CIRCLE EQUATIONS

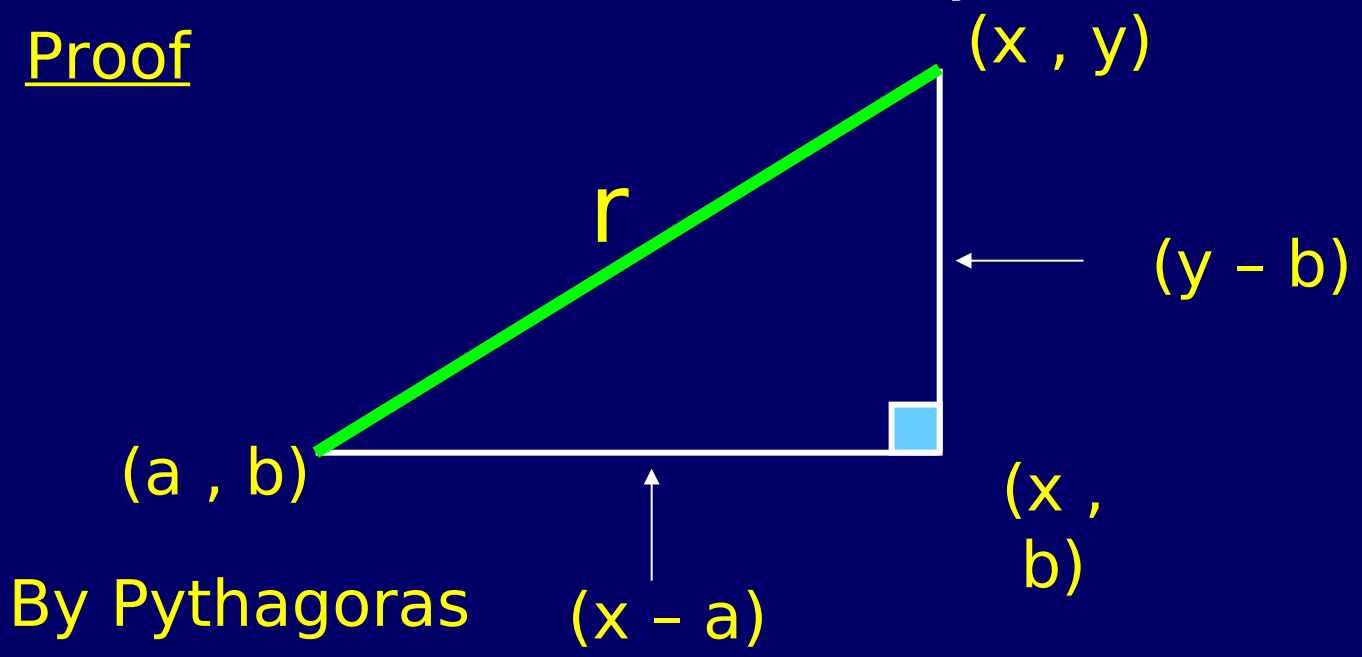
- The Graphical Form of the Circle Equation
- Inside , Outside or On the Circle
- Intersection Form of the Circle Equation
- Finding distances involving circles and lines
- Find intersection points between a Line & Circle
- Tangency (& Discriminant) to the Circle
- Equation of Tangent to the Circle
- Mind Map of Circle Chapter
- Exam Type Questions

