

Chapter Two

2. LINEAR PROGRAMMING: Application and Model

Formulation

Unit Objectives:

Up on completion of this unit, the learner is expected to:

- Recognize problems that can be solved using LP models.
- Formulate an LP model in mathematical terms.
- Solve Linear Programming Problems (LPP) using both graphic and simplex approach.
- Explain special cases in both graphic and simplex techniques.

INTRODUCTION

In 1947, George Danzig developed the use of algebra for determining solutions to problems that involved the optimal allocation of scarce resources. In spite of numerous potential applications in business, response to this new technique was low due to substantial computational burden, which is now removed with subsequent advances in computer technology and related software during the last three decades.

The term linear implies that all the mathematical relations used in the problem are linear or straight-line relations, while the term programming refers to the method of determining a particular program or plan of action, i.e., the use of algorithms that is a well defined sequence of steps that will lead to an optimal solution. Taken as a whole, the term linear programming refers to a family of mathematical techniques for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the limited or constrained resources.

The technique of linear programming is applicable to problems in which the total effectiveness can be expressed as linear function of individual allocations and the limitations on resources give rise to linear equation or inequalities of the individual allocations.

The usefulness of this technique is enhanced by the availability of several user-friendly soft wares such as STORM, TORA, QSB+, LINDO, etc. However, there is no general package for building an LP model. Model building is an art of practice.

? Dear learner, can you make differences between solutions of LP models solved manually and using computers utilizing soft wares?

2.1. LINEAR PROGRAMMING MODELS

Linear programming models are mathematical representations of LP problems. Linear programming models have certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models, and to be able to correctly formulate an LP model. These characteristics can be

grouped as components and assumptions. The components relate to the structure of a model, where as the assumptions reveal the conditions under which the model is valid.

2.1.1. COMPONENTS OF LP MODELS

There are four major components of LP models including: Objective function, decision variables, constraints and parameters.

Objective and Objective Function

The objective in problem solving is the criterion by which all decisions are evaluated. It provides the focus for problem solving. In linear programming models, a single, quantifiable objective must be specified by the decision maker. Because we are dealing with optimization, the objective will be either maximization or minimization. Hence, every LP problem will be either maximization or a minimization problem. Once the objective is specified, it becomes the measure of effectiveness against which alternate solutions are judged. An LP model consists of a mathematical statement of the objective called the objective function.

Decision variables

They represent unknown quantities to be solved for. The decision maker can control the value of the objective, which is achieved through choices in the levels of decision variables. For example, how much of each product should be produced in order to obtain the greatest profit?

Constraints

However, the ability of a decision maker to select values of the decision variables in an LP problem is subject to certain restrictions or limits coming from a variety of sources. The restrictions may reflect availabilities of resources (e.g., raw materials, labor time, etc.), legal or contractual requirements (e.g., product standards, work standards, etc.), technological requirements (e.g., necessary compressive strength or tensile strength) or they may reflect other limits based on forecasts, customer orders, company policies, and so on. In LP model, the restrictions are referred to as constraints. Only solutions that satisfy all

constraints in a model are acceptable and are referred to as **feasible solutions**. The optimal solution will be the one that provides the best value for the objective function.

Generally speaking, a constraint has **four** elements:

- A right hand side (RHS) quantity that specifies the limit for that constraint. It must be a constant, not a variable.
- An algebraic sign that indicates whether the limit is an upper bound that cannot be exceeded, a lower bound that is the lowest acceptable amount, or an equality that must be met exactly.
- The decision variables to which the constraint applies.
- The impact that one unit of each decision variable will have on the right-hand side **quantity of the constraint**.

Constraints can be arranged into three groups:

- *System constraints* - involve more than one decision variable,
- *Individual constraints* - involve only one variable, and
- *Non-negativity constraints* - specify that no variable will be allowed to take on a negative value. The non-negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

Parameters

The objective function and the constraints consist of symbols that represent the decision variables (e.g., X_1 , X_2 , etc.) and numerical values called parameters. The parameters are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as the numerical value of each constraint.

