Chapter 5: Duality in LP

- Duality deals with pairs of related linear programs.
- One problem is called **the primal (P)** the other **the dual (D)**.

For the primal, we take initially an LP in **canonical form**:

$$\max c^T x \quad \text{s.t } Ax \le b, \quad x \ge 0$$

A is ***m×n,*** ***c∈R^n^,*** ***b∈R^m^*** (both column vectors)

- In fact, any LP can be written in canonical form; because, denoting the ***i-th*** row of matrix ***A*** by ***A^i^***:
  - $A^i x \ge b_i \iff -A^i x \le -b_i$
  - $A^i x = b_i \iff A^i x \le b_i$ and $-A^i x \le -b_i$

where ***A^i^*** is ***i-th*** row of ***A.***

5.1 Definition of the Dual Problem

Definition: Given an LP, called **primal**,

(P)
$$\begin{aligned} \max\ & c^T x \\ \text{s.t}\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$c \in R^n$, A is $m \times n$, $b \in R^m$

the following LP is called its dual.

(D)
$$\begin{aligned} \min\ & b^T y \\ \text{s.t}\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

where $y = (y_1, y_2, \ldots, y_m)^T \in R^m$

- y_i is called **dual variable of (P)** corresponding to its *i-th* constraint.
- **(D)** has ***n*** constraints, called **dual constraints** of (P).
- Any feasible solution of **(D)** is called **dual feasible;**

  and a feasible solution of **(P)** is called **primal feasible;**

(P)

$$\max\ c_1x_1 + \dots + c_jx_j + \dots + c_nx_n$$

$$\text{s.t.}\quad a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n \le b_i,\quad i=1,\dots,m$$

$$x_j \ge 0 \quad \text{for all } j.$$

Primal Variables

x_1	…	x_j	…	x_n		
c_1	…	c_j	…	c_n		
a_{11}	…	a_{1j}	…	a_{1n}	$\le$	b_1
:		:			:	:
a_{i1}	…	a_{ij}	…	a_{in}	$\le$	b_i
:		:			:	:
a_{m1}	…	a_{mj}	…	a_{mn}	$\le$	b_m

Dual Variables

y_1 : y_i : y_m

(D)

$$\min\ b_1y_1 + \dots + b_iy_i + \dots + b_my_m$$

$$\text{s.t.}\quad a_{1j}y_1 + \dots + a_{ij}y_i + \dots + a_{mj}y_m \ge c_j,\quad j=1,\dots,n$$

$$y_i \ge 0 \quad \text{for all } i.$$

Examples: Find the dual of the following

(P) **(D)**

(1)

$$\max 2x_1 - x_2 + 7x_3$$
$$\text{s.t. } 4x_1 + 5x_2 \le 3$$
$$x_1 - x_2 + 2x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

$$\min 3y_1 + 6y_2$$
$$\text{s.t. } 4y_1 + y_2 \ge 2$$
$$5y_1 - y_2 \ge -1$$
$$2y_2 \ge 7$$
$$y_1, y_2 \ge 0$$

(2)

$$\max 3x_1 + 7x_2$$
$$\text{s.t. } -x_1 + 2x_2 \ge 5$$
$$4x_1 + x_2 = 6$$
$$x_1, x_2 \ge 0$$

$$\min -5y_1 + 6y_2$$
$$\text{s.t. } y_1 + 4y_2 \ge 3$$
$$-2y_1 + y_2 \ge 7$$
$$y_1 \ge 0, \quad y_2 \in R$$

Note : The dual of $\max c^T x \ \text{s.t}\ Ax = b,\ x \geq 0$ is $\min b^T y^T \ \text{s.t.}\ A^T y \geq c,\ y \in R^m$ (**y** - unrestricted)

- **Justification**: The given LP in canonical form is

(P) $\max c^T x \ \text{s.t.}\ Ax \leq b,\ -Ax \leq -b,\ x \geq 0$

Now, taking dual variables $(u, v) \geq 0$, where $u, v \in R^m$, the dual of this problem is:

