

CHAPTER

3

DIFFERENTIAL CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES

- Notations, examples and graphs
- Limit and continuity
- partial Derivatives
- Directional Derivatives and Gradient
- Total Differential and tangent plane approximation
- Chain Rule and Implicit Differentiation
- Relative Extreme of functions of two variables
- Largest and smallest values of function on a given set
- Extreme values under constraint

FUNCTIONS OF TWO VARIABLES

A real-valued **function of two variables** f , consists of

1. A set A of **ordered pairs** of real numbers (x, y) called the **domain** of the function.
2. A **rule** that associates with each ordered pair in the domain of f **one and only one** real number, denoted by $z = f(x, y)$.

$$\text{i.e } \{f(x,y): (x, y) \in D\}$$

EXAMPLE 1

- Let f be the function defined by

$$f(x, y) = x^2 + xy + y^2$$

- Compute $f(0, 0)$, $f(1, 2)$, and $f(2, 1)$.

Solution

$$f(0, 0) = 0^2 + (0)(0) + 0^2 = 0$$

$$f(1, 2) = 1^2 + (1)(2) + 2^2 = 9$$

$$f(2, 1) = 2^2 + (2)(1) + 1^2 = 7$$

- The **domain** of a function of **two variables** $f(x, y)$, is a set of **ordered pairs** of real numbers and may therefore be viewed as a **subset of the xy -plane**.

EXAMPLE 2

- Find the **domain** of the function

$$f(x, y) = x^2 + y^2$$

Solution

- $f(x, y)$ is defined for all real values of x and y , so the **domain** of the function f is the set of all points (x, y) in the xy -plane.

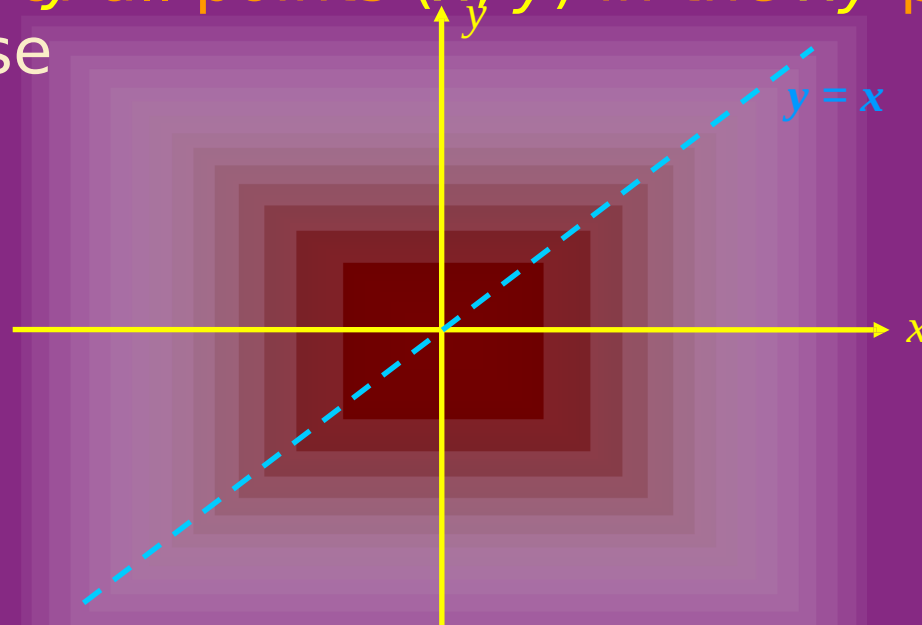
EXAMPLE 3

Find the **domain** of the function $g(x, y) = \frac{2}{x - y}$

Solutio

$g(x, y)$ is defined for all $x \neq y$, so the **domain** of the function

is the set of all points (x, y) in the xy -plane **except** those



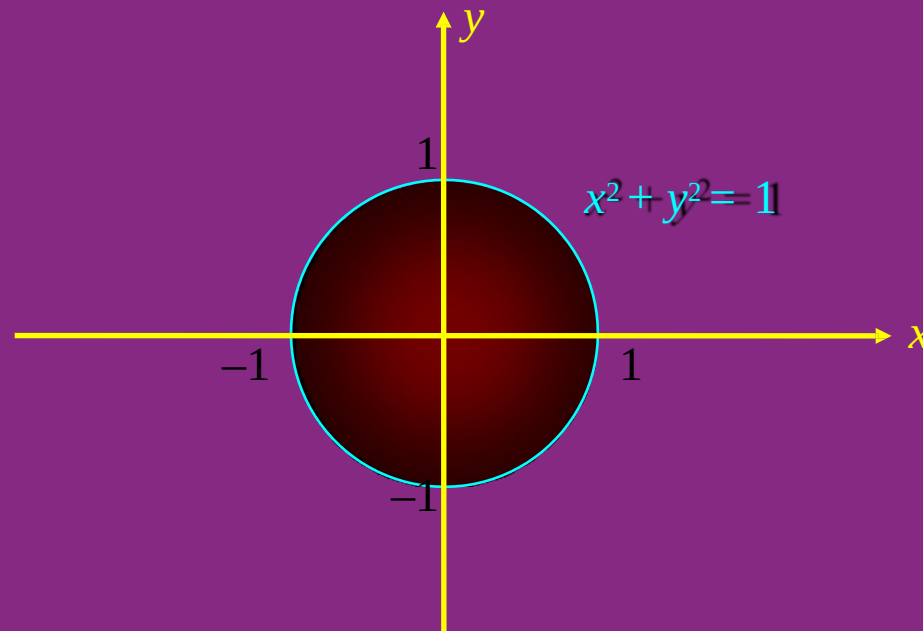
EXAMPLE 4

- Find the **domain** of the function

$$h(x, y) = \sqrt{1 - x^2 - y^2}$$

Solution

- We require that $1 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 1$ which is the **set of all points** (x, y) lying **on and inside** the **circle** of **radius 1** with **center** at the **origin**:



