

PART II

Discrete Programming Problems :

Integer programming,
Binary Programming,
Transportation and Assignment
Problems

- Integer programming (values of decision variables, only **integer**, i.e., **not fractional**)

Include in the model: $x_1, \dots, x_n \geq 0$
and Integer

- Binary programming (values of decision variables can be **either 0 or 1** only).

Include in the model: $x_1, \dots, x_n \geq 0$
and bin

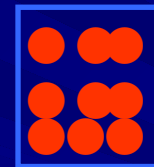
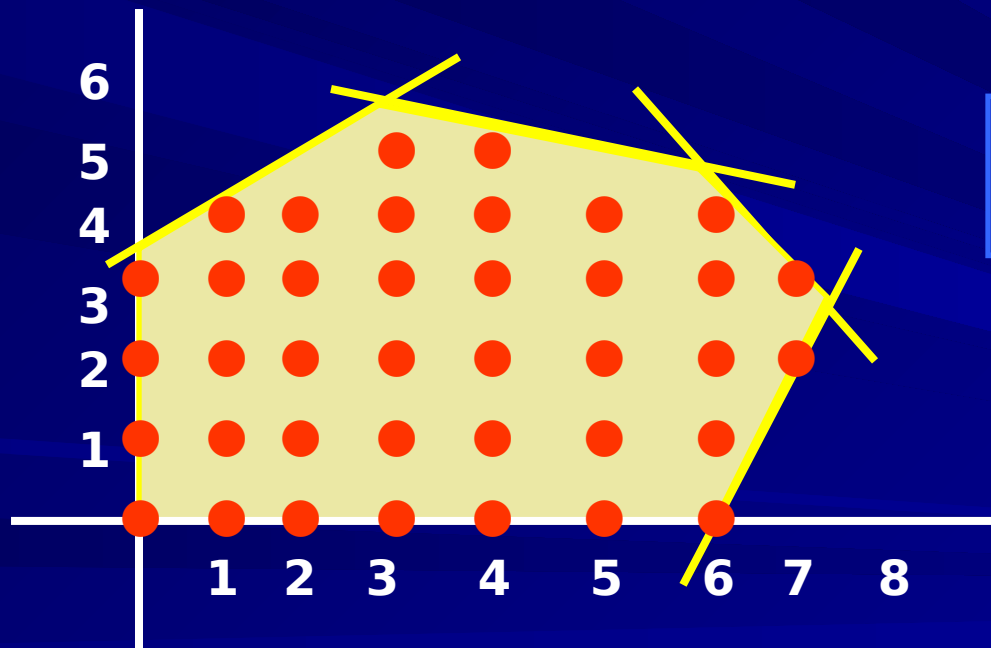
- Transpiration Problems, and
- Assignment Problems

1. Integer Programming (IP)

- **General form :**

$$\begin{aligned} \text{Max } & c^T x \\ \text{s.t } & Ax \leq b \\ & x \geq 0 \text{ and Integer} \end{aligned}$$

✱ Thus, **Feasible set of an IP** is the set of **integer points** in a **polyhedral** given by $Ax \leq b$.



**Feasible
Points
(Integer
Points)**

**Solution : Excel
solver**

**Same as LP except
adding the
constraint:**

Example

• Use Excel to solve the following IP problem

$$\begin{array}{ll}\max & x_1 + 4x_2 \\ \text{s.t.} & -x_1 + x_2 \leq 1 \\ & x_1 + 3x_2 \leq 14 \\ & x_1, x_2 \geq 0 \text{ \& Integer}\end{array}$$

Optimal Solution: $x_1 = 5, x_2 = 3, z^* = 17$

IP Problem Formulation Examples:

1. Community service center of a town is open for a service

every day from 8:00 am to 6: 00 pm.

Regulation of the center requires that every employee works

three days

Mon d.	Tues d	Wed.	Th.	Frida y	Sat.	Sund .
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Problem: To determine the minimum number of total employees needed at the service center that meets the daily minimum number of required employees.

Write an IP formulation (optimization model) of the

Decision Variable: Let x_i = number of employees who start duty on Day i,

$i = 1, 2, \dots, 7.$

➤ Hence the **IP** formulation of the problem of the service center is:

$$\min x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

$$\text{s.t. } x_1 + x_5 + x_6 + x_7 \geq b_1$$

$$x_1 + x_2 + x_6 + x_7 \geq b_2$$

$$x_1 + x_2 + x_3 + x_7 \geq b_3$$

$$x_1 + x_2 + x_3 + x_4 \geq b_4$$

$$x_2 + x_3 + x_4 + x_5 \geq b_5$$

$$x_3 + x_4 + x_5 + x_6 \geq b_6$$

$$x_4 + x_5 + x_6 + x_7 \geq b_7$$

$$x_1, x_2, \dots, x_7 \geq 0 \text{ \& integer}$$

Assignment 1: (Nurse Scheduling)

Each nurse in a hospital is **on duty for five consecutive days** and **off for two consecutive days**. The five days of the duty can start on any day of the week and the schedule rotates indefinitely. The **minimum number of nurses required to be on duty on each day** is given in the following table.

Mon d.	Tues d	Wed.	Th.	Frida y	Sat.	Sund .
25	30	28	30	24	15	12

The management wants to determine the minimum number of total nurses required to be employed in the hospital while satisfying the daily requirements.

- Write an integer programming formulation of the problem
- Find the optimal solution of the problem .
- According to your solution, on which days do you have excess number of nurses beyond the day's requirement and by how many?

2. **Binary (0-1) Programming problems:**
LP in which the values of the **decision variables** can be only **0 or 1** is called **Binary (or 0-1) Programming problem**.