The Circle

The distance from (a,b) to (x,y) is given by

$$r^2 = (x - a)^2 + (y - b)^2$$

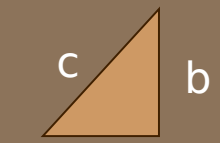
Proof



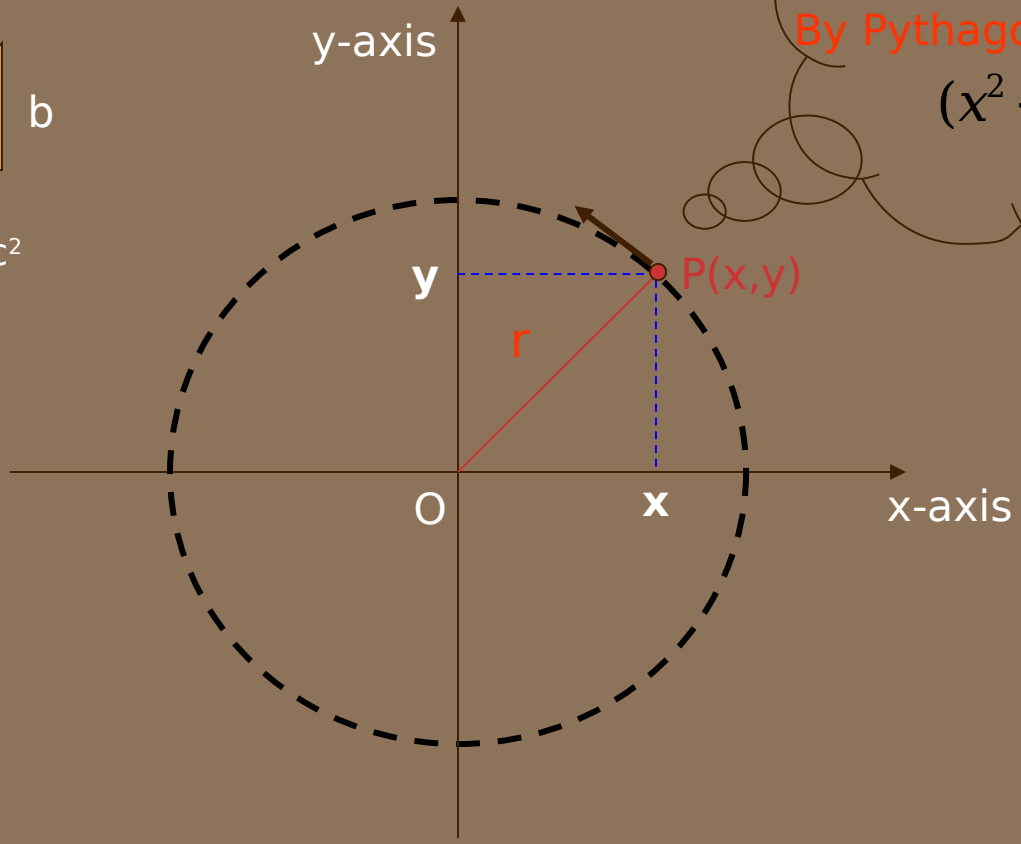
By Pythagoras

$$r^2 = (x - a)^2 + (y - b)^2$$

Equation of a Circle Centre at the Origin



$a^2+b^2=c^2$



OP has length r

r is the radius of the circle

By Pythagoras Theorem

$$(x^2+y^2)=r^2$$

The Circle

Find the centre and radius of the circles below

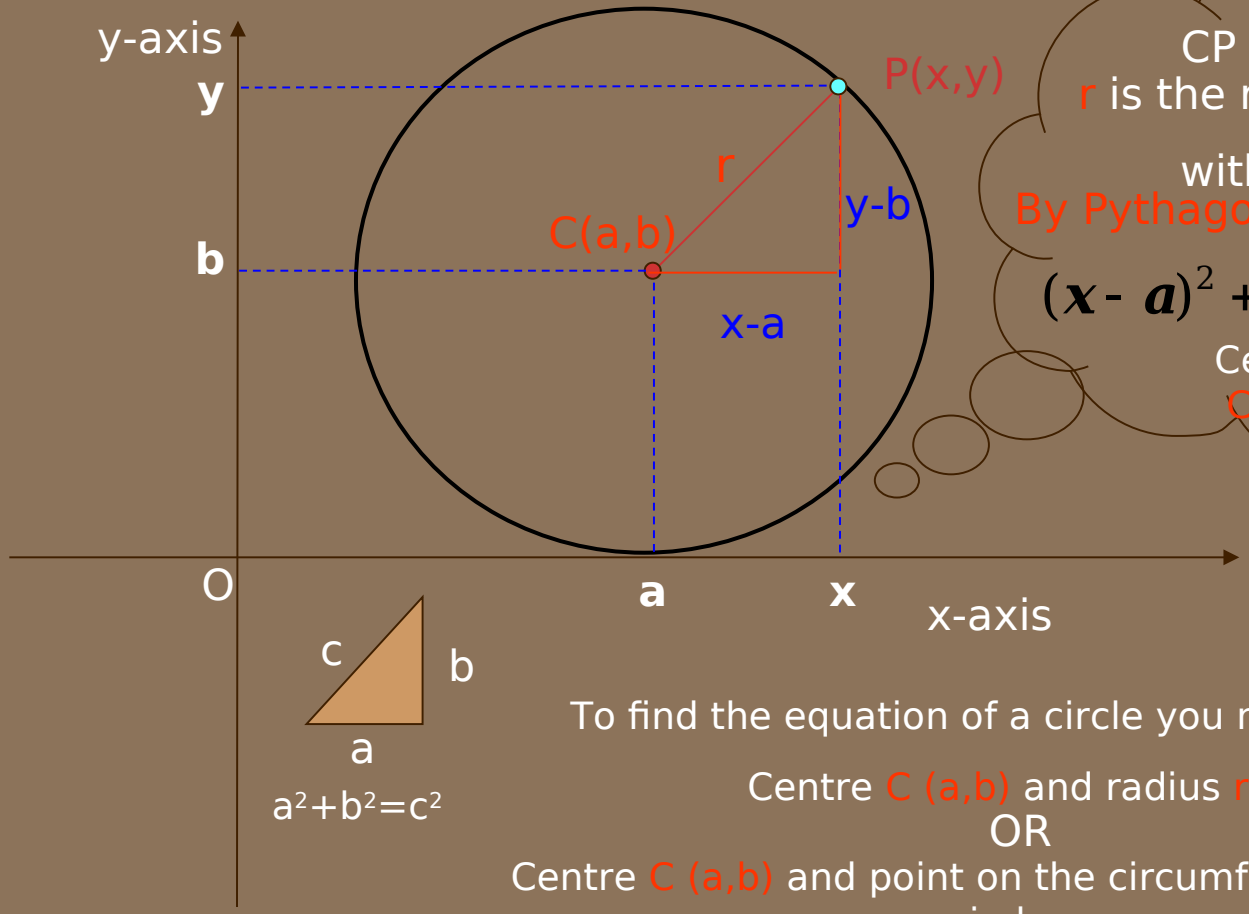
$x^2 + y^2 = 7$

centre (0,0) & radius = $\sqrt{7}$

$x^2 + y^2 = {}^1/_9$

centre (0,0) & radius = ${}^1/_3$

General Equation of a Circle



CP has length r
r is the radius of the circle

with centre (a,b)
By Pythagoras Theorem

$$(x-a)^2+(y-b)^2=r^2$$

Centre
C(a,b)

To find the equation of a circle you need to know

Centre C (a,b) and radius r
OR
Centre C (a,b) and point on the circumference of the circle

The Circle

Example

$(x-2)^2 + (y-5)^2 = 49$ centre $(2,5)$ radius $= 7$

$(x+5)^2 + (y-1)^2 = 13$ centre $(-5,1)$ radius $= \sqrt{13}$

$(x-3)^2 + y^2 = 20$ centre $(3,0)$ radius $= \sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$

Centre $(2,-3)$ & radius $= 10$

Equation is $(x-2)^2 + (y+3)^2 = 100$

NAB

Centre $(0,6)$ & radius $= 2\sqrt{3}$

Equation is $x^2 + (y-6)^2 = 12$

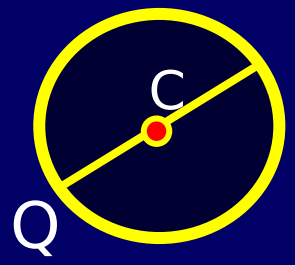
$r^2 = 2\sqrt{3} \times 2\sqrt{3}$

$= 4\sqrt{9}$

$= 12$

The Circle

Example



Find the equation of the circle that has PQ as diameter where P is(5,2) and Q is(-1,-6).

