

Logarithms 2



Indices

- **Definition** - Any expression written as $\mathbf{a^n}$ is defined as the variable $\mathbf{a}$ raised to the power of the number $\mathbf{n}$
- $\mathbf{n}$ is called a power, an index or an exponent of $\mathbf{a}$

- **Example** - where n is a positive whole number,

$$a^1 = a$$

$$a^2 = a \times a$$

$$a^3 = a \times a \times a$$

$$a^n = a \times a \times a \times a \dots n \text{ times}$$

Indices satisfy the following rules:

1) where n is *positive whole* number

$$a^n = a \times a \times a \times a \dots n \text{ times}$$

- e.g. $2^3 = 2 \times 2 \times 2 = 8$

2) Negative powers.....

$$a^{-n} = \frac{1}{a^n}$$

e.g. $a^{-2} = \frac{1}{a^2}$

- e.g. where $a = 2$
- $2^{-1} = \frac{1}{2}$ **or** $2^{-2} = \frac{1}{2 \times 2} = \frac{1}{4}$

- **3)** A Zero power

$$\mathbf{a^0 = 1}$$

$$\text{e.g. } 8^0 = 1$$

- **4)** A Fractional power

$$\mathbf{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

$$\text{e.g. } 9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt{9} = 3$$

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

All indices satisfy the following rules in mathematical applications

Rule 1 $a^m \cdot a^n = a^{m+n}$

e.g. $2^2 \cdot 2^3 = 2^5 = 32$

e.g. $5^1 \cdot 5^1 = 5^2 = 25$

e.g. $5^1 \cdot 5^0 = 5^1 = 5$

Rule 2 $\frac{a^m}{a^n} = a^{m-n}$

e.g.1 $\frac{2^3}{2^2} = 2^{3-2} = 2^1 = 2$

e.g.2 $\frac{2^3}{2^0} = 2^{3-0} = 2^3 = 8$

Rule 2 notes...

$$\textbf{e.g.} \quad \frac{2^3}{2^3} = 2^{3-3} = 2^0 = 1$$

$$\textbf{e.g.} \quad \frac{2^3}{2^{-2}} = 2^{3-(-2)} = 2^5 = 32$$

$$\textbf{e.g.} \quad \frac{2^{-3}}{2^2} = 2^{-3-2} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

Rule 3

$$(a^m)^n = a^{mn}$$

e.g. $(2^3)^2 = 2^6 = 64$

Rule 4

$$a^n \cdot b^n = (ab)^n$$

e.g. $3^2 \times 4^2 = (3 \times 4)^2 = 12^2 = 144$

Likewise,

$$\frac{a^n}{b^n} = \left(\frac{a}{b} \right)^n \quad \text{if } b \neq 0$$

e.g.

$$\frac{6^2}{3^2} = \left(\frac{6}{3} \right)^2 = 2^2 = 4$$

Simplify the following using the above Rules:

1) $b = x^{1/4} \times x^{3/4}$

2) $b = x^2 \div x^{3/2}$

3) $b = (x^{3/4})^8$

4) $b = \frac{x^2 y^3}{x^4 y}$

These are practice questions for you to try at home!

Logarithms

A Logarithm is a mirror image of an index

If $m = b^n$ then $\log_b m = n$

The log of m to base b is n

If $y = x^n$ then $n = \log_x y$

The log of y to the base x is n

e.g.

$1000 = 10^3$ then $3 = \log_{10} 1000$

$0.01 = 10^{-2}$ then $-2 = \log_{10} 0.01$

Evaluate the following:

1) $x = \log_3 9$

the log of m to base b = n then $m = b^n$

the log of 9 to base 3 = x then

$$\Rightarrow 9 = 3^x$$

$$\Rightarrow 9 = 3 \times 3 = 3^2$$

$$\Rightarrow x = 2$$

2) $x = \log_4 2$

the log of m to base b = n then $m = b^n$

the log of 2 to base 4 = x then

$$\Rightarrow 2 = 4^x$$

$$\Rightarrow 2 = \sqrt{4} = 4^{1/2}$$

$$x = 1/2$$

The following rules of logs apply

1) $\log_b(x \times y) = \log_b x + \log_b y$

eg. $\log_{10}(2 \times 3) = \log_{10} 2 + \log_{10} 3$

2) $\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$

eg. $\log_{10} \left(\frac{3}{2} \right) = \log_{10} 3 - \log_{10} 2$

3) $\log_b x^m = m \cdot \log_b x$

e.g. $\log_{10} 3^2 = 2 \log_{10} 3$

From the above rules, it follows that

(1) $\log_b 1 = 0$
(since $\Rightarrow 1 = b^0$.)

