

Introduction to Optimization: Linear Programming

Course Outline:

1. Introduction

General Introduction, Basic definitions and terminologies, Problem formulations, Linear Programming (LP), Canonical form of LP problems, Some Examples of LP formulations

2. Geometric Solution Method for LP

Graphical solution for LP, polyhedral set, level lines, direction of increase/decrease, Corner Point Theorem

3. Basic Feasible Solutions & Fundamental Theorem of LP

LP in standard form, Basic Feasible Solutions, Algebraic Analysis of the Constraint Set, The

4. The Simplex Method

Fundamental Theorem of LP
Introduction, Optimality Condition, The Simplex Algorithm, Starting Basic Feasible Solution (Two phase simplex method)

5. Duality in Linear Programming

Dual problem, Duality theorems, Dual Simplex Method

✱ **Karak;** Linear Programming and
References:
Theory of
Games

✱ **Bazaara;** Linear Programming and
Network
Flows

Chapter 1.

Introduction

1.1 What does optimization mean?

- ❖ **Literally:** Among various possibilities, it is to find the best possible solution of a problem under a set of constraints.
- ❖ **In Mathematics:** To optimize means -
to **maximize or minimize a function $f(x)$** , where the values of **x have to satisfy a set of prescribed mathematical relations.**

Thus, given a real valued function f of several variable, i.e.,

$$f: R^n \rightarrow R \quad \text{and a set}$$

$$S \subseteq R^n$$

a problem of optimization is either a **maximization** or a **minimization** problem :

- **Maximization Problem:** Find a point $x^* \in S$, at which the value of f is **maximum** over the set S .

That is, find $x^* \in S$ such that $f(x^*) \geq f(x)$ for all $x \in S$.

This problem is written as
$$\begin{array}{ll} \max & f(x) \\ \text{s.t} & x \in S \end{array}$$

Note: $x = (x_1, x_2, \dots, x_n)$ in R^n .

- **Minimization Problem:** Find a point $x^* \in S$, at which the value of f is **minimum** over the set S .

That is, find $x^* \in S$ such that $f(x^*) \leq f(x)$ for all $x \in S$.

This problem is written as
$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & x \in S \end{array}$$

1.2 Problem Formulation and Basic

Terminologies
(i) Formulate the following problem as an optimization problem
(Min surface area) Find the dimension, l, w, h , of a top-open rectangular box (container) that should have base area of at least 6 m^2 and volume of at least 12 m^3 that can be constructed from the least amount of material (metal) .

Let x_1 = length, x_2 = width & x_3 = height of the box

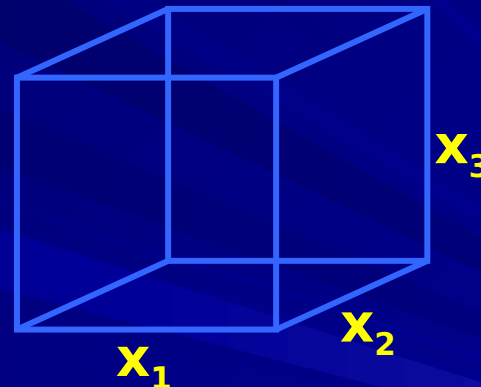
Base area:

Volume:

Aim: Surface area: $\rightarrow$
Min

The Problem is

$$\begin{aligned} \min & x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ \text{s.t } & x_1x_2 \geq 6 \\ & x_1x_2x_3 \geq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$



x_1	x_2	x_3	SA
2	3	2	26
2.4	2.5	2	25.6
2.5	2.6	1.85	25.37

(ii) Basic Definitions and Terminologies

Consider an optimization problem

- The components of variable x , i.e., $x_1, x_2, \dots, x_n$, are called **decision variables**.
(P) $\max$ (or $\min$) $f(x)$ s.t. $x \in S$
- The function $f(x)$ is called **objective function** of (P).
- The conditions (inequalities/equations) describing S are called **constraints**
- The conditions that $x_j \geq 0$ in optimization problem is called **non-negativity constraints**

Problem Formulation:

- Item B occupies M_1 for 8 minutes and M_2 for 4 minutes.