C is $(^{(5+(-1))}/_{2}, ^{(2+(-6))}/_{2})$ = (2,-2) = (a,b)

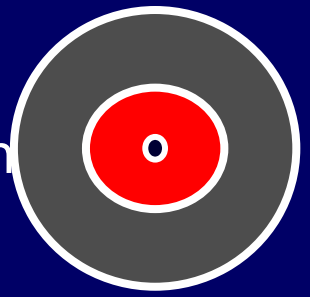
$CP^2 = (5-2)^2 + (2+2)^2$ = 9 + 16 = 25 = r^2

Using $(x-a)^2 + (y-b)^2 = r^2$

Equation is $(x-2)^2 + (y+2)^2 = 25$

The Circle

Example



Two circles are concentric. (ie have same cen

The larger has equation $(x+3)^2 + (y-5)^2 = 12$

The radius of the smaller is half that of the larger. Find its equation.

Using $(x-a)^2 + (y-b)^2 = r^2$

Centres are at (-3, 5)

Larger radius = $\sqrt{12}$ $= \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Smaller radius = $\sqrt{3}$ so $r^2 = 3$

Required equation is $(x+3)^2 + (y-5)^2 = 3$

Inside / Outside or On Circumference

When a circle has equation $(x-a)^2 + (y-b)^2 = r^2$

If (x,y) lies **on** the circumference then $(x-a)^2 + (y-b)^2 = r^2$

If (x,y) lies **inside** the circumference then $(x-a)^2 + (y-b)^2 < r^2$

If (x,y) lies **outside** the circumference then $(x-a)^2 + (y-b)^2 > r^2$

Example

Taking the circle $(x+1)^2 + (y-4)^2 = 100$

Determine where the following points lie;

K(-7,12) , L(10,5) , M(4,9)

Inside / Outside or On Circumference

At K(-7,12)

$(x+1)^2 + (y-4)^2 = (-7+1)^2 + (12-4)^2 = (-6)^2 + 8^2 = 36 + 64 = 100$

So point K is on the circumference.

At L(10,5)

$(x+1)^2 + (y-4)^2 = (10+1)^2 + (5-4)^2 = 11^2 + 1^2 = 121 + 1 = 122 > 100$

So point L is outside the circumference.

At M(4,9)

$(x+1)^2 + (y-4)^2 = (4+1)^2 + (9-4)^2 = 5^2 + 5^2 = 25 + 25 = 50 < 100$

So point M is inside the circumference.

Intersection Form of the Circle Equation

1. $(x-a)^2+(y-b)^2=r^2$ Centre C(a,b) Radius r

$(x-a)(x-a)+(y-b)(y-b)=r^2$

$(x^2-2xa+a^2)+(y^2-2yb+b^2)=r^2$

$x^2+y^2-2xa-2yb+a^2+b^2=r^2$

$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$

Let $g=-a$, $f=-b$, $c=a^2+b^2-r^2$

$c=a^2+b^2-r^2$

$c=(-g)^2+(-f)^2-r^2$

$c=g^2+f^2-r^2$

$r^2=g^2+f^2-c$

$r=\sqrt{g^2+f^2-c}$

2. $x^2+y^2+2gx+2fy+c=0$ Centre C(-g,-f) Radius $r=\sqrt{g^2+f^2-c}$

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

Write the equation $(x-5)^2 + (y+3)^2 = 49$ without brackets.

$$(x-5)^2 + (y+3)^2 = 49$$

$$(x-5)(x+5) + (y+3)(y+3) = 49$$

$$x^2 - 10x + 25 + y^2 + 6y + 9 - 49 = 0$$

$$x^2 + y^2 - 10x + 6y - 15 = 0$$

This takes the form given above where

$2g = -10$, $2f = 6$ and $c = -15$

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

Show that the equation $x^2 + y^2 - 6x + 2y - 71 = 0$ represents a circle and find the centre and radius.

$x^2 + y^2 - 6x + 2y - 71 = 0$

$x^2 - 6x + y^2 + 2y = 71$

$(x^2 - 6x + 9) + (y^2 + 2y + 1) = 71 + 9 + 1$

$(x - 3)^2 + (y + 1)^2 = 81$

This is now in the form $(x-a)^2 + (y-b)^2 = r^2$

So represents a circle with centre (3,-1) and radius

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

We now have 2 ways on finding the centre and radius of a circle depending on the form we have.

$x^2 + y^2 - 10x + 6y - 15 = 0$

$2g = -10$, $g = -5$

$2f = 6$, $f = 3$

$c = -15$

centre = $(-g, -f) = (5, -3)$

radius = $\sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(25 + 9 - (-15))}$

$= \sqrt{49}$

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

$x^2 + y^2 - 6x + 2y - 71 = 0$

$2g = -6$ $g = -3$

$2f = 2$ $f = 1$

$c = -71$

centre = $(-g, -f)$ = $(3, -1)$

radius = $\sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(9 + 1 - (-71))}$

$= \sqrt{81}$

$= 9$

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

Find the centre & radius of $x^2 + y^2 - 10x + 4y - 5 = 0$

$x^2 + y^2 - 10x + 4y - 5 = 0$

NAB

$2g = -10$

$g = -5$

$2f = 4$

$f = 2$

$c = -5$

centre = $(-g, -f)$ = $(5, -2)$

radius = $\sqrt{(g^2 + f^2 - c)}$

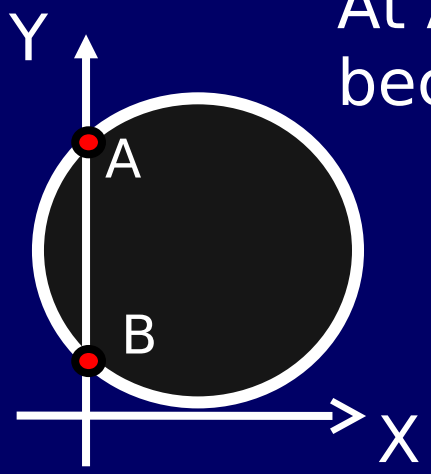
$= \sqrt{(25 + 4 - (-5))}$

$= \sqrt{34}$

Equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Example

The circle $x^2 + y^2 - 10x - 8y + 7 = 0$ cuts the y- axis at A & B. Find the length of AB.