The following simple example illustrates the components of LP models:

x_1 = quantity of product 1
 x_2 = quantity of product 2
 x_3 = quantity of product 3

} Decision variables

Maximize $Z = 4x_1 + 7x_2 + 5x_3$ (profit) ← Objective function

Subject to

$2x_1 + 3x_2 + 6x_3 \leq 300$ labor hours
 $5x_1 + 4x_3 \leq 200$ lb. raw material A
 $3x_1 + 5x_2 + 2x_3 \leq 360$ lb. raw material B

} System constraints

$x_1 = 30$ units
 $x_2 \geq 40$ units

} ← Individual Constraints

$x_1, x_2, \text{ and } x_3 \geq 0$ ← Non-negativity constraints

? Dear learner, can you give some of the examples of LP models which consist of unit and system constraint and discuss the components of each constraint in the model?

2.1.2. ASSUMPTIONS OF LP MODELS

Linearity (proportionality)

The linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. In terms of a mathematical model, a function or equation is linear when the variables included are all to the power 1 (not squared, cubed, square root, etc.) and no products (e.g., x_1x_2) appear. On the other hand, the amount of each resource used (supplied) and its contribution to the profit (or cost) in the objective function must be proportional to the value of each decision variable. For example, if production of one unit requires 5 hours of a particular resource, then making 3 units of that product requires 15 hours (3x5) of that resource.

Divisibility (Continuity)

The divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. However, if the problem concerns, for example, the optimal number of houses to construct, 3.5 do not appear to be acceptable. Instead, that type of problem would seem to require strictly integer solutions. In such cases, integer-programming methods should be used. It should be noted, however, that some obvious integer type situations could be handled under the assumption of divisibility. For instance, suppose 3.5 to be the optimal number of television sets to produce per hour, which is unacceptable, but it would result in 7 sets per two hours, which would then be acceptable.

Certainty

This requirement involves two aspects of LP models. One aspect relates to the model parameters, i.e., the numerical values. It is assumed that these values are known and constant. In practice, production times and other parameters may not be truly constant. Therefore, the model builder must make an assessment as to the degree to which the certainty requirement is met. Large departures almost surely will have a significant effect on the

model. The other aspect is the assumption that all relevant constraints have been identified and represented in the model.

Additivity

The value of the objective function and the total amount of each resource used (or supplied), must be equal to the sum of the respective individual contributions (profit or cost) by decision variables. For example, the total profit earned from the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed for producing A and B must be equal to the sum of resources used for A and B respectively.

Non-negativity

It assumes that negative values of variables are unrealistic and, therefore, will not be considered in any potential solutions. Only positive values and zero will be allowed and the non-negativity assumption is inherent in LP models.

? Dear learner, can you give examples for each of the above assumptions? What will happen if the assumptions are not met?

2.1.3. Advantages of Linear Programming

Following are certain advantages of linear programming.

- Linear programming helps in attaining the optimum use of **scarce productive resources**. It also indicates how a decision maker can employ productive resources effectively by selecting and allocating these resources.
- LP techniques improve **quality** of decisions. The decision making approach of the user of this technique becomes more objective and less subjective.
- Highlighting of bottlenecks in the production process is one of the most advantages of this technique. For example, when a bottleneck occurs, some machines can not meet demand while others remain idle for some of the time.
- LP also helps in re-evaluation of a basic plan for changing conditions. If conditions change when the plan is carried out, they can be determined so as to adjust the remainder of the plan for best results.

2.1.4. Limitations of Linear Programming

In spite of having many advantages and wide area of applications, there are some limitations associated with this technique. These are:

- LP treats all relationships among decision variables as linear. However, generally neither the objective functions nor the constraints in real-life situations concerning business and industrial problems are linearly related to variables.
- While solving an LP model, there is **no guarantee** that we will get integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, a non integer valued solution will be meaningless. Rounding off the solution to the nearest integer will not yield an optimal solution.
- An LP model does not take in to consideration the **effect of time and uncertainty**. Thus, the LP model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
- It deals with only with a single objective, whereas in real-life situations we may come across conflicting multi-objective problems.
- Parameters appearing in the model are assumed to be constant but, in real life situation they are frequently neither known nor constant.

? Dear learner, can you mention some other advantages and disadvantages of LP?

2.1.5. FORMULATING LP MODELS

Just as it is to define a problem, careful formulation of the model that will be used to solve the problem is important. Linear programming algorithms (solution techniques) are widely used and understood and computer packages are readily available for solving LP problems. Consequently, obtaining solutions is not the real issue, what is very important to note is failure to check that all constraints have been accounted for and have been correctly formulated results in ill-structuring of the model that can easily lead to poor decisions.

