

"Teach A Level Maths"

**Vol. 1: AS Core
Modules**

**51: The Trapezium
Rule**

The Trapezium Rule

Module C2

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The Trapezium Rule

To find an area bounded by a curve, we need to evaluate a definite integral.

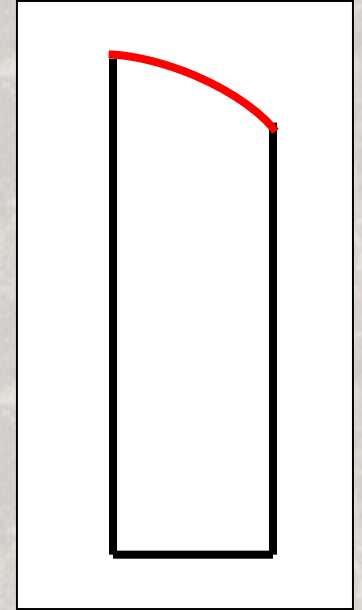
If the integral cannot be evaluated, we can use an approximate method.

This presentation uses the approximate method called the **Trapezium Rule**.



The Trapezium Rule

The area under the curve is divided into a number of strips of equal width. The top edge of each strip . . .
. . . is replaced by a straight line so the strips become trapezia.



The Trapezium Rule

The area under the curve is divided into a number of strips of equal width. The top edge of each strip . . .

. . . is replaced by a straight line so the strips become trapezia.

The total area of the trapezia gives an approximation to the area under the curve.

The formula for the area of a trapezium is:

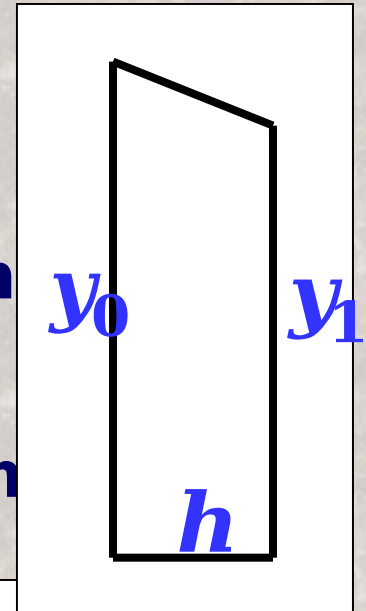
the average of the parallel

sides

×

the distance apart

$$= \frac{1}{2} (y_0 + y_1) \times h$$



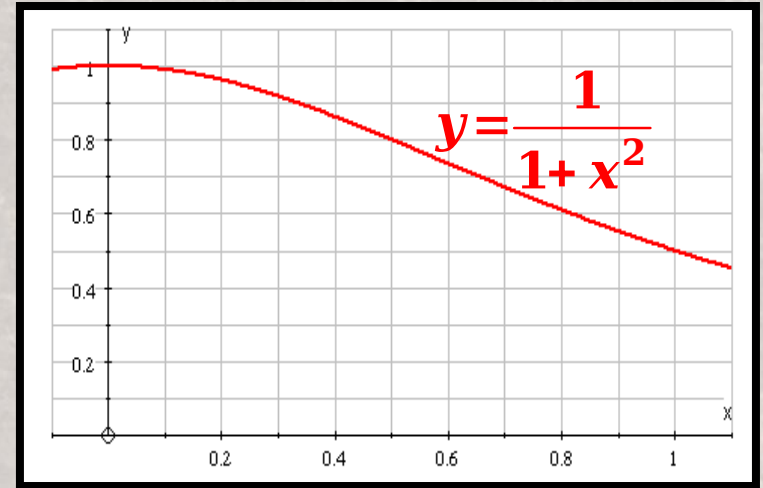
h seems a strange letter to use for width but it is always used in the trapezium rule

The Trapezium Rule

e.g.1

$$\int_0^1 \frac{1}{1+x^2} dx$$

Suppose we have 5 strips.