At A & B $x = 0$ so the equation becomes

$y^2 - 8y + 7 = 0$

$(y - 1)(y - 7) = 0$

$y = 1$ or $y = 7$

A is (0,7) & B is (0,1)

So AB = 6 units

Application of Circle Theory

Frosty the Snowman's lower body section can be represented by the equation

$x^2 + y^2 - 6x + 2y - 26 = 0$



His middle section is the same size as the lower but his head is only $^1/_3$ the size of the other two sections. Find the equation of his head !

$x^2 + y^2 - 6x + 2y - 26 = 0$

$2g = -6$ $\quad g = -3$

$2f = 2$ $\quad f = 1$

$c = -26$

centre $= (-g, -f) = (3, -1)$

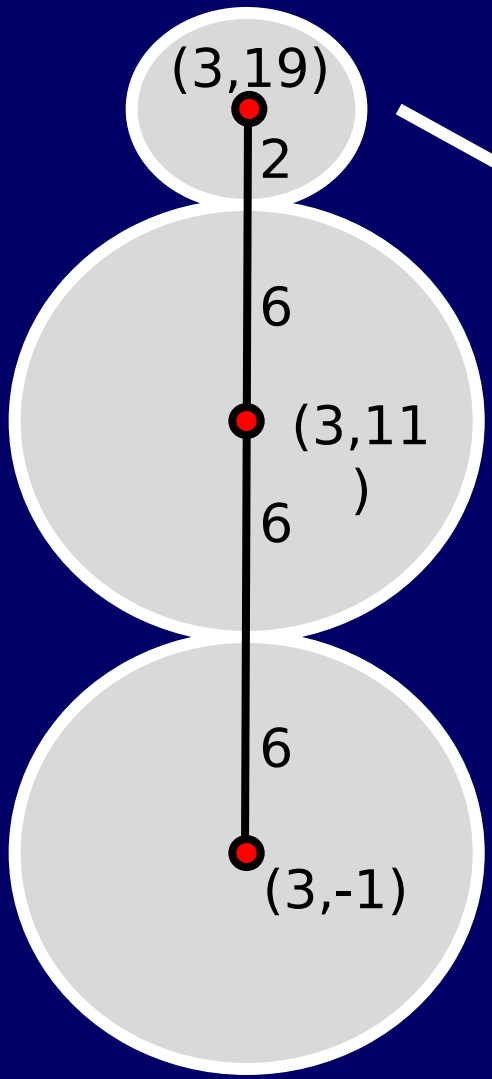
radius $= \sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(9 + 1 + 26)}$

$= \sqrt{36}$

$= 6$

Working with Distances



radius of head = 1/3 of 6 = 2

Using $(x-a)^2 + (y-b)^2 = r^2$

Equation is $(x-3)^2 + (y-19)^2 = 4$

Working with Distances

Example

By considering centres and radii prove that the following two circles touch each other.

Circle 1 $x^2 + y^2 + 4x - 2y - 5 = 0$

Circle 2 $x^2 + y^2 - 20x + 6y + 19 = 0$

Circle 1

$2g = 4$ so $g = 2$

$2f = -2$ so $f = -1$

$c = -5$

centre $= (-g, -f) = (-2, 1)$

radius $= \sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(4 + 1 + 5)}$

$= \sqrt{10}$

Circle 2

$2g = -20$ so $g = -10$

$2f = 6$ so $f = 3$

$c = 19$

centre $= (-g, -f) = (10, -3)$

radius $= \sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(100 + 9 - 19)}$

$= \sqrt{90}$

$= \sqrt{9} \times \sqrt{10} = 3\sqrt{10}$

Working with Distances

If d is the distance between the centres then

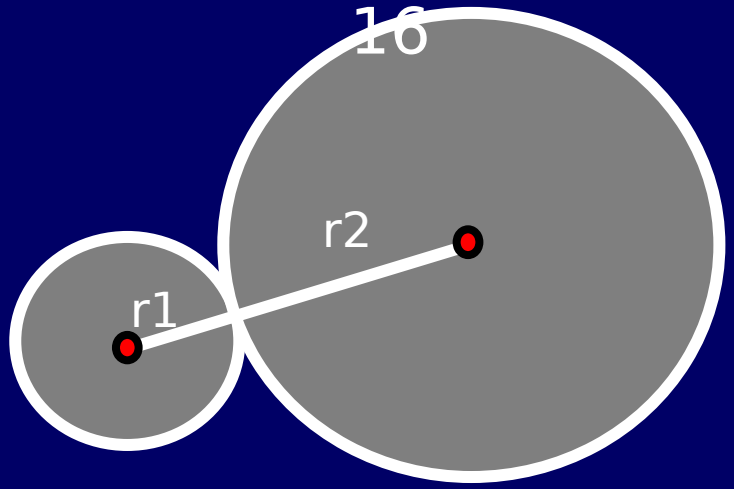
$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$ $= (10+2)^2 + (-3-1)^2$ $= 144 + 16$

$d = 160$

$\sqrt{160} = \sqrt{16} \times \sqrt{10} = 4\sqrt{10}$

radius1 + radius2 $= \sqrt{10} + 3\sqrt{10} = 4\sqrt{10}$

= distance between centres



It now follows that the circles touch !

