

Unit 2

Polynomial Functions

Main Content

- Introduction of polynomial functions
- Operations on polynomial functions
- Theorem on polynomial functions
- Zeros of polynomial functions
- Graphs of polynomial functions
- Applications

2.1. Introduction to Polynomial Functions

Activity 2.1

Classify the following functions as constant function, linear function, quadratic function or none of these:

a. $f(x) = 2x + 4$

b. $g(x) = \frac{2}{3} - x$

c. $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

d. $h(x) = -x^2 + 5x + 9$

e. $f(x) = x^4 + 2x - 5$

f. $k(x) = 5$

g. $h(x) = (x - 2)(x + 2)$

h. $6 + \frac{9}{5}x + 4x^2$

i. $f(x) = -\sqrt{5}x + \sqrt{2}$

j. $l(x) = 3\sqrt{x} + \sqrt{3}$

1. constant function

A function f is a **constant function**, if it can be written in the form $f(x) = b$, where, b is a real number.

➤ The domain is the set of all real numbers.

➤ The range is

2. linear function

A function f is a **linear function**, if it can be written in the form $f(x) = ax + b$, $a \neq 0$ where, a and b are real numbers.

- The domain is the set of all real numbers.
- The range is the set of all real numbers

3. Quadratic function

A function f is a **quadratic function**, if it can be written in the form $f(x) = ax^2 + bx + c$, $a \neq 0$ where, a , b and c are real numbers.

- The domain of f is the set of all real numbers
- the range is **not** the set of all real numbers and it depends on the values of a , b and c .

Definition of Polynomial Function

Definition 2.1:

Let n be non-negative integer and let $a_0, a_1, \dots, a_n$ be real numbers with

$a_n \neq 0$. The function

is called a **polynomial function** in one variable x

of degree n .

In the above definition of polynomial function

1. $a_0, a_1, \dots, a_n$ are called the **coefficients** of the polynomial function
2. The number a_n is called the **leading coefficient** of the polynomial
3. $a_n x^n$ is the **leading term**.
4. The number n (the exponent of the **highest power** of x) is the **degree** of the polynomial.
5. The number a_0 is called the constant term of the polynomial

Activity 2.2

1. Are constant, linear and quadratic function polynomial functions?
2. What is the degree of linear function?
3. What is the degree of quadratic function?
4. What do you think for the degree of a **non-zero** constant function?

Note:

- The constant function $f(x) = 0$ is called the zero polynomial with **no degree** assigned to it
- The domain of a polynomial function is **all real numbers**.

Example 1

Find the degree, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 3x^3 - 9x^2 + 5x + \frac{3}{2}$

2. $h(x) = -5x^4 + 8x^3 + 2x^2 - 3x + 7$

3. $g(x) = -2 + 3x^3 + \frac{2}{5}x^2 - x^4 + 4x^5 + 5x$

4. $g(x) = 2(x^4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1$

Note:

➤ There are two ways to check **whether** or **not** a given function is **not** a polynomial function.

1. Its domain is **not** all real numbers.

2. It does **not** take the form or unable to transform it to the form

where n is **non negative integer**

(0 or positive integers) and

, are real numbers

with $\neq 0$

Example 2

Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term

A, $g(x) = 4 + 3 - 7$

B, $f(x) = 2 - 9 + 25x$

C, $g(x) = (2 - x)(2 + x)$

D,

E,

F,

$$2 + 5 - x + 5$$

$$\mathbf{H}, k(\mathbf{x}) =$$

$$\mathbf{I}, f(x) = + 5 -$$

$$\mathbf{L}, f(x) =$$

$$\mathbf{M}, f(x) =$$

$$\mathbf{N}, f(x) =$$

polynomial expression

Definition 2.2:

A polynomial expression is an expression of the form $\sum_{i=0}^n a_i x^i$ where n is non negative integer and $a_n \neq 0$. Each individual expression making up the polynomial is called a term.

Example 3

Consider the expression

$$-3 + 5$$

- a. Is it a polynomial expression?**
- b. Find the degree, leading coefficient and the constant term.**
- c. What is the coefficient of**

Definition 2.3

A polynomial function $p(x)$ is said to be

1. a polynomial function over **integers** if the coefficients of $p(x)$ are **all integers**.
2. a polynomial function over **rational** numbers if the coefficients of $p(x)$ are **all rational** numbers.
3. a polynomial function over real numbers if the coefficients of $p(x)$ are **all real** numbers.

