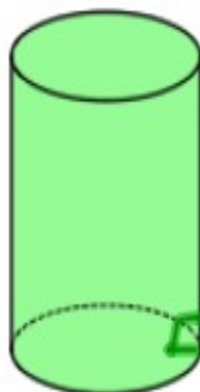
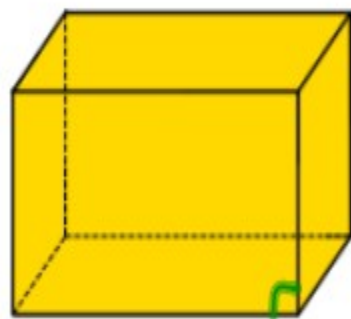


➤ 5.2 VOLUME OF REVOLUTION

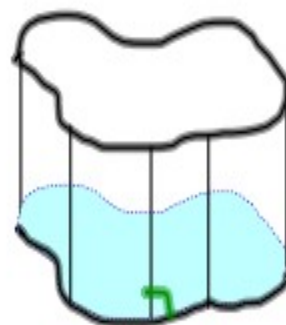
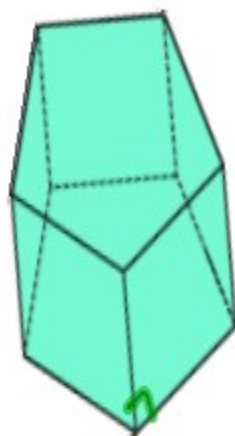
➤ OBJECTIVES

- ✓ FIND THE VOLUME OF SOLID OF REVOLUTION

The volume of a solid right prism or cylinder is the area of the base times the height.



right
circular
cylinder

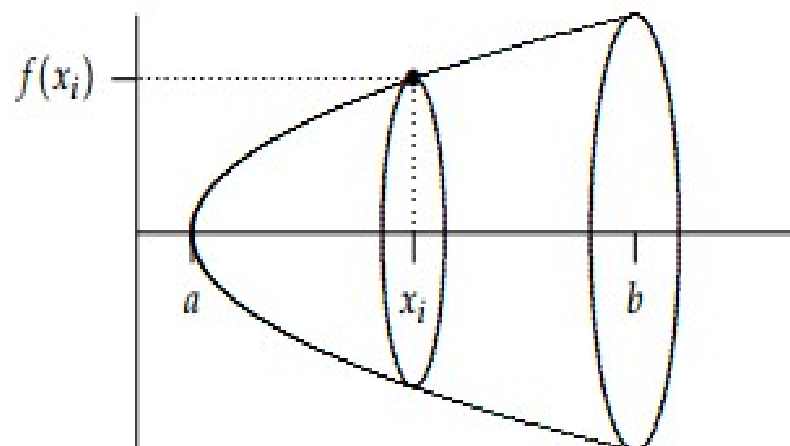
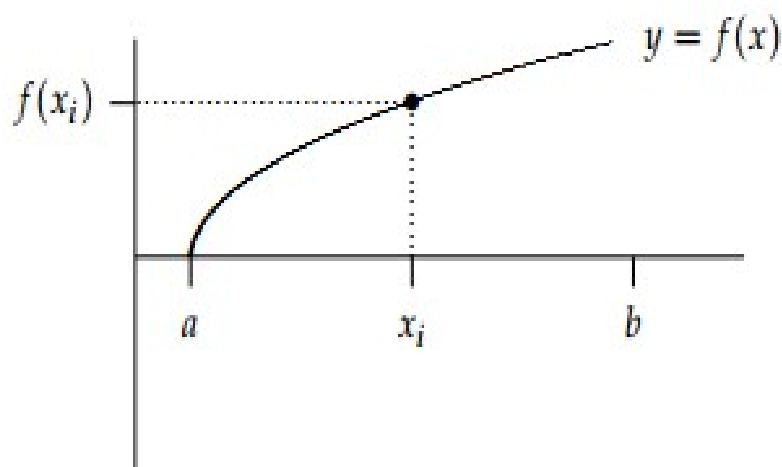


cylinder

Solids of Revolution

If a region in the plane is revolved about a line in the same plane, the resulting object is a **solid of revolution**, and the line is called the **axis of revolution**. The following situation is typical of the problems we will encounter.

