



Unit 7

Vectors and transformation

1. Vectors and calculation with vectors

Definition: A quantity that have both magnitude and direction is called a **vector**.

Example: acceleration, velocity, force, . . .

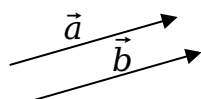
- Vector is denoted by arrow

E.g. vector $a = \vec{a}$ vector $v = \vec{v}$

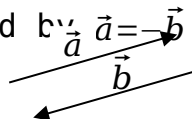
$\vec{V} = \overrightarrow{AB}$

Note:

- If $\vec{v} = x_1\vec{i} + y_1\vec{j}$ then $|\vec{v}|$ denotes magnitude or norm of $\vec{v}$, defined as $|\vec{v}| = \sqrt{x_1^2 + y_1^2}$
- Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if and only if they have the same direction and equal magnitude and denoted by $\vec{a} = \vec{b}$.



- Two vectors $\vec{a}$ and $\vec{b}$ are said to be opposite if they have the same magnitude but opposite in direction and denoted by $\vec{a} = -\vec{b}$



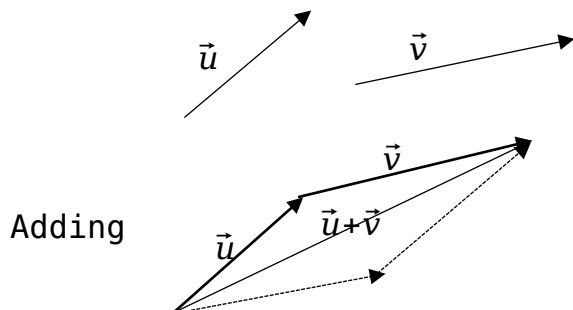
E.g. If $\vec{a} = 2\vec{i} - 3\vec{j}$ and $\vec{b} = -2\vec{i} + 3\vec{j}$, then $\vec{a} = -2\vec{i} + 3\vec{j} = -(-2\vec{i} + 3\vec{j}) = -\vec{b}$ and $|\vec{a}| = |\vec{b}|$ tells us they are opposite direction and equal in magnitude resp.

$\therefore \vec{a}$ and $\vec{b}$ are opposite.

Addition of vectors

Tail-to-tip rule

E.g. Add $\vec{u}$ and $\vec{v}$.



Note: Tail of $\vec{v}$ is fixed to tip of $\vec{u}$ keeping its dnxn.

Definition: If $\vec{u}=(x_1, y_1)$ and $\vec{v}=(x_2, y_2)$, then $\vec{u}+\vec{v}=(x_1+x_2, y_1+y_2)$ and $k\vec{u}=(kx_1, ky_1)$ for $k \in \mathbb{R}$.

Example 1: If $\vec{u}=(-3,2)$ and $\vec{v}=(4,3)$, then find:

- (a) $\vec{u}+\vec{v}$ (b) $2\vec{u}+3\vec{v}$ (c) $3\vec{u}+\frac{1}{2}\vec{v}$

Solution: (a) $\vec{u}+\vec{v} = (-3,2)+(4,3) = (-3+4, 2+3) = (1,5)$

(b) $2\vec{u}+3\vec{v} = 2(-3,2)+3(4,3) = (-6,4)+(12,9) = (-6+12, 4+9) = (6,13)$

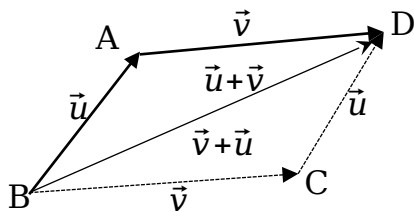
(c) $3\vec{u}+\frac{1}{2}\vec{v} = 3(-3,2)+\frac{1}{2}(4,3) = (-9,6)+\left(2, \frac{3}{2}\right) = \left(-9+2, 6+\frac{3}{2}\right) = \left(7, \frac{15}{2}\right)$

Theorem 8.1-8.2: For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$...Commutativity

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$...Associativity

Proof:

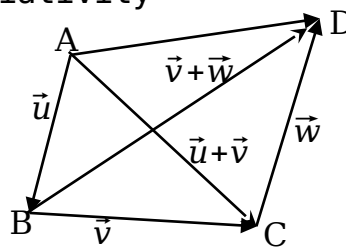


Commutativity

$$\vec{u}+\vec{v}=\overrightarrow{BA}+\overrightarrow{AD}=\overrightarrow{BD}$$

$$\vec{v}+\vec{u}=\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BD}$$

$$\Rightarrow \vec{u}+\vec{v}=\vec{v}+\vec{u}$$



Associativity

$$\vec{u}+(\vec{v}+\vec{w})=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$$

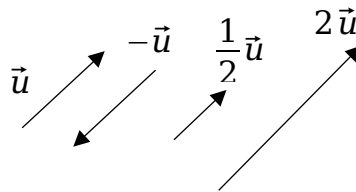
$$(\vec{u}+\vec{v})+\vec{w}=(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$$

$$\Rightarrow \vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$$

Multiplication of a vector by a scalar

Definition: Let $\vec{a}$ be any vector and k be scalar. Then, the vector $k\vec{a}$ is a vector whose magnitude is $|k\vec{a}|=|k||\vec{a}|$ where:

- i. If $\vec{a} \neq \vec{0}$ then (a) for $k>0$, $k\vec{a}$ has the same direction as $\vec{a}$
 (b) for $k<0$, $k\vec{a}$ has opposite direction with $\vec{a}$
 ii. $k\vec{a}=\vec{0}$ if $k=0$ or $\vec{a}=\vec{0}$



Theorem 8.3: For all vectors $\vec{a}$ and $\vec{b}$ and $k, l \in \mathbb{R}$

$$\text{i. } k(\vec{a}+\vec{b}) = k\vec{a}+k\vec{b} \quad \text{ii. } (k+l)\vec{a} = k\vec{a}+l\vec{a} \quad \text{iii. } (kl)\vec{a}=k(l\vec{a})=l(k\vec{a})$$

Homework 8.1: Do Exercise 8.2 on page 301, #1,2,4

Coordinate representation of vectors

- Vector $\vec{v} = (a,b) \leftrightarrow \vec{v} = a\vec{i} + b\vec{j}$ where $\vec{i}=(1, 0)$ and $\vec{j}=(0, 1)$. $\vec{i}$ & $\vec{j}$ are called **unit vectors**.
- $a\vec{i} + b\vec{j} = a(1,0) + b(0,1)$

E.g., $\vec{v} = 2\vec{i} - 3\vec{j} = (2, -3) = (2, 0) + (0, -3) = 2(1, 0) - 3(0, 1)$

Definition: (i) a vector with magnitude 1 is called a unit vector.

(ii) unit vector in the direction of $\vec{v}$ can be given by $\frac{\vec{v}}{|\vec{v}|}$.

Example 1:

Example 2: Let $\vec{a} = \vec{i} - 3\vec{j}$ and $\vec{b} = 2\vec{i} + 2\vec{j}$. Then find

a) unit vector in the direction of $\vec{a}$

b) unit vector in opposite direction to $\vec{b}$

c) unit vector in the direction of $\vec{a} - \vec{b}$

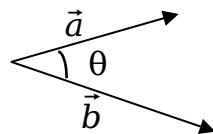
Solution: a. $\vec{v} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} - 3\vec{j}}{\sqrt{(1)^2 + (-3)^2}} = \frac{\vec{i} - 3\vec{j}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\vec{i} - \frac{3}{\sqrt{10}}\vec{j}$

b. $\vec{v}_{opp} = \frac{-\vec{b}}{|\vec{b}|} = \frac{-(2\vec{i} + 2\vec{j})}{\sqrt{(2)^2 + (2)^2}} = \frac{-2\vec{i} - 2\vec{j}}{\sqrt{8}} = \frac{-2\vec{i} - 2\vec{j}}{2\sqrt{2}} = \frac{-\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$

c. $\{\vec{v}_{\text{dixn of } \{\vec{a}-\vec{b}\}} = \frac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|} = \frac{(\vec{i}-3\vec{j})-(2\vec{i}+2\vec{j})}{|\vec{i}-3\vec{j}-(2\vec{i}+2\vec{j})|}$

Scalar (Dot) product of vectors

Definition: suppose $\vec{a}$ and $\vec{b}$ be two vectors and θ ($0 \leq \theta \leq \pi$) is an angle between $\vec{a}$ and $\vec{b}$ (let $\vec{a} = x_1\vec{i} + y_1\vec{j}$ & $\vec{b} = x_2\vec{i} + y_2\vec{j}$), then the dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and defined as



(i) $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

(ii) or equivalently $\vec{a} \cdot \vec{b} = (x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$

(iii) $\cos \theta = \frac{x_1 x_2 + y_1 y_2}{|\vec{a}| |\vec{b}|}$ (derived from $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = x_1 x_2 + y_1 y_2$)

Note: - If $\vec{a} \parallel \vec{b}$ then $\theta = 0 \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$ maximum

- If $\vec{a} \perp \vec{b}$, then $\theta = 90^\circ \Rightarrow \vec{a} \cdot \vec{b} = 0$

- If angle between $\vec{a}$ and $\vec{b}$ is 180° , that is if $\vec{a}$ is anti-parallel with $\vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos 180^\circ = -|\vec{a}| |\vec{b}|$

- $\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| \cos 0 = |\vec{a}|^2$

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative property)

- $(\vec{a} + \vec{b})^2 = \vec{a}^2 + 2\vec{a} \cdot \vec{b} + \vec{b}^2$

- $(\vec{a} - \vec{b})^2 = \vec{a}^2 - 2\vec{a} \cdot \vec{b} + \vec{b}^2$

- $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 = |\vec{a}|^2 - |\vec{b}|^2$

$$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$$

Theorem 8.4: Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and k be a scalar. Then

(i) $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v}$ associative property

(ii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ distributive property

$\vec{u} \cdot \vec{u} > 0$ if $\vec{u} \neq \vec{0}$ and $\vec{u} \cdot \vec{u} = 0$ if $\vec{u} = \vec{0}$. Here, $\vec{0} = (0, 0) = 0i + 0j$.