Steps in formulating LP models:

- **Identify the decision variables.**
- **Determine the objective function.**
- **Identify the constraints.**
- **Determine appropriate values for parameters and determine whether an upper limit, lower limit or equality is called for.**
- **Use this information to build a model.**

➤ **Validate the model.**

In many cases, the decision variables are obvious; in others it might require brief discussion with the appropriate manager. However, identifying the constraints and determining appropriate values for the parameters can require considerable time and effort. Potential sources of information include historical records, interviews with managers and staff, and data collection. Validating the model will involve a critical review of the output, perhaps under a variety of inputs, in order to decide if the results are reasonable.

2.2. LINEAR PROGRAMMING APPLICATIONS

There is a wide range of problems that lend themselves to solution by linear programming techniques. This discussion is only meant to give an indication of the LP techniques for managerial decision making and the apparent diversity of situations to which linear programming can be applied. Some of these include production management (product mix, blending problems, production planning, Assembly line balancing...), Marketing management (media selection, traveling sales man problem, physical distribution), Financial management (portfolio selection, profit planning), agricultural application, military applications, personnel management (staffing problem, Determination of equitable salary and etc.

Product Mix

Organizations often produce similar services that use the same resources. For example, labor, material cost, etc. because of limited resources available during any time period a decision must be made concerning how much of each product to produce or make available. Linear programming answers what mix of output (or service) will maximize profit given the availability of scarce resources

Diet problem

It usually involves the mixing of raw materials or other ingredients to obtain an end product that has certain characteristics. For example, what mix of inputs will achieve the desired results in the output for the least cost? Other applications that fall into this category include mixing feed for livestock, mixing pet foods, mixing building materials (concrete, mortar, paint), and so on.

Blending problems

They are very similar to diet problems. Strictly speaking, however, blending problems have additional requirement, i.e. to achieve a mix that have specific consistency. For example, how many quarts of the different juices each with different sugar content proportion must be mixed together to achieve one gallon that has a sugar content of 17 percent?

Portfolio selection

These problems generally involve allocating a fixed dollar amount among a variety of investments, such as bonds, stocks, real states, etc. The goal usually is to maximize income or total return. The problems take on an added dimension when certain other requirements are specified (for example, no more than 40 percent of the portfolio can be invested in bonds).

Dear learner, the section below presents some examples that demonstrate how a linear programming model is formulated. Although these examples are simplistic, they are realistic and represent the type of problem to which linear programming can be applied. By carefully studying each of these examples, the learner can become familiar with the process of formulating linear programming models.

Product Mix

ABC private limited company is engaged in the production of power and traction transformers. Both of these categories of transformers pass through three basic processes: core preparation, core to coil assembly, and vapor phase drying. A power transformer yields a contribution of Birr 50,000 and traction transformer contributes Birr 10,000. The time required in the production of these two products in terms of hours for each of the processes is as follows.

	Power transformer
Traction Transformer	
Core preparation	75
15	
Core to Coil Assembly	160
30	
Vapor Phase Drying	45
10	

If the capacities available are 1000, 1500, and 750 machine hours in each processes

respectively, formulate the problem as LP?

Solution

Step1. Identify decision variables

Since the products to be produced are power and traction transformers using the available resource to attain the objective set, we consider them as decision variables. This is because the organization's problem here is how many of each product to produce in order to attain the objective, which requires the management decision.

Let X_1 = the no of power transformers to be produced.

X_2 = the no of traction transformer to be produced

Step2. Determine Objective Function

From the problem above, we understand that the problem is maximization problem.

Hence,

$$Z_{\max} = 50,000X_1 + 10,000X_2$$

This is because, each unit of X_1 contributes Birr 50,000 and X_2 contributes Birr

10,000 to objective function.

Step 3. Identify constraints

Here we have two constraints: structural and non-negativity constraints. The

structural constraint is the amount of machine hours available in each process.

Step 4. Determining Parameters

Parameters are already identified in the table of the problem.

