The Binomial Expansion

The Binomial Expansion

Powers of $a + b$

In this presentation we will develop a formula to enable us to find the terms of the expansion of

$$(a+b)^n$$

where n is any positive integer.

We call the expansion binomial as the original expression has 2 parts.



The Binomial Expansion

Powers of $a + b$

We know that

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + 2ab + b^2$$

We can write this as

$$= 1a^2 + 2ab + 1b^2$$

so the coefficients of the terms are 1, 2 and 1

The Binomial Expansion

Powers of $a + b$

$$(a+b)^3 = (a+b)(a+b)^2$$

$$=(a+b)(a^2+2ab+b^2)$$

$$=1a^3+2a^2b+1ab^2$$



The Binomial Expansion

Powers of $a + b$

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$$1a^3+3a^2b+3ab^2+1b^3$$

The Binomial Expansion

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$$1a^3+3a^2b+3ab^2+1b^3$$

so the **coefficients** of the expansion of $(a+b)^3$ are 1, 3, 3 and 1

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$$(a+b)^4 = (a+b)(a+b)^3$$

$$=(a+b)(a^3+3a^2b+3ab^2+b^3)$$

$$=1a^4+3a^3b+3a^2b^2+1ab^3$$

$$+\quad\quad +1a^3b+3a^2b^2+3ab^3+1b^4$$

$$1a^4+4a^3b+6a^2b^2+4ab^3+1b^4$$



The Binomial Expansion

Powers of $a + b$

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$$1a^4+4a^3b+6a^2b^2+4ab^3+1b^4$$

This coefficient . . .

. . . is found by adding 3 and 1; the coefficients that are in $(a+b)^3$



The Binomial Expansion

Powers of $a + b$

$$(a+b)^4 = (a+b)(a+b)^3$$

$$=(a+b)(a^3+3a^2b+3ab^2+b^3)$$

$$=1a^4+3a^3b+3a^2b^2+1ab^3$$

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The Binomial Expansion

Powers of $a + b$

So, we now have

Expression	Coefficients
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1



The Binomial Expansion

Powers of $a + b$

So, we now have

Expression	Coefficients
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1



Each number in a row can be found by adding the 2 coefficients above it.



The Binomial Expansion

Powers of $a + b$

So, we now have

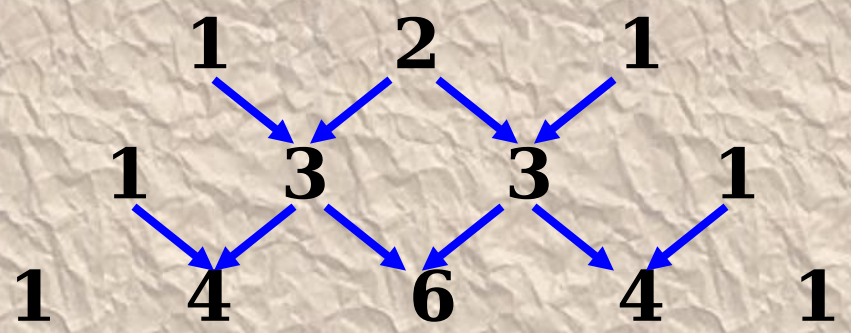
Expression

$(a+b)^2$

$(a+b)^3$

$(a+b)^4$

Coefficients



Each number in a row can be found by adding the 2 coefficients above it.

The 1st and last numbers are always 1.

The Binomial Expansion

Powers of $a + b$

So, we now have

Expression	Coefficients
$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1



To make a triangle of coefficients, we can fill in the obvious ones at the top.

The Binomial Expansion

Powers of $a + b$

The triangle of binomial coefficients is called Pascal's triangle, after the French mathematician.

Notice that the 4^{th} row gives the coefficients of

$$(a+b)^3$$

. . . but it's easy to know which row we want as, for example,

$(a+b)^3$ starts with 1 3 .
. .

$(a+b)^{10}$ will start 1 10 . . .



The Binomial Expansion



Exercise

Find the coefficients in the expansion of $(a+b)^6$

Solution: We need 7 rows

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Coefficients →

The Binomial Expansion

Powers of $a + b$

We usually want to know the complete expansion not just the coefficients.

e.g. Find the expansion of $(a+b)^5$

Solution: Pascal's triangle gives the coefficients

1 5 $\binom{1}{0}$ $\binom{1}{0}$ 5 1

The full expansion is

$$1a^5 + 5a^4b^1 + \binom{1}{0} a^3b^2 + \binom{1}{0} a^2b^3 + 5ab^4 + 1b^5$$

Tip: The powers in each term sum to 5

The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1 - x)^4$ in **ascending** powers of x.