The Trapezium Rule

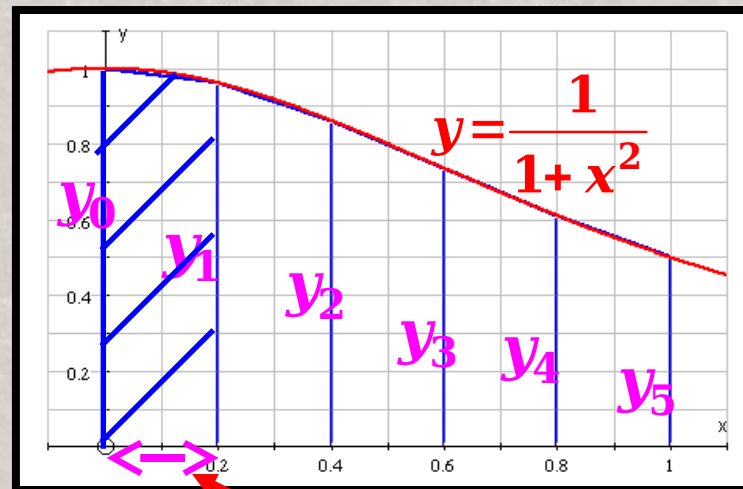
e.g.1

$$\int_0^1 \frac{1}{1+x^2} dx$$

Suppose we have 5 strips.

The parallel sides of the trapezia are the y -values of the function.

The area of the 1st trapezium



$$= \frac{1}{2}(y_0 + y_1) \times 0.2$$

The Trapezium Rule

e.g.1

$$\int_0^1 \frac{1}{1+x^2} dx$$

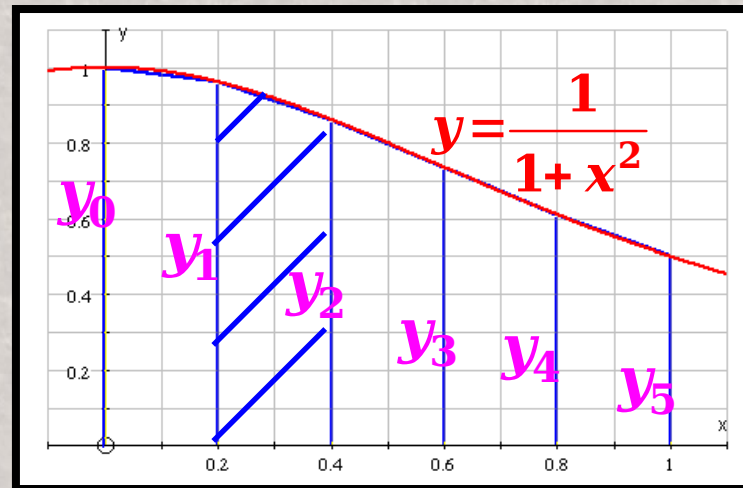
Suppose we have 5 strips.

The parallel sides of the trapezia are the y -values of the function.

The area of the 1st trapezium

The area of the 2nd trapezium

etc.