EXERCISE

1. Find $g(x, y) = \sqrt{9 - x^2 - y^2}$

2. For each of the following functions, evaluate $f(3, 2)$ and find the domain.

a. $f(x, y) = \frac{\sqrt{x - y}}{1}$

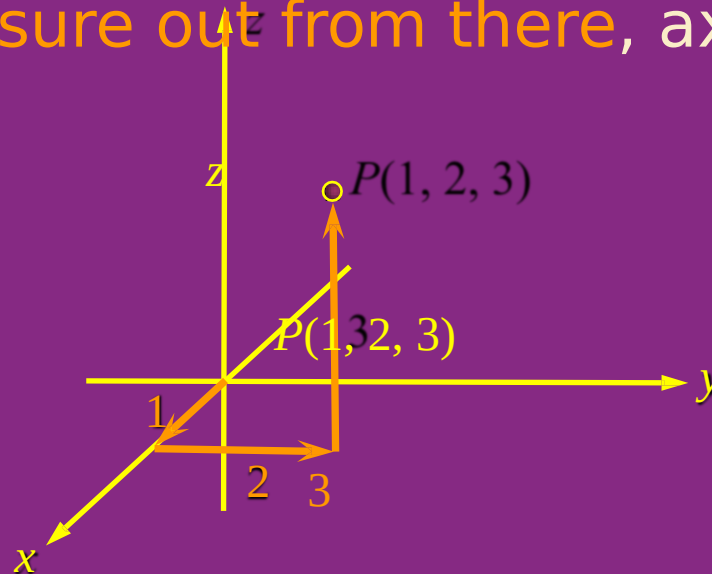
b. $f(x, y) = x \ln(y^2 - x)$

3. Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

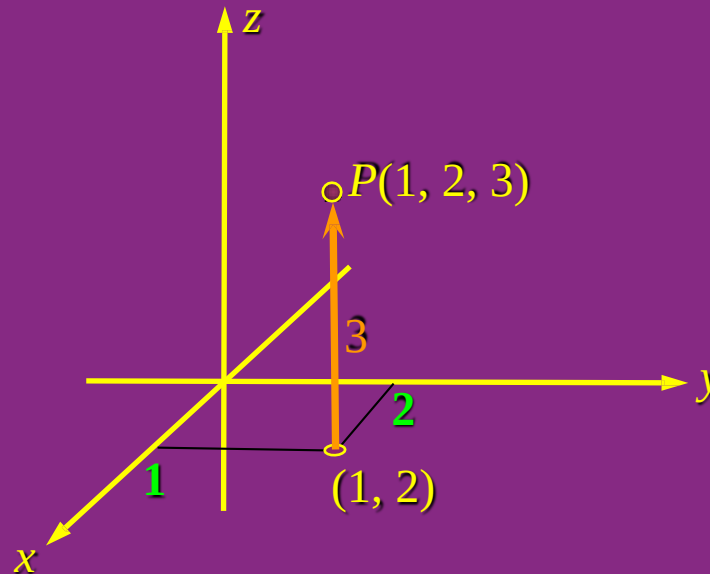
GRAPHS OF FUNCTIONS OF TWO VARIABLES

- ▮ Consider the task of locating $P(1, 2, 3)$ in 3-space:
- ▮ One method to achieve this is to start at the origin and measure out from there, axis by axis:



GRAPHS OF FUNCTIONS OF TWO VARIABLES

- ◆ Consider the task of locating $P(1, 2, 3)$ in 3-space:
- ◆ Another common method is to find the xy coordinate and from there elevate to the level of the z value:

