

Chapter 4: The Simplex Method

4.1 Introduction

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4.1 Introduction

- The procedure, called **simplex method (algorithm)**, is designed by **Dantzig (1947)** to solve any LP problem:
 $\max cx$ s.t. $Ax=b, x \geq 0$
- Recall that an optimal solution of this problem, if exist, occurs at a **basic feasible solution (BFS)** of the system **$Ax=b$** .
- Thus, the simplex procedure searches an optimal solution of an LP from among its BFSs iteratively (step by step).
- **Generic form of Simplex Procedure, to solve $\max cx$**

$Ax=b, x \geq 0$ **s.t**

1. Start with an initial BFS
2. **Optimality Test:** Is the current BFS an optimal solution?
 - ❖ If Yes, Stop (The current BFS is optimal)
 - ❖ If No, **move to adjacent BFS** with better value;

4.2 The Simplex Algorithm

- This section presents the simplex algorithm to solve

$$\begin{array}{ll}\max & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0\end{array}$$

- We assume that a BFS, called **initial BFS**, is known at the beginning. Other case is considered in the next section.
- If each constraint has slack variables, then **the slack variables are the initial BFS**.
- In particular, Simplex procedure is applied on an LP with **perfect standard form (PSF)** as defined below

Definition: An LP in standard form is said to be in **perfect standard form (PSF)** if it is of the following form

$$\begin{aligned} \max \quad & \sum_{j \in N} c_j x_j + z_B \\ \text{s.t} \quad & \sum_{j \in N} A_j x_j + I x_B = b \\ & x \geq 0 \end{aligned}$$

and $b \geq 0$.

- I.e.,
1. every constraint equation contains only one BV with coefficient 1.
 2. Objective function is expressed in terms of NBVs only
 3. The RHS of each constraint, b_i , is nonnegative.

For instance, if an LP is

$$\begin{array}{ll}\max & cx \\ \text{s.t.} & Ax \leq b \\ & x \geq 0\end{array}$$

with $b \geq 0$, then we use slack variables $x_{n+1}, \dots, x_{n+m}$ to change the problem into equivalent

PSF:

$$\begin{array}{ll}\max & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m \\ & x_i \geq 0 \quad \forall i.\end{array}$$

where $b_i \geq 0$ for all i , and z_B is a constant.

❖ If so, the simplex method starts with the initial BFS

$$x_B = \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad x_N = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the value of objective function at this BFS is z_B .

❖ $z = C_1 x_1 + C_2 x_2 + \dots + C_k x_k + z_B$

Setting $\Leftrightarrow -z + C_1 x_1 + C_2 x_2 + \dots + C_k x_k = -z_B$

❖ The LP in PSF can be formulated as

$$\max z$$

$$-z + C_1 x_1 + C_2 x_2 + \dots + C_k x_n = -z_B$$

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + x_{n+1} = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n + x_{n+2} = b_2$$

$$\vdots \quad \ddots$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n + x_{n+m} = b_m$$

- ❖ After the data of an LP is placed in a table as above, the simplex procedure applies special elementary row operations, called **Gauss-Jordan Elimination (GJE)**, to generate BFS.
- ❖ In a general case, GJE with arbitrary pivot element can produce a non feasible basic solution that has no use for LP.
- ❖ Thus, **Simplex Procedure** incorporates important rules to the Gauss-Jordan pivoting operations so that
 1. **Only BFSs** are generated at every pivoting steps.
 2. Check whether a BFS is optimal or not
 3. If a BFS at hand is not optimal, generate **adjacent BFS that improves objective value**.
 4. Identify the case when the LP is **unbounded**.

- ❖ In particular, starting at a tableau with a PSF, we apply GJE in such a way that the structure of **the PSF is preserved**
- ❖ Since pivoting changes each entry in the table, we put **bar** on each “**symbol**” for generality.