$$= \frac{1}{2} (y_0 + y_1) \times 0.2$$

$$= \frac{1}{2} (y_1 + y_2) \times 0.2$$

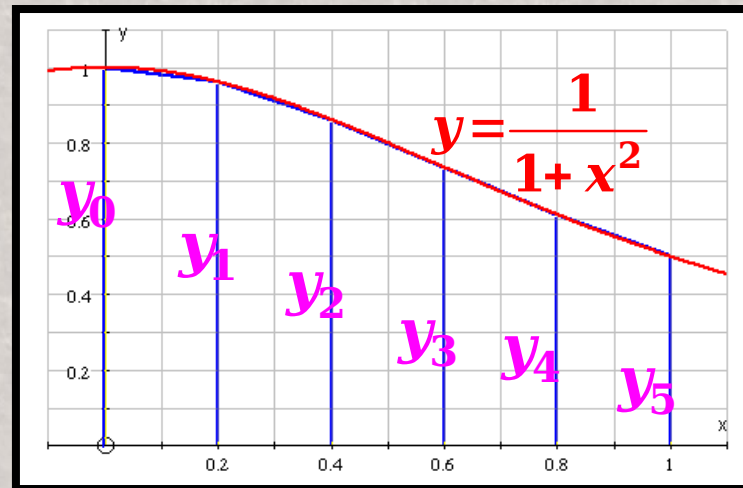
The Trapezium Rule

e.g.1

$$\int_0^1 \frac{1}{1+x^2} dx$$

Suppose we have 5 strips.

Adding the areas of the trapezia, we get



$$\begin{aligned} & \frac{1}{2}(y_0 + y_1) \times 0.2 + \frac{1}{2}(y_1 + y_2) \times 0.2 + \dots + \frac{1}{2}(y_4 + y_5) \times 0.2 \\ &= \frac{1}{2} \times 0.2 (y_0 + y_1 + y_1 + y_2 + y_2 + \dots + y_5) \end{aligned}$$

So, Since y_1 is the side of 2 trapezia, it occurs twice in the formula. The same is true for y_2, y_3, y_4 (removing common factors)

$$\int_0^1 \frac{1}{1+x^2} dx \approx \frac{1}{2} \times 0.2 (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$$

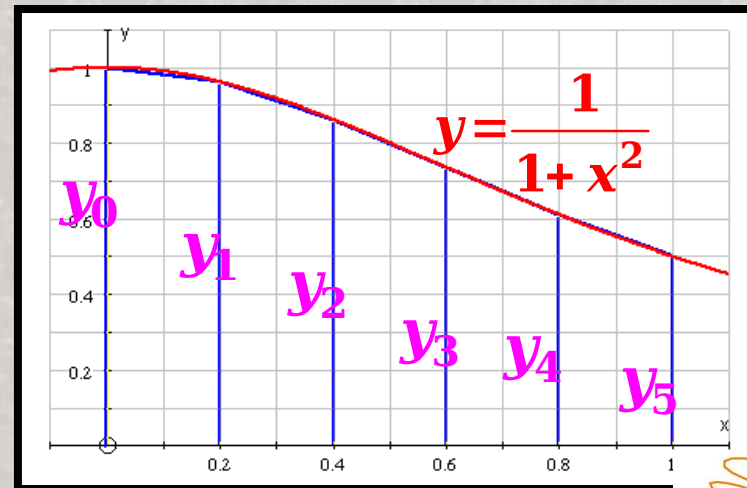


The Trapezium Rule

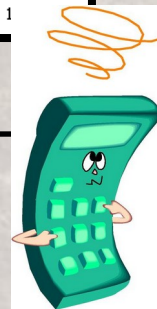
e.g.1

$$\int_0^1 \frac{1}{1+x^2} dx$$

For each value of x , we calculate the y -values, using the function, and writing the values in a table.



x	0	0.2	0.4	0.6	0.8	1.0
y	1	0.9615	0.8621	0.7353	0.6098	0.5



$$\int_0^1 \frac{1}{1+x^2} dx \approx \frac{1}{2} \times 0.2 (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$$

$$\approx 0.1 (1 + 2(0.9615 + 0.8621 + 0.7353 + 0.6098) + 0.5)$$

$$\approx 0.784 \text{ (3.s.f.)}$$



The Trapezium

The general formula for the trapezium rule is

$$\int_a^b y dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n)$$

where n is the number of strips

The y -values are called ordinates.

There is always 1 more ordinate than the number of strips.

The width, h , of each strip is given by

$$h = \frac{b - a}{n}$$

To improve the accuracy we just need to use more strips.