Intersection of Lines & Circles

There are 3 possible scenarios



2 points of contact

discriminant $(b^2 - 4ac > 0)$

1 point of contact

discriminant $(b^2 - 4ac = 0)$

0 points of contact

discriminant $(b^2 - 4ac < 0)$

To determine where the line and circle meet we use simultaneous equations and the discriminant tells us how many solutions we have

Intersection of Lines & Circles

Example

Find where the line $y = 2x + 1$ meets the circle $(x - 4)^2 + (y + 1)^2 = 20$ and comment on the answer

Replace y by $2x + 1$ in the circle equation

$(x - 4)^2 + (y + 1)^2 = 20$

becomes $(x - 4)^2 + (2x + 1 + 1)^2 = 20$

$(x - 4)^2 + (2x + 2)^2 = 20$

$x^2 - 8x + 16 + 4x^2 + 8x + 4 = 20$

$5x^2 = 0$

$x^2 = 0$

$x = 0$ one solution tangent point

Using $y = 2x + 1$, if $x = 0$ then $y = 1$ Point of contact is (0,1)

Intersection of Lines & Circles

Example

Find where the line $y = 2x + 6$ meets the circle $x^2 + y^2 + 10x - 2y + 1 = 0$

Replace y by $2x + 6$ in the circle equation $x^2 + y^2 + 10x - 2y + 1 = 0$

becomes $x^2 + (2x + 6)^2 + 10x - 2(2x + 6) + 1 = 0$

$x^2 + 4x^2 + 24x + 36 + 10x - 4x - 12 + 1 = 0$

$5x^2 + 30x + 25 = 0$ ($\div 5$)

$x^2 + 6x + 5 = 0$

$(x + 5)(x + 1) = 0$

$x = -5$ or $x = -1$

Using $y = 2x + 6$

if $x = -5$ then $y = -4$

if $x = -1$ then $y = 4$

Points of contact are

(-5,-4) and (-1,4).

Tangency

Example

Prove that the line $2x + y = 19$ is a tangent to the circle $x^2 + y^2 - 6x + 4y - 32 = 0$, and also find the point of contact.

NAB

$2x + y = 19$ so $y = 19 - 2x$

Replace y by $(19 - 2x)$ in the circle equation.

$x^2 + y^2 - 6x + 4y - 32 = 0$

$x^2 + (19 - 2x)^2 - 6x + 4(19 - 2x) - 32 = 0$

$x^2 + 361 - 76x + 4x^2 - 6x + 76 - 8x - 32 = 0$

$5x^2 - 90x + 405 = 0$ $(\div 5)$

$x^2 - 18x + 81 = 0$

$(x - 9)(x - 9) = 0$

$x = 9$ only one solution hence tangent

Using $y = 19 - 2x$

If $x = 9$ then $y = 1$

Point of contact is $(9, 1)$

Using Discriminants

At the line $x^2 - 18x + 81 = 0$ we can also show there is only one solution by showing that the discriminant is zero.

For $x^2 - 18x + 81 = 0$, $a = 1$, $b = -18$ and $c = 9$

So $b^2 - 4ac$ $(-18)^2 - 4 \times 1 \times 81 = 364 - 364 =$

$= 0$

Since disc = 0 then equation has only one root so there is only one point of contact so line is a tangent.

The next example uses discriminants in a slightly different way.

Using Discriminants

Example

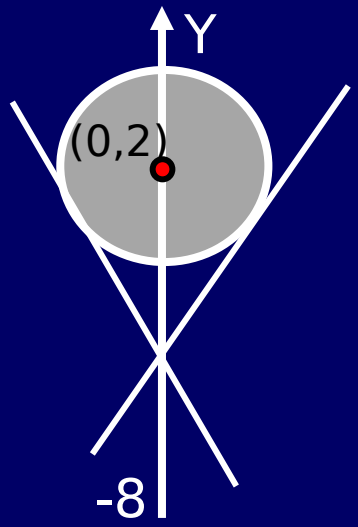
Find the equations of the tangents to the circle $x^2 + y^2 - 4y - 6 = 0$ from the point $(0,-8)$.

$x^2 + y^2 - 4y - 6 = 0$

$2g = 0$ so $g = 0$

$2f = -4$ so $f = -2$

Centre is $(0,2)$



Each tangent takes the form $y = mx - 8$

Replace y by $(mx - 8)$ in the circle equation to find where they meet. This gives us ...