(D) $\min b^T (u - v) \ \text{s.t.}\ A^T (u - v) \geq c,\ u, v \geq 0$

(Put $y := u - v \Rightarrow$ unrestricted)

$\Leftrightarrow \min b^T y^T \ \text{s.t.}\ A^T y \geq c,\ y \in R^m$

5.2 Duality Theorems

Theorem 3.1: **The dual of the dual is the primal.**

- **Hence, it doesn't matter which problem is taken as primal.**

❖ **Let z^* and w^* represent maximum value of (P) and minimum value of (D), respectively; i.e.,**

$$z^* = \max \{ c^T x : \; Ax \le b \,,\, x \ge 0 \} \quad \ldots \text{(P)}$$

$$w^* = \min \{ b^T y : \; A^T y \ge c, \; y \ge 0 \} \quad \ldots \text{(D)}$$

Theorem 3.2: **(Weak Duality)**

If $\bar{x}$ is primal feasible and $\bar{y}$ is dual feasible then

$$c^T \bar{x} \le z^* \le w^* \le b^T \bar{y}$$

Proof: $c^T \le \bar{y}^T A \Rightarrow \; c^T \bar{x} \le \bar{y}^T A \bar{x} \le \bar{y}^T b = b^T \bar{y}$

Corollary 3.3:

If (D) is unbounded, then (P) is infeasible.

Similarly, If (P) is unbounded, then (D) is infeasible.

Proof: (D) is unbounded $\Rightarrow w^* = -\infty \Rightarrow$ (P) should be infeasible. Otherwise, if there were a primal feasible point, say x_0, then $c^T x_0$ is a finite value, and by Theorem 5.2,

$c^T x_0 \leq w^* = -\infty$; which is not possible.

Theorem 3.4 (Strong Duality)

Let $\bar{x}$ & $\bar{y}$ be primal and dual feasible points, respectively. If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal solution of (P) and $\bar{y}$ is optimal solution of (D).

Proof: Follows directly from Week Duality Theorem.

Theorem 3.5: (Fundamental Theorem of Duality (FTD))

If z^* or w^* is finite, then both (P) and (D) have finite value and $z^* = w^*$.

Proof: W.L.O.G let z^* be finite and let B be optimal basis of (P).

⇒ $z^* = C_B B^{-1} b$; $C - C_B B^{-1} A \leq 0$ (Optimality condition)

⇒ $\bar{y}^T = C_B B^{-1}$ is a feasible solution of (D); and

⇒ $\bar{y}^T b = z^*$ is optimal solution of (D) ; and

$w^* = b^T \bar{y} = z^*$ ∎

- From the proof of FTD: an optimal solution of (D) is directly available if optimal solution of (P) is known
- In fact, $y^* = C_B B^{-1}$ is readily available in the optimal simplex tableau of (P) as displayed below.

- When you solve (P) using simplex tableau, suppose the last *m* columns are the staring basis (unit submatrix).

$x_1, x_2, \ldots, x_n$	$s_1, s_2, \ldots, s_m$	RHS
c	0	0
A	I	b

Starting simplex tableau for (P)

- Let B be an optimal basis; and WLOG let $A = (B, N)$:

x_B	x_N	$s_1, s_2, \ldots, s_m$	RHS
c_B	c_N	0	0
B	N	I	b

Pivots ≡ multiply the row 1– m by B^{-1} and eliminate c_B

x_B	x_N	$s_1, \ldots, s_m$	RHS
0	$c_N - c_B B^{-1} N$	$-c_B B^{-1}$	$-c_B B^{-1} b$
I	$B^{-1}N$	I	$B^{-1}b$

$y^* = c_B B^{-1}$

That is, $y_i = c_{n+i}$, $i = 1, \ldots m$, is optimal solution of (D).