	$x_B \bar{c}_1 \cdots \bar{c}_j \cdots \bar{c}_n$			
$-z$	$\bar{a}_{11} \cdots \bar{a}_{1j} \cdots \bar{a}_{1n}$	0		
	$\bar{a}_{m1} \cdots \bar{a}_{mj} \cdots \bar{a}_{mn}$			
		$\bar{b}_1$		
		$\bar{b}_m$		

Initially

$$z_B = 0$$

$\bar{c}_j$ is called the **net gain** per unit of x_j

for all j ,

all

then, the current BFS is optimal

If $\bar{c}_j > 0$, for some j , basis exchange! Pivot column?

❖ If $\bar{c}_j > 0$, for some **non-basic column j** , then an increase of x_j from 0 to **positive increases the objective value**.
So, we let x_j enters basis.

- That is, we choose this column for **pivot column**.
- But, how far x_j can increase without violating the constraints?

❖ If $x_j = \delta > 0$, while the other NBV are 0, the constraints are:

$$x_B + \bar{A}_j \delta = \bar{b} \quad \dots \dots \dots (3)$$

where $\bar{A}_j$ is the current j -th column of the constraints

❖ Denoting the BV in i -th row be x_i , (3) can be written as $x_i + \bar{a}_{ij} \delta = \bar{b}_i$, $i = 1, 2, \dots, m$

❖ Thus, we have to choose a positive δ such that

$$x_i = \bar{b}_i - \bar{a}_{ij}\delta \geq 0, \quad \forall i = 1, 2, \dots, m \quad \dots \dots \dots (4)$$

❖ Here, two different cases can occur

• **Case 1:** Suppose $\bar{a}_{ij} \leq 0, \quad \forall i = 1, 2, \dots, m$

In this case, δ can be arbitrarily large without violating any of the non-negativity condition (4).

Consequently, $Cx = z_B + \bar{c}_j\delta \rightarrow \infty$

Thus, we obtained the following result

Theorem: (Criterion for unbounded LP)

Given a BFS with respect a basis B , if there is a non-basic column j such that

$$\bar{c}_j > 0 \text{ and } B^{-1}A_j = \bar{A}_j \leq 0,$$

then the LP $\max\{ cx \mid Ax = b, x \geq 0 \}$ is unbounded.

• **Case 2:** Suppose, in the pivot column j , there is $\bar{a}_{ij} > 0$, for some $i \in \{1, 2, \dots, m\}$.

Then, from (4), $x_i = \bar{b}_i - \bar{a}_{ij}\delta \geq 0$, $\forall i = 1, 2, \dots, m$

provided that $\delta \leq \bar{b}_i / \bar{a}_{ij}$, for every i s.t. $\bar{a}_{ij} > 0$.

- Thus, in order to choose δ as large as possible we take

$$\delta = \min\{ \bar{b}_i / \bar{a}_{ij} \mid \bar{a}_{ij} > 0 \} \quad (\text{Min Ratio})$$

- If the minimum of these ratio occurred in r -th row, then $\delta = \bar{b}_r / \bar{a}_{rj}$

- That is, for old BV $x_r = \bar{b}_r - (\bar{a}_{rj}\bar{b}_r / \bar{a}_{rj}) = 0$
and, for the new $x_j = \delta = \bar{b}_r / \bar{a}_{rj}$

This is a basis exchange: x_j enters basis and x_r leaves the basis.

- That is, row r is a pivot row that yields a new BFS.

- Thus we have the following result.

Theorem 3: (Basis Exchange)

In the simplex tableau of an LP, suppose that

- $\bar{c}_j > 0$ for some non-basic column (pivot column) j ; $\bar{c}_j$ $\bar{A}_j$
- column under $\bar{A}_j$, i.e., $\bar{A}_j$, has at least one positive entry;

- and let the $\bar{b}_r / \bar{a}_{rj}$ be occurred in row r . i.e., $\bar{b}_r / \bar{a}_{rj} = \min \{ \bar{b}_i / \bar{a}_{ij} \mid \bar{a}_{ij} > 0 \}$

Then, choosing r -th row as pivot row, pivoting with $\bar{a}_{rj}$ yields a new BFS (i.e., x_j enters basis & x_r leaves basis).