Corollary 8.1: If $\vec{u} = (a_1, a_2)$ and $\vec{v} = (b_1, b_2)$, then $\vec{u} \cdot \vec{v} = a_1b_1 + a_2b_2$.

Proof: see page 308.

Application of Scalar (Dot) product of vectors

Example 1:

Example 2:

Example 3:

1. Vectors $\vec{a}$ and $\vec{b}$ makes an angle $\theta = \frac{2\pi}{3}$, if $|\vec{a}| = 3$ and $|\vec{b}| = 4$ calculate
a. $\vec{a} \cdot \vec{b}$ b. $\vec{a} \cdot \vec{a}$ c. $(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b})$

Solution: a) $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = 3 \cdot 4 \cdot \cos 120^\circ = 12 \times -\frac{1}{2} = -6$

b) $\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| \cos \theta = |\vec{a}|^2 = 3^2 = 9$

c) $(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b}) = 2\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$
 $= 2|\vec{a}|^2 + \vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{a} - |\vec{b}|^2$
 $= 2(3)^2 - 6 - 2(-6) - 4^2 = 18 + 6 - 16 = 8$

2. Given two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = |\vec{b}| = 1$. If the angle between $\vec{a}$ and $\vec{b}$ is 60° , find $|\vec{a} + \vec{b}|$

Solution: $|\vec{a} + \vec{b}|^2 = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = |\vec{a}|^2 + 2|\vec{a}| |\vec{b}| \cos 60^\circ + |\vec{b}|^2$

$$\Rightarrow |\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + 2|\vec{a}| |\vec{b}| \cos \theta + |\vec{b}|^2}$$

$$= \sqrt{1 + 2(1)(1) \cos 60^\circ + 1} = \sqrt{1 + 1 + 1} = \sqrt{3}$$

Homework 8.2: Do Exercise 8.4 on page 310, #1, 2.

Corollary 8.2: Let $\vec{u}, \vec{v}$ be vectors. Then

(i) $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 - \vec{v}^2$.

(ii) $(\vec{u} \pm \vec{v})^2 = \vec{u}^2 \pm 2\vec{u} \cdot \vec{v} + \vec{v}^2$ where $\vec{u}^2 = \vec{u} \cdot \vec{u} = |\vec{u}|^2$ and $\vec{v}^2 = \vec{v} \cdot \vec{v} = |\vec{v}|^2$

Example 4: If $\vec{u}, \vec{v}$ make an angle

$\theta = \frac{2\pi}{3}$, $|\vec{u}| = 3$ and $|\vec{v}| = 4$, then calculate:

(a) $\vec{u} \cdot \vec{v}$ (b) $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v})$ (c) $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$ (d) $|2\vec{u} + \vec{v}|$

Solution: (a) $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = (3)(4) \cos \frac{2\pi}{3} = (3)(4)(-0.5) = -6$

(b) $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 - \vec{v}^2 = |\vec{u}|^2 - |\vec{v}|^2 = 3^2 - 4^2 = -7$

$$(c) (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + 2\vec{u} \cdot \vec{v} + \vec{v}^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v} = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$$

$$= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|\cos\frac{2\pi}{3} = 3^2 + 4^2 + 2(6) = 37$$

$$(d) |2\vec{u} + \vec{v}| = \sqrt{|2\vec{u} + \vec{v}|^2} = \sqrt{(2\vec{u} + \vec{v}) \cdot (2\vec{u} + \vec{v})} = \sqrt{4(\vec{u}^2) + 4\vec{u} \cdot \vec{v} + \vec{v}^2} = \sqrt{4|\vec{u}|^2 + |\vec{v}|^2 + 4\vec{u} \cdot \vec{v}}$$

$$= \sqrt{4|\vec{u}|^2 + |\vec{v}|^2 + 4|\vec{u}||\vec{v}|\cos\frac{2\pi}{3}} = \sqrt{4(3^2) + 4^2 + 4(3)4(0.5)} = \sqrt{76}$$

Example 5:

If $|\vec{u}|=13$, $|\vec{v}|=19$ and

$|\vec{u} + \vec{v}|=24$, then calculate:

(a) $\vec{u} \cdot \vec{v}$ (b) $|\vec{u} - \vec{v}|$ (c) $|3\vec{u} + 4\vec{v}|$

Solution: (a) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ will not work out b/c θ is not given. So, find $\vec{u} \cdot \vec{v}$ from $|\vec{u} + \vec{v}|^2$. But, $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + 2\vec{u} \cdot \vec{v} + \vec{v}^2$.

$$\Rightarrow |\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v} \Rightarrow 2\vec{u} \cdot \vec{v} = |\vec{u} + \vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2$$

$$\Rightarrow 2\vec{u} \cdot \vec{v} = 24^2 - 13^2 - 19^2 = 46 \Rightarrow \vec{u} \cdot \vec{v} = 23.$$

$$(b) |\vec{u} - \vec{v}| = \sqrt{|\vec{u} - \vec{v}|^2} = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = \sqrt{\vec{u}^2 - 2\vec{u} \cdot \vec{v} + \vec{v}^2}$$

$$= \sqrt{|\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}} = \sqrt{13^2 + 19^2 - 2(23)}$$

$$= \sqrt{484} = 22 \text{ where } \vec{u} \cdot \vec{v} = 23 \text{ is taken from (a).}$$

$$(c) |3\vec{u} + 4\vec{v}| = \sqrt{|3\vec{u} + 4\vec{v}|^2} = \sqrt{(3\vec{u} + 4\vec{v}) \cdot (3\vec{u} + 4\vec{v})} = \sqrt{9(\vec{u}^2) + 24\vec{u} \cdot \vec{v} + 16\vec{v}^2}$$

$$= \sqrt{9|\vec{u}|^2 + 16|\vec{v}|^2 + 24\vec{u} \cdot \vec{v}} = \sqrt{9(13^2) + 16(19^2) + 24(23)} = \sqrt{7849} = 88.59$$

Corollary 8.3: Let $\vec{u} \wedge \vec{v}$ be non-zero vectors. If θ is the angle b/n them, then

(i) θ is acute if and only if $\vec{u} \cdot \vec{v} > 0$

(ii) θ is obtuse if and only if $\vec{u} \cdot \vec{v} < 0$

(iii) $\theta = \frac{\pi}{2}$ ($\vec{u} \wedge \vec{v}$ are $\perp$ or orthogonal) if and only if $\vec{u} \cdot \vec{v} = 0$

Example 6:

Determine k so that the

angle b/n $\vec{u} = (6, -5k)$ and $\vec{v} = (-3, 2)$ is

(a) $\acute{\angle}$ (b) obtuse (c) 90°

Solution: Let θ be the angle b/n $\vec{u}$ and $\vec{v}$.

$$(a) \theta \text{ is acute iff } \vec{u} \cdot \vec{v} = (6, -5k) \cdot (-3, 2) > 0 \Leftrightarrow 6(-3) + (-5k)2 > 0$$

$$\Leftrightarrow 10k < -18 \Leftrightarrow k < -1.8$$

$$(b) \theta \text{ is obtuse iff } \vec{u} \cdot \vec{v} = (6, -5k) \cdot (-3, 2) < 0 \Leftrightarrow k > -1.8$$

$$(c) \theta = \frac{\pi}{2} \text{ (}\vec{u} \wedge \vec{v} \text{ are } \perp \text{ or orthogonal) iff } \vec{u} \cdot \vec{v} = (6, -5k) \cdot (-3, 2) = 0 \Leftrightarrow k = -1.8$$

Example 7:

Show that:

(a) $\vec{u} = \vec{0}$ (i.e., $\vec{u} = (0, 0)$) if $\vec{u} \cdot \vec{v} = 0$ for all non-zero $\vec{v}$. ($(6, -5k)$ and $\vec{v} = (-3, 2)$)

(b) $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular (or orthogonal) iff $|\vec{u}| = |\vec{v}| = 0$

Solution: Let θ be the angle b/n $\vec{u}$ and $\vec{v}$.

(a) Suppose $\vec{u} \neq \vec{0}$. Then by **Corollary 8.3** ($\vec{u} \neq \vec{0}, \vec{v} \neq \vec{0} \wedge \vec{u} \cdot \vec{v} = 0$ gives) $\vec{u} \perp \vec{v}$

But, this is impossible b/c a vector u cannot be $\perp$ all non-zero vectors ($\vec{v}$)

Thus, the supposition (that $\vec{u} \neq \vec{0}$ is **wrong** and its negation will be true.

$$\therefore \vec{u} = \vec{0} \text{ iff } \vec{u} \cdot \vec{v} = (6, -5k) \cdot (-3, 2) > 0 \iff 6(-3) + (-5k)2 > 0$$

(b) By **Corollary 8.3** $\vec{u} + \vec{v}$ & $\vec{u} - \vec{v}$ are $\perp$ iff $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$.

$$\iff (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u}^2 - \vec{v}^2 = |\vec{u}|^2 - |\vec{v}|^2 = 0 \iff |\vec{u}|^2 = |\vec{v}|^2 \iff |\vec{u}| = |\vec{v}|$$

Applications of vectors

1. Applications of the dot product of vectors

Example 8: Find the angle between $\vec{u} = 2\mathbf{i} - \mathbf{j}$ and $\vec{v} = 6\mathbf{i} + 2\mathbf{j}$.