$x^2 + y^2 - 4y - 6 = 0$

$x^2 + (mx - 8)^2 - 4(mx - 8) - 6 = 0$

$x^2 + m^2x^2 - 16mx + 64 - 4mx + 32 - 6 = 0$

$(m^2 + 1)x^2 - 20mx + 90 = 0$

In this quadratic $a = (m^2 +$ $b = -20m$ $c = 90$

Tangency

For tangency we need discriminate $b^2 - 4ac = 0$

$$(-20m)^2 - 4 \times (m^2 + 1) \times 90 = 0$$

$$400m^2 - 360m^2 - 360 = 0$$

$$40m^2 - 360 = 0$$

$$40m^2 = 360$$

$$m^2 = 9 \qquad m = -3 \text{ or } 3$$

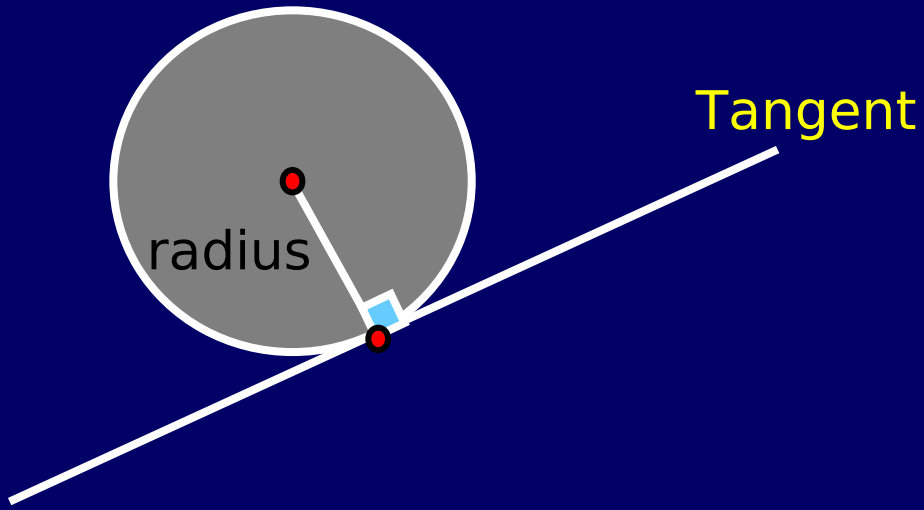
So the two tangents $y = -3x - 8$ and $y = 3x - 8$ are

and the gradients are reflected in the symmetry of the diagram.

Equations of Tangents

NB: At the point of contact

a tangent and radius/diameter are perpendicular.



This means we make use of $m_1 m_2 = -1$.

Equations of Tangents

Example

Prove that the point (-4,4) lies on the circle

$x^2 + y^2 - 12y + 16 = 0$

Find the equation of the tangent here.

NAB

At (-4,4) $x^2 + y^2 - 12y + = 16 + 16 - 48 + = 0$

16 16

So (-4,4) must lie on the circle.

$x^2 + y^2 - 12y + 16 = 0$

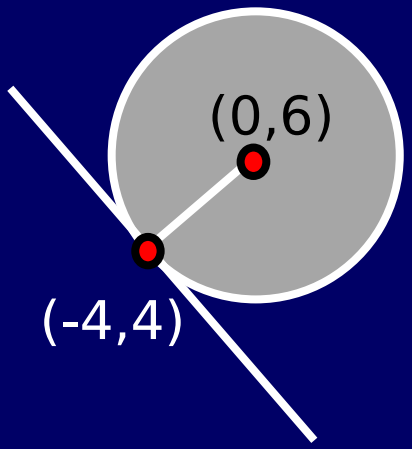
$2g = 0$ so $g = 0$

$2f = -12$ so $f = -6$

Centre is $(-g,-f) =$

(0,6)

Equations of Tangents



Gradient of radius $= \frac{y_2 - y_1}{x_2 -} = {}^{(6-4)}/_{(0+4)}$

$= x_1\ {}^{2}/_{4}$

$= {}^{1}/_{2}$

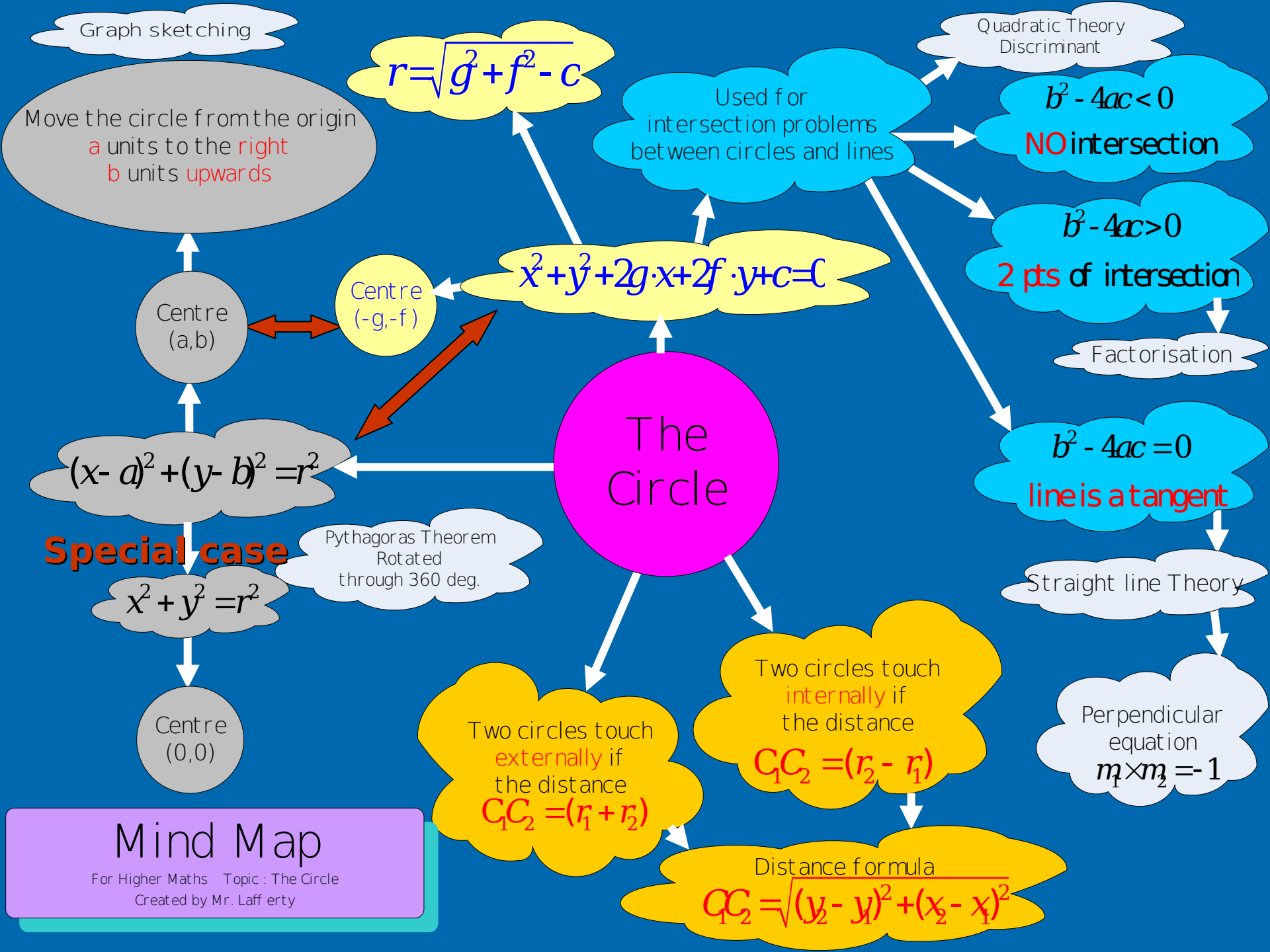
So gradient of tangent = - 2 ($m_1 m_2 = -1$)