Solids of Revolution from Areas Under Curves. Suppose that $y = f(x)$ is a continuous (non-negative) function on the interval $[a, b]$. Rotate the region under the f between $x = a$ and $x = b$ around the the x -axis and determine the volume of the resulting solid of revolution. See Figure 6.10



VOLUME CONT.....

to determine the volume. But as Figure 6.10 shows, when the point $(x_i, f(x_i))$ on the curve is rotated about the x -axis, it forms a circular cross-section of radius $R = f(x_i)$. Therefore, the cross-sectional area at x_i is

$$A(x_i) = \pi R^2 = \pi[f(x_i)]^2.$$

Since f is continuous, so is $\pi[f(x)]^2$ and consequently Theorem 6.1.1 applies.

$$\text{Volume of Solid of Revolution} = \int_a^b A(x) dx = \int_a^b \pi[f(x)]^2 dx.$$

Of course, we could use this same process if we rotated the region about the y -axis and integrated along the y -axis. We gather these results together and state them as a theorem.

THEOREM 6.1.2 (THE DISK METHOD). If V is the volume of the solid of revolution determined

Volume cont.....

THEOREM 6.2.1 (The Disk Method). If V is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a, b]$ about the x -axis, then

$$V = \pi \int_a^b [f(x)]^2 dx. \quad (6.2)$$

If V is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c, d]$ about the y -axis, then

$$V = \pi \int_c^d [f(y)]^2 dy. \quad (6.3)$$

II/ REVOLVING THE REGION BETWEEN TWO CURVES (THE WASHER METHOD)

- Finally, we present a formula for the volume of the solid region generated by revolving a more general plane region about the axis. Let f and g be functions such that
$$0 \leq g(x) \leq f(x) \text{ for } a \leq x \leq b$$
- Here, the cross section at x is an annular ring (or washer) bounded by two circles. The ring has inner radius $r_{in} = g(x)$ and outer radius $r_{out} = f(x)$ so the formula for the cross sectional area at x is
- $A(x) = (r_{out})^2 - (r_{in})^2 = [y_{top})^2 - y_{bot})^2] = [(f(x))^2 - (g(x))^2]$
- We write $y_{top} = f(x)$ and $y_{bot} = g(x)$ for the top and bottom curves for the plane region

- The region between two positive graphs then, the volume

$$V =$$

$$\Rightarrow V = \int_a^b (f(x)^2 - g(x)^2) dx, \text{ about the } x \text{ axis}$$

- Similarly, $f(x) \geq 0$, for $c \leq y \leq d$, then the volume of the solid obtained by revolving around the y axis, the region between

$$X_{\text{right}} = f(y)$$

$$X_{\text{left}} = g(y)$$

$$V = \int_c^d (f(y)^2 - g(y)^2) dy$$

$$\Rightarrow V = \int_c^d (f(y)^2 - g(y)^2) dy \text{ about the } y \text{ axis}$$

➤ EXAMPLES

A/ Solids of revolution about x axis

Example1: Find the volume v of sphere of radius r

- Solution:

A sphere is generated by revolving a semicircle about its diameter

If we let $f(x) = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$

We take $a=-r$ and $b=r$ this gives

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r \Rightarrow V = \frac{4}{3} \pi r^3$$

- **Example2:** Find the volume v of solid generated by revolving the graph of $x=y^2$, above the x axis over the interval $[0, 2]$ about the x axis .
- **Solution:**

$$v = \int_0^2 dx = 2$$

Example3: find the volume generated when the area bounded by the line $y = x$ and the x axis from $x=0$ to $x=3$ is rotated about the x axis.

- **Solution:**

$$V = \int_0^3 x^2 dx = 9$$

0

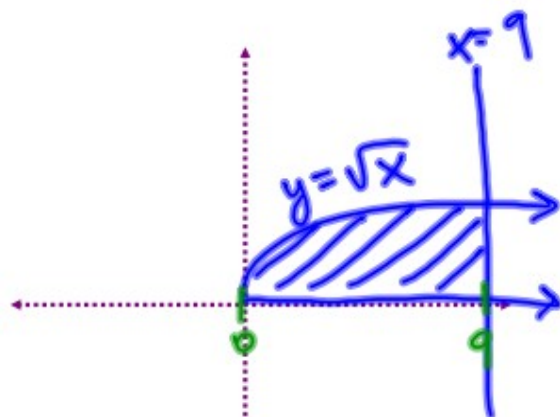
Example4: Find the volume of the solid generated by revolving region bounded by the graph of $y=x^2$ and the x axis between $x=0$ and $x=1$ about the x axis