👉 **Note:** Step 3 and 4 can be performed simultaneously as:

$$75X_1 + 15X_2 \leq 1000 \text{ hrs- Core preparation process}$$

$$160X_1 + 30X_2 \leq 1500 \text{ hrs- Core to Coil Assembly} \quad \text{Machine}$$

hour constraint

$$45X_1 + 10X_2 \leq 750 \text{ hrs- Vapor Phase Drying}$$

$$X_1, X_2 \geq 0$$

Step 5. Building and validating the model

$$Z_{\max} = 50,000X_1 + 10,000X_2$$

Subject to:

$$75X_1 + 15X_2 \leq 1000 \text{ hrs}$$

$$160X_1 + 30X_2 \leq 1500 \text{ hrs}$$

$$45X_1 + 10X_2 \leq 750 \text{ hrs}$$

$$X_1, X_2 \geq 0$$

Investment Application

1. An individual investor has Birr 70,000 to divide among several investments. The alternative investments are municipal bonds with an 8.5% return, certificates of deposits with a 10% return, Treasury bill with a 6.5% return, and income bonds with a 13% return. The amount of time until maturity is the same for each alternative. However, each investment alternative has a different perceived risk to the investor; thus it is advisable to diversify. The investor wants to know how much to invest in each alternative in order to maximize the return. The following guidelines have been established for diversifying the investment and lessening the risk perceived by the investor.

1. No more than 20% of the total investment should be in an income bonds.
2. the amount invested in certificates of deposit should not exceed the amount invested in other three alternatives.
3. At least 30% of the investment should be in treasury bills and certificates of deposits.
4. The ratio of the amount invested in municipal bonds to the amount invested in treasury bills should not exceed one to three.

The investor wants to invest the entire Birr 70,000.

Required: Formulate a LP model for the problem.

Solution

Decision variables

Four decision variables represent the monetary amount invested in each investment alternative.

Let X_1 , X_2 , X_3 and X_4 represent investment on municipal bonds, certificates of deposits, Treasury bill, and income bonds respectively.

Objective function

The problem is maximization because from the word problem we know that the objective of the investor is to maximization return from the investment in the four alternatives.

Therefore, the objective function is expressed as:

$$Z_{\max} = 0.085 X_1 + 0.1X_2 + 0.065X_3 + 0.13X_4$$

Where,

Z = total return from all investment

$0.085 X_1$ = return from the investment in municipal bonds.

$0.100 X_2$ = return from the investment in certificates in deposit

$0.065 X_3$ = return from the investment in treasury bills

$0.130 X_4$ = return from the investment in income bonds

Model Constraints

In this problem the constraints are the guide lines established for diversifying the total investment. Each guideline is transformed in to mathematical constraint separately.

Guideline 1 above is transformed as:

$$X_4 \leq 20\% (70,000) \Rightarrow X_4 \leq 14,000$$

Guideline 2

$$X_2 \leq X_1 + X_3 + X_4 \Rightarrow X_2 - X_1 - X_3 - X_4 \leq 0$$

Guideline 3

$$X_2 + X_3 \geq 30\% (70,000) \Rightarrow X_2 + X_3 \geq 21,000$$

Guideline 4

$$X_1 / X_3 \leq 1/3 \Rightarrow 3X_1 \leq X_3 \Rightarrow 3X_1 - X_3 \leq 0$$

Finally,

$X_1 + X_2 + X_3 + X_4 = 70,000$, because the investor's money equals to the sum of money invested in all alternatives.

This last constraint differs from $\leq$, $\geq$ and $=$ inequalities previously developed in that there is a specific requirement to invest an exact amount. Therefore, the possibility to invest more or less than Birr 70,000 is not considered.

This problem contains all three types of constraints possible in LP problems: $\leq$, $\geq$ and $=$. As this problem demonstrates there is no restriction on mixing these types of constraints.

The complete LPM for this problem can be summarized as:

$$Z_{\max} = 0.085 X_1 + 0.1X_2 + 0.065X_3 + 0.13X_4$$

Subjected to:

$$\begin{array}{rcl} & & X_4 \leq 14,000 \\ X_2 & - & X_1 - X_3 - X_4 \leq 0 \\ & X_2 + X_3 & \geq 21,000 \end{array}$$

$$3X_1 - X_3 \leq 0$$

$$X_1 + X_2 + X_3 + X_4 = 70,000,$$

$$X_1 + X_2 + X_3 + X_4 \geq 0$$

Marketing Application

2. Supermarket store chain has hired an advertising firm to determine the types and amount of advertising it should have for its stores. The three types of advertising available are radio and television commercials, and news papers advertisements. The retail chain desires to know the number of each type of advertisement it should purchase in order to maximize exposure. It is estimated that each ad or commercial will reach the following potential audience and cost the following amount.