e.g. $\log_{10} 1 = 0$

and therefore,

$$\log_b \left(\frac{1}{x} \right) = -\log_b x$$

e.g. $\log_{10} (1/3) = -\log_{10} 3$

And.....

$$(2) \quad \log_b b = 1$$

(since $\Rightarrow b = b^1$)

e.g. $\log_{10} 10 = 1$

$$(3) \quad \log_b \left(\sqrt[n]{x} \right) = \frac{1}{n} \log_b x$$

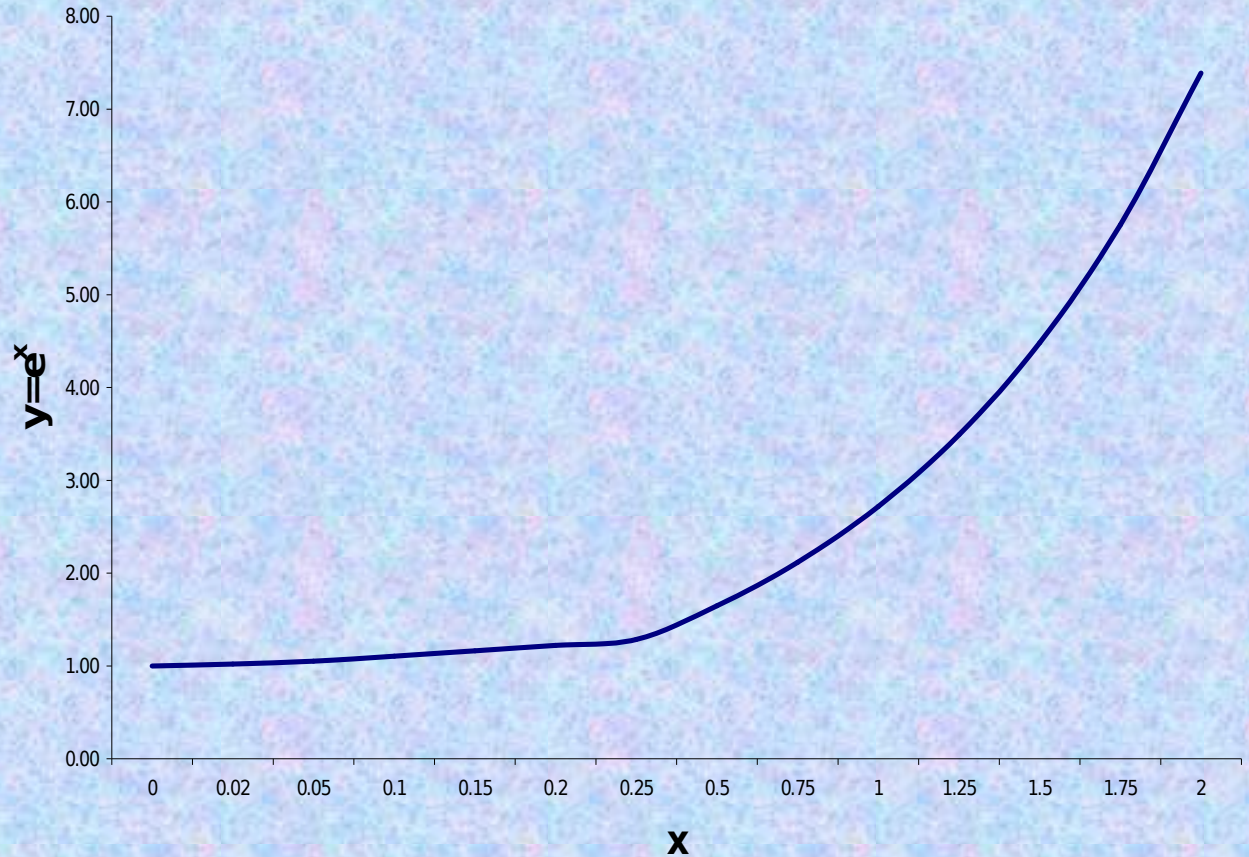
A Note of Caution:

- All logs must be to the same base in applying the rules and solving for values
- The most common base for logarithms are logs to the base 10, or logs to the base **e** ($e = 2.718281\dots$)
- Logs to the base **e** are called Natural Logarithms
- $\log_e x = \ln x$
- **If** $y = \exp(x) = e^x$
then $\log_e y = x$ or $\ln y = x$