• **Solution:** **1**

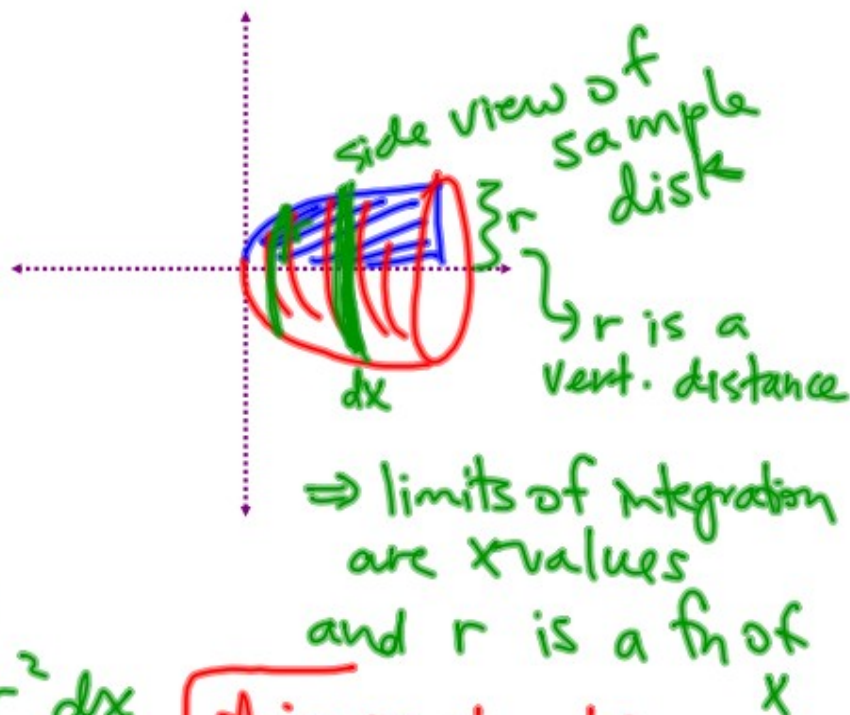
• $V = \int_0^1 \pi x^4 dx$

0

EX 1 Find the volume of the solid of revolution obtained by revolving the region bounded by $y = \sqrt{x}$, the x -axis and the line $x=9$ about the x -axis.



$$\begin{aligned} V &= \int_a^b \pi r^2 dx \\ &= \int_0^9 \pi r^2 dx \\ &= \int_0^9 \pi (\sqrt{x})^2 dx \end{aligned}$$



★important:

r is always measured from

$$\begin{aligned}
 &= \pi \int_0^9 x \, dx \\
 &= \pi \left(\frac{x^2}{2} \right) \Big|_0^9 = \pi \left(\frac{81}{2} - 0 \right) \\
 &= \boxed{\frac{81\pi}{2}} \text{ units}^3
 \end{aligned}$$

axis of rotation

Examples

Let's start with a couple of easy ones.

EXAMPLE 6.2.2. Let $y = f(x) = x^2$ on the interval $[0, 1]$. Rotate the region between the curve and the x -axis around the x -axis and find the volume of the resulting solid.

Solution. Using Theorem 6.2.1

$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_0^1 [x^2]^2 dx = \frac{\pi x^5}{5} \Big|_0^1 = \frac{\pi}{5}.$$

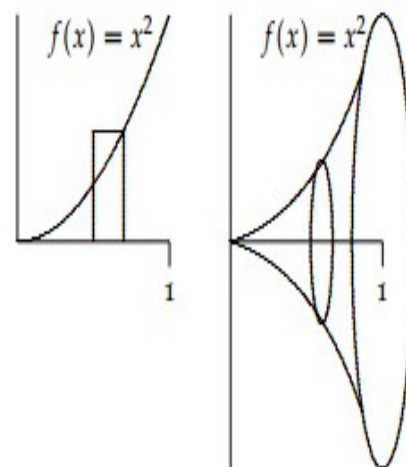


Figure 6.13: Left: A representative rectangle for the curve $y = x^2$. Right: A representative circular slice for the curve $y = x^2$ rotated about the x -axis.

✓ Solid revolution about the y axis

$$V = \int dy = \int dy$$

- **Example1:** let R be the region between the graph of f and the y axis on the given interval. Find the volume of the solid obtained by revolving R about the y axis.