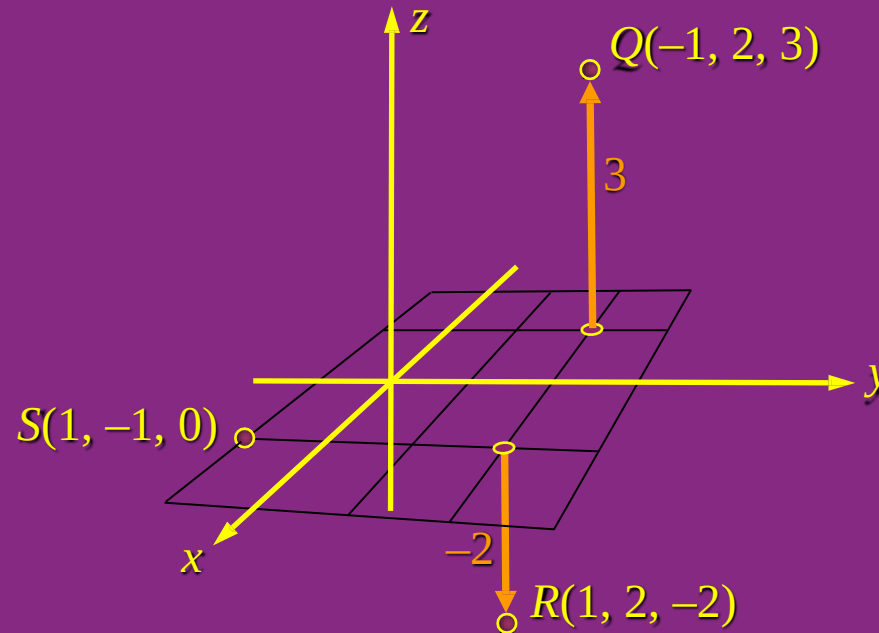


GRAPHS OF FUNCTIONS OF TWO VARIABLES

◆ **Locate the following points in 3-space:**

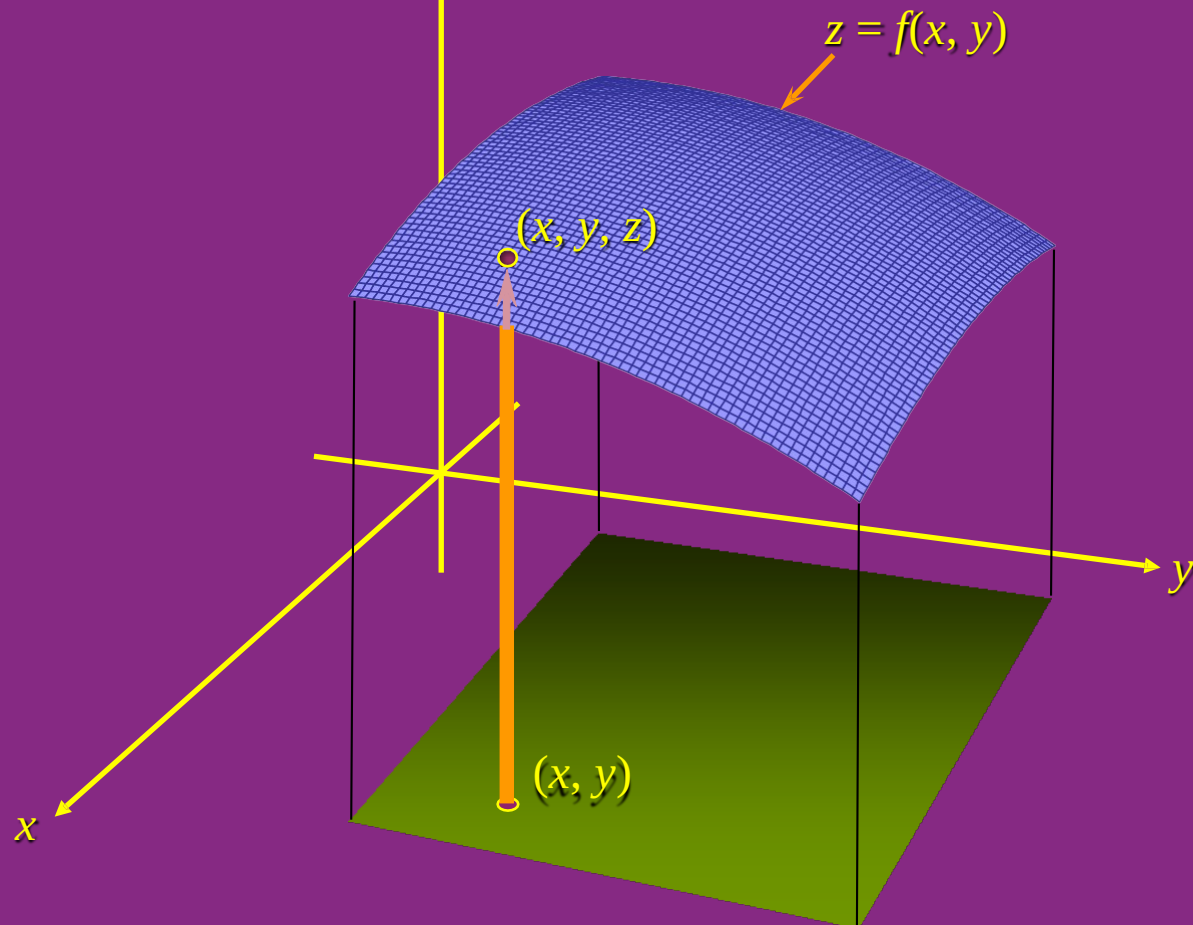
$Q(-1, 2, 3)$, $R(1, 2, -2)$, and $S(1, -1, 0)$.

Solution



GRAPHS OF FUNCTIONS OF TWO VARIABLES

- ◆ The **graph** of a function in **3-space** is a **surface**.
- ◆ For every (x, y) in the **domain** of f , there is a z value **on the surface**.

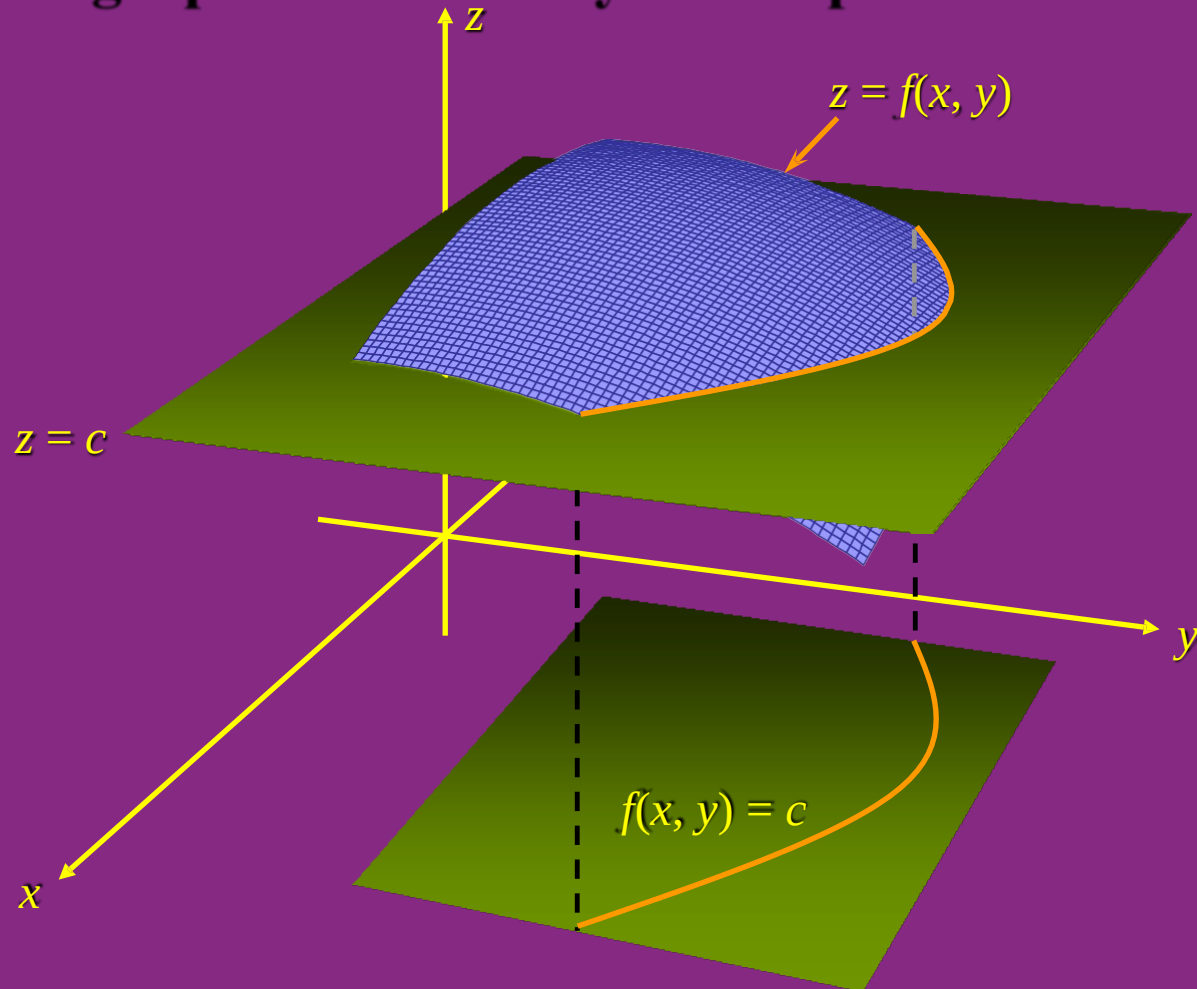


LEVEL CURVES

- ▮ The graph of a function of two variables is often difficult to sketch.
- ▮ It can therefore be useful to apply the method used to construct topographic maps.
- ▮ This method is relatively easy to apply and conveys sufficient information to enable one to obtain a feel for the graph of the function.