Solution: The coefficients are

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

(Ascending powers just means that the 1st term must have the lowest power of x and then the powers must increase.)

We know that

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

To get we need to replace a by 1

The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1 - x)^4$ in **ascending** powers of x.

Solution:

The coefficients are

1 4 6 4 1

We know that

$$(1 + b)^4 = 1^4 + 4(1)^3 b + 6(1)^2 b^2 + 4(1) b^3 + b^4$$

To get $(1 - x)^4$ we need to replace a by 1



The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1-x)^4$ in **ascending** powers of x.

Solution:

The coefficients are

1 4 6 4 1

We know that $(1+(-x))^4 = 1^4 + 4(1)^3(-x) + 6(1)^2(-x)^2 + 4(1)(-x)^3 + (-x)^4$

To get $(1-x)^4$ we need to replace a by 1 and b by $(-x)$

Simplifying gives

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

This is packed with information. Brackets are vital, otherwise the signs will be wrong!

Be careful! The minus sign is squared as well as the x.

The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1-x)^4$ in **ascending** powers of x.

Solution:

The coefficients are

1 4 6 4 1

We know that

$$(1+(-x))^4 = 1 + 4(1)^3(-x) + 6(1)^2(-x)^2 + 4(1)(-x)^3 + (-x)^4$$

Since we know that any power of 1 equals 1, we could have written 1 here . . .

Simplifying gives

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

The Binomial Expansion

Powers of $a + b$

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The Binomial Expansion

Powers of $a + b$

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Solution:

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1 4 6 4 1

We know that

$$(1+(-x))^4 = 1 + 4(1)^3(-x) + 6(1)^2(-x)^2 + 4(1)(-x)^3 + (-x)^4$$

To get [illegible] we need to replace [illegible] by $-x$

. . . and missed these 1s out.

Simplifying gives

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1-x)^4$ in **ascending** powers of x.

Solution: The coefficients are

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

We could go straight to

$$(1+(-x))^4 = 1 + 4(-x) + 6(-x)^2 + 4(-x)^3 + (-x)^4$$

Simplifying gives

$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$



The Binomial Expansion



Exercise

1. Find the expansion of $(1+2x)^5$ in ascending powers of x.

Solution: The coefficients are

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

So, $(1+2x)^5 =$

$$1+5(2x)+10(2x)^2+10(2x)^3+5(2x)^4+(2x)^5$$

$$=1+10x+40x^2+80x^3+80x^4+32x^5$$

The Binomial Expansion

Powers of $a + b$

If we want the first few terms of the expansion of, for example, $(a+b)^{20}$, Pascal's triangle is not helpful.

We will now develop a method of getting the coefficients without needing the triangle.



The Binomial Expansion

Powers of $a + b$

Let's consider $(a+b)^6$

We know from Pascal's triangle that the coefficients are

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

There is a pattern here:

$$\times\frac{6}{1} \quad \times\frac{5}{2} \quad \times\frac{4}{3} \quad \times\frac{3}{4} \quad \times\frac{2}{5} \quad \times\frac{1}{6}$$

Each coefficient can be found by multiplying the previous one by a fraction. The fractions form an easy sequence to spot.

So if we want the 4th coefficient without finding the others, we would need

$$\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}$$

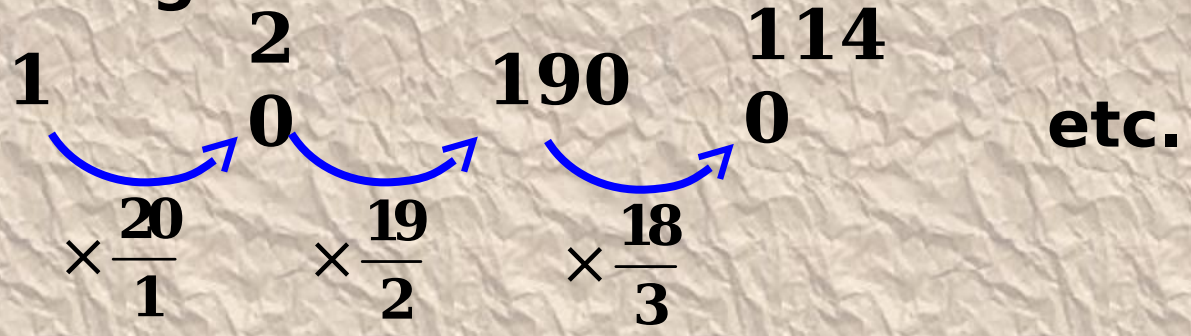
(3 fractions)



The Binomial Expansion

Powers of $a + b$

For $(a+b)^{20}$ we get



The 9th coefficient of $(a+b)^{20}$ is

$$\frac{20\times19\times18\times17\times16\times15\times14\times13}{1\times2\times3\times4\times5\times6\times7\times8}$$

Even using a calculator, this is tedious to simplify. However, there is a shorthand notation that is available as a function on the calculator.