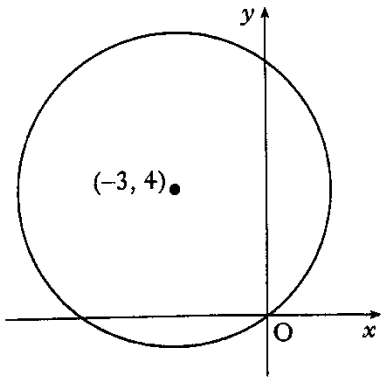
Using $y - b = m(x - a)$

We get $y - 4 = -2(x + 4)$

$y - 4 = -2x - 8$

$y = -2x - 4$


Graph sketching
$r=\sqrt{g^2+f^2-c}$
Quadratic Theory
Discriminant
Move the circle from the origin
a units to the right
b units upwards
Used for
intersection problems
between circles and lines
$b^2-4ac<0$
NO intersection
$b^2-4ac>0$
2 pts of intersection
Factorisation
$x^2+y^2+2g\cdot x+2f\cdot y+c=0$
Centre
(-g,-f)
Centre
(a,b)
The
Circle
$b^2-4ac=0$
line is a tangent
$(x-a)^2+(y-b)^2=r^2$
Special case
Pythagoras Theorem
Rotated
through 360 deg.
Straight line Theory
$x^2+y^2=r^2$
Centre
(0,0)
Two circles touch
externally if
the distance
$C_1C_2=(r_1+r_2)$
Two circles touch
internally if
the distance
$C_1C_2=(r_2-r_1)$
Perpendicular
equation
$m_1\times m_2=-1$
Distance formula
$C_1C_2=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$
Mind Map
For Higher Maths Topic : The Circle
Created by Mr. Lafferty



Find radius (distance formula): $r = 5$

You know the centre: $(-3, 4)$

Write down equation: $(x+3)^2 + (y-4)^2 = 25$ ✓

$$x^2 + y^2 + 2x + 3y + 5 = 0$$

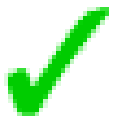
Consider the 2 conditions

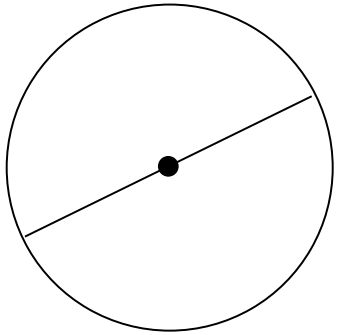
i.e. $g^2 + f^2 - c > 0$

Calculate g and f: $g = -1, \quad f = -\frac{3}{2}$

Evaluate $g^2 + f^2 - c$ $\quad (-1)^2 + \left(-\frac{3}{2}\right)^2 - 5 \quad \Rightarrow \quad 1 + 2\frac{1}{4} - 5 < 0$

Deduction: $g^2 + f^2 - c < 0$ *so* $\sqrt{g^2 + f^2 - c}$ *not real*





Make a sketch

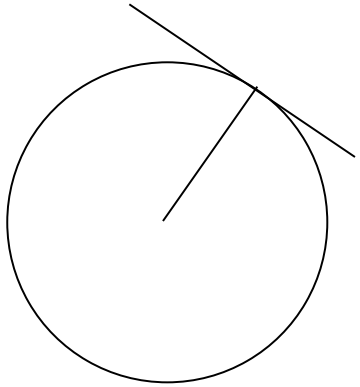
Calculate mid-point for centre: $(1, 2)$

Calculate radius CQ: $r = \sqrt{18}$

Write down equation; $(x - 1)^2 + (y - 2)^2 = 18$

$$x^2 + y^2 + 2x - 4y - 15 = 0$$

Calculate centre of circle: $(-1, 2)$



Make a sketch

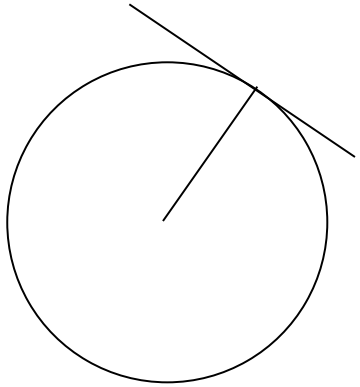
Calculate gradient of OP (radius to tangent) $m = \frac{1}{2}$

Gradient of tangent: $m = -2$

Equation of tangent: $y + 2x = 10$ ✓

$$(x+1)^2+(y-1)^2=13$$

Find centre of circle: $(-1, 1)$



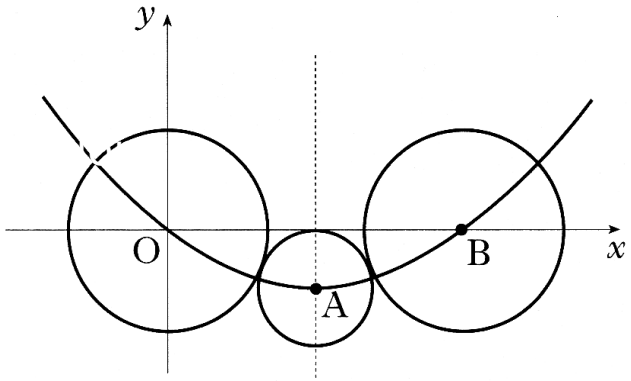
Make a sketch

Calculate gradient of radius to tangent $m=\frac{2}{3}$

Gradient of tangent: $m=-\frac{3}{2}$

Equation of tangent: $2y+3x=12$ ✓

$(x-12)^2+(y+5)^2=25$



$y=px(x+q)$

ii) Hence find the coordinates of centre B.