Example :

$$(P)\quad \max\ x_1 + 3x_2 \quad \text{st } -2x_1 + 3x_2 \le 6,\ 8x_1 + 6x_2 \le 48,\ x_1, x_2 \ge 0$$

$$(D)\quad \min\ 6y_1 + 48y_2 \quad \text{st } -2y_1 + 8y_2 \ge 1,\ 3y_1 + 6y_2 \ge 3,\ y_1, y_2 \ge 0$$

Solving (P) by Simplex:

Starting tableau

	x_1	x_2	x_3	x_4	RHS
-z	1	3	0	0	0
x_3	-2	3	1	0	6
x_4	8	6	0	1	48

Final (optimal) tableau

	x_1	x_2	x_3	x_4	RHS
-z	0	0	-1/2	-1/4	-15
x_2	0	1	4/18		4
x_1	2/36				3
	1	0	-1/6		
	1/12				

Optimal solution of (P) : $x_1 = 3$, $x_2 = 4$, $z^* = 15$

Optimal solution of (D) : $y_1 = 1/2$, $y_2 = 1/4$, $w^* = 15$

Remark: Given an LP, if optimal solution of its dual is known, then the solution of the LP can be obtained from that of its dual using the above result (Theorem 5.1).

This is useful particularly when the number of constraints are greater than the number of variables.

Exercise: Consider the following LP

$$
\begin{aligned}
\min \quad & 5x_1 + 6x_2 + 3x_3 \\
\text{st} \quad & 5x_1 + 5x_2 + 3x_3 \geq 50 \\
& x_1 + x_2 - x_3 \geq 20 \\
& 7x_1 + 6x_2 - 9x_3 \geq 30 \\
& 5x_1 + 5x_2 + 5x_3 \geq 35 \\
& 2x_1 + 4x_2 - 15x_3 \geq 10 \\
& 12x_1 + 10x_2 \geq 90 \\
& x_2 - 10x_3 \geq 20 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

1) Solve the problem using the simplex method (Two-phase)

2) Solve the dual of this problem; and find its optimal solution from that of the dual.

Does the procedure in (2) offer computational advantage?

5.3 The Dual Simplex Method

- The original simplex method given in Chapter 4 is called primal **simplex method**.
- Recall that the primal simplex method requires PSF:

$$\begin{aligned} \max \quad & 0x_B + \bar{c}_N x_N + z_B \\ \text{s.t} \quad & Ix_B + A_N x_N = \bar{b} \\ & x_B, x_N \geq 0 \end{aligned}$$

x_B	x_N	RHS
0	.. $\bar{c}_j$ ..	$-z_B$
I	: $\bar{A}_j$:	$\bar{b}$

- It requires also $\bar{b} \geq 0$, called for **primal feasibility**, and $\bar{c}_j \leq 0, \ \forall j$ works towards **optimality**
- The condition $\bar{c}_j \leq 0, \ \forall j$ is called **dual feasibility**

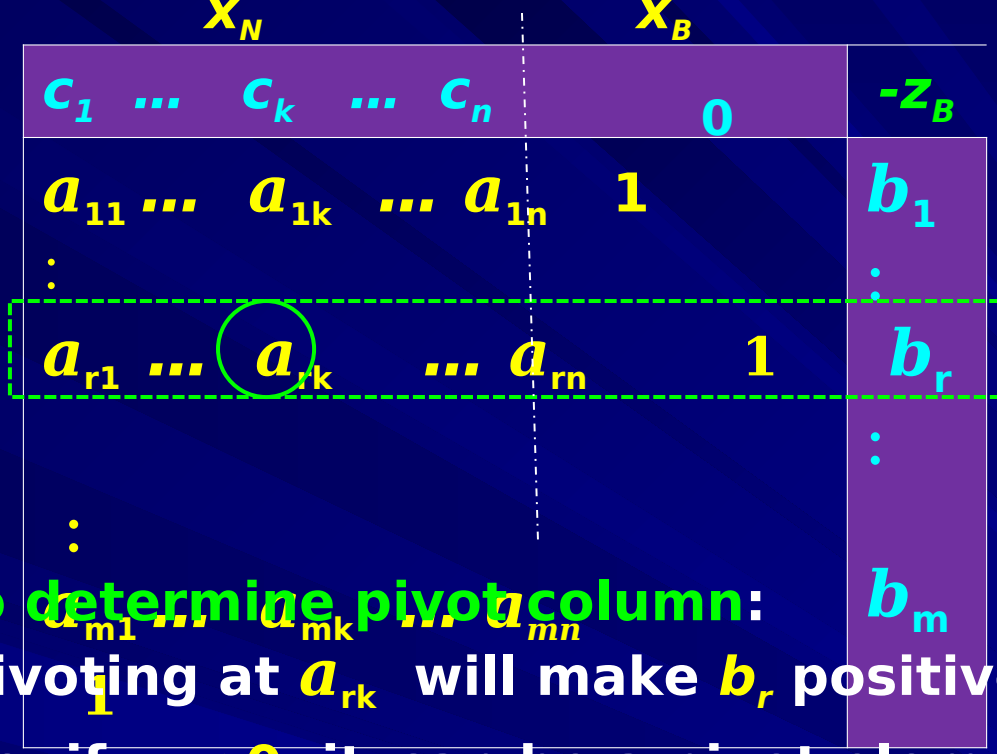
So, a BS is an optimal if it satisfies both **the primal feasibility(** $\bar{b}_i \geq 0, \ \forall i$ **)** and the **dual feasibility(** $\bar{c}_j \leq 0, \ \forall j$ **)**