Solution: Let the angle between $\vec{u}$ and $\vec{v}$ be θ . Then $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$

$$\Rightarrow \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{(2, -1) \cdot (6, 2)}{|(2, -1)| |(6, 2)|} = \frac{(2)(6) + (-1)(2)}{\sqrt{2^2 + (-1)^2} \sqrt{6^2 + 2^2}} = \frac{10}{\sqrt{5} \sqrt{40}} = \frac{10}{\sqrt{5} \sqrt{4} \sqrt{5} \sqrt{2}} = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = 45^\circ.$$

Self-task: Do/see example 2&3 on page 312-313.

Remark: $\vec{AB} = -\vec{BA}$. i.e., if $\vec{v}$ has initial pt A and terminal pt B, then $\vec{v}$ can be given by $\vec{v} = \vec{AB}$ or $\vec{v} = -\vec{BA}$.

Example 9: If $\vec{u}, \vec{v}, \vec{w}, \vec{z}$ are vectors from the origin to the points A, B, C, D

respectively and $\vec{v} - \vec{u} = \vec{w} - \vec{z}$, then show that ABCD is a parallelogram.

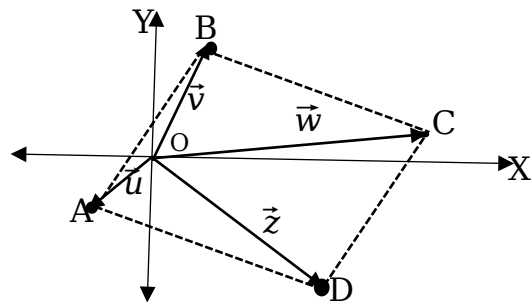
Solution: Assume the points on the coordinate plane look the ff.

You can see below that $\vec{AB} = \vec{v} - \vec{u}$ and in a similar way $\vec{DC} = \vec{w} - \vec{z}$, $\vec{BC} = \vec{w} - \vec{v}$ and $\vec{DA} = \vec{u} - \vec{z}$.

But, it is given that $\vec{v} - \vec{u} = \vec{w} - \vec{z}$.

$$\Rightarrow \vec{AB} = -\vec{CD} \Rightarrow |\vec{AB}| = |\vec{CD}| \text{ (See how,}$$

from



the figures (Δ s

of the //gram) given below)

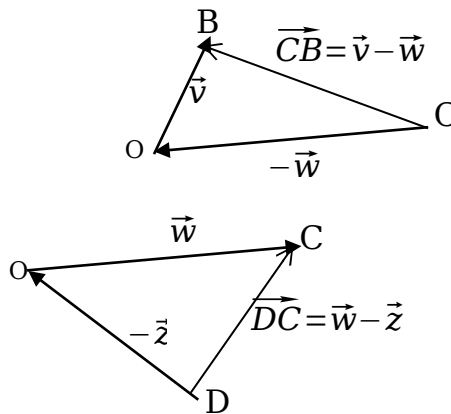
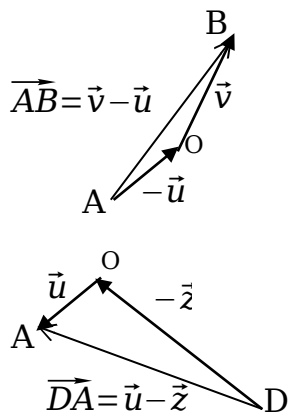
From $\vec{v} - \vec{u} = \vec{w} - \vec{z}$, we have $\vec{v} - \vec{w} = \vec{u} - \vec{z}$.

But, $\vec{v} - \vec{w} = -\vec{BC}$ and $\vec{u} - \vec{z} = \vec{DA}$.

$$\Rightarrow -\vec{BC} = \vec{DA} \Rightarrow |\vec{BC}| = |\vec{DA}|$$

$\Rightarrow$ The two pairs of opposite sides AB & CD and BC & AD are equal/congruent.

$\therefore$ ABCD is a parallelogram



Homework 8.1:

Homework 8.2:

Homework 8.3:

Homework 8.4: Do Exercise 8.5 on page 319-20, #4, 8, 9, 10

1.

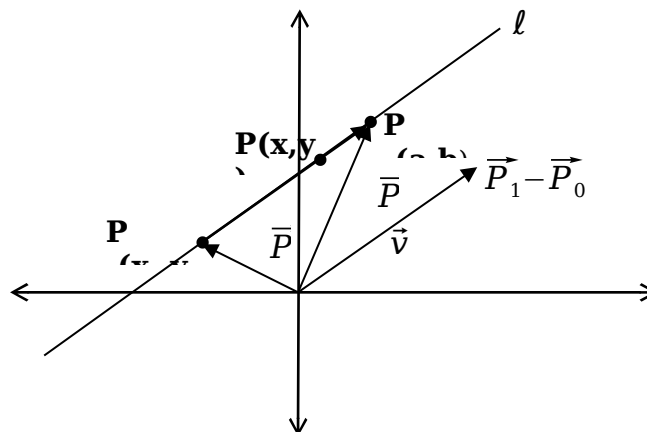
2. Vectors and lines

If a line passes through (x_0, y_0) and parallel to the vector $\vec{a} = (a_1, a_2)$ then the equation

i. $(x, y) = (x_0, y_0) + \lambda(a, b)$ for $\lambda \in \mathbb{R}$ is **vector equation** of the line.

ii. $x = x_0 + \lambda a$ and $y = y_0 + \lambda b$ is **parametric equation** of the line.

iii. $\frac{x - x_0}{a} = \frac{y - y_0}{b}$ is rectangular (standard) form of equation of the line.



$\vec{P}_0$ and $\vec{P}_1$ are position vectors corresponding to P_0 and P_1 resp.

Here, $\vec{v} \parallel l$ and $|\vec{v}| = P_0P_1$

$$\vec{P}_1 - \vec{P}_0 = (a - x_0, b - y_0) = \vec{v}$$

$\vec{P} - \vec{P}_0 = \vec{P}_0\vec{P} = \lambda(\vec{P}_1 - \vec{P}_0)$, where λ is a scalar (i.e., $\lambda \in \mathbb{R}$)

$$\vec{P} - \vec{P}_0 = \lambda \vec{v} \Rightarrow \vec{P} = \vec{P}_0 + \lambda \vec{v} \Rightarrow (x, y) = (x_0, y_0) + \lambda(a, b)$$

$$\Rightarrow \begin{cases} x = x_0 + \lambda a \\ y = y_0 + \lambda b \end{cases} \quad (a, b) \neq (0, 0) \dots \text{parametric equation of the line}$$

$$\Rightarrow \lambda = \frac{x - x_0}{a} = \frac{y - y_0}{b} \dots \text{standard equation of the line}$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{y_0}{b} - \frac{x_0}{a} = 0 \Rightarrow \frac{1}{a}x + \frac{1}{b}y + \left(\frac{y_0}{b} - \frac{x_0}{a}\right) = 0 \dots \text{General linear equation form}$$

$$\Rightarrow Ax + By + C = 0 \text{ where } A = \frac{1}{a}, B = \frac{1}{b} \text{ and } C = \frac{y_0}{b} - \frac{x_0}{a}.$$

Example 10:

1. Find the vector, parametric & rectangular form of equation of a line ℓ that passes through $(-3, 2)$ and parallel to the vector $(5, 3)$

Solution: $(x_0, y_0) = (-3, 2)$, $\vec{a} = (a_1, a_2) = (5, 3)$ then $(x, y) = (x_0, y_0) + t(a_1, a_2)$

$$\Rightarrow (x, y) = (-3, 2) + t(5, 3) \text{ is vector equation}$$

$$\Rightarrow (x, y) = (-3 + 5t, 2 + 3t)$$

$$\Rightarrow x = -3 + 5t \text{ \& } y = 2 + 3t \text{ is parametric equation } \Rightarrow t = \frac{x+3}{5} \text{ and } t = \frac{y-2}{3}$$

$$\Rightarrow \frac{x+3}{5} = \frac{y-2}{3} \text{ is rectangular form of equation of } \ell$$

$$\Rightarrow 3x + 9 = 5y - 10 \Rightarrow 3x - 5y + 19 = 0 \text{ is general form of equation of } \ell$$

3. Vectors and Circles

A circle with center $C(x_0, y_0)$ and radius $r > 0$ is the set of all points $P(x, y)$ in the plane such that $|\vec{OP} - \vec{OC}| = r \Rightarrow |\vec{OP} - \vec{OC}|^2 = r^2 \dots \text{squaring both sides}$

$(\vec{OP} - \vec{OC}) \cdot (\vec{OP} - \vec{OC}) = r^2 \dots \text{vector form representation of equation of the circle.}$

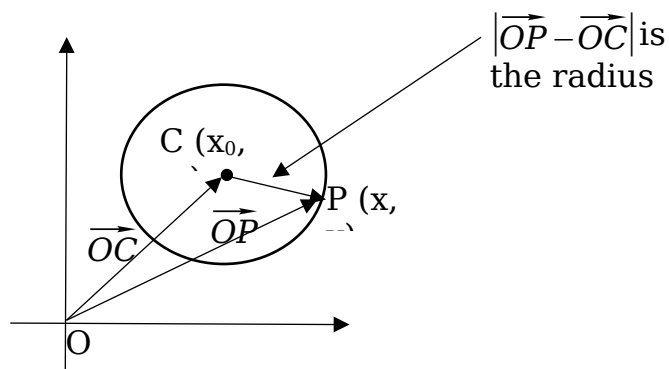
$$\vec{OP} \cdot \vec{OP} - 2\vec{OP} \cdot \vec{OC} + \vec{OC} \cdot \vec{OC} = r^2$$

Substituting the corresponding components of $\vec{OP}$ and $\vec{OC}$,

$$(\vec{OP} - \vec{OC}) \cdot (\vec{OP} - \vec{OC}) = r^2 \text{ gives } (x - x_0, y - y_0) \cdot (x - x_0, y - y_0) = r^2$$

Which is equivalent to the form:

$$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2 \dots \text{standard equation of a circle centered at } (x_0, y_0) \text{ with radius } r.$$



Example 11: Find an equation of a circle centered at $(-3, 2)$ with radius 3.