- a/ $f(x) = x^2$, on $[0,1]$

- **Solution:**

- $f(x) = x^2$, on $[0,1]$, $y=x^2$ on $[0, 1]$, $x=$, on $[0, 1]$, $x=$, on $[0,1]$

1

- $v=$ = = cubic unit

0

Wow, that was easy!

EXAMPLE 6.2.3. Let $y = f(x) = x^2$ on the interval $[0, 1]$. Rotate the region between the curve and the y -axis *around the y -axis* and find the volume of the resulting solid.

Solution. The region is not the same one as in Example 6.2.2. It lies between the y -axis and the curve, not the x -axis. See Figure 6.14.

Since the rotation is about the y -axis, we need to solve for x as a function of y . Since $y = x^2$, then $x = \sqrt{y}$. Notice that the region lies over the interval $[0, 1]$ on the y -axis now. Using Theorem 6.2.1

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_0^1 [\sqrt{y}]^2 dy = \frac{\pi y^2}{2} \Big|_0^1 = \frac{\pi}{2}.$$

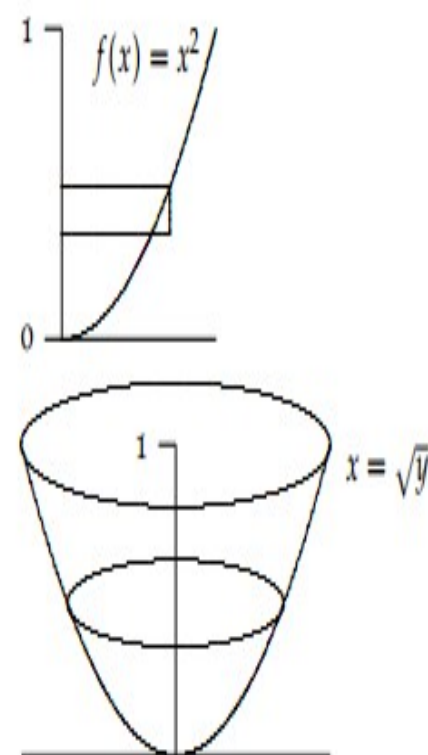
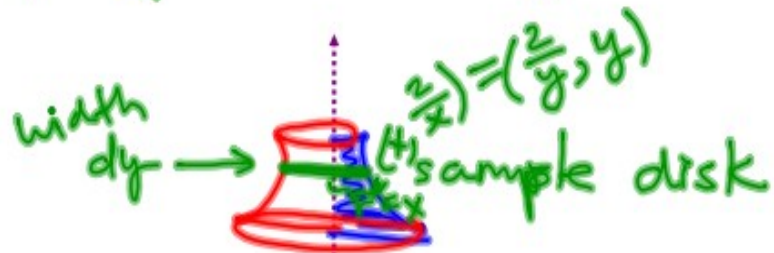
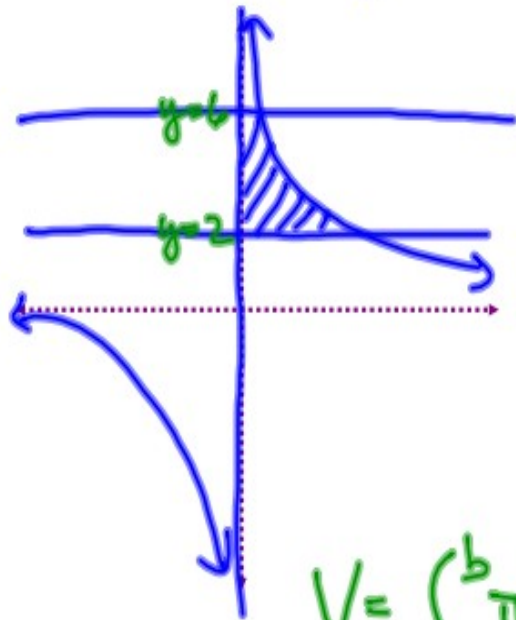


Figure 6.14: Left: A representative rectangle for the region between the curve $y = x^2$ ($x = \sqrt{y}$) and the y -axis. Right: A representative circular slice for the curve $x = \sqrt{y}$ rotated about the x -axis.