- It is common also to find LPs with PSF, except for $\bar{b}_i < 0, \text{ for some } i$ (i.e., **primal infeasible**),

  but $\bar{c}_j \leq 0, \ \forall j$ (**dual feasible** – which is desirable)

- In such cases, we use the method, called **dual simplex:**
  - **Start with a dual feasibility** ($\bar{c}_j \leq 0, \ \forall j$), and maintain this at every step.
  - **Do basis exchange(pivoting) until primal feasibility**

- So, ignoring the bars on symbols for simplicity, suppose $c_j \leq 0$ for all j but $b_r < 0$ for some r in the simplex tableau.
- In this case, row r is selected for pivot row to modify b_r.



To determine pivot column:

- Pivoting at a_{rk} will make b_r positive if $a_{rk} < 0$.
- So, if $a_{rk} < 0$, it can be a pivot element; i.e., k is pivot column
- If more than one entry is negative in the pivot row, the pivot column k is selected so as the dual feasibility is maintained.

$$\min\left\{ c_j / a_{rj} \mid a_{rj} < 0,\ j = 1, \ldots, n \right\} = c_k / a_{rk}$$

This is the column in which the following min ratio occurs.

Note: If $a_{rj} \geq 0$, $\forall j$, the LP is infeasible . (Why ?)

- Putting these together, we get the following

Dual Simplex Algorithm: To solve max $\{cx \mid Ax=b, x \geq 0\}$

1. Start with $c \leq 0$ (where $c_B = 0$,for basic columns)
2. **Optimality Test**: If $b_i \geq 0$, $\forall i$, **STOP**. The current BFS is optimal solution.

   Else;

   Choose a **pivot row** r that contains the most negative b_r.
3. **Infeasibility**: If $a_{rj} \geq 0$, $\forall j$, then **STOP** (The LP is infeasible)

   Else do **basis** exchange as follows:

   - Choose a **pivot column** k in which the ff **min** occurs:

$$\min\left\{ c_j / a_{rj} \mid a_{rj} < 0,\ j = 1,\ldots,n \right\} = c_k / a_{rk}$$

and perform pivot operation **pivoting at** a_{rk}

Repeat the process from 2

Example:
Consider

$$\begin{aligned} \min \quad & x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & -x_1 + x_2 \geq 3 \\ & x_1 - x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

This is equivalent to:

$$\begin{aligned} \Leftrightarrow \quad \max \quad & -x_1 - x_2 - 2x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_4 = -3 \\ & -x_1 + x_2 - x_3 + x_5 = -1 \\ & x_1, \ldots, x_5 \geq 0. \end{aligned}$$

- **Solving this (max problem) by the dual simplex method, we get an optimal solution:**
  $(x_1, x_2, x_3) = (0, 3, 4)$ with maximum value $z^* = -11$.
- **Hence, the optimal solution of the given problem is:**
  $(x_1, x_2, x_3) = (0, 3, 4)$, with minimum value $z^* = 11$.

END