Solution: since $CP=r$, $\overrightarrow{CP}=\vec{r}$.

Then $\overrightarrow{OC}+\overrightarrow{CP}=\overrightarrow{OP} \Rightarrow \overrightarrow{OC}+\vec{r}=\overrightarrow{OP}$

$$\Rightarrow \vec{r}=\overrightarrow{OP}-\overrightarrow{OC} \Rightarrow |\vec{r}|=|\overrightarrow{OP}-\overrightarrow{OC}|$$

$$\Rightarrow |\vec{r}|^2=|\overrightarrow{OP}-\overrightarrow{OC}|^2, |\vec{r}|^2=r^2$$

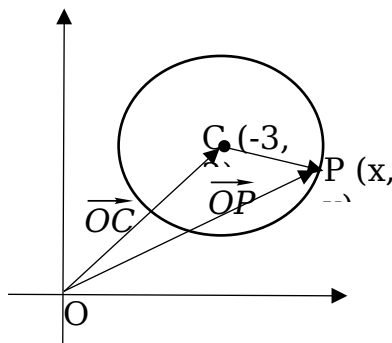
Here, $\overrightarrow{OP}-\overrightarrow{OC}=(x, y)-(-3, 2)=(x+3, y-2)$

$$\Rightarrow r^2=|(x+3, y-2)|^2=(\sqrt{(x+3)^2+(y-2)^2})^2$$

$$\Rightarrow r^2=(x+3)^2+(y-2)^2$$

In short, the circle can be described by:

$$r=|(x+3, y-2)|$$



4. Tangent line to a circle

It is known that a radius drawn to the point of tangency is perpendicular (orthogonal) to the tangent line.

That is, $\overrightarrow{P_1C} \perp \overrightarrow{P_1P}$ and the circle is $(x_1-h)^2+(y_1-k)^2=r^2$

$$\Rightarrow \overrightarrow{P_1C} \cdot \overrightarrow{P_1P} = 0 \Rightarrow (\mathbf{a}-\mathbf{h}, \mathbf{b}-\mathbf{k}) \cdot (\mathbf{x}-\mathbf{a}, \mathbf{y}-\mathbf{b}) = 0 \Rightarrow (\mathbf{a}-\mathbf{h})(\mathbf{x}-\mathbf{a}) + (\mathbf{b}-\mathbf{k})(\mathbf{y}-\mathbf{b}) = 0$$

Then by adding $(x_1-h)^2+(y_1-k)^2=r^2$ to it we have:

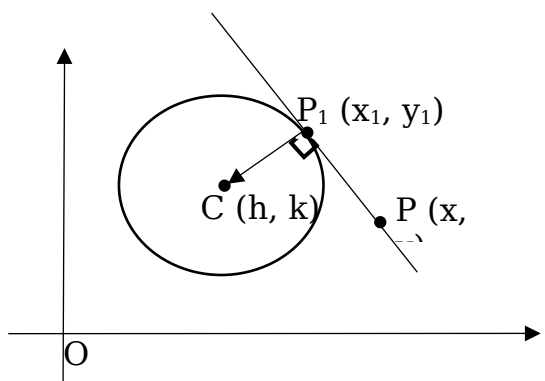
$$(x_1-h)(x-x_1) + (y_1-k)(y-y_1) + (x_1-h)^2 + (y_1-k)^2 = r^2$$

$$\Rightarrow (x_1-h)(x-x_1) + (y_1-k)(y-y_1) + (x_1-h)(x_1-h) + (y_1-k)(y_1-k) = r^2$$

$$\Rightarrow (x_1-h)[(x-x_1)+(x_1-h)] + (y_1-k)[(y-y_1)+(y_1-k)] = r^2$$

Thus, equation of the tangent ℓ $(x_1-h)(x-h) + (y_1-k)(y-k) = r^2$

If the center is $(0, 0)$, ℓ is $x_1x+y_1y = r^2$



Example 12: Equation of the tangent line at $(-1, 2)$ to the circle $x^2+y^2=5$ is $-1x+2y=5$

Example 13: Equation of the tangent line at $(2, -3)$ to the circle $(x-5)^2+(y-1)^2=25$ is $(x-5)(2-5)+(y-1)(-3-1)=25$

In general to find tangent at (a, b) to a circle $(x-h)^2+(y-k)^2=r^2$

$$(x-h)(x-h) + (y-k)(y-k) = r^2$$

$\uparrow$ $\uparrow$
 a k

Transformation of the plane

Definition: A mapping of a plane on to itself is called a rigid motion if it preserves distance.

Types of rigid motion

1. Translation

Definition: If a every point of a figure is mapped along the same direction through the same distances then the transformation is called a **translation** or **parallel movement**.

Note: 1 - In rigid motion
 - point is mapped to point
 - line is mapped to line
 - triangle mapped to triangle

Note: 2 - If a translation T takes the origin to the point (h,k) , then each point (x,y) is taken by T to the point $(x+h,y+k)$

- If a translation T takes the point (a,b) to the point (c,d) then each point (x,y) is taken by T to the point $(x+c-a,y+d-b)$. i.e., the translation vector is $(c-a, d-b)$.

Example 14:

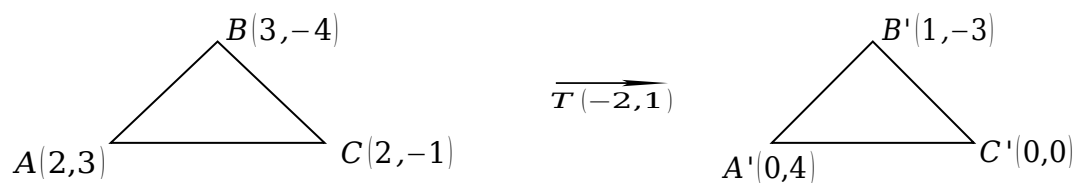
1. A translation takes the origin to the point $(-2,1)$. Find where it takes the point $A(2,3), B(3,-4), C(2,-1)$

Solution: $(0,0) \xrightarrow{T} (-2,1) \Rightarrow T = (-2, -0, 1-0) = (-2,1)$

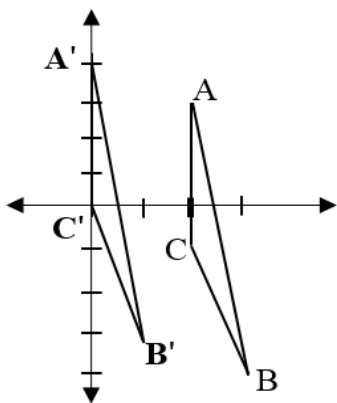
A. $(2,3) \xrightarrow{T(-2,1)} A'(0,4)$

B. $(3,-4) \xrightarrow{T(-2,1)} B'(1,-3)$

C. $(2,-1) \xrightarrow{T(-2,1)} C'(0,0)$



Here, the real rigid motion (in this case, translation) looks the following.