Example 4

Identify the following polynomial functions as polynomial function over **integers**, polynomial function over **rational** numbers or

1. $f(x) = \sqrt{5}x^3 - 6x^2 + \frac{10}{11}x - 9$

2. $g(x) = 3x^5 + 2x^4 - 23x^3 + 13x^2 + 25x - 45$

3. $k(x) = 3.5x^4 + \frac{4}{5}x^3 + 2x^2 + 33x - 55$

2.2, Operations on Polynomial Functions

Activity 2.3

1. Which of the following pairs contain like terms?
 - a. $2x$ and $5x$
 - b. $5a^2$ and $6a^2$
 - c. x^2 and $2x^3$
 - d. $3x^2$ and $2y^2$
 - e. 3 and y
 - e. x^5 and $6x^5$
2. For any three real numbers a, b and c , determine whether each of the following statements is true or false. Give reason for your answer.
 - a. $a + b = b + a$
 - b. $a - b = b - a$
 - c. $(a + b) + c = a + (b + c)$
 - d. $(ab)c = a(bc)$
 - e. $a(b + c) = ab + ac$
 - f. $a - (b + c) = a - b + c$
 - g. $a - (b - c) = a - b + c$
 - h. $(a - b) + c = a + (-b + c)$

A, Addition of polynomial Functions

Definition 2.4:

The sum of two polynomial functions f and g is written as $f + g$ and is defined as:

$(f + g)(x) = f(x) + g(x)$ for all real numbers x .

Note:

➤ **The sum of two polynomial functions is found by adding the coefficients of like terms.**

Example

Find the sum of the polynomial functions $f(x)$ and $g(x)$ were

a. $f(x) = 2x^3 - 3x - 5$ and $g(x) = 5x^4 - 7x^2 + 3$.

b. $f(x) = -x^4 + 2x^3 - 3x^2 - 3x + 2$ and $g(x) = 5 + 7x - 2x^2 - x^3 + x^4$.

Note:

- If $f(x)$ and $g(x)$ have different degrees, the degree of $f(x) + g(x)$ is the same as the degree of $f(x)$ or the degree of $g(x)$ whichever has the **highest** degree.
- If $f(x)$ and $g(x)$ have the same degree, the degree of the sum may be **lower** or **equal** to the common degree.
- The sum of two polynomial functions is a polynomial function.

B, Subtraction of Polynomial Functions

Definition 2.5:

The difference of two polynomial functions f and g is written as $f - g$, and is defined as

$(f - g)(x) = f(x) - g(x)$ for all real numbers x

Example

Find $f(x) - g(x)$ where

a. $f(x) = -x^3 - 5x^2 + 3x + 12$ and $g(x) = 9x^3 - x^2 - 6x + 3$.

b. $f(x) = -\frac{1}{3}x^4 + 3x^3 - 3x^2 - \frac{3}{5}x + 2$ and $g(x) = 5 + \frac{7}{5}x - 2x^2 - x^3 + \frac{1}{2}x^4$.

Activity 2.4 .1

- 1. The degree of $f(x) - g(x)$ is equal to the degree of $f(x)$ or $g(x)$ whichever has the highest degree. (True/False)**
- 2. Is there a possibility for the degree of $f(x) - g(x)$ to be lower than the degree of $f(x)$ or the degree of $g(x)$? When?**
- 3. Is the difference of two polynomial functions a**

C, Multiplication of Polynomial Functions

Definition 2.6:

➤ The product of two polynomial functions $f(x)$ and $g(x)$ is written as $f \cdot g$, and is defined as:

$(f \cdot g)(x) = f(x) \cdot g(x)$ for all real numbers x .

➤ The product of two polynomials $f(x)$ and $g(x)$ is found by multiplying each term of one by every term of the

Example

Let $f(x) = 3 + 2x - 4$,

$g(x) = x - 2$

a. Find $f(x) \cdot g(x)$.

b. Find the degree of f , g and $f \cdot g$.

c. Is the degree of $f \cdot g$ equal to the sum of the degrees of f and g ?

Note:

1. For any two non-zero polynomial functions f and g , the degree of $f \cdot g$ is $m + n$ if the degree of f is m and the degree of g is n .
2. If either f or g is the zero polynomial then $f \cdot g$ becomes the zero polynomial and has no degree.
3. The product of two polynomial functions is a polynomial

D, Division of polynomial Functions

Activity 2.5

1. For each of the following divide the number a by the number b to find two numbers q (quotient) and r (remainder) with $r < b$ such that $a = bq + r$ or if

A, $a = 97$ and $b = 8$.

B, $a = 168$ and $b = 5$.

Definition 2.7:

- The division (Quotient) of two polynomial functions f and g is written as $f \div g$, and is defined as:

$$(f \div g)(x) = f(x) \div g(x), \text{ and for all real numbers } x \text{ and } g(x) \neq 0$$

- It is possible to divide one polynomial by another using a **long division process**.
- The long division process **stop** when the degree of the **remainder** polynomial less than the degree of the **divisor** polynomial.