The Trapezium

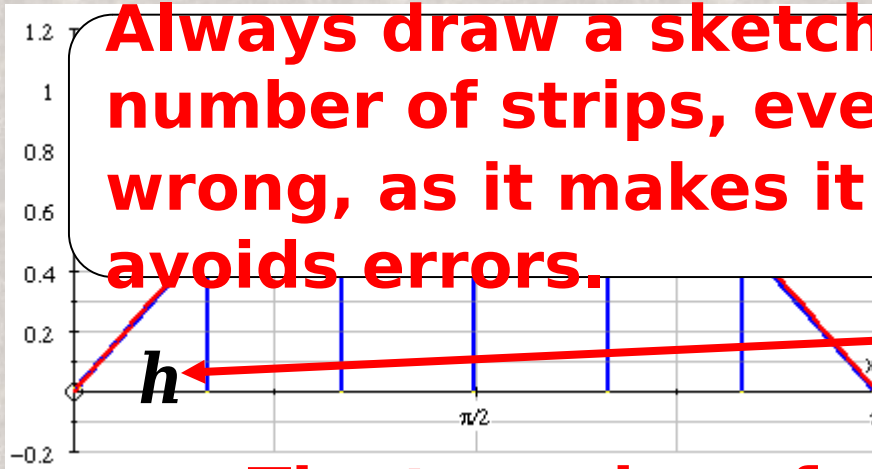
Rule

e.g.2 Find the approximate value of $\int_0^{\pi} \sin x \, dx$ using 6 strips and giving the answer to 2

Solution:

$$\int_a^b y \, dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6)$$

Always draw a sketch showing the correct number of strips, even if the shape is wrong, as it makes it easy to find h and avoids errors.



$$\Rightarrow h = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

The top edge of every trapezium in this example lies below the curve, so the rule will underestimate the answer.

the
re



The Trapezium

Rule

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6)$$

**Radian
s!**

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	0
y	0	0.5	0.8660	1	0.8660	0.5	0

To make sure that we have the required accuracy we must use at least 1 more than the answer requires. It is better to store the values in the calculator's memories.



⇒

$$\frac{0.5236}{2} (0 + 2(0.5 + 0.8660 + 1 + 0.8660 + 0.5) + 0)$$

$$\approx 1.95 \text{ (2 d. p.)}$$

The Trapezium Rule

The exact value of $\int_0^{\pi} \sin x \, dx$ is 2.

The percentage error in our answer is found as follows:

$$\text{Percentage error} = \frac{\text{error}}{\text{exact value}} \times 100$$



(where, error = exact value - the approximate value).

$$\begin{aligned} &\approx \frac{2 - 1.95}{2} \times 100 \\ &= 2.5\% \end{aligned}$$

The Trapezium Rule

SUMMARY

- The trapezium law for estimating an area

is

$$\int_a^b y dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n)$$

where n is the number of

strips

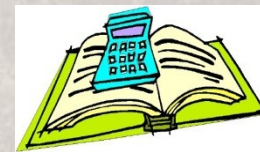
- The width, h , of each strip is given by $h = \frac{b - a}{n}$ (but should be checked on a sketch)

- The number of ordinates is 1 more than the number of strips.

- The trapezium law underestimates the area if the tops of the trapezia lie under the curve and overestimates it if the tops lie above the curve.
- The accuracy can be improved by increasing n .



The Trapezium Rule



Exercise

s

Use the trapezium rule to estimate the areas given by the integrals, giving the answers to 3 s. f.

1. $\int_0^1 \sqrt{1+x} \, dx$ using 4 strips
How can your answer be improved?

2. $\int_0^{\frac{\pi}{2}} \cos x \, dx$ using 4 ordinates
Find the approximate percentage error in your answer, given that the exact value is 1.