Note: Each point of the triangle is moved 2 units left 1 unit up by $T_{(-2,1)}$

2. Find the equation of the image of the circle $(x-1)^2 + y^2 = 4$ when translated by the vector $\overrightarrow{AB}$ where $A=(2,3)$ and $B=(3,-4)$

Solution: Translating vector $T = \overrightarrow{AB} = (3, -4) - (2, 3) = (1, -7)$

- Then translate only the center keeping constant the radius $r=2$

Center $(1, 0) \xrightarrow{(1, -7)} (2, -7)$

$\therefore$ equation of the image becomes

$$(x-2)^2 + (y+7)^2 = 4$$

3. A translation T takes $(-1, 3)$ to $(4, 2)$. Find the image of the line $l: y=2x-3$ under T .

Translating vector $= T = (4, 2) - (-1, 3) = (5, -1)$

Change any point from l and translate it by T . $(0, -3)$ is on l

$(0, -3) \xrightarrow{T(5, -1)} (5, -4)$ is on the image

NB. Translation is parallel movement, therefore slope of l and its image are the same $\Rightarrow m_l = m_{l'}$

Equation of l' becomes $y - (-4) = 2(x - 5) \Rightarrow y + 4 = 2x - 10$

$$\Rightarrow y = 2x - 14$$

Shortcut: To translate any geometric figure $F(x, y)$ by $T(a, b)$, simply substitute $(x-a, y-b)$ in place of (x, y) .

so, $l: y=2x-3 \xrightarrow{T(5, -1)} l': y-(-1)=2(x-5)-3$

Summary		
Translate by vector	Translate	Translated to
(h, k)	a point (a, b)	point (a+h, b+k)
	a figure F(x, y)	$F' = F(x-h, y-k)$

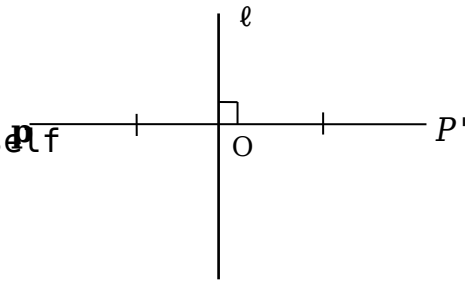
2. Reflections

Definition: If P is a point not on L , then its mirror image is in L is the point P' such that

i. $PP' \perp L$

ii. $\overline{PO} = \overline{P'O}$

iii. If P is on L , its image is P itself



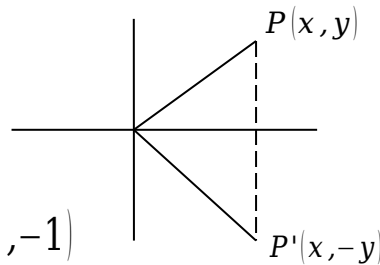
Types of reflection

1. Reflection in an axes

i. Reflection along the x-axis

For any point $P(x, y) \xrightarrow{x\text{-axis}} P'(x, -y)$

E.g. $(2, 3) \xrightarrow{x\text{-axis}} (2, -3)$ $(-5, 1) \xrightarrow{x\text{-axis}} (-5, -1)$

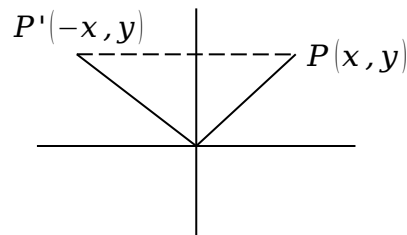


ii. Reflection along the y-axis

For any point $P(x, y) \xrightarrow{y\text{-axis}} P'(-x, y)$

E.g. 1. $(-2, 4) \xrightarrow{y\text{-axis}} (2, 4)$

2. $(4, 3) \xrightarrow{y\text{-axis}} (-4, 3)$



2. Reflection is the line $y=x$

For any point $P(x, y) \xrightarrow{y=x} P'(y, x)$

E.g. i) $P'(2, 1) \xrightarrow{y=x} P'(-1, 2)$ ii) $P'(-1, 2) \xrightarrow{y=x} P'(2, 1)$

3. Reflection along the line $x=a$

For any point $P(x, y) \xrightarrow{x=a} P'(x', y') = ?$

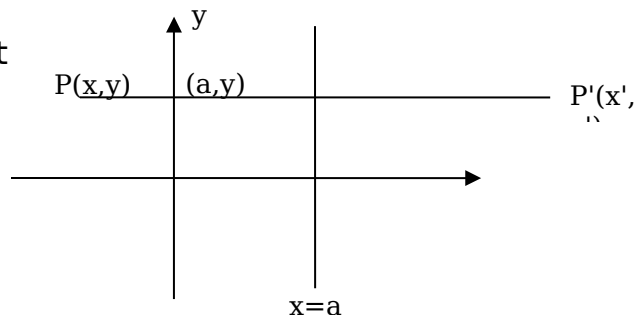
$(x', y') = ?$

$y = y'$ because in horizontal movement y is constant and

(a, y) is the midpoint of P and P'

$$\Rightarrow a = \frac{x+x'}{2} \Rightarrow x' = 2a - x \quad y = \frac{y+y'}{2} \Rightarrow y' = y$$

$$P(x, y) \xrightarrow{\text{along the line } x=a} P'(2a-x, y)$$



E.g. 1. $p(-2,-4) \xrightarrow{x=2} (2(2)+2,-4)=(6,-4)$
 2. $P(5,3) \xrightarrow{x=-3} (2(-3)-5,3)=(-11,3)$

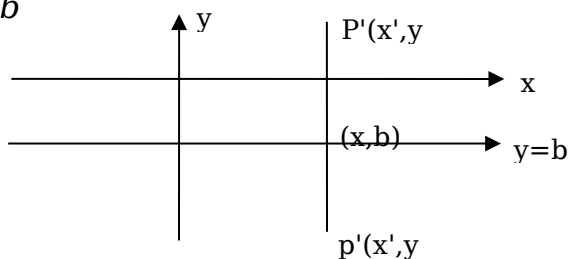
4. Reflection along the line $y=b$

A similar proof of no.3 results:

The image of the point $p(x,y)$

the line $y=b$ is $p'(x,2b-y)$

E.g. 1. $(-2,3) \xrightarrow{y=1} (-2,2(1)-3)=(-2,-1)$
 2. $(-4,-3) \xrightarrow{y=-5} (-4,2(-5)-(-3))=(-4,-7)$



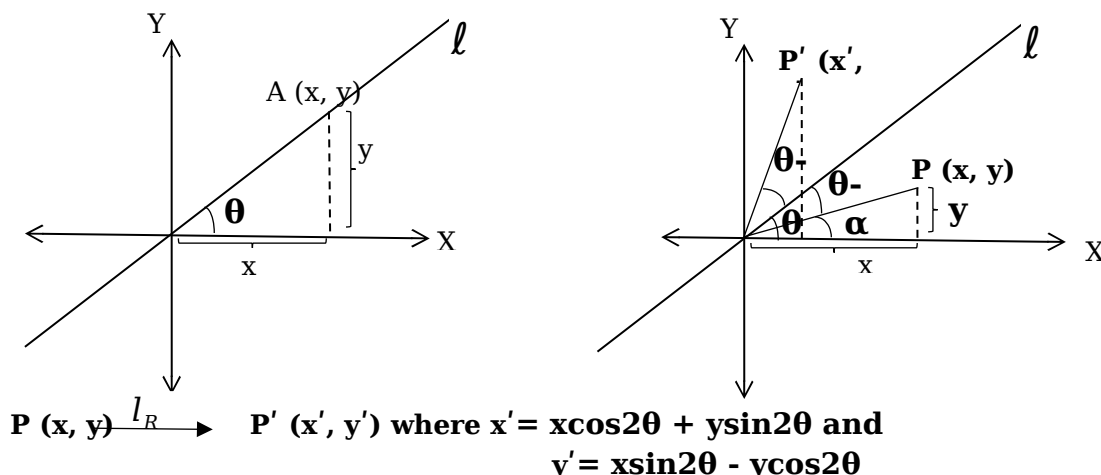
along

Shortcut:

Figure $F(x,y)$	Reflect on	(To find Image F') Substitute the ff in place of (x,y) in formula $F(x,y)$	No need of selecting & reflecting particular points from the figure.
	x-axis	$(x, -y)$	
	$y=k$	$(x, -y+2k)$	
	y-axis	$(-x, y)$	
	$x=k$	$(-x+2k, y)$	
	$y=x$	(y, x)	
	$y=-x$	$(-y, -x)$	

5. Reflection in the line $y=mx$, where

$$m=\tan\theta$$



For easy remembering, rewrite it as dot product: $x' = (x, y) \cdot (\cos 2\theta, \sin 2\theta)$

$$y' = (-y, x) \cdot (\cos 2\theta, \sin 2\theta)$$

Here, $\cos 2\theta = \frac{1-m^2}{1+m^2}$ and $\sin 2\theta = \frac{-2m}{1+m^2}$. This will be summarized in table later.

N.B: see proof on page 328.

6. Reflection in the line $y=mx+b$

E.g. Find the image of the point

$(1, -2)$ along the line $l: y=x+1$

$$\text{Here, } \overline{PP'} \perp l \Rightarrow m_{\overline{PP'}} = \frac{-1}{m_l} = -1$$

$$\Rightarrow \text{equation of } \overline{PP'} \text{ is } y - (-2) = -1(x - 1)$$

$$\Rightarrow y = -x - 1 \text{ is equation of } \overline{PP'}$$

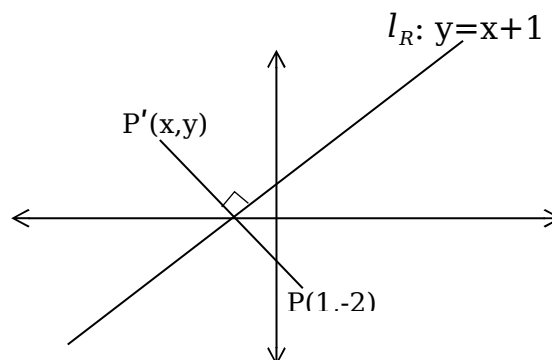
- Find intersection point of $\overline{PP'} \cap l$.

