UNIT 5: ELEMENTS AND APPLICATIONS OF CALCULUS

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5.0 AIMS AND OBJECTIVES

After reading this unit students must be able to:

- to explain the concepts of limits and continuity
- to provide an understanding of average rate of change
- to provide an understanding of derivative
- to illustrate a wide variety of applications of optimization procedures
- to understand skills in problem formulation
- to reinforce skills of interpretation of mathematical skills

5.1 INTRODUCTION

This unit examines the calculus and its application to business, economics and other areas of problem solving. The major areas of study within the calculus are differential calculus and integral calculus. Differential calculus focuses on rates of change in analyzing a situation.

Integral calculus involves summation of a special type. Graphically the concepts of area in two dimensions or volume in three dimensions are important in integral calculus. The goal in this chapter is to provide an appreciation for what the calculus is and where it can be applied. Though it would take several semester of intensive study to understand most of the finer points of the calculus, your coverage will enable you to understand the tools for conducting analysis at an elementary level.

Calculus is a mathematical tool used to solve problems in business, Economics and other areas.

- differential calculus and
- integral calculus

1) * Differential calculus focuses on rates of change in analyzing a situation. It broadens the concept of slope.

   - Limits
   - Continuity
   - Derivatives

   Application areas:

- Optimization problems i.e minimizing cost and /or maximizing profit, revenue---

2) Integral calculus involves summation of a special type, total change -------

It is the inverse of Differential calculus or vice versa like that of log and antilog.

Scope of this chapter = Differential calculus, derivative and its application in solving optimization problems.

5.2 THE DERIVATIVE

The process of finding a derivative is called differentiation. A set of rules of differentiation exists for finding the derivatives of many common functions.

5.2.1 The rules of differentiation:

The rules of differentiation have been developed using the limit approach. The mathematics involved in providing these rules can be finally complicated. For our purposes it will suffice the rules without proof.

The rules of differentiation apply to functions, which have specific structural characteristics. A rule will state that if a function has specific characteristics, then the derivative of the function will have resulting form.

⇒ Each function can be graphed and that the derivative is a general expression for the slope of the function.

Notation: $\frac{dy}{dx}$⇒ "which read the derivative of Y with respect to X"

: This notation can be used interchangeably with the notation "**F' (X)** –read **f prime** of **X**" – which represent the derivative of the function **f** at **X**. That is given f(X)

$\frac{dy}{dx} = f'(X)$

⇒ Differentiation by direct application of the limit formula is time consuming and difficult, there fore, functions should be classified into certain groups according to their behavior so that the process of finding a derivative when applied to these groups follows a definite pattern (formula)

1. **Derivative of a constant function**.

   If f(x) = K Eg. F(x) = 10,

   f' (x) = 0 f'(x) = 0 slop = 0

   If you consider what the function (f(x) = 10) looks like graphically, this result seems reasonable. The function f(x) = 10 is horizontal line intersecting the Y axis at (x,10). The slope at all points along such function equals 0.

2. **Power functions**

   If f(x) = X^n, where n is a real number $f'(X) = nx^{n-1}$

Example

1) If (X) = X = X^1 where, n = 1

$f'(X) = nx^{n-1}$

$= 1X^{1-1}$

$= \underline{\underline{1}}$

This implies that for the function f(X) = X, the slope equals 1 at all points. You should recognize that f(X) = X is a linear function with slope 1.

2) f(X) = X5

$f'(X) = \underline{\underline{5x^4}}$

3) f(X) = $\frac{1}{X^3}$

4) f(X) = $\sqrt[3]{X^2}$

$= X^{2/3}$

$f'(X) = 2/3\ X^{-1/3}$

$= X^{-3}$ $= \frac{2x^{\frac{-1}{3}}}{3}$

$f'(X) = -3X^{-4}$ $= \frac{2}{3X^{\frac{1}{3}}}$

$= \frac{-3}{X^4}$ $= \frac{2}{3\sqrt[3]{X}}$

3. A constant times a function

If f (X) = K. g(X), where K is a constant and g is a differentiable function,

$f'(X) = K.g'(X)$

a) $f(X) = 5X^3$ b) $f(X) = X^2 - 5X$

$f'(X) = 15X^2$ f(X) can be expressed as

f(X) = g(X) – h(X), where $g(X) = X^2$

h(X) = 5x

$f'(X) = g'(X) - h'(X)$

$= \underline{2x - 5}$

4. **Sum or differences of functions**.

This implies that the derivative of function formed by the sum (difference) of two or more component functions is the sum (difference) of the derivatives of the component functions.