The Trapezium Rule

Solution

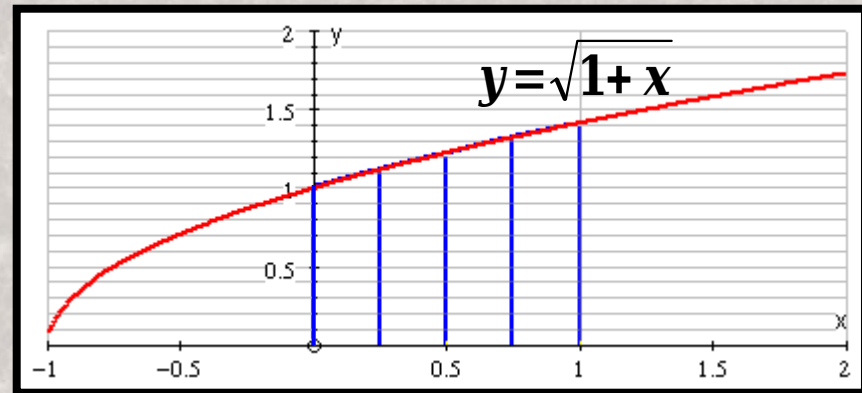
S

1.

$$\int_0^1 \sqrt{1+x} \, dx$$

using 4

strips



$$n=4,$$

$$h=0.25$$

x	0	0.25	0.5	0.75	1
y	1	1.118	1.225	1.323	1.414

$$A \approx \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3) + y_4)$$

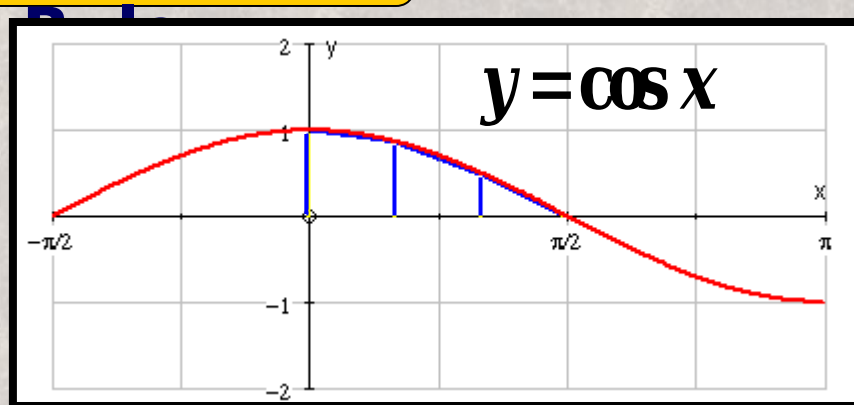
$$\approx 1.22 \text{ (3 s. f.)}$$

The answer can be improved by using more strips.



The Trapezium

Solution



2. $\int_0^{\frac{\pi}{2}} \cos x \, dx$
using 4

ordinates

$n=3,$

$h = \frac{\pi}{6}$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{\pi}{2}$
y	1	0.8660	0.5	0

$$A \approx \frac{h}{2} (y_0 + 2(y_1 + y_2) + y_3) \approx 0.977 \text{ (3 s. f.)}$$

The exact value is 1, so the percentage error

$$\approx \frac{1 - 0.977}{1} \times 100 = 2.3 \%$$



The Trapezium Rule



The Trapezium Rule

The following slides contain repeats of information on earlier slides, shown without colour, so that they can be printed and photocopied. For most purposes the slides can be printed as “Handouts” with up to 6 slides per sheet.