The Binomial Expansion

Powers of $a + b$

$20 \times 19 \times 18 \times \ldots 3 \times 2 \times 1$ is called 20 factorial.

We write $20!$

(20 followed by an exclamation mark)

We can write

$$\frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$= \frac{20 \times 19 \times 18 \times \ldots 14 \times 13 \times 12 \times \ldots 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 12 \times \ldots 3 \times 2 \times 1}$$

$$= \frac{20!}{8!\,12!}$$

The 9^{th} term of $(a+b)^{20}$ is $\frac{20!}{8!\,12!} a^{12} b^{8}$

The Binomial Expansion

Powers of $a + b$

$\frac{20!}{8!\,12!}$ can also be written as ${}^{20}C_8$ or $\binom{20}{8}$

The 9th term of $(a+b)^{20}$ is then ${}^{20}C_8 a^{12} b^8$

This notation. . .

. . . gives the number of ways that 8 items can be chosen from 20.

In the expansion, we are choosing the letter b 8 times from the 20 sets of brackets that make up $(a+b)^{20}$. (a is chosen 12 times).

${}^{20}C_8$ is read as “20 C 8” or “20 choose 8” and can be evaluated on our calculators.



The Binomial Expansion

Powers of $a + b$

The binomial expansion of $(a+b)^{20}$ is

$$(a+b)^{20} = a^{20} + 20a^{19}b + {}^{20}C_2\, a^{18}b^2 + {}^{20}C_3\, a^{17}b^3 + \ldots + b^{20}$$

We know from Pascal's triangle that the 1^{st} two coefficients are 1 and 20, but, to complete the pattern, we can write these using the C notation:

$$1 = {}^{20}C_0 \quad \text{and} \quad 20 = {}^{20}C_1$$

Since ${}^{20}C_0 = \dfrac{20!}{0!\,20!} = 1$ we must define $0!$ as equal to 1.



The Binomial Expansion

Powers of $a + b$

Any term of $(a+b)^{20}$ can then be written as

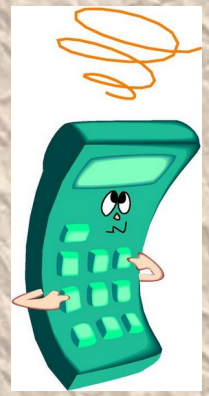
$$^{20}C_r a^{20-r} b^r$$

where r is any integer from 0 to 20.

Tip: When finding binomial expansions, it can be useful to notice the following:

$$\frac{20!}{8!\,12!} = \frac{20!}{12!\,8!}$$

So, $^{20}C_8$ is equal to $^{20}C_{12}$



The Binomial Expansion

Generalizations

- The binomial expansion of $(a+b)^n$ in ascending powers of x is given by

$$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + \ldots + b^n$$

- Any term of $(a+b)^n$ can be written in the form

$${}^nC_r a^{n-r}b^r$$

where r is any integer from 0 to n.



- The expansion of $(1+x)^n$ is

$$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots + x^n$$

The Binomial Expansion

Powers of $a + b$

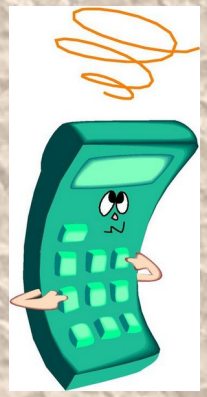
e first 4 terms in the expansion of $(1- x)^{18}$ in ascending powers of x.

Solutio

n

$$(1- x)^{18} \equiv {}^{18}C_0 + {}^{18}C_1(- x) + {}^{18}C_2(- x)^2 + {}^{18}C_3(- x)^3 + \ldots$$

$$= 1 - 18x + 153x^2 - 816x^3 + \ldots$$





The Binomial Expansion

Powers of $a + b$

e.g.4 Find the 5th term of the expansion of $(2+x)^{12}$ in ascending powers of x.