LEVEL CURVES

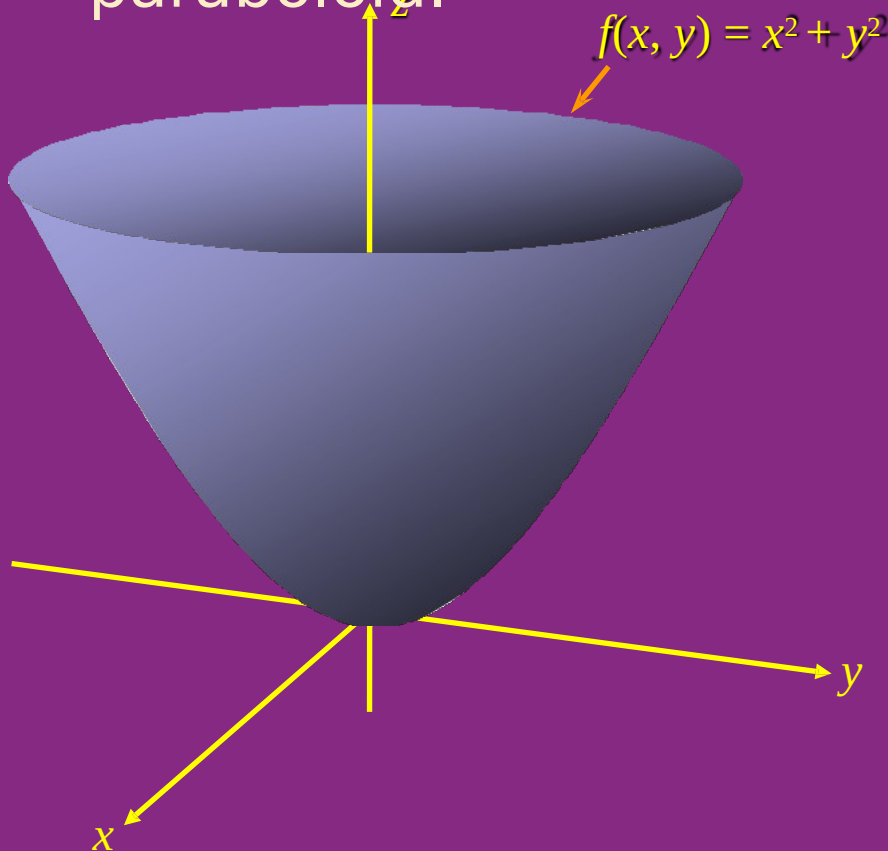
- ◆ In the 3-space graph we just saw, we can delineate the contour of the graph as it is cut by a $z = c$ plane:



EXAMPLE 1

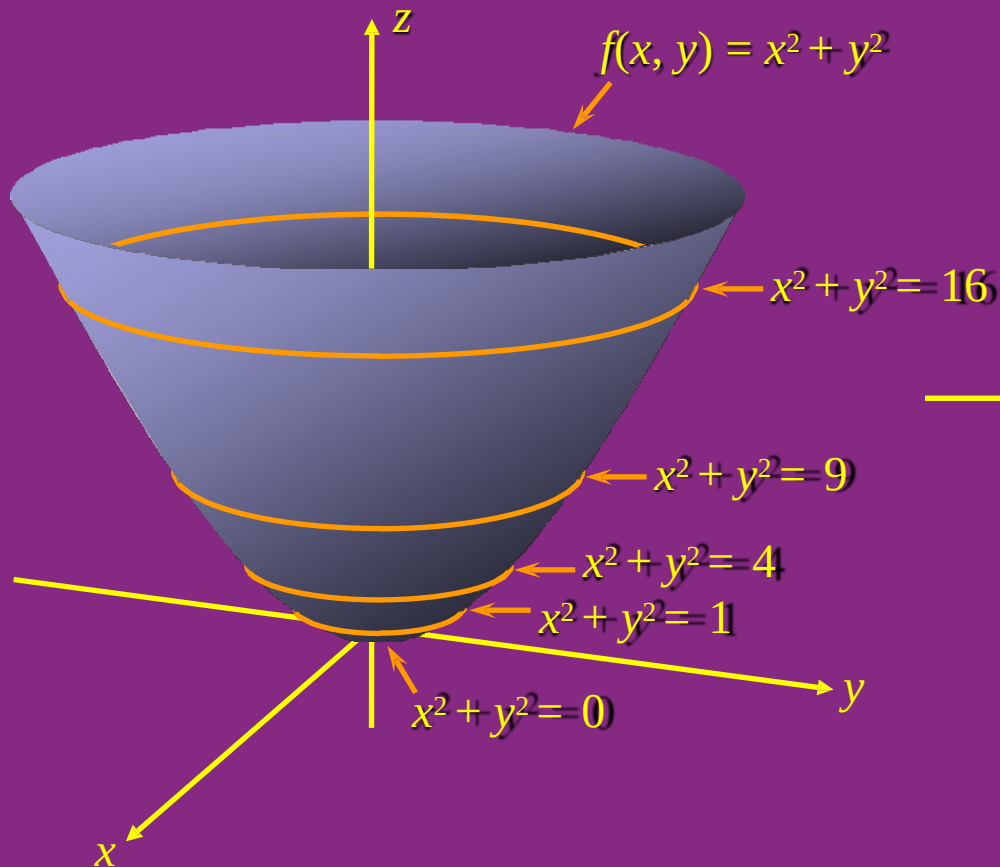
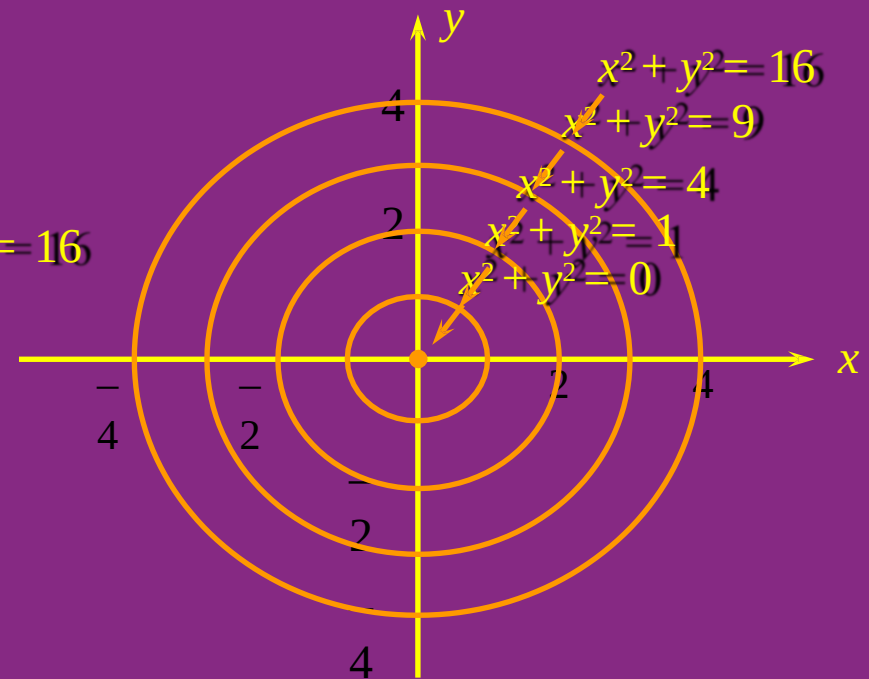
▮ Sketch a **contour map** of the function $f(x, y) = x^2 + y^2$. Solution

▮ The function $f(x, y) = x^2 + y^2$ is a revolving parabola called a paraboloid.