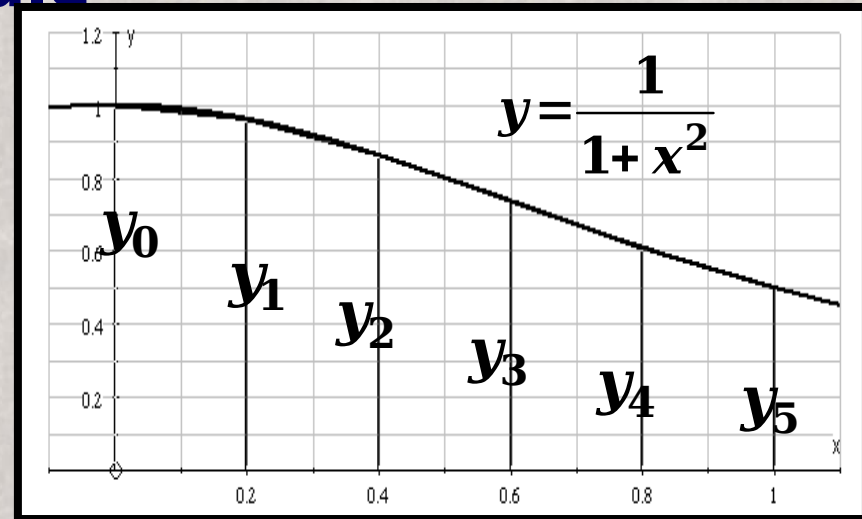
The Trapezium Rule

e.g.1

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So, Since y_1 is the side of 2 trapezia, it occurs twice in the formula. The same is true for y_2, y_3, y_4 (removing common factors)

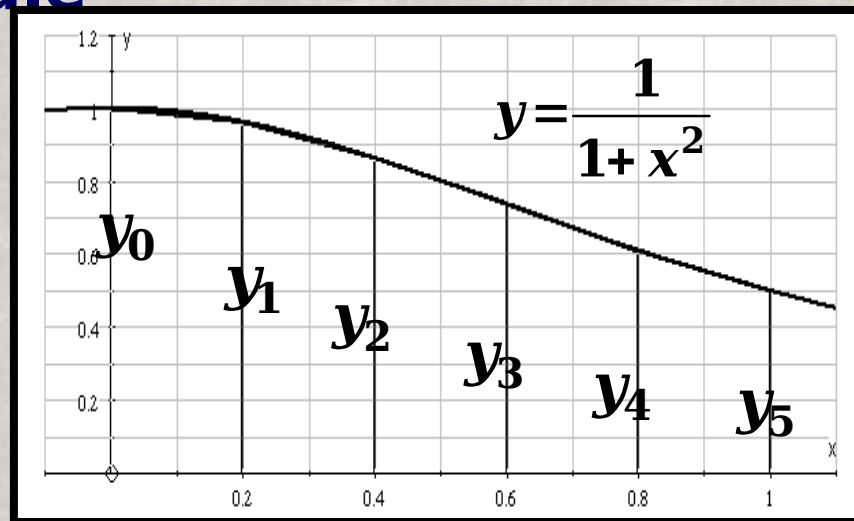
$$\text{to } \int_0^1 \frac{1}{1+x^2} dx \approx \frac{1}{2} \times 0.2 (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$$

The Trapezium Rule

e.g.1 cont.

$$\int_0^1 \frac{1}{1+x^2} dx$$

For each value of x , we calculate the y -values, using the function, and writing the values in a table.



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$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &\approx \frac{1}{2} \times 0.2 (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5) \\ &\approx 0.1 (1 + 2(0.9615 + 0.8621 + 0.7353 + 0.6098) + 0.5) \\ &\approx 0.784 \text{ (3.s.f.)} \end{aligned}$$