Solution: The 5th term contains x^4

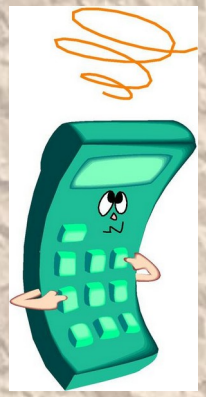
It is

$$^{12}C_4(2)^8x^4$$

These numbers will always be the same.

$$= 495(2)^8x^4$$

$$= 126720x^4$$



The Binomial Expansion

SUMMARY

- The binomial expansion of $(a+b)^n$ in ascending powers of x is given by

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots + b^n$$

- The $(r+1)^{th}$ term is ${}^nC_r\, a^{n-r}b^r$

- The expansion of $(1+x)^n$ is

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + x^n$$

The Binomial Expansion



Exercise

1. Find the 1st 4 terms of the expansion of $(2+3x)^8$ in ascending powers of x.

Solution:

$$^8C_0 2^8 + {}^8C_1 2^7(3x) + {}^8C_2 2^6(3x)^2 + {}^8C_3 2^5(3x)^3$$

$$= 256 + 3072x + 16128x^2 + 48384x^3$$

---

2. Find the 6th term of the expansion of $(1-x)^{13}$ in ascending powers of x.

Solution:

$$^{13}C_5(-x)^5$$

$$= -1287x^5$$

The Binomial Expansion



The Binomial Expansion

The following slides contain repeats of information on earlier slides, shown without colour, so that they can be printed and photocopied.

For most purposes the slides can be printed as “Handouts” with up to 6 slides per sheet.



The Binomial Expansion

Powers of $a + b$

Pascal's Triangle

Expression	Coefficients
$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1

The Binomial Expansion

Powers of a + b

We usually want to know the complete expansion not just the coefficients.

e.g. Find the expansion of $(a+b)^5$

Solution: **Pascal's triangle gives the coefficients**

$$1 \quad 5 \quad \binom{1}{0} \quad \binom{1}{0} \quad 5 \quad 1$$

The full expansion is

$$1a^5 + 5a^4b^1 + \binom{1}{0}a^3b^2 + \binom{1}{0}a^2b^3 + 5ab^4 + 1b^5$$

Tip: The powers in each term sum to 5

The Binomial Expansion

Powers of $a + b$

e.g. 2 Write out the expansion of $(1 - x)^4$ in ascending powers of x.

Solution: The coefficients are

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So,

$$(1 + (-x))^4 = 1 + 4(-x) + 6(-x)^2 + 4(-x)^3 + (-x)^4$$

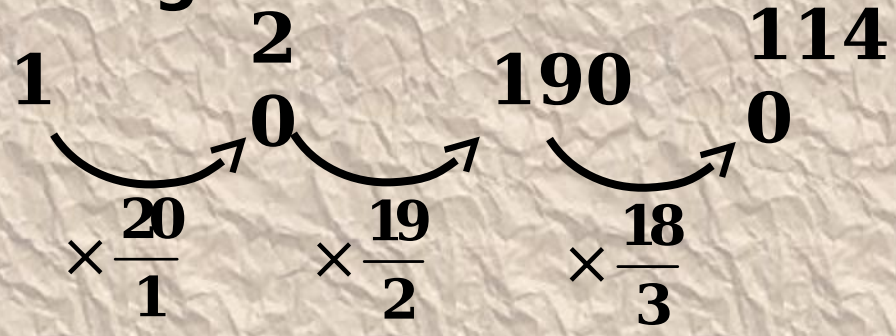
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$$(1 - x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

The Binomial Expansion

Powers of $a + b$

For $(a+b)^{20}$ we get



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The Binomial Expansion

Powers of $a + b$

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The Binomial Expansion

Powers of a

$+\ b$

$\frac{20!}{8!\,12!}$ **can also be written as** ${}^{20}C_8$ **or** $\binom{20}{8}$

The 9th term of $(a+b)^{20}$ **is then** ${}^{20}C_8 a^{12} b^8$

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The Binomial Expansion

Powers of a + b

Any term of $(a+b)^{20}$ can then be written as

$$^{20}C_r a^{20-r} b^r$$

where r is any integer from 0 to 20.

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The Binomial Expansion

Powers of $a \pm b$

e.g. 3 Find the first 4 terms in the expansion of $(1-x)^{18}$ in ascending powers of x.

Solution

$$(1-x)^{18} \equiv {}^{18}C_0 + {}^{18}C_1(-x) + {}^{18}C_2(-x)^2 + {}^{18}C_3(-x)^3 + \ldots$$

$$= 1 - 18x + 153x^2 - 816x^3 + \ldots$$

The Binomial Expansion SUMMARY

- The binomial expansion of $(a+b)^n$ in ascending powers of x is given by

$$(a+b)^n =$$

$$^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + \ldots + b^n$$

- The $(r+1)^{th}$ term is $^nC_r\,a^{n-r}b^r$

- The expansion of $(1+x)^n$ is

$$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots + x^n$$