(X) = [g(X) ± h(X)]

$f'(X) = [g'(X) \pm h'(X)]$

$f(X) = 6X^4 - 5X^2$ g(X) = 6X4 $g'(X)$ = 24X3

$f'(X) = 24X^3 - 10x$ $h(X) = 5x^2$ $h'(X) = 10^x$

$g'(X) - h'(X) = 24x^3 - 10^x$

$f(X) = 3x^2 + 2x + 1$

$f'(X) = \underline{\underline{6x + 2}}$

5. Product of functions

If $h(X) = f(X) \cdot g(X)$

$h'(X) = f(X) \cdot g'(X) + g(X) \cdot f'(X)$

$h(X) = (2x^2 - 5)(x + 3)$ $\qquad f(X) = 2x^2 - 5$

$= (2x^2 - 5) \times 1 + (X + 3)(4x)$ $\qquad g(X) = X + 3$

$= \underline{\underline{6x^2 + 12x - 5}}$ $\qquad f'(X) = 4x$

$g'(X) = 1$

$h'(X) = f(X) \cdot g'(X) + g(X) \cdot f'(X)$

$\underline{2x^2 - 5 \times 1 (X + 3)(4x)}$

Eg. 2. $f(X) = (x^2 - 5)(X - X3)$

$g(X) = X^2 - 5 \quad g'(X) = 2x$

$h(X) = X - X^3 \quad h'(X) = 1 - 3X^2$

$f'(X) = g(X) \cdot h'(X) + h(X) \cdot g'(X)$

$= (x^2 - 5)(1 - 3x^2) + (X - X^3)(2x)$

$= \underline{\underline{-5x^4 + 18x^2 - 5}}$

6. Derivative of the Quotient of function

$$h(X) = \frac{f(X)}{g(X)}$$

$$h'(X) = \frac{g(X).f'(X) - f(X).g'(X)}{[g(X)]^2}$$

Eg. 1. $$h(X) = \frac{1 - X}{X^2}$$

$$h'(X) = \frac{(-1)(X^2) - (1 - X)(2X)}{(X^2)^2}$$

$$= \frac{-x^2 - 2x + 2x^2}{(X^2)^2} = \frac{X^2 - 2x}{X^4}$$

Eg. 2. f(X) = $\frac{1}{X^3}$ $f'(X) = \frac{-3}{X^4}$

3. F(X) = $(3X^2 / 5)\ 1 - X^3$ $f'(X)$ =

$$\frac{3x4 - 15x^2 - 6x}{(1 - x^3)^2}$$

7. Other rules-----

5.3 APPLICATION OF CALCULUS IN BUSINESS DECISIONS

5.3.1 Revenue, Cost, And Profit Applications

5.3.1.1 Revenue Applications:

5.3.1.2 Marginal analysis

Examines Incremental Effects

C(X) = total cost function $\Rightarrow c'(X)$ = marginal Cost

R(X) = total Revenue function $\Rightarrow R'(X)$ = Marginal revenue

P(X) = R(X) – (C(X)) = Profit $\Rightarrow p'(X)$ = marginal Profit

$$p'(X) = R'(X) - c'(X)$$

Marginal Cost = is the additional cost incurred as a result of producing and selling one more unit of a product or service. Linear cost functions assume that the variable cost per unit is constant for such functions the marginal cost is the same at any level of output.

- A non-linear cost function is characterized by variable marginal costs.
- For the total cost function C(X), the derivative C'(X) represents
  i)The instantaneous rate of change in TC given a change in the number of units produced.

ii) A general expression for the slope of the graph of the TC function

iii) The marginal cost, MC = $c'(X)$

$\therefore$ $c'(X)$ Can be used to approximate the marginal cost associated with producing the next unit.

→ It is the rate of change in total cost per unit change in production at an out put level of X unit. It is also an optimization to the actual cost of making one more unit at any production level X (non-linear functions).

Example: Suppose the total cost C(X) in thousands of dollars for manufacturing X unit is given by the function

$C(X) = 575 + 25x \frac{X^2}{4}; 0 < X \leq 50$

[0 50]

Required:

1) Find the MC at a production level of X units
2) Find the MC at a production level of 40 unit and interpret the result
3) Find the actual cost of producing the 41st unit and compare this cost with the result found in question number 2

Solution

1) $c'(X) = 25 – x/2$

2) $c'(40) = 25 – 40/2$

= 5 Br. = 5, 000Br. ⇒ marginal cost of 41st unit.