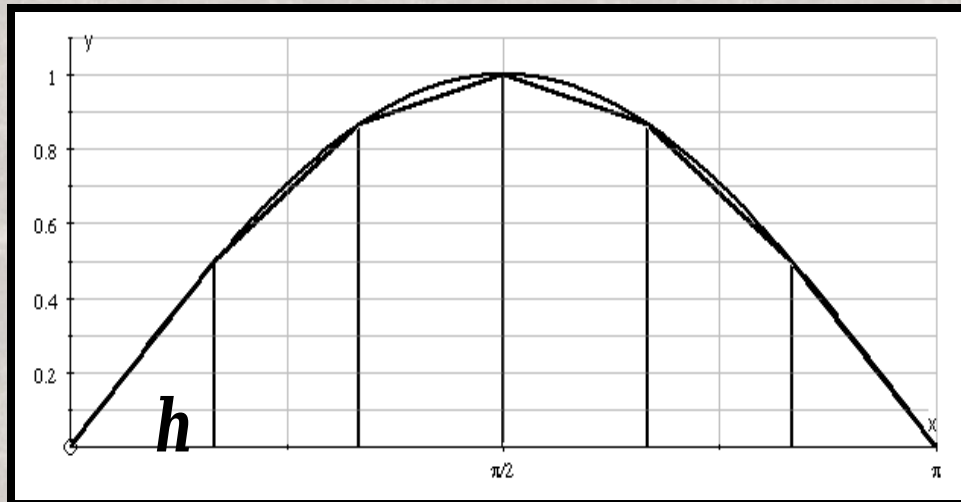
The Trapezium

e.g.2 Find the approximate value of $\int_0^{\pi} \sin x \, dx$ using 6 strips and giving the answer to 2

Solution:

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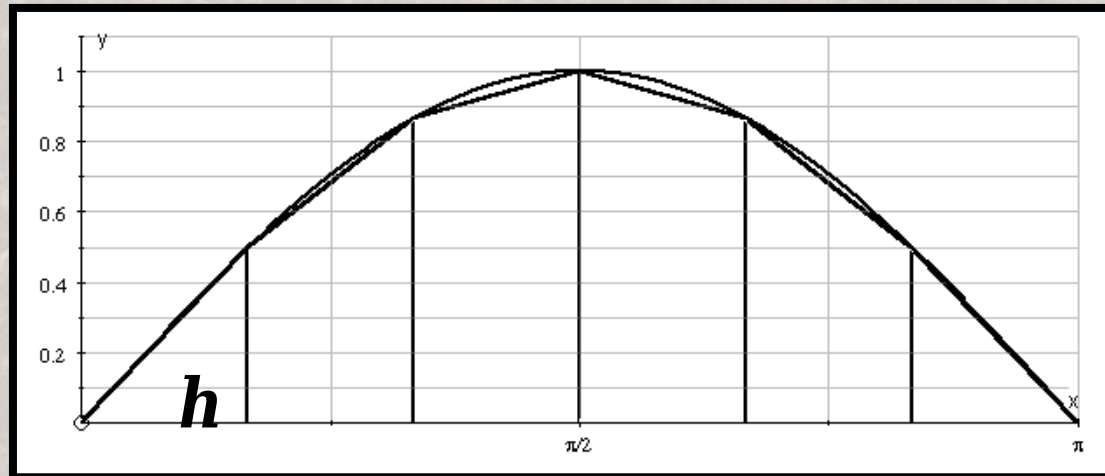
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$$\Rightarrow h = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

The Trapezium Rule



The top edge of every trapezium in this example lies below the curve, so the rule will underestimate the answer.

(It doesn't matter that the 1st and last "trapezia" are triangles)

**Radian
s!**

The Trapezium Rule

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6)$$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	0
y	0	0.5	0.8660	1	0.8660	0.5	0

$$h = \frac{\pi}{6} \approx 0.5236$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \, dx \approx$$

$$\frac{0.5236}{2} (0 + 2(0.5 + 0.8660 + 1 + 0.8660 + 0.5) + 0)$$

$$\approx 1.95 \text{ (2 d. p.)}$$

To make sure that we have the required accuracy we must use at least 1 more d. p. than the answer requires. It is better still to store the values in the calculator's memories.

The Trapezium Rule

SUMMARY

➤ The trapezium law for estimating an area is

$$\int_a^b y dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n)$$

where n is the number of

strips

➤ The width, h , of each strip is given by $h = \frac{b - a}{n}$ but should be checked on a sketch)

➤ The number of ordinates is 1 more than the number of strips.

➤ The trapezium law underestimates the area if the tops of the trapezia lie under the curve and overestimates it if the tops

lie above the curve.
➤ The accuracy can be improved by increasing n .