Type of Advertisement	Exposure (people /ad or commercial)	
Cost		
Television commercial	20,000	Birr
15,000		
Radio commercial	12,000	
6,000		
News paper advertisement	9,000	
4,000		

The company must consider the following resource constraints.

1. The budget limit for advertising is Birr 100,000
2. The television station has time available for 4 commercials.
3. The radio station has time available for 10 commercials.
4. The news paper has space available for 7 ads.
5. The advertising agency has time and staff available for producing no more than a total of 15 commercials and/or ads.

Decision variables

Let X_1 = number of television commercials

X_2 = number of radio commercials

X_3 = number of news paper ads

Objective function

It is not only profit which is to be maximized. In this problem the objective to be maximized is audience exposure.

$$Z_{\max} = 20,000x_1 + 12,000x_2 + 9,000x_3, \text{ where}$$

Z = total level of audience exposure

$20,000x_1$ = estimated number of people reached by television commercials

$12,000x_2$ = estimated number of people reached by radio commercials

$9,000x_3$ = estimated number of people reached by news paper ads

Model constraints

Budget constraint

$$15,000x_1 + 6,000x_2 + 4,000x_3 \leq \text{Birr } 100,000$$

Capacity constraint

Television commercials and radio commercials are limited to 4 and 10 respectively and news paper ads are limited to 7.

$$X_1 \leq 4 \text{ television commercials}$$

$$X_2 \leq 10 \text{ radio commercials}$$

$$X_3 \leq 7 \text{ news paper ads}$$

Policy constraint

The total number of commercials and ads can not exceed 15

$$X_1 + X_2 + X_3 \leq 15$$

The complete linear programming model for this problem is summarized as:

$$Z_{\max} = 20,000X_1 + 12,000X_2 + 9,000X_3$$

Sub. to:

$$15,000X_1 + 6,000X_2 + 4,000X_3 \leq \text{Birr } 100,000$$

$$X_1 \leq 4$$

$$X_2 \leq 10$$

$$X_3 \leq 7$$

$$X_1 + X_2 + X_3 \leq 15$$

$$X_1, X_2, X_3 \geq 0$$

Chemical mixture

A chemical corporation produces a chemical mixture for the customer in 1000- pound batches. The mixture contains three ingredients- Zinc, mercury and potassium. The mixture must conform to formula specifications (i.e., a recipe) supplied by a customer. The company wants to know the amount of each ingredient to put in the mixture that will meet all the requirements of the mix and **minimize** total cost.

The customer has supplied the following formula specifications for each batch of mixture.

1. The mixture must contain at least 200 lb of mercury
2. The mixture must contain at least 300 lb of zinc
3. The mixture must contain at least 100 lb of potassium

The cost per pound of mixture is Birr4, of zinc Birr 8 and of potassium Birr 9.

Required: Formulate LPM for the problem

Solution

Decision variables

The model of this problem consists of three decision variables representing the amount of each ingredient in the mixture.

X_1 = number of lb of mercury in a batch

X_2 = number of lb of zinc in a batch

X_3 = number of lb of potassium in a batch

Objective function

$$Z_{\min} = 4x_1 + 8x_2 + 9x_3$$

Constraints

$X_1 \leq 200$ lb... specification 1

$X_2 \leq 300$ lb... specification 2

$X_3 \leq 100$ lb... specification 3

Finally, the sum of all ingredients must equal 1000 pounds.

$$x_1 + x_2 + x_3 = 1000 \text{ lb}$$

The complete LPM of the problem is:

$$Z_{\min} = 4x_1 + 8x_2 + 9x_3$$

Subject to:

$$X_1 \leq 200 \text{ lb}$$

$$X_2 \leq 300 \text{ lb}$$

$$X_3 \leq 100 \text{ lb}$$

$$x_1 + x_2 + x_3 = 1000 \text{ lb}$$

$$x_1, x_2, x_3 \geq 0$$