3) C(41) – C(40) = ?

C(41) = 575 + 25(41) – (41)2/4

= 1179.75

C(40) = 575 + 25(40) – (40) – (40) 2/4

= 1175

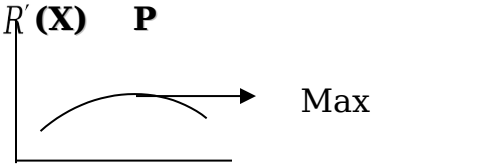
C(41) – C(40) = 4750Br. ⇒ 1179.75 – 1175 = 4.75 x 1000

At a production level of 40 unit the rate of change of TC relative to production is Br. 5,000. In Other words, the cost of producing 1 more unit at this level of production (40) is approximately 5, 000Br.

Marginal Revenue = Marginal revenue (MR) is the additional revenue derived from selling one more unit of a product or service. If each unit of a product sells at the same price, the MR is always equal to the price. Eg. R = 10x MR= 10Br.

Marginal revenue for non-linear total revenue function is not constant. For a total revenue function R (X), the derivative $R'(X)$ represents the instantaneous rate of change in total revenue given a change in the number of units sold. For the purpose of marginal analysis, the derivative is used to represent the

Marginal revenue or **MR = R'(X)**



It represent / approximately the marginal revenue from selling the next unit

Eg. R(X) = 500x – 0.005x^2 X = number of units

$R'(X)$ = 500 – 0.005X

Marginal Profit= Marginal revenue –Marginal cost

$P'(X) = R'(X) - C'(X)$

- Marginal revenue (Profit) analysis is concerned with the effect on profit if one-additional unit of a product is produced and sold. As long as the additional revenue brought in by the next unit exceeds the cost of producing and selling that unit, there is a net profit from producing and selling that unit and total profit increases.

I. If **MR > MC**, produce the next unit

Ⅱ. If **MR < MC**, do not produce the next unit.

5.3.2 Profit maximization criterion

If MR = MC, for the last unit produced and sold, total profit will be maximized.

$P(X) = R(X) - C(X) = 0$

$P'(X) = R'(X) - C'(X) = 0$

$P'(X) = 0$

$R'(X) = C'(X)$

Example: The market research department of a Company recommends that the Company to manufacture and market a new transistor radio after suitable test. The marketing department also presents the following demand equation.

$X=10{,}000-1000P$ i.e. $P=10-X/1000$

Furthermore, the financial department provides the following cost equation:

⟶ $C(X) = 7,000 + 2x$

Conduct a marginal analysis for the company.

1) $C'(X) = 2$

2) $TR = P.X = 10x - \frac{X^2}{1{,}000}$

$R'(X) = 10 - \frac{X}{500}$

3) $P(X) = R(X) - C(X)$

$= (10x - \frac{X^2}{1{,}000}) - (7,000 + 2x)$

$= 8x - \frac{X^2}{1{,}000} - 7,000$

$P'(X) = 8 - \frac{X}{500}$ or

$P'(X) = R'(X) - C'(X)$

$= 10 - \frac{X}{500} - 2$

$= 8 - \frac{X}{500}$

Average cost, Average revenue, Average profit

→ Average Cost = $\bar{C}$(X) = $\frac{C(X)}{X}$ ⇒ Cost per unit

Marginal Average cost = $\bar{C}'$(X) ⇒ the rate of change of average cost

→ Average revenue = $\bar{R}$(X) = $\frac{R(X)}{X}$ ⇒ Revenue/unit

Marginal Average revenue = $\bar{R}'$(X) ⇒ R'(X)\X

→ Average Profit = $\bar{P}$(X) = $\frac{p(X)}{X}$

Marginal Average profit = $\bar{P}'$(X)

Suppose: C(X) = 1, 000 + 25x - $\frac{X^2}{10}$

$C'(X) = 25 - \frac{X}{5}$

$\bar{C}(X) = \frac{1{,}000}{X} + 25 - \frac{X}{10}$

$\bar{C}'(X) = \frac{-1{,}000}{X^2} - 1/10$

$C'(10) = \frac{1{,}000}{100} - \frac{1}{10}$

= Br. –10.10 ⇒ Shows that a unit increase in production will decrease the average cost by approximately Br. 10.10 at a production level of 10 units.