EX 2 Find the volume of the solid generated by revolving the region enclosed by $x = \frac{2}{y}$, $y = 2$, $y = 6$, and $x = 0$ about the y -axis.

$$y = \frac{2}{x}$$

$$\text{Disk } V = \int_a^b \pi r^2 dx \text{ or } dy$$



$$V = \int_a^b \pi r^2 dy \quad (\text{setup})$$

$$= \int_2^6 \pi r^2 dy$$

r must be
a fn of y
 $r = \underline{\text{horiz. dist}}$
from y -axis

Cont....

$$\begin{aligned} &= \pi \int_2^6 \left(\frac{2}{y}\right)^2 dy && | \text{ To curve } y = \frac{2}{x} \text{ or } \boxed{x = \frac{2}{y}} \\ &= 4\pi \int_2^6 y^{-2} dy \\ &= 4\pi \left(\frac{y^{-1}}{-1} \right) \Big|_2^6 = -4\pi \left(\frac{1}{y} \Big|_2^6 \right) \\ &= -4\pi \left(\frac{1}{6} - \frac{1}{2} \right) \\ &= -4\pi \left(-\frac{1}{3} \right) = \boxed{\frac{4\pi}{3}} \text{ units}^3 \end{aligned}$$

ii/ Revolving the region between two curves (the washer method)

$V = \int_a^b \pi (R^2 - r^2) dx$, about the x axis

$V = \int_c^d \pi (R^2 - r^2) dy$ about the y axis

- **Example1:** Let $f(x) = 5x$ and $g(x) = x^2$ and let R be region between the graphs of f and g on $[0, 3]$. Find the volume v of the solid obtained by revolving R about the x axis.

- **Solution:**

Since $5x \geq x^2$ for $0 \leq x \leq 3$, imply that 3

$$V = \int_0^3 \pi (5x^2 - x^4) dx = \pi \left[\frac{5x^3}{3} - \frac{x^5}{5} \right]_0^3 = \pi \left(\frac{5 \cdot 27}{3} - \frac{243}{5} \right) = \pi \left(45 - \frac{243}{5} \right) = \pi \left(\frac{225 - 243}{5} \right) = \pi \left(-\frac{18}{5} \right)$$

0

- **Example2:** Find the volume of solid generated by rotating graph around the indicated axis.

a/ $y^2 = x$ and $y = x^3$, about x axis

b/ $y^2 = x$ and $y = x^3$ about y axis

- **Solution:**

a/ The two plane intersect at the points (0, 0) and (1,1)

$Y_{\text{top}} = \sqrt{x}$, $y_{\text{bot}} = x^3$, then $v = \pi (y_{\text{top}}^2 - y_{\text{bot}}^2) dx$

$= \pi (\sqrt{x}^2 - x^6) dx$

1

$= \pi \left[\frac{2}{3} x^{3/2} - \frac{1}{7} x^7 \right]_0^1$

$\Rightarrow V =$

0

- b/ if the same region is revolved around the y axis then each cross section perpendicular to the y axis is an annular ring with outer radius inner radius , $x_{\text{left}} = y^2$.