CONT'D

- ▣ A **level curve** is the **graph** of the equation $x^2 + y^2 = c$, which is a **circle** with $\sqrt{c}$
- ▣ **Taking** different values of c we obtain:

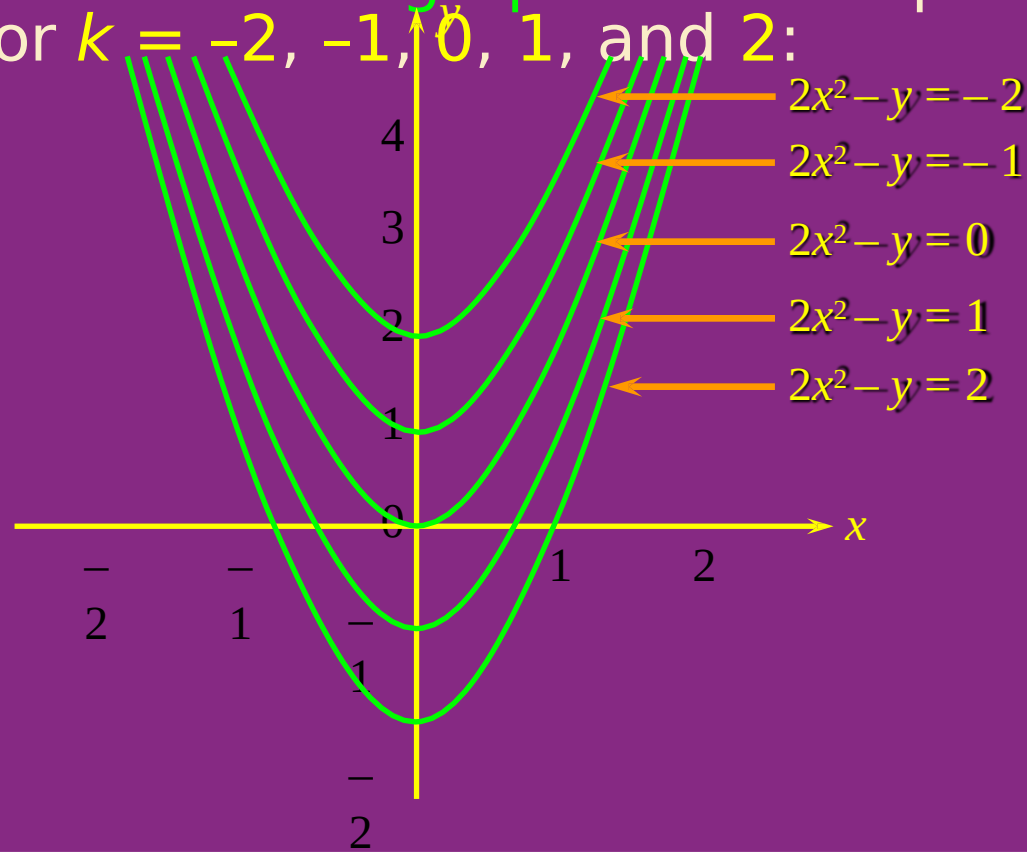


EXAMPLE 2

- ▮ Sketch **level curves** of the function $f(x, y) = 2x^2 - y$ corresponding to $z = -2, -1, 0, 1$, and 2 .

Solution

- ▮ The **level curves** are the **graphs** of the equation $2x^2 - y = k$ or for $k = -2, -1, 0, 1$, and 2 :



EXERCISE

1. Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y$$

2. Sketch

$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

3. Find the domain and range and sketch the graph of

$$h(x, y) = 4x^2 + y^2$$

LIMITS AND CONTINUITY

- Goals

- Define limits of functions of two variables
- Learn to use two paths to show that a limit does not exist.
- Define and discuss continuity.

Example: We want to compare the behaviors of

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

- The next two slides show tables of values of $f(x, y)$ and $g(x, y)$ for points (x, y) near the origin:

CONT'
D**TABLE 1** Values of $f(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

CONT'
D**TABLE 2** Values of $g(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

CONT'D

- It appears that as (x, y) approaches $(0, 0)$,
 - the values of $f(x, y)$ approach 1, whereas
 - the values of $g(x, y)$ aren't approaching any number.
- These guesses based on numerical evidence are correct, and we

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \text{ does not exist}$$

CONT'D

▮ In general, we use the notation:

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f :

THE LIMIT OF A FUNCTION OF TWO VARIABLES

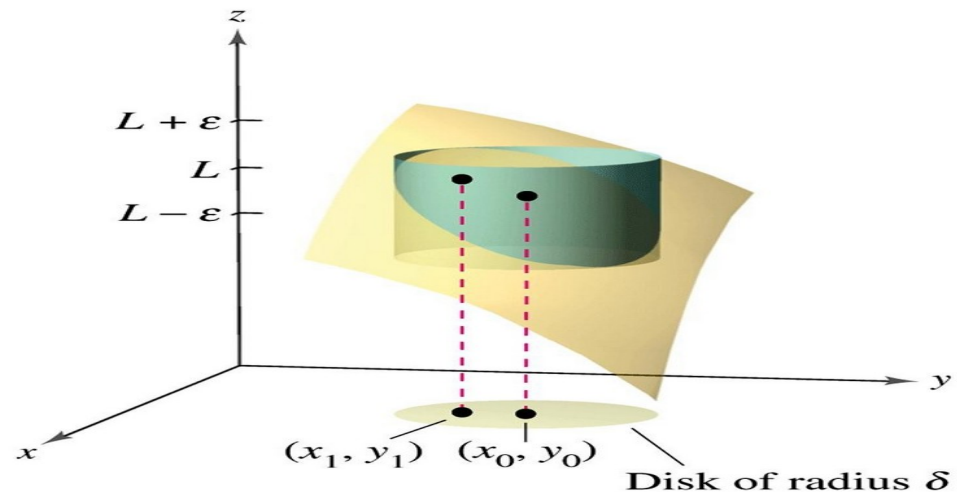
Definition of the Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0, y_0) , on an open disk centered at (x_0, y_0) , and let L be a real number. Then