$$\Rightarrow -x - 1 = x + 1 \Rightarrow -2x = 2 \Rightarrow x = -1$$

$$\Rightarrow y = x + 1 = -1 + 1 = 0$$

$$\Rightarrow \frac{1+x}{2} = -1 \text{ and } \frac{-2+y}{2} = 0 \Rightarrow x = -3 \text{ and } y = 2$$

$$\therefore P(1, -2) \xrightarrow{y=x+1} P'(-3, 2)$$

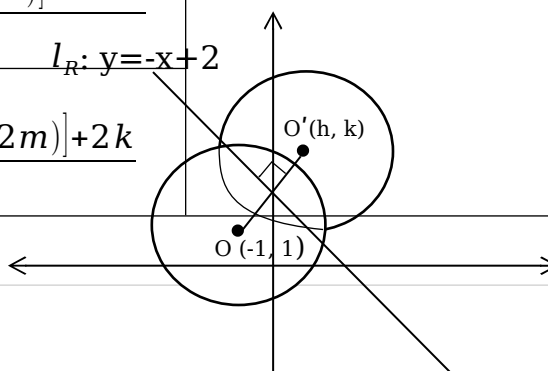


In short, $P(1, -2) \xrightarrow{l_R} P'(a', b')$ where $(a', b') = (b - k, a + k)$ (see the table below).

$$\Rightarrow (a', b') = (-2 - 1, 1 + 1) = (-3, 2).$$

Shortcut: The ff works in same way for both a fixed pt (a, b) & (x, y) of $F(x, y)$

Figure $F(x, y)$	Reflect on	$(x, y) \rightarrow (x', y')$ where	
	$y = x + k$	$(x', y') = (y - k, x + k)$	No need of selecting & reflecting particular points from the figure. — multiply as dot product
	$y = -x + k$	$(x', y') = (-y + k, -x + k)$	
	$y = mx$	$x' = \frac{(x, y) \cdot (1 - m^2, 2m)}{(1 + m^2)}$	
		$y' = \frac{(-y, x) \cdot (1 - m^2, 2m)}{(1 + m^2)}$	
	$y = mx + k$	$x' = \frac{[(x, y) \cdot (1 - m^2, 2m)] - 2mk}{(1 + m^2)}$	
		$y' = \frac{[(-y, x) \cdot (1 - m^2, 2m)] + 2k}{(1 + m^2)}$	



Example 15:

1. Find the image of the

circle

$C: x^2 + y^2 + 2x - 2y - 2 = 0$ under reflection
on the line $y = -x + 2$

Solution: $x^2 + y^2 + 2x - 2y - 2 = 0$

$$\Rightarrow C: (x+1)^2 + (y-1)^2 = 4$$

Center: $(-1, 1)$ $(-1, 1) \xrightarrow{y = -x + 2} ?$

Slope of $\ell = -1$ & $\overline{PP'} \perp \ell \Rightarrow m_{\overline{OO'}} = \frac{-1}{m_\ell} = 1$

Equation of OO' is $y - 1 = 1(x - (-1)) \Rightarrow y = x + 2$.

- intersection point of $\ell \wedge OO'$ is $x + 2 = -x + 2 \Rightarrow x = 0$ & $y = x + 2 = 0 + 2 = 2$

And using mid-point formula, $0 = \frac{x-1}{2} \Rightarrow x = 1$ & $2 = \frac{y+1}{2} \Rightarrow y = 3$

Hence, the center of the image is $(1, 3)$. But, the radius is unchanged.

$\therefore$ the equation of the image is $(x-1)^2 + (y-3)^2 = 2^2$

$$\Rightarrow \boxed{(x-1)^2 + (y-3)^2 = 4} \text{ or } x^2 + y^2 - 2x - 6y + 6 = 0$$

Shortcut: $C: (x+1)^2 + (y-1)^2 = 4$ $\ell_R: y = -x + 2$ (here, $k=2$)

$$C': (-y+2+1)^2 + (-x+2-1)^2 = 4 \quad (x, y) \mapsto (-y+k, -x+k)$$

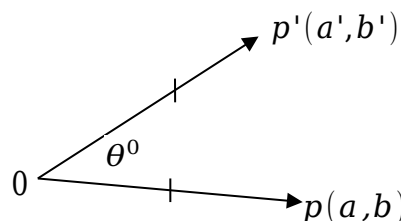
$$: (-y+3)^2 + (-x+1)^2 = 4$$

$$: (y-3)^2 + (x-1)^2 = 4$$

3. Rotation

Definition: Rotation about a fixed point O through an angle θ^0 transforms a point P into the point P' which is the same distance as P from O and such that

$$m(\angle POP') = \theta^0 \quad (\text{anti clock wise } \theta^0 > 0), \quad (\text{clock wise } \theta^0 < 0)$$



Note: 1 – Rotation through an angle θ^0 may denoted by $R\theta^0$

Rotation through an angle θ^0 about the origin

If R is a rotation of the plane about the origin through an angle θ , then the point $P(a, b)$ is mapped on to the point $P(a', b')$.

i.e., $P(a, b) \xrightarrow{R_\theta(0,0)} P'(a', b')$ where

$$a' = a \cos \theta - b \sin \theta = (-b, a) \cdot (\sin \theta, \cos \theta) \text{ and}$$

$$b' = a \sin \theta + b \cos \theta = (a, b) \cdot (\sin \theta, \cos \theta)$$

Example 16:

- Find the image of $(2,5)$ obtained by rotating through 90°

a) anti-clock wise about $(0,0)$

b) clock wise about $(0,0)$

Solution: a) $P(2, 5) \xrightarrow{R_{90^\circ}(0,0)} P'(a', b')$

$$a' = 2\cos 90^\circ - 5\sin 90^\circ = 2(0) - 5(1) = -5$$

$$\text{or } = (-5, 2) \cdot (\sin 90^\circ, \cos 90^\circ) = (-5, 2) \cdot (1, 0) = -5$$

$$b' = 2\sin 90^\circ + 5\cos 90^\circ = 2(1) + 5(0) = 2$$

$$\text{or } = (2, 5) \cdot (\sin 90^\circ, \cos 90^\circ) = (2, 5) \cdot (1, 0) = 2$$

$$\Rightarrow P'(a', b') = (-5, 2)$$

Note: To rotate clockwise, replace $-\theta$ in place of θ & do it the same way.

a) $P(2, 5) \xrightarrow{R_{-90^\circ}(0,0)} P'(a', b')$

$$a' = (-5, 2) \cdot (\sin(-90^\circ), \cos(-90^\circ)) = (-5, 2) \cdot (-1, 0) = 5$$

$$b' = (2, 5) \cdot (\sin(-90^\circ), \cos(-90^\circ)) = (2, 5) \cdot (-1, 0) = -2$$

$$\Rightarrow P'(a', b') = (5, -2)$$

Shortcut: shows a fixed pt (a, b) & (x, y) have the image of rotations (a', b') & (x', y') resp. (x', y') is obtained by directly substituting the variables of $F(x, y)$ by appropriate forms.

a fixed point (a, b) or Figure $F(x, y)$	Rotate anti-clockwise by $\theta =$	$(a', b') =$	$(x', y') =$
	π or 180°	$(-a, -b)$	$(-x, -y)$
	$\frac{\pi}{2}$ or 90°	$(-b, a)$	$(y, -x)$
	$\frac{3\pi}{2}$ or 270°	$(b, -a)$	$(-y, x)$

Rotation about any point (h, k) other than the origin

- To rotate point $P(a, b)$ under the rotation of the given angle θ° about point (h, k) , follow the following steps:

$$a' = -(b - k) \cdot \sin \theta + (a - h) \cdot \cos \theta + h$$

$$b' = (a - h) \cdot \sin \theta + (b - k) \cdot \cos \theta + k$$

Example 17:

Find the image of the point $(-3, 4)$ obtained by rotating through 90° anti-clock wise about the point $(1, 2)$.

Solution:

$$a' = -(b - 2) \cdot \sin 90^\circ + (a - 1) \cdot \cos 90^\circ + 1$$

$$= -(4 - 2) \cdot (1, 0) + 1 = (-2, -4) \cdot (1, 0) + 1 = [-2(1) + (-4)(0)] + 1 = -1$$

$$b' = (a - 1) \cdot \sin 90^\circ + (b - 2) \cdot \cos 90^\circ + 2$$

$$= (-3 - 1) \cdot (1, 0) + 2 = (-4, 2) \cdot (1, 0) + 2 = [-4(1) + 2(0)] + 2 = -2$$

$$\Rightarrow (a', b') = (-1, -2)$$

Homework 8.2:

Homework 8.3: Do Exercise 8.8 on page 342, #2,3

Solved problems

1. Given the equation of a line $\ell: 3x - y - 2 = 0$. If it is translated by the vector with initial point $(4, 1)$ and end point $(-5, 3)$, find the image line ℓ' .
 2. The image of the line $3x + 7y - 2 = 0$ after it has been rotated 180° anti-clockwise about the origin is: _____.
 3. Suppose a circle C has equation $x^2 + y^2 - 4x - 6y + 9 = 0$. What is the image of the circle C after it has been reflected about the line $y = x + 3$?
 4. What is the image of the line $x - 3y - 2 = 0$ after a rotation about the origin through an angle of 270° in a counter clock wise direction?
 5. In reflection the image of the point $P(3, 10)$ is $P'(7, 2)$. The equation of the line of reflection is _____.
 6. What is the equation of the image of a line $\ell: y = 2x - 1$ when it is reflected about the line $y = -x + 3$?
 7. A translation T takes the point $(2, -4)$ to the point $(0, -2)$. Then what is the image of point $P(0, 1)$ under T ?
 8. What is the equation of the image of the parabola $y = x^2 - 1$ after rotation about the origin through 90° in counter clock wise direction?
 9. Let $\vec{u} = i + 3j$ and $\vec{v} = mi + 2j$, then for what value of m the angle between $\vec{u}$ and $\vec{v}$ is 45° ?
 10. Find a parametric equation of the line that passes through $(-1, 2)$ and $(4, 3)$.
 11. Let $\vec{a}$ and $\vec{b}$ are two perpendicular vectors. Then show that:
 - (i) $|\vec{a} + \vec{b}|^2 = \sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
 - (ii) $(\vec{a} - \vec{b}) \cdot \vec{b} = -|\vec{b}|^2$
 - (iii) $(\vec{a} - \vec{b}) \cdot (2\vec{a} + 3\vec{b}) = 2|\vec{a}|^2 - 3|\vec{b}|^2$
 - (iv) $\vec{a} \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2$
- 1.
 - 2.
 - 3.
 - 4.
 - 5.
 - 6.
 - 7.
 - 8.
 - 9.
 - 10.
 - 11.