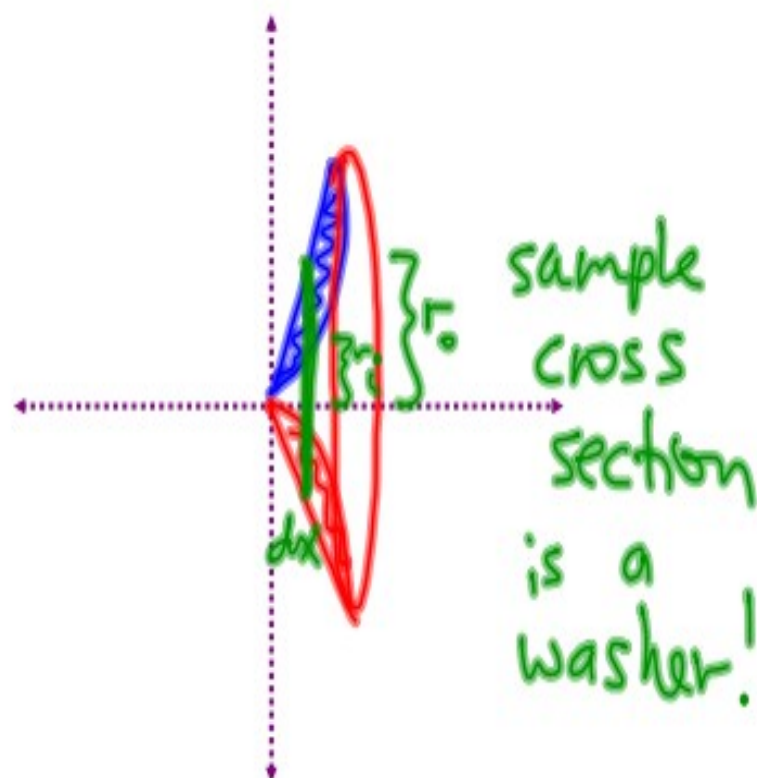
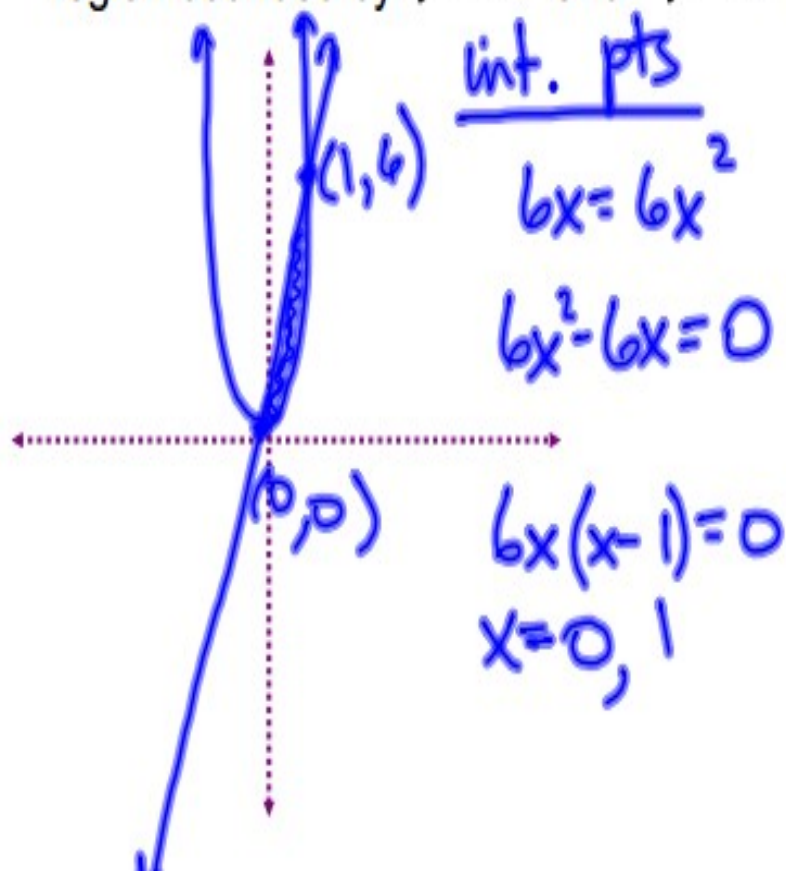
$$V = \int dy$$

$$V =$$

$$1$$

$$= \left[- \right] = \Rightarrow V =$$

EX 3 Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 6x$ and $y = 6x^2$.



CONT...

$$V = \pi \int_a^b (r_o^2 - r_i^2) dx$$

$$= \pi \int_0^1 (r_o^2 - r_i^2) dx$$

$$= \pi \int_0^1 ((6x)^2 - (6x^2)^2) dx$$

r_o = vert dist. from x-axis to
line $y = 6x$

r_i = vert dist. from x-axis to
curve $y = 6x^2$

$$\rightarrow = \pi \int_0^1 (36x^2 - 36x^4) dx$$

washer/disk method

① if rotate about
horiz line, each
disk/washer will have
 dx width

② if rotate about
a vert. line, each
disk/washer will
have dy width

Cont...

$$\begin{aligned} &= \pi \left(\frac{36x^3}{3} - \frac{36x^5}{5} \right) \Big|_0^1 \\ &= \pi \left(\left(12(1^3) - \frac{36}{5}(1^5) \right) - (0 - 0) \right) \\ &= \pi \left(12 - \frac{36}{5} \right) = \boxed{\frac{24\pi}{5}} \text{ units}^3 \end{aligned}$$



Motion of a particle in a straight line

- Suppose a particle p moves along a straight line
the velocity v is the rate at which the
displacement s increases with respect to time t

$$v = \frac{ds}{dt}$$

$$\Rightarrow s = \int v dt$$

- The acceleration a is the rate at which the
velocity increases with respect to time t

$$\Rightarrow a = \frac{dv}{dt}$$

$$\Rightarrow v = \int a dt$$

- **Example1:**
- **Suppose a particle p moves along a straight line with an acceleration of $3.5t$. when $t=2\text{sec}$, p has a displacement of 10m from the initial point and a velocity of 15m/sec . find the velocity v and the displacement when $t=5\text{sec}$.**