Moreover, the value of the objective function at the new BFS is increased by $\bar{c}_j \bar{b}_r / \bar{a}_{rj}$

- Collecting the above results together, we have the following algorithm (solution procedure).

The Simplex Algorithm: To solve $\max \{cx \mid Ax=b, x \geq 0\}$

1. Construct the simplex tableau of the LP in PSF and start with initial BFS

2. **Optimality Test:** If $\bar{c}_j \leq 0, \forall j$ **STOP**. The current BFS is optimal solution, and the optimal objective value is

(the negative of the entry at RHS of row 0)

Else;

$$\bar{c}_j > 0$$

Choose a non-basic column j with largest

(Column j is the pivot column)

3. **Unboundedness:** If, , then

STOP - the

LP is unbounded.

Else do basis exchange as follows:

- Choose a pivot row r in which the following $\min$ occurs:

Example : Using the simplex algorithm solve the following LP problems.

1)
$$\begin{aligned} \max \quad & -2x_1 + 4x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 - 2x_3 \leq 2 \\ & 4x_1 + 2x_3 \leq 4 \\ & -x_1 + 2x_2 - x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Optimal solution:

$$x^* = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix},$$
$$z^* = 14$$

2)
$$\begin{aligned} \max \quad & 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 - 2x_3 \leq 7 \\ & -3x_1 + x_2 + 2x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(Unbounded)

4.3 Starting Basic Feasible Solution

• **Recall:** Simplex method starts with an initial BFS (in PSF).

• For the LP with type $Ax \leq b$, where $b \geq 0$, the slack variables are used for starting basic variables.

• In general, however, starting BFS may not be obvious.

• **Example:** Let the constraints of an LP (with all $x_i \geq 0$) be

$$\begin{array}{rcl} x_1 + x_2 & \leq & 2 \\ -x_1 + x_2 & \geq & 1 \\ x_2 & \leq & 3 \end{array} \quad \begin{array}{l} x_3, x_4 : \text{surplus;} \\ x_5 : \text{slack} \end{array} \quad \begin{array}{rcl} x_1 + x_2 - x_3 & = & 2 \\ -x_1 + x_2 - x_4 & = & 1 \\ x_2 + x_5 & = & 3 \end{array}$$

• To get a starting BV, for each constraint that has no slack variable, we introduce a so called **artificial variable**.

• Thus, with artificial variables $x_6, x_7 \geq 0$, the above becomes:

$$\begin{array}{rcl} x_1 + x_2 - x_3 + x_6 & = & 2 \\ -x_1 + x_2 - x_4 + x_7 & = & 1 \\ x_2 + x_5 & = & 3 \end{array}$$

• Thus, x_5, x_6, x_7 are starting basic variables.

- In general, if a BFS is not directly available in

$$Ax = b, \quad x \geq 0, \quad \dots \dots \dots (12)$$

then, to each i -th equation, we add a non-negative artificial variable x_i^a

$$\Rightarrow Ax + Ix^a = b, \quad x, x^a \geq 0, \quad \dots \dots \dots (13)$$

where x^a is vector of artificial variables.

So, at the beginning $x_B = x^a = b$.

- Note :** The original system, (12), is consistent (feasible) iff

(13) has a BFS with $x^a = 0$

- Hence, starting with (13), a sequence of basis exchange (pivoting) should find a BFS in which all artificial variables are 0 (non basic).

At this stage, we get a BFS for (12)

- There are two ways of solving an LP with artificial variables, namely, the big-M methods and the two-phase

We will discuss the former.

Notation : $e^T = (1, 1, \dots, 1)$. (So, $e^T y = y_1 + y_2 + \dots + y_m$ where $y = (y_1, y_2, \dots, y_m)^T$)

I. The big-M Method: To solve the LP

$$\max cx \quad (1)$$

$$s.t, Ax=b, x \geq 0$$

- Let the constraints be altered to $Ax + x_a = b, x, x_a \geq 0$,

and let the objective function be $f(x) = cx - Me^T x_a$

where M is a large positive number (or $M \rightarrow \infty$), called

penalty cost,

• Thus, **the big-M** is to solve:
 $\Rightarrow c_j x_j > -M x_a$ for any feasible value of x_j and any $x_a > 0$.