12. Find the area of the parallelogram having adjacent sides with the given points of vectors $2\vec{i}+3\vec{j}$ and $5\vec{i}+\vec{j}$.
13. If $\vec{a}=2\vec{i}-3\vec{j}$, $\vec{b}=3\vec{i}-\vec{j}$ and $\vec{c}=\vec{i}-\vec{j}$, then find $\vec{a} \cdot (2\vec{b}-\vec{c})$.
14. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}|=4$, $|\vec{b}|=\sqrt{3}$ and $\vec{a} \cdot \vec{b}=-3$, then the magnitude of $2\vec{a}+3\vec{b}$ is _____.

Solutions

1. Given equation of line $\ell: 3x-y-2=0$ and

$$\text{vector } (4, 1) \xrightarrow{T_v} (-5, 3) \Rightarrow T_v = (-5-4, 3-1) = (-9, 2)$$

$$\text{Then } \ell: 3x-y-2=0 \xrightarrow{T_v} \ell': 3(x-(-9))-(y-2)-2=0 \equiv 3(x+9)-y+2-2=0$$

applying the

$$\text{shortcut: } (x, y) \xrightarrow{\text{Replacing by}} (x-(-9), y-2)$$

$$\text{Hence, } \ell': 3x-y+27=0$$

Self-exercise: Take any two points on ℓ , translate them & find the line containing the translated points (you'll get the same answer).

2. Apply the shortcut: $(x, y) \xrightarrow{\text{Replacing by}} (-x, -y)$ while a figure is rotated 180° about the origin.

$$\Rightarrow \ell: 3x+7y-2=0 \xrightarrow{R_{180^\circ}(0,0)} \ell': 3(-x)-7(-y)-2=0 \equiv -3x+7y-2=0 \equiv 3x-7y+2=0$$

$$\text{Hence, } \ell': 3x-7y+2=0$$

3. Given the circle $x^2+y^2-4x-6y+9=0$ it is reflected by $\ell_R: y = x+3$,

$$(x, y) \xrightarrow{R_{y=x+3}} (y-k, x+k) \text{ (shortcut for reflection about } y=x+k \text{)}$$

$$x^2+y^2-4x-6y+9=0 \equiv (x^2-4x+4)+(y^2-6y+9)=0+4 \equiv (x-2)^2+(y-3)^2=4$$

$$\text{Then } C: (x-2)^2+(y-3)^2=4 \xrightarrow{R_{y=x+3}} C': ((y-3)-2)^2+((x+3)-3)^2=4$$

$$\equiv (y-5)^2+x^2=4. \text{ This is done by:}$$

$$(x, y) \xrightarrow{\text{Replacing by}} (y-3, x+3)$$

$$\text{Hence, } C': x^2+(y-5)^2=4$$

Self-exercise: Reflect the center $(2, 3)$ by $\ell_R: y = x+3$ & replace it in place of (h, k) in $(x-h)^2+(y-k)^2=4$. (check: you get the same answer).

4. Apply the shortcut: $(x, y) \xrightarrow{\text{Replacing by}} (-y, x)$ while a figure is

$$\text{rotated } 270^\circ \text{ or } \frac{3\pi}{2} \text{ about the origin.}$$

$$\Rightarrow \ell: x-3y-2=0$$

$$\ell': (-y)-3(x)-2=0 \equiv -y-3x-2=0 \equiv 3x+y+2=0$$

Hence, $\ell': 3x+y+2=0$

5. Given an unknown line of reflection ℓ_R , $(3, \xrightarrow{\ell_R} (7, 2)$

$\Rightarrow l_R: (y-b)=m_{l_R}(x-a)$ where M (a, b) is midpoint of $\overline{PP'}$ and m_{l_R} is slope of l_R .

$$M(a, b) = \left(\frac{3+7}{2}, \frac{10+2}{2} \right) = (5, 6) \text{ and}$$

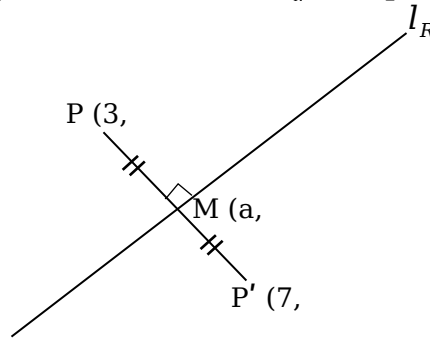
$$m_{l_R} = -1 \div \frac{2-10}{7-3} = -1 \div \frac{-8}{4} = \frac{1}{2}$$

$$\text{Thus, } l_R: (y-6) = \frac{1}{2}(x-5)$$

$$\equiv y = \frac{1}{2}x - \frac{7}{2}$$

$$\equiv x-2y-7=0$$

$\therefore l_R: x-2y-7=0$ is equation of the line of reflection.



6. Given the $\ell: y=2x-1$ when it is reflected about the line $\ell_R: y = -x+3$,

$(x, y) \xrightarrow{R_{y=-x+k}} (-y+k, -x+k)$ (shortcut for reflection about $y = -x+k$)

$$x^2+y^2-4x-6y+9=0 = (x^2-4x+4)+(y^2-6y+9)=0+4 \equiv (x-2)^2+(y-3)^2=4$$

$$\text{Then } \ell: y=2x-1 \xrightarrow{R_{y=-x+3}} \ell': (-x+3)=2(-y+3)-1$$

$$\equiv \ell': -x+3=-2y+6-1. \text{ This is done by: } (x, y) \xrightarrow{\text{Replacing}} (-y+3, -x+3)$$

$$\equiv \ell': 0=x-2y+2$$

Hence, $\ell': x-2y+2=0$

Self-exercise: Reflect any two points of $\ell: y=2x-1$ by $\ell_R: y = -x+3$ & find the equation of the line containing the two image points. (check: you get the same answer).

7. Recall the shortcut: translating by(c, d),

✓ substitute variable (x, y) by (x-c, y-d)

✓ substitute fixed point (a, b) by (a+c, b+d)

$$(2, -4) \xrightarrow{T_v} (0, -2) \Rightarrow T_v = (0-2, -2+4) = (-2, 2)$$

$$\Rightarrow (0, 1) \xrightarrow{T_{(-2,2)}} (0-2, 1+2) = (-2, 3)$$

$$\therefore P(0, 1) \xrightarrow{T_{(-2,2)}} (-2, 3) \dots \dots \text{fixed pt. translation}$$

8. Apply the shortcut: $(x, y) \xrightarrow{\text{Replacing}} (y, -x)$ while a figure is rotated 90° or $\frac{\pi}{2}$ about the origin.

$$\Rightarrow \text{parabola } P: y = x^2 - 1 \xrightarrow{R_{90^\circ}(0,0)} P': P: (-x) = (y)^2 - 1 \equiv x = 1 - y^2$$

upward Left opening parabola

9. $\vec{u} = i + 3j = (1, 3)$ and $\vec{v} = m\vec{i} + 2\vec{j} = (m, 2)$ and $\theta = 45^\circ$

$$\text{From } \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$$

$$\Rightarrow \sqrt{1^2 + 3^2} \sqrt{m^2 + 2^2} \cos 45^\circ = 1m + 3(2)$$

$$\Rightarrow \sqrt{10} \sqrt{m^2 + 4} \left(\frac{1}{\sqrt{2}} \right) = m + 6 \Rightarrow \sqrt{5} \sqrt{m^2 + 4} = m + 6 \Rightarrow 5m^2 + 20 = m^2 + 12m + 36$$

$$\Rightarrow 4m^2 - 12m - 16 = 0 \Rightarrow m^2 - 3m - 4 = 0 \Rightarrow (m - 4)(m + 1)$$

$$\Rightarrow m = 4 \text{ or } m = -1$$

10. Let l be a line that passes through $(-1, 2)$ and $(4, 3)$.

$$\text{Then } \ell: (x, y) = (-1, 2) + (4 - (-1), 3 - 2)t \equiv (x, y) = (-1, 2) + (5, 1)t \text{ using fixed } (-1, 2)$$

$$\text{or } \ell: (x, y) = (4, 3) + (4 - (-1), 3 - 2)t \equiv (x, y) = (4, 3) + (5, 1)t$$

or replacing the fixed point by any point of ℓ and/or replacing the direction vector $(5, 1)$ by any multiple of it, its parametric equation can be represented by infinitely many ways. But, all are equation of the same line.

