2 + 0.2V_3 + V_2 + V_3 = 1$

add similar variables together

$0.32V_2 + 0.4V_3 = 1- 0.8$

$\underline{0.32V_2 + 0.4V_3 = 0.2}$ this is a summarized equation from the first equation. And do the same for the second equation.

$0.15V_1 + 0.85V_2 + 0.15V_3 = V_2$

$0.15 (1 - V_2 - V_3) + 0.8V_2 + 0.15V_3 = V_2$

$0.15 - 0.15V_2 - 0.15V_3 + 0.85V_2 + 0.15V_3 = V_2$

add together similar variables and take the constant values to the right side.

$-0.3V_2 = -0.15$

$V_2 = \underline{-0.15}$

-0.30

$\underline{V_2 = 0.5}$ this implies the long run share for the purchasers of brand B will be 50%. Then using this value and the first summarized equation ($0.32V_2 + 0.4V_3 = 0.2$)

Find the value of V_3 in the long run

That is, $0.32\ (0.5) + 0.4V_3 = 0.2$

$0.16 + 0.4V_3 = 0.2$

$0.4V_3 = 0.20 - 0.16$

$V_3 = \underline{0.04}$

0.04

$\underline{V_3 = 0.1}$

This implies the share of neither brand purchases in the long run will be 10%. Hence the share of Brand A purchasers will be 0.4 (1 − 0.5 − 0.1)

Because we said above $V_1 = 1 - V_2 - V_3$

Therefore $V_1 = 1 - 0.5 - 0.1$

$\underline{V_1 = 0.4}$

That is the long run share of Brand A purchasers will be 40% of the total population

i.e. $V_A \quad V_B \quad V_N = \quad (0.4 \quad 0.5 \quad 0.1)$

3) A) let X be the proportion of brand X users

X^1 be the proportion of other brand users

To the next week

Shopping period

From this		X	X^1	
Week	X	0.80	0.20	= P
Shopping	X^1	0.20	0.80	
period				

B) Current users of brand X = 20% then, the users of other brand will be 80%

V_{XX}^1 (0) = (0.20 0.80)

V_{XX}^1 (1) = V_{XX}^1 (0) [P]

= (0.20 0.8) $\begin{pmatrix} 0.80 & 0.20 \\ 0.20 & 0.80 \end{pmatrix}$

= <u>(0.32 0.68)</u>

The proportion of brand X and other brand users after one week period is expected to be 32% and 68% respectively. Then the expected users in the 2nd week will be

V_{XX}^1 (2) = V_{XX}^1 (2 − 1) (P)

= V_{XX}^1 (1) P

= (0.32 0-.68) $\begin{pmatrix} 0.80 & 0.20 \\ 0.20 & 0.80 \end{pmatrix}$

= <u>0.392 0.608</u>

The expected share of brand X and other brand users is 39.2% and 60.8% in the second week.