$\Rightarrow$ Simplex method removes all artificial variables from basis (for $S \neq \emptyset$)

$$\begin{aligned} \max & cx - Me^T x_a \\ s.t. & Ax + Ix_a = b \\ & x, x_a \geq 0, \end{aligned} \quad (2)$$

using the simplex method starting with initial BFS $x_a = b$, $x = 0$.

- **Remark :** To solve the problem with Big-M:

$$\begin{aligned} \max \quad & cx - Me^T x_a \\ \text{s.t.} \quad & Ax + Ix_a = b \\ & x, x_a \geq 0, \end{aligned}$$

1. Execute the simplex method without a need to specify a value for **M**, but recall that **-M** is less than any number (and **M** > any number)
2. **Adjust the obj. row to PSF** (i.e, coeff of **$x_a \rightarrow 0$**) before starting the simplex method (**Add M times rows of artificial variables to objective row**)
3. A non-basic artificial variable is never considered for re-entry into the basis.
Therefore, **ignore the column of artificial variable once it leaves the basis.**

4. **If all artificial variables are zero in the optimal tableau, then**

● **Example: Solve** -

$$\begin{aligned} \max \quad & -x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & -x_1 + x_2 \geq 1 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\Rightarrow$
(standard
form)

$$\begin{aligned} \max \quad & -x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 = 2 \\ & -x_1 + x_2 - x_4 = 1 \\ & x_2 + x_5 = 3 \\ & x_1, \dots, x_5 \geq 0 \end{aligned}$$

$\Rightarrow$
(Big-M)
 x_6, x_7 artif.

$$\begin{aligned} \max \quad & -x_1 + 2x_2 - Mx_6 - Mx_7 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 + x_6 = 2 \\ & -x_1 + x_2 - x_4 + x_7 = 1 \\ & x_2 + x_5 = 3 \\ & x_1, \dots, x_7 \geq 0 \end{aligned}$$

Initial basic var. : $x_B = (x_6, x_7, x_5)$, $x_a = (x_1, x_2, x_3, x_4)$.

The **Simplex** procedure of the problem with **big M** :

	x_1	x_2	x_3	x_4	x_5		RHS
	-1	$2+2M$	$-M$	$-M$	0	0	$3M$
	x_6	0	x_7				
M	-z	-1	2	0	0	0	$-M$
M		$-M$					0 (IT)
	x_6	1	1	-1	0	0	1
	-z	$1+2M$	0	$-M$	$2+M$	0	$-2+M$
		0					
	x_6	2	0	-1	1	0	1
	x_2	-1	1	0	-1	0	0
	-z	-3	0	2	0	0	-4
	x_4	2	0	-1	1	0	1
	x_2	1	1	-1	0	0	2
	x_5	-1	0	1	0	1	1

Initial basis: $B = (A_6, A_7, A_5)$

Artificial variables : x_6, x_7 .

First adjust the obj row of x_a to PSF by adding M times 1st and 2nd rows to obj. row.

Continuing in this way, we will get optimal solution

$x^* = (x_1, x_2) = (0, 3)$,
 $z^* = 6$.

🔴 **Exercises:** Using the big-M method solve the following LP

1)
$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 = 10 \\ & x_1 + x_2 + x_3 = 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2)
$$\begin{aligned} \min \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\ & 3x_1 - x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

3)
$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \\ & x_1 - x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

4)
$$\begin{aligned} \min \quad & x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & -x_1 + x_2 \geq 3 \\ & x_1 - x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

5)
$$\begin{aligned} \max \quad & 4x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \geq 5 \\ & 6x_1 + 4x_2 - 3x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

6)
$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \\ & -3x_1 + 2x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

END