11. Given $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos 90^\circ = 0$

$$(i) |\vec{a} + \vec{b}|^2 = \sqrt{|\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2} = \sqrt{|\vec{a}|^2 + 0 + |\vec{b}|^2} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2}$$

$$(ii) (\vec{a} - \vec{b}) \cdot (2\vec{a} + 3\vec{b}) = 2\vec{a}^2 + 3\vec{a} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} - 3\vec{b}^2$$

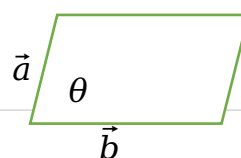
$$= 2|\vec{a}|^2 + 0 - 0 - 3|\vec{b}|^2 = 2|\vec{a}|^2 - 3|\vec{b}|^2$$

$$(iii) (\vec{a} - \vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - \vec{b}^2 = 0 - |\vec{b}|^2 = -|\vec{b}|^2$$

$$(iv) \vec{a} \cdot (\vec{a} + \vec{b}) = \vec{a}^2 + \vec{a} \cdot \vec{b} = |\vec{a}|^2 + 0 = |\vec{a}|^2$$

12. Given $\vec{a} = 2\vec{i} + 3\vec{j} = (2, 3)$ and $\vec{b} = 5\vec{i} + \vec{j} = (5, 1)$,

$$\text{Area} = (\text{base1})(\text{base2}) \cos \theta$$



$$=|\vec{a}||\vec{b}|\cos\theta = (2,3).(5,1)=2(5)+3(1)=13$$

$\therefore$ Area=13 sq. units.

13. If $\vec{a}=2i-3j=(2,-3)$, $\vec{b}=3i-j=(3,-1)$ and $\vec{c}=i-j=(1,-1)$, then

$$\vec{a} \cdot (2\vec{b}-\vec{c}) = (2,-3) \cdot [2(3,-1)-(1,-1)] = (2,-3) \cdot [(6,-2)-(1,-1)]$$

$$= (2,-3) \cdot (5,-1) = 2(5)+(-3)(-1) = 10+3=13$$

$$\therefore \vec{a} \cdot (2\vec{b}-\vec{c})=13.$$

14. Given $|\vec{a}|=4$, $|\vec{b}|=\sqrt{3}$ and $\vec{a} \cdot \vec{b}=-3$, then

$$\text{the magnitude of } 2\vec{a}+3\vec{b} = |2\vec{a}+3\vec{b}| = \sqrt{|2\vec{a}+3\vec{b}|^2} = \sqrt{(2\vec{a}+3\vec{b}) \cdot (2\vec{a}+3\vec{b})}$$

$$= \sqrt{4\vec{a}^2+12\vec{a} \cdot \vec{b}+9\vec{b}^2} = \sqrt{4|\vec{a}|^2+12\vec{a} \cdot \vec{b}+9|\vec{b}|^2} = \sqrt{4(4)^2+12(-3)+9(\sqrt{3})^2} = \sqrt{55}$$

Summary Exercise

1. let $\vec{a}=3i-7j$ and $\vec{b}=-2i+3j$, $\vec{c}=i+2j$ then find

A. $2\vec{a}-3\vec{b}$ C. $|2\vec{c}-\vec{a}|$

B. $(\vec{c}-\vec{a})+2\vec{b}$ D. $\vec{a} \cdot (2\vec{b}-\vec{c})$

2. if $\vec{a}=i-2j$ and $\vec{b}=-2i+3j$ find

A. a unit vector in the direction of $\vec{a}-\vec{b}$

B. a unit vector in opposite direction to $\vec{a}+2\vec{b}$

3. Let $|a|=4, |b|=5$, and $a \cdot b=-2$, then find

A. $(a-2b) \cdot (2a+3b)$ B. $(3a+b) \cdot (\vec{a}-2b)$ C. $|\vec{a}-2\vec{b}|$ D. $|2\vec{a}+\vec{b}|$

4. Find the cosine of an angle between $\vec{a}$ and $\vec{b}$ if

A. $\vec{a}=2i-j$, $\vec{b}=i-3j$ B. $\vec{a}=-i-2j$, $\vec{b}=2i+5j$

5. Let $\vec{a}=3i+4j$ and $\vec{b}=mi+2j$, for what value of $\vec{a}$ and $\vec{b}$ are

A. Perpendicular B. Parallel

6. Let $a=6, b=4$, and $a+b=10$, then find

A. $a \cdot b$ B. $|\vec{a}-2\vec{b}|$ C. $|2a+3b|$

7. What is the angle between the vectors $\vec{a}=i-j$ and $v=2j$

8. Find the image of $(-2,-1)$ after it has been rotated 300°

A. anti-clockwise about the origin B. Clockwise about the origin

C. Anti-clockwise about $(4,-1)$ D. Clockwise about $(4,-1)$

9. What is the image of the point $(3,5)$ after it has been reflected about the line
- A. $x=-2$ C. $y=x+3$ E. $y=-x$
 B. $y=3$ D. $y=x$ F. the x -axis G. the y -axis
10. What is the image of the circle $x^2+y^2+2x-4y-4=0$, after it has been reflected about the line
- A. $x=1$ B. $y=-3$ C. $y=-x+1$ D. $y=x$
11. If a translation T takes the point $(-1,2)$ to $(3,5)$. What is the image of the following points and lines under T ?
- A. $(-4,8)$ B. $(3,-2)$ C. $y=x+2$
12. Find the image of the line $\ell: y=2x+3$ if it is reflected on the line whose equation is
- A. $y=2x-4$ B. $y=-x+1$
13. What is the image of the circle $x^2+y^2+4x-4y=0$ after it has been rotated 270° anti clock wise about the origin
14. Find the image of the line $y=2x+6$, after it has been rotated 90° clock wise about the origin.
15. Suppose that the equation of the line $\ell: 2x-3y-7=0$. Find the different possible vector & parametric equation of the line.
16. The end points of the diameters of a circle are (a,b) and $(1-a,1-b)$. Then find the equation of the circle after it has been rotated 30° counter clock wise about the origin.
17. Find the image of the line $2y+3x=1$ after it has been reflected by the line $y+3x=2$
18. Let ℓ be line that passes through $(-2,3)$ and parallel to the vector $\vec{d}=(4,1)$, then find
- A. The vector equation of ℓ
 B. The parametric equation of ℓ
 C. The rectangular form of equation of ℓ
19. The coordinate of the image of $(-4,2)$ after a rotation about the origin through 135° is _____.
20. Let $\vec{U}=(-1,10)$, $\vec{V}=(1,3)$. A unit vector in the direction of $\vec{U}-2\vec{V}$
21. The image of the point $(-1,3)$ when reflected along the line $y=x$ is ____.
22. If the line $3x-4y+10=0$ is translated by T to $3x-4y+27=0$ and the image of the point $(0,1)$ by T is $(3,t)$ then the value of t is _____.
23. Let $\vec{a}=i-j$ and $\vec{b}=2i-3j$. What is the unit vector in opposite direction with $\vec{a}-\vec{b}$?

24. Find the vector $\vec{a}$ perpendicular to $b=2i+3j$ and satisfying the condition $\vec{a} \cdot (2i-j) = -6$?
25. The angle between the two vectors $\vec{V} = -3i-2j$ and $\vec{V} = 5i-j$ is ____.
26. Let $T_1(x+5, y-16) = (x+7, y-8)$ and $T_2(x-3, y-1) = (x+3, y-1)$ and $p = (7, 4)$ then $T_1(T_2(p))$ is equal to ____.
27. The image of $p(-3, 4)$ in the line $x=2$ is ____.
28. What is the image of the point $(15, -8)$ when rotated 90° clockwise about the point $(5, 3)$?
29. If two vectors $\vec{a} = (x+y, 1)$ and $\vec{b} = (2, 2x-y)$ are equal vectors for some real numbers x and y then what should be the values of x and y respectively.
30. The equation of the image of the line $x+y=\sqrt{2}$ when rotated $\frac{\pi}{4}$ rad anticlockwise about the origin is ____.
31. The parametric vector equation of the line through $(2, 3)$ parallel to the line $3x+y-4=0$ is ____.
32. What translation vector takes the parabola $y=x^2+2x$ to a parabola $y=x^2-4x+6$?
33. If $\vec{U}, \vec{V}$ and $\vec{W}$ are three vectors such that $\vec{U} = 3i-4j$, $\vec{V} = -2i+2j$ and $\vec{W} = 2i+j$, then which of the following is not true?
34. If $A = (-2, 3)$, $B = (1, -1)$ and $C = (2, 0)$ are vertices of a triangle, then what is $\cos \angle A$ where $\angle A$ is the interior angle of $\triangle ABC$?
35. A translation T takes $(1, -1)$ to $(3, 2)$. If a line whose parametric equation is given by $(x, y) = (2, -1) + t(1, 1)$ for all $t \in \mathbb{R}$ is translated by T . What is the image of ℓ ?
36. What is the image of the hyperbola $\frac{(x-1)^2}{36} - \frac{(y+2)^2}{64} = 1$, when it is reflected by the line $y = -x$?
37. A parametric vector equation of the line through $(1, 3)$ and perpendicular to the line containing the vector $(-1, 2)$ is ____.
38. Let $|\vec{a}|$ denote the length of vector $\vec{a}$. Suppose $|\vec{a}| = 3$, $\vec{b}$ is a unit vector and $\vec{a}$ and $\vec{b}$ makes an angle $\theta = \frac{\pi}{3}$ rad what is $|\vec{a} - 2\vec{b}|$?