If f' set equal to zero

$\bar{C}'(X) = \frac{1000}{X} + 25 - \frac{-X}{10} = 0$

X = represents the minimum value of f, If $f'(X) > 0$ i.e.-Cost

$f'(X) < 0$ i.e. –Profit

5.3.3 Higher-order Derivatives

If a function F has a derivative for each value of X in some specified interval, then the derivative function f' is defined for that interval. If in turn the derivative function itself has a derivative for points in that interval this new derivative function is called the second derivative of the original function f or the first derivative of f'.

→ **The first derivative test**

* Locate all critical values X* $f'(X) = 0$

* For any critical value X^*, determine the value of X and right

→**Second derivative test**

For critical points, where $f'(X) = 0$, the most expedient test is the second –derivative test. Intuitively the 2nd derivative test attempts to determine the concavity of the function at a critical point.

Eg. $f(X) = \frac{X^4}{4} - \frac{9x^2}{2}$

$f'(X) = X^3 - 9x$

$f'(X) = 0$

X3 – 9x = 0

X (X2 – 9) = 0

X (X + 3) (X – 3) = 0

X = 0, X + 3 = 0 or X – 3 = 0

X = 0, X = -3, or X = 3

↓ $f''(X)$ = -ve Concave upward

Concave downward

f (0, -3, or 3) ⇒ (0,0), (-3, $-\frac{81}{4}$), (-3, $-\frac{81}{4}$)

f'' (X) = 3x2 – 9 max min min

Rule

1 → find $f'(X)$, set it equal to zero, and solve for candidate values, X.

2. Find $f''(X)$ and evaluate $f''(X)$

a) If $f''(X)$ is negative, a local maximum occurs at X

b) If $f''(X)$ is positive, a local minimum occurs at X.

c) If $f''(X)$ is zero; the test fails to determine what happens at X.

* $f'(X) = 0$ is called stationary points. The slope of the line is tangent to a curve.

g. $f(X) = 3x^3 + 5x^2 + 2x + 3$

$f'(X) = 9x2 + 10x + 2$

$f''(X) = 18x + 10$

$f'''(X) = 18$

$f^{iv}(X) = 0$

Where:

f = The original function

f = Relates information about the behavior of f

f'' = Relates information about f'

Giv

If $f'(X) > 0$ f is increasing

If $f'(X) < 0$ f is decreasing

If $f'(X) = 0$ f is constant

If $f''(X) > 0$ f' is increasing

$f''(X) < 0$ f' is increasing

$f''(X) = 0$ f' is constant

5.3.4 Optimization Problems

Ex. 1. A company manufactures and sales X units of transistor radios per week. If the weekly cost and demand equations are:

C(X) = 5, 000 + 2x

P = 10 - $\frac{X}{1,000}$ [0, 8000]

Finding for each week

a) The production level that leads to maximum revenue and the maximum revenue.

b) The production levels that leads to maximum profit and the maximum profit.

c) The production level that leads to minimum cost and the minimum cost.

Solution

1) R(X) = 10x - $\frac{X^2}{1,000}$ = (10 - $\frac{X}{1,000}$) . (X)

R'(X) = 10 - $\frac{X}{500}$ R'(X) = 0

= 10 - $\frac{X}{500}$ = 0

X = 5, 000 units

R''(X) = $\frac{-1}{500}$ = -0.02, so X = 5, 000 units leads to maximum revenue

TR = XP = 5, 000 x 5 (5, 000 25, 000)

= 25, 000 Br R(5, 000) = 10 x 5, 000 - $\frac{(5,000)^2}{1,000}$

= 25, 000

2) P(X) = 10x - $\frac{X^2}{1,000}$ - 5, 000 – 2x

= 8x - $\frac{X^2}{1,000}$ - 500 P''(X) = $\frac{1}{500}$

P' (X) = 8 $\frac{X}{500}$ Negative, so it is an optimal solution (4, 000 unit Br. 21, 000)

P' (X) = 0 P(4, 000) = 4, 000 x 8 - $\frac{(4,000)^2}{1,000}$ - 50,000

$8 - \frac{X}{500} = 0$

$\underline{\underline{4,000 = X}}$

3) Cost is minimum at zero production level. If there is non-linear function, we can use the 2nd derivative test.

$C(X) = 5,000 + 2x$

$C'(X) = 2$

$C'(X) = 0 \neq 2$

Ex. 2. When X gallons of alcohol are produced, the average cost per gallons is given by the following function.

$C(X) = \frac{200}{0.1x+5} + 0.05X$, $X > 0$

Required:

1) Find the gallon of alcohol production level that leads to minimum cost.
2) Prove that this value (1) of X occurs at a local minimum of C(X)
3) Compute the minimum average cost per gallon.