- **Solution:**
- **Using the given information you have**
- **$V = \dots = +c$ but $v(2) = 15 \Rightarrow 15 = +c \Rightarrow c = 8$
 $v = + 8$**
- **Also $s = \dots \Rightarrow s = \dots + 8)dt = \dots + 8t + c$**
- **But $S(2) = 10 \Rightarrow 10 = \dots + 8(2) + c$**

Cont.....

$$\Rightarrow c = -\frac{32}{3} \Rightarrow s = \frac{7}{12}t^3 + 8t - \frac{32}{3}$$

Therefore, when $t=5$

a/ the velocity, $v = \frac{7}{4}(5)^2 + 8 = 51.75\text{m/sec}$

b/ the displacement, $s = \frac{7}{12}5^3 + 8(5) - \frac{32}{3} = 102.25\text{m}$

Exercise

1. FIND THE VOLUME GENERATED WHEN THE AREA BOUNDED BY THE LINE $Y=X$ AND THE X AXIS FROM $X=0$ TO $X=3$ IS ROTATED ABOUT THE X AXIS .
2. FIND THE VOLUME OF THE SOLID GENERATED BY REVOLVING THE REGION BOUNDED BY THE GRAPH OF $Y=X^2$ AND THE X AXIS BETWEEN $X=0$ AND $X=1$ ABOUT THE X AXIS.
3. THE AREA BOUNDED BY THE GRAPH OF $Y=X^2 +1$ AND THE LINE $Y=4$ ROTATES ABOUT THE Y AXIS, FIND THE VOLUME OF THE SOLID GENERATED.

4.FIND THE VOLUME OF THE SOLID OF REVOLUTION ABOUT THE X AXIS WHEN THE REGION ENCLOSED BY $Y = e^x - 1$ AND THE X AXIS FROM $x = \ln(1/2)$ TO $x = \ln(2)$ ROTATES ABOUT THE X AXIS.

5. USING THE VOLUME OF A SOLID OF REVOLUTION,SHOW THAT THE VOLUME OF A SPHERE OF RADIUS r IS $\frac{4}{3}$

Cont....

6. Find the volume of the solid generated when the region enclosed by the x axis and the given curves and the lines is rotated about the x axis.

a/ $y=4x-x^2$

b/ $y=x^3+1; x=-1, x=2$

7. Find the volume of the solid of revolution generated when the region enclosed by the curve $y=e^x$, the y axis and the line $y=e$ rotates about the x axis.

8. let R be the region between the graph of f and the y axis on the given interval. Find the volume of the solid obtained by revolving R about the y axis.

a/ $f(y) = e^y$, $[1, 2]$

b/ $f(y) = \cos y$, $[0, \pi]$

9. Find the volume of the solid of revolution about the x axis, when the region enclosed by $y = e^x - 1$ and the x axis from $x = 0$ to $x = \ln(2)$ rotates about the x axis.

10. The area bounded by the graph of $y = x^2 + 1$ and the line $y = 4$ rotates about the y axis, find the volume of the solid generated.

11. Using the volume of a solid of revolution, show that the volume of a sphere of radius r is .

12. Find the volume of water in spherical bowl of radius 5m, if its maximum depth is 2m.

13. Find the volume of the solid of revolution about the y axis generated by revolving the region enclosed by the curve $x = \sqrt{y}$ and the y axis from $y=0$ to $y=4$.

14. If the region bounded by the curve $y=x^2$ and the line $y=4$ rotates about the x axis, find the volume of the solid of revolution.

Exercise cont....

15. If the region enclosed by the graphs of $f(x)=x$ and $g(x)=x^2$ from $x=0$ to $x=1$ rotates about the x axis. Find the volume of the solid of revolution.
16. Find the volume of the solid generated when the region
17. Find the volume of solid of revolution about the x axis generated by revolving the area between the lines $y=x$ and $y=4$ from $x=1$ to $x=3$.

18. Bounded by $y=x^2+2$ rotates about the x axis.

19. A particle p starts from a point A with velocity 4m/sec. if it is moving along a straight line AB with an acceleration of $-1.5t^2$ at a time t seconds, find

a/the acceleration

b/the velocity

c/the displacement after ten sec.