The machines can be used for at most 3 hour per day.

The profit per unit of item A and B is 29 and 45, resp.

The problem is to find the number of daily production of each item that maximizes the net profit per day.

Dec. Let $x_1 :=$ number of item A

Variables: ($x_1 \geq 0$)

Constraints: Due to M_1 : $2x_1 + 8x_2 \leq 180$
Due to M_2 : $4x_1 + 4x_2 \leq 180$
($x_2 \geq 0$) $x_2 :=$ number of item B

Objective function: Profit: $29x_1 + 45x_2 \rightarrow \max$

Thus, the mathematical (optimization) formulation of the above production planning problem is:

$$\begin{array}{ll}\max & 29x_1 + 45x_2 \\ \text{s.t} & 2x_1 + 8x_2 \leq 180 \\ & 4x_1 + 4x_2 \leq 180 \\ & x_1, x_2 \geq 0\end{array}$$

x_1	x_2	profit
0	22	990
45	0	1305
24	16	1416
30	15	1545

Note: Techniques of calculus cannot directly help us to solve such problems (There is no critical point as

$\nabla f \neq 0$.)

Further Terminologies: Feasible and Optimal Solution
 $x = (x_1, x_2, \dots, x_n)$ is called a **feasible solution**

Consider an Opt. Problem (P), if $\max_{x \in S} \min_{x \in S} f(x)$ if the values of its components, $x_1, x_2, \dots, x_n \in S$ satisfy the constraints.

(**Note:** Optimization problem may have infinitely many feasible solutions. Indeed, every $x \in S$ is a feasible solution of (P))

● **Optimal solution and Optimal Value:**

- $x^* \in S$ is called **optimal solution** for a min problem if

$$f(x^*) \leq f(x) \quad \text{for all } x \in S.$$
- If $x^* \in S$ is optimal solution of (P), then $f(x^*)$ is called its **optimal (objective) value** of (P).

- (P) is said to be **infeasible** problem if $S = \emptyset$.

Example: Let (P) be

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ \text{s.t} & x_1 - x_2 \geq 1 \\ & -x_1 + x_2 \geq 1 \end{array}$$

There is no value for x_1 & x_2 that satisfy the

constraints, i.e., $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 \geq 1, -x_1 + x_2 \geq 1, x_1, x_2 \geq 0\} = \emptyset$

- Thus, the problem is **infeasible**.
- The opt. problem (P) is said to be **unbounded**, if
 - $\max f(x)$ tends to ∞ on S for max problem
 - $\min f(x)$ tends to $-\infty$ on S for min problem

Example:

$$\begin{array}{ll} \max & 5x_2 \\ \text{s.t} & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array} \quad (\text{unbounded problem})$$

An optimization problem $\max$ (or $\min$) $f(x)$
1.4 Linear Programming (LP): s.t $x \in S$

is said to be a **linear programming (LP) problem** if

the objective function $f(x)$ is **linear**,

$$\text{i.e., } f(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Example: - The Production Planning problem and also every function involving in the given description of S is **linear** (of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n \{ \leq, \geq, = \} b$) above is a linear programming problem.

- The surface area problem given above is not an LP. It is a nonlinear programming prob.

● The subject matter of this course is only LP

Canonical (common) form of an LP Problem:

$$\max C_1x_1 + C_2x_2 + \dots + C_nx_n$$

$$\text{s.t. } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_j \geq 0, j = 1, \dots, n$$

Objective
function

Constrain
ts

Non negativity

Concise form:

$$\max c^T x$$

$$\text{s.t. } Ax \leq b$$

$$x \geq 0$$

$c \in R^n$, $b \in R^m$, and A is an $m \times n$ matrix.
(m constraints & n variables. Usually $m < n$)

END