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



DIRECTION OF APPROACH

- For functions of **a single variable**, when we let x approach a , there are only two possible directions of approach, from the left or from the right.
- If the limits from these two directions are **different**, then the function itself has **no limit** as x approaches a .
- ▮ For functions of two variables the situation is more complicated because (x, y) can approach (a, b) ...
 - from **an infinite** number of directions,
 - in any manner whatsoever, as long as (x, y) remains in the domain of f .
- ▮ This is illustrated on the next slide:

CONT'D

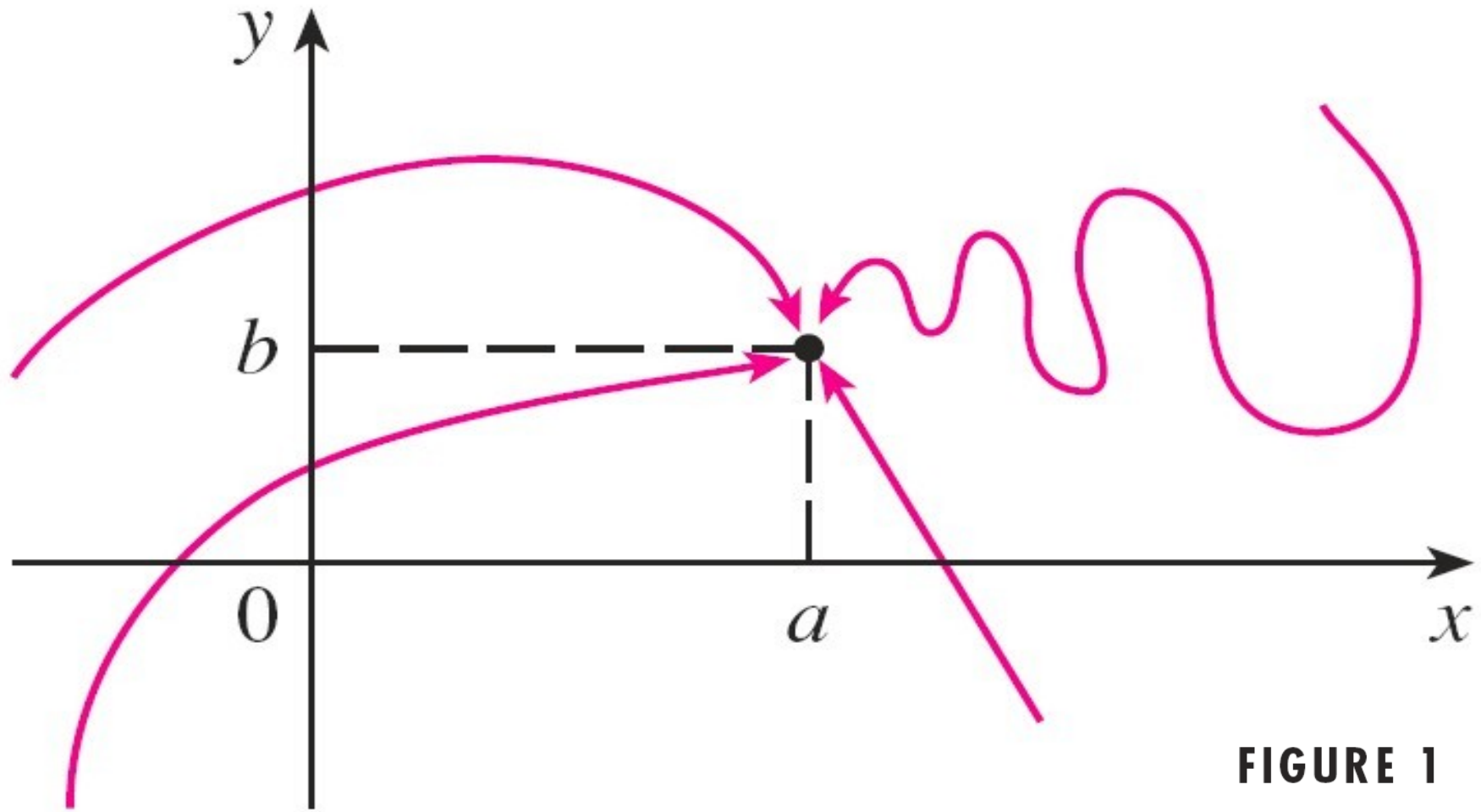


FIGURE 1

CONT'D

- This can be summarized as follows:

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

Example 1: show that

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$

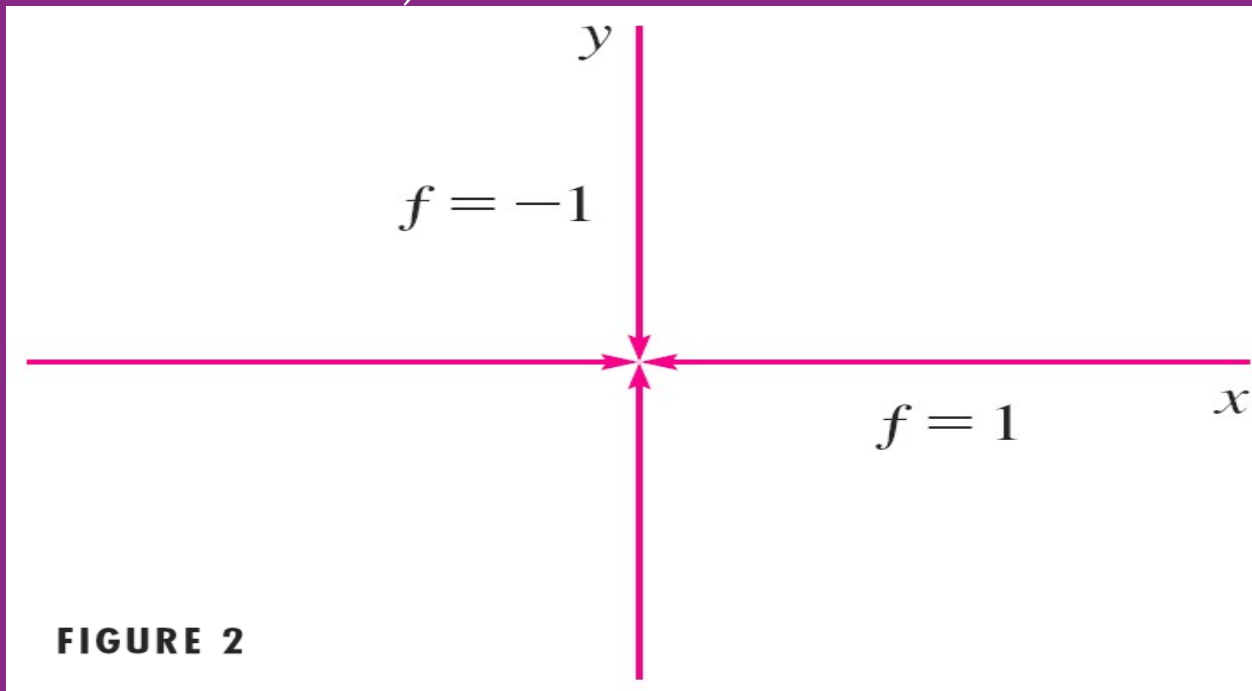
does not exist:

SOLUTION

- Let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$.
 - First we approach $(0, 0)$ along the x -axis:
 $f(x, 0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$, so $f(x, y)$ approaches 1 as (x, y) approaches $(0, 0)$ along the x -axis.
 - Next we approach $(0, 0)$ along the y -axis:
 $f(0, y) = -\frac{y^2}{y^2} = -1$ for all $y \neq 0$, so $f(x, y)$ approaches -1 as (x, y) approaches $(0, 0)$ along the y -axis.

CONT'D

- Since f has two different limits along two different lines, the given limit does not exist.
- (This confirms the conjecture we made earlier on the basis of numerical evidence.)



EXAMPLE 2

■ If $f(x, y) = \frac{xy}{x^2 + y^2}$ does $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exist?

■ Solution: Again we consider various paths:

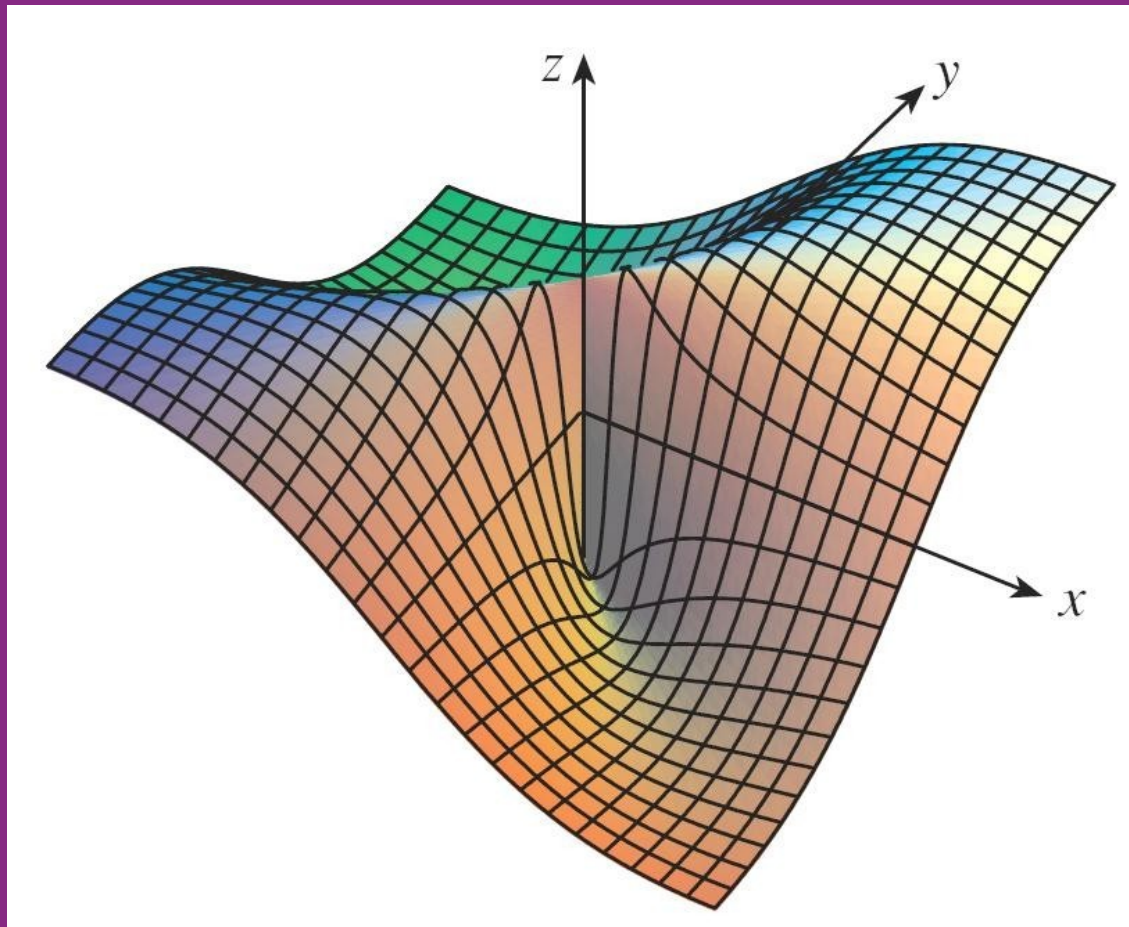
- $f(x, 0) = \frac{0}{x^2} = 0$ for all $x \neq 0$, so $f(x, y)$ approaches 0 as (x, y) approaches $(0, 0)$ along the x -axis.
- $f(0, y) = \frac{0}{y^2} = 0$ for all $y \neq 0$, so $f(x, y)$ approaches 0 as (x, y) approaches $(0, 0)$ along the y -axis as well.

SOLUTION (CONT'D)

- Although we have obtained identical limits along the axes, that **does not show** that the given limit is **0**.
- Let's now approach **(0, 0)** along another line, say **$y = x$** :
For all $x \neq 0$,
$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$
- Therefore **$f(x, y)$** approaches **$\frac{1}{2}$** as **(x, y)** approaches **$(0, 0)$** along **$y = x$** .
- Since we have obtained **different limits along different paths**, the given limit **does not exist**, as the next slide illustrates.

SOLUTION (CONT'D)

- The ridge above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all (x, y) on that line except the origin.



Example 3

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2}$ if it exists:

Solution: As in Example 2, we could show that the limit along any line through the origin is 0.

- But, this **doesn't prove** that the given limit is 0.

- However, the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0.

- So, we begin to suspect that the limit does exist and is equal to 0.

CONT'D

- Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{that is, if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \frac{3x^2 |y|}{x^2 + y^2} < \varepsilon$$

- To prove this we look at the distance from $f(x, y)$ to 0:

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| = \left| \frac{3x^2 y}{x^2 + y^2} \right| = \frac{3x^2 |y|}{x^2 + y^2}$$

SOLUTION (CONT'D)

- Notice that $\frac{x^2}{x^2 + y^2} \leq 1$ because $y^2 \geq 0$.