A is centre of small circle

$A(12,-5)$

Find OA (Distance formula)

13

Use symmetry, find B

$B(24,0)$

Find radius of circle A from eqn.

5

Find radius of circle B

$13-5=8$

Eqn. of B

$(x-24)^2+y^2=64$

Points O, A, B lie on parabola – subst. A and B in turn

$0=24p(24+q)$

$-5=12p(12+q)$

Solve:

$p=\frac{5}{144},\ q=-24$ ✓

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$x^2 + y^2 - 8x - 10y + 9 = 0$

$a \pm b\sqrt{3}$

ordinates of the poi

Find centre of circle P: $(4, 5)$

Find radius of circle :P: $\sqrt{4^2 + 5^2 - 9} = \sqrt{32} = 4\sqrt{2}$

Find distance between centres: $\sqrt{72} = 6\sqrt{2}$

Deductions:

Gradient of radius of Q to tangent: $m = -1$

Gradient tangent at Q: $m = 1$

Equation of tangent: $y = x + 5$

Solve eqns. simultaneously: $x^2 + y^2 - 8x - 10y + 9 = 0$

$y = x + 5$

Soln: $2 \pm 2\sqrt{3}$ ✓



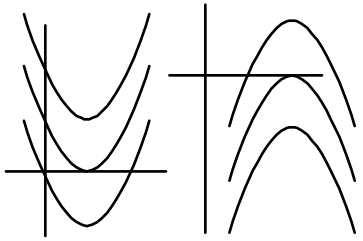
$$x^2 + y^2 + 4kx - 2ky - k - 2 = 0$$

Determine g, f and c: $g = -2k, \quad f = k, \quad c = -k - 2$

State condition $g^2 + f^2 - c > 0$ Put in values $(-2k)^2 + k^2 - (-k - 2) > 0$

Simplify $5k^2 + k + 2 > 0$

Need to see the position of the parabola



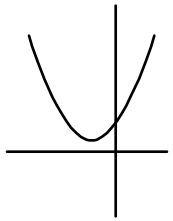
Complete the square

$$5\left(k^2 + \frac{1}{5}k\right) + 2$$

$$5\left[\left(k + \frac{1}{10}\right)^2 - \frac{1}{100}\right] + 2$$

$$5\left(k + \frac{1}{10}\right)^2 + \frac{195}{100}$$

$\frac{195}{100}$ *when* $k = -\frac{1}{10}$



$$x^2 + y^2 - 6x + 4y + c = 0$$

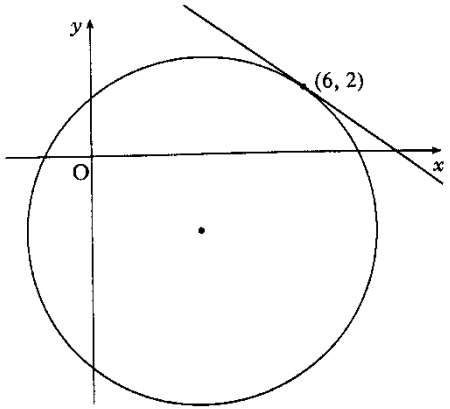
Determine g, f and c: $g = 3, \quad f = -2, \quad c = ?$

State condition $g^2 + f^2 - c > 0$ Put in values $3^2 + (-2)^2 - c > 0$

Simplify $9 + 4 - c > 0$

Re-arrange: $c < 13$ ✓

$$(x-3)^2+(y+2)^2=25$$



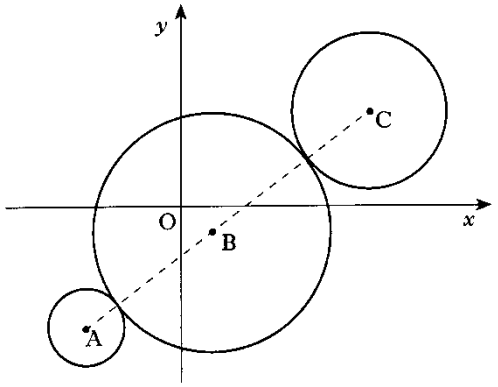
Calculate centre of circle: $(3, -2)$

Calculate gradient of radius (to tangent) $m=\frac{4}{3}$

Gradient of tangent: $m=-\frac{3}{4}$

Equation of tangent: $4y+3x=26$ ✓

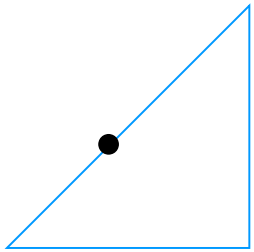
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$(x+12)^2+(y+15)^2=25$ and $(x-24)^2+(y-12)^2=100$

Find centre and radius of Circle A $(-12,-15)$ $r=5$

Find centre and radius of Circle C $(24,12)$ $r=10$



Find distance AB (distance formula) $\sqrt{36^2+27^2}=45$

Find diameter of circle B $45-(5+10)=30$ so radius of B $=15$

Use proportion to find B $\frac{25}{45}\times 27=15,\quad \frac{25}{45}\times 36=20$ relative to C

Centre of B $(4,-3)$ Equation of B $(x-4)^2+(y+3)^2=225$

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