Example 1: Project Selection (Knapsack Problem)
Consider n investment projects $P_1, \dots, P_n$ that can be undertaken only fully, with expected return $r_1, r_2, \dots, r_n$, respectively. Suppose the running cost of the projects are $a_1, a_2, \dots, a_n$, resp.; and the available capital (budget) for the investment is b .
Problem: To choose a set of projects that can be undertaken with the available budget that can maximize the total return.

Mathematical Formulation of the problem:.

Decision variables: $x_j = 1$, if P_j is undertaken, 0

$$\begin{aligned} \Rightarrow \text{Optimal } \max & r_1x_1 + r_2x_2 + \dots + r_nx_n \\ \text{s.t } & a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b \\ & x_j \in \{0, 1\}, \text{ for all } j \end{aligned}$$

Example:

Consider 5 investment projects, $P_1, P_2, \dots, P_5$, each of which can be undertaken only fully (i.e., not in a portion).

The **return** from each is: **9, 8, 6, 4** and **3** units of money,

and the **running cost** for each is: **7, 6, 3, 2** and **1**, resp.

The **available budget** for the investment is **10** units of money. It is required to choose some of the projects that **maximize the total return**. Write the mathematical formulation of the problem and solve.

Assignment 2: (Cargo Loading

Problem) Some of the following eight indivisible items that maximizes total value is to be transported by a cargo plane. The value, unit weight (in ton) and unit volume (in m^3) of each item is given in the following table.

	I1	I2	I3	I4	I5	I6	I7	I8
Value	5	6	4	8	3	3	4	5
weight	3	5	4	7	2	1	4	3
Volume	1	3	2	4	2	2	3	2

The cargo plane can carry at most **20 ton** and has space (volume) of **15 m^3** . Which of the items should be loaded (transported) to maximize the total value of the transported items without exceeding the capacity of the cargo plane? Formulate the problem and solve

3. Transportation Problem (TP)

Consider m supply centers $S_1, \dots, S_m$, where a_i units of indivisible items are available at S_i ; and n demand centers $D_1, \dots, D_n$, where D_j wants b_j units of the items.

Let unit cost of shipping (transporting) item from S_i to D_j be c_{ij}

The problem is to find a transportation plan so that

all demands are satisfied and all items at every supply center is shipped; and do so with minimum transportation cost.

Balanced TP: If $\sum a_i = \sum b_j$ (Total demand = Total Supply)

Mathematical formulation for balanced TP:

• **Decision variables:**

Let x_{ij} = Unit of items shipped from S_i to D_j , for each i, j

❖ **Constraints:**

I. Supply : Ship all items at S_i to any of the demand

$$x_{i1} + x_{i2} + \cdots + x_{in} = a_i, \quad i = 1, 2, \dots, m$$

II. Demand: Meet the demand at each D_j .

$$x_{1j} + x_{2j} + \cdots + x_{mj} = b_j, \quad j = 1, 2, \dots, n$$

❖ **Objective function:** Total cost of shipping:

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij}$$

❖ **Hence the TP is:**

$$\begin{aligned} \min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} \\ \text{s.t} \quad & x_{i1} + x_{i2} + \cdots + x_{in} = a_i, \quad i = 1, 2, \dots, m \\ & x_{1j} + x_{2j} + \cdots + x_{mj} = b_j, \quad j = 1, 2, \dots, n \\ & x_{ij} \geq 0, \quad \text{for all } i, j \end{aligned}$$

Example:

Determine the optimal transportation plan of the following TP problem (from three sources to four destinations) with the following unit cost matrix, supply capacities and demands:

from/to	D1	D2	D3	D4	Supply
S1	100	150	120	140	100
S2	50	70	60	65	160
S3	40	90	100	150	140
Demand	80	130	90	100	

(See the separate Note for the Excel solution of this problem)

4. The Assignment Problem (AP)

There are m people to carryout n jobs ($m \geq n$). Each job has to be done by one person; and a person is assigned to at most one job.

- Let c_{ij} is the value if person i is assigned to job

The problem is to find a Maximum value assignment.
Definition of the variables:

$$x_{ij} = \begin{cases} 1, & \text{if person } i \text{ does(assigned to) job } j \\ 0, & \text{otherwise} \end{cases}$$

So, the variables are binary:

$$x_{ij} \in \{0, 1\}$$

for $i, j = 1, 2, \dots, n$

❖ **Constraints:**

I. Each person is assigned to at most one job:

$$x_{i1} + x_{i2} + \cdots + x_{in} \leq 1, \quad i = 1, 2, \dots, m$$

II. Each job is done by exactly one person:

$$x_{1j} + x_{2j} + \cdots + x_{mj} = 1, \quad j = 1, 2, \dots, n$$

❖ **Objective function:** Total value
(maximize)

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij}$$

❖ Hence the Assignment Prob
is:

$$\begin{aligned} \min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} \\ \text{s.t} \quad & x_{i1} + x_{i2} + \cdots + x_{in} \leq 1, \quad i = 1, 2, \dots, m \\ & x_{1j} + x_{2j} + \cdots + x_{mj} = 1, \quad j = 1, 2, \dots, n \\ & x_{ij} \in \{0, 1\} \quad \text{for all } i, j \end{aligned}$$

Example:

The following table shows the mark (value) of assigning person P_i to job J_k , $i, k = 1, 2, 3, 4$. Find out the optimal assignment that yields maximum value for the assignment.

	J1	J2	J3	J4
P1	9	8	6	5
P2	8	7	5	6
P3	8	6	5	7
P4	9	9	8	7

(Excel Solver: Similar format with Transportation problem)

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