Features of $y = e^x$

non-linear
always
positive

as $\uparrow x$ get
 $\uparrow y$ and
 $\uparrow$ slope
of graph
(gets
steeper)



Logs can be used to solve algebraic equations where the unknown variable

appears as a power

An Example : Find the value of x

$$(4)^x = 64$$

1) rewrite equation so that it is no longer a power

- Take logs of both sides
 $\log(4)^x = \log(64)$
- rule 3 $\Rightarrow x \cdot \log(4) = \log(64)$

2) Solve for x

- $x = \frac{\log 64}{\log 4}$

Does not matter what base we evaluate the logs, providing the same base is applied both to the top and bottom of the equation

3) Find the value of x by evaluating logs using (for example) base 10

- $x = \frac{1.8062}{0.6021} = 3$

Check the solution

- $(4)^3 = 64$

Logs can be used to solve algebraic equations where the unknown variable appears as a power

An Example : Find the value of x
 $200(1.1)^x = 20000$

Simplify

- divide across by 200
 $(1.1)^x = 100$

to find x, rewrite equation so that it is no longer a power

- Take logs of both sides
 $\log(1.1)^x = \log(100)$
- rule 3 $\Rightarrow x \cdot \log(1.1) = \log(100)$

Solve for x

- $x = \frac{\log(100)}{\log(1.1)}$

no matter what base we evaluate the logs, providing the same base is applied both to the top and bottom of the equation

Find the value of x by evaluating logs using (for example) base 10

- $x = \frac{2}{0.0414} = 48.32$

Check the solution

- $200(1.1)^x = 20000$
- $200(1.1)^{48.32} = 20004$

Another Example:

Find the value of x

$$5^x = 2(3)^x$$

1. rewrite equation so x is not a power

- Take logs of both sides
 $\log(5^x) = \log(2 \times 3^x)$
- rule 1 $\Rightarrow \log 5^x = \log 2 + \log 3^x$
- rule 3 $\Rightarrow x \cdot \log 5 = \log 2 + x \cdot \log 3$

» Cont.....

2. **Solve for x**

$$x [\log 5 - \log 3] = \log 2$$

$$\text{rule 2} \Rightarrow x \left[\log \left(\frac{5}{3} \right) \right] = \log 2$$

$$x = \frac{\log(2)}{\log\left(\frac{5}{3}\right)}$$

3. **Find the value of x by evaluating logs using (for example) base 10**

$$x = \frac{\log(2)}{\log\left(\frac{5}{3}\right)} = \frac{0.30103}{0.2219} = 1.36$$

4. **Check the solution**

$$5^x = 2(3)^x \Rightarrow 5^{1.36} = 2(3)^{1.36} \Rightarrow 8.92$$

An Economics Example 1

$$Y = f(K, L) = A K^{\alpha} L^{\beta}$$

$$Y^* = f(\lambda K, \lambda L) = A (\lambda K)^{\alpha} (\lambda L)^{\beta}$$

$$Y^* = A K^{\alpha} L^{\beta} \lambda^{\alpha} \lambda^{\beta} = Y \lambda^{\alpha + \beta}$$

$\alpha + \beta = 1$ **Constant Returns to Scale**

$\alpha + \beta > 1$ **Increasing Returns to Scale**

$\alpha + \beta < 1$ **Decreasing Returns to Scale**

Homogeneous of Degree r if:

$$f(\lambda X, \lambda Z) = \lambda^r f(X, Z) = \lambda^r Y$$

Homogenous function if by scaling all variables by λ , can write Y in terms of λ^r

An Economics Example 2

Y = National Income = £30,000 mill in 1964. It grows at 4% p.a. In 1984, Y = ?

$$1964: Y = 3$$

$$1965: Y = 3(1.04)$$

$$1966: Y = 3(1.04)^2$$

$$1984: Y = 3(1.04)^{20}$$

$$1984: \log Y = \log\{3 \times (1.04)^{20}\}$$

$$\log Y = \log 3 + \log\{(1.04)^{20}\}$$

$$\log Y = \log 3 + 20 \cdot \log(1.04)$$

Evaluate to the base 10

$$\log Y = 0.47712 + 20(0.01703)$$

$$\log Y = 0.817788$$

Find the anti-log of the solution:

$$Y = 6.5733$$

In 1984, Y = £65733 mill

Good Learning Strategy!

- Up to students to revise and practice the rules of indices and logs using examples from textbooks.
- These rules are very important for remaining topics in the course.