Solution

1. We can rewrite C(X) as

$C(X) = 200 (0.1x + 5)^{-1} + 0.05X$

$C'(X) = -200 (0.1x + 5)^{-2(0.1)} + 0.05$

$= -20 (0.1x + 5)^{-2} + 0.05$

$C'(X) = 0$

$-20 (0.1x + 5)^{-2} + 0.05 = 0$

$\frac{-20}{(0.1X+5)^2} + 0.05 = 0$

$\left[(0.1x+5)^2\right]^{1/2} = (400)^{1/2}$

$0.1x + 5 = \pm 200$

$X = 150$ or $X = -250$

We discard X = -250 since it is negative.

2. To show X = 150 yields a minimum cost, we start with,

$A'(X) = -20\,(0.1x + 5)^{-2} + 0.05$

Find $A''(X)$

$A''(X) = -20\,(-2)\,(0.1x + 5)^{-3}\,(0.1)$

$= \frac{4}{(0.1x+5)^3}$ $A''(X) > 0$ Minimum point.

$A''(150)$ = Positive

3. Minimum average cost / gallon

$C(X) = \frac{200}{0.1x+5} + 0.05x$

$C(150) = \frac{200}{0.1(150)+5} + 0.05(150)$

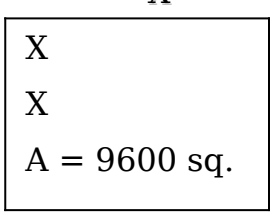
= <u>17.5 Br/ gallon</u>

Example: 3

A rectangular warehouse with a flat roof is to have a floor area of 9600 square feet. The interior is to be divided into storeroom and office space by an interior wall parallel to one pair of the sickles of the building (as shown below). The roof and floor areas will be of 600 square feet for any building, but the total wall length will vary for different dimensions.

Required:

Find the dimensions that minimize the total amount of wall.



XY = 9600 Sq.ft

X $Y = \frac{960(}{X}$

W = 3X + 2Y i.e total wall

$W(X) = 3X + 2\ \frac{960(}{X}$

$= 3X + \frac{1920(}{X}$

$W'(X) = 3 - \frac{1920(}{X^2}$

Set $W'(X)$ equal to zero.

$3 - \frac{1920(}{X^2} = 0$

$3X^2 - 19200 = 0$

$X^2 = \frac{19200}{3} = 640($

$X = \pm\sqrt{640(}$

$= \pm 80$

$Y = 120$

X = 80 X = -80 is discarded because X must be positive in this problem.

$Y = \frac{960(}{80} = 120$

⇒ Note that the second derivative

$W(X) = 3 - 19200\ X^{-2}$

$W''(X) = -2\ (-19200)\ X^{-3}$

$= \frac{3840(}{X^3}$

* $W''(80) > 0+$ positive. So we have a minimum value.

* The minimum length is

W = 3X + 2Y = 3(80) + 2 (120)

= 240 + 240

= 486 feet.

Ex2. Fence is required on three sickles of a rectangular plot. Fence for the two ends costs Br. 1.25 per running foot; fence for the third side costs Br. 2 per running foot. Find the maximum area that can be enclosed with Br. 100 worth of fence.

Let



2 (1.25X) + 2Y = 100

2.50X + 2Y = 100

A = XY

$$A(X) = X\left(\frac{100 - 2.50X}{2}\right)$$

$$Y = \frac{100 - 2.50X}{2}$$

$$A(X) = X\,\frac{100X - 2.5X^2}{} = 50X - \frac{}{20}$$

2Y = 100 – 2.50X

$$A'(X) = 50 - 2\left(\frac{25}{20}\right)X$$

$A'(X) = 0$

50 – 2.5X = 0

50 = 2.5X

$$X = \frac{50}{2.5} = 20$$

A(20) = 500 sq.feet

X = 20 ⇒ leads to max area.

Y = 25

$$Y = \frac{100 - 2.5X}{2}$$

$$= \frac{100 - 2.5(20)}{2}$$

$$= \frac{50}{2} = 25$$

2.5X + 2Y = 100

2.5X = 100 – 2Y

$$X = \frac{100 - 2Y}{2.5}$$

$$A(Y) = \left(\frac{100 - 2Y}{2.5}\right)(Y)$$

$$= \frac{100Y - 2Y^2}{2.5}$$

$= 40Y - 0.8Y^2$

$A'(Y) = 40 - 1.6Y$

$A'(Y) = 0$

$40 - 1.6Y = 0$

$$Y = \frac{40}{1.6}$$

$= \underline{\underline{25}}$