So

$$\frac{x^2}{x^2 + y^2} \leq 1$$

Thus,

$$\frac{3x^2 |y|}{x^2 + y^2} \leq 3 |y| \sqrt{y^2} \leq 3 \sqrt{x^2 + y^2} = 3$$

So we choose $\sigma = \frac{1}{3}$, and
let

$$0 \leq \sqrt{x^2 + y^2}$$

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| \leq \sqrt{x^2 + y^2} \leq 3 \sqrt{\frac{1}{9} + \frac{1}{9}} = 3 \sqrt{\frac{2}{9}} = \sqrt{2}$$

CONT'
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$$\left| \frac{3x^2 y}{x^2 + y^2} \right| \leq \frac{3\sqrt{x^2 + y^2}}{2} \leq \frac{3}{2} \sqrt{x^2 + y^2} \leq \frac{3}{2} \sqrt{\delta} < \delta$$

Hence by the definition:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$$

CONTINUITY

- Continuous functions of two variables are defined by the direct substitution property, just as with functions of one variable:

3 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D .

- The **intuitive meaning** of continuity is that
 - if the point (x, y) changes by a small amount,
 - then the value of $f(x, y)$ changes by a small amount.

CONT'D

- This means that a surface that is the graph of a continuous function **has no hole** or **break**.
- Using the properties of limits, we can see that
 - sums,
 - differences,
 - products, and
 - quotients

of continuous functions are continuous on their domains.

- For instance, $f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$ is **a polynomial**, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is **a rational function**.

EXAMPLE 1

- Evaluate

$$\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$$

Solution: Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial,
it is continuous everywhere.

So, we can find the limit by direct substitution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) &= \\ &= 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 \\ &= 2 - 2 + 3 + 4 = 7 \end{aligned}$$

EXAMPLE 5

- Where is the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

continuous?

Solution: The function f is **discontinuous** at $(0, 0)$ because it is not defined there. Since f is a **rational function**, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$

- Let

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- Here g is defined at $(0, 0)$ but g is still **discontinuous** at $(0, 0)$ because $\lim_{(x, y) \rightarrow (0, 0)} g(x, y)$ does not exist, as shown above.

EXAMPLE 3

- Let

$$f(x, y) = \begin{cases} \frac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there.

- Also by preceding example

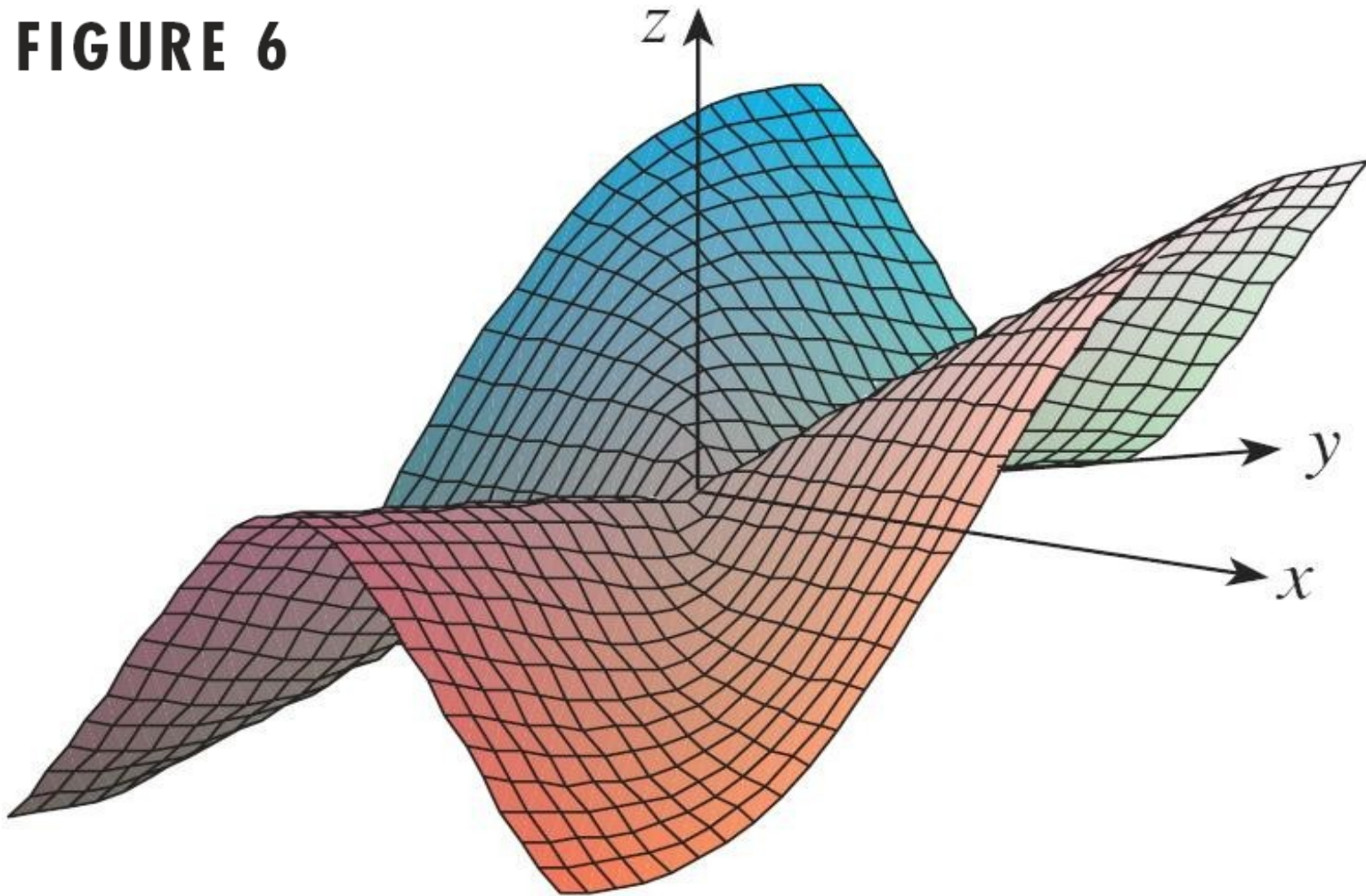
$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

- Therefore, f is continuous at $(0, 0)$, and so it is continuous throughout the plane.

- The next slide shows the graph of f :

EXAMPLE (CONT'D)

FIGURE 6



COMPOSITION FUNCTION

- Composition is another way of combining two continuous functions to get a third.
- We can show that if
 - f is a continuous function of two variables and
 - g is a continuous function of a single variable that is defined on the range of f ,then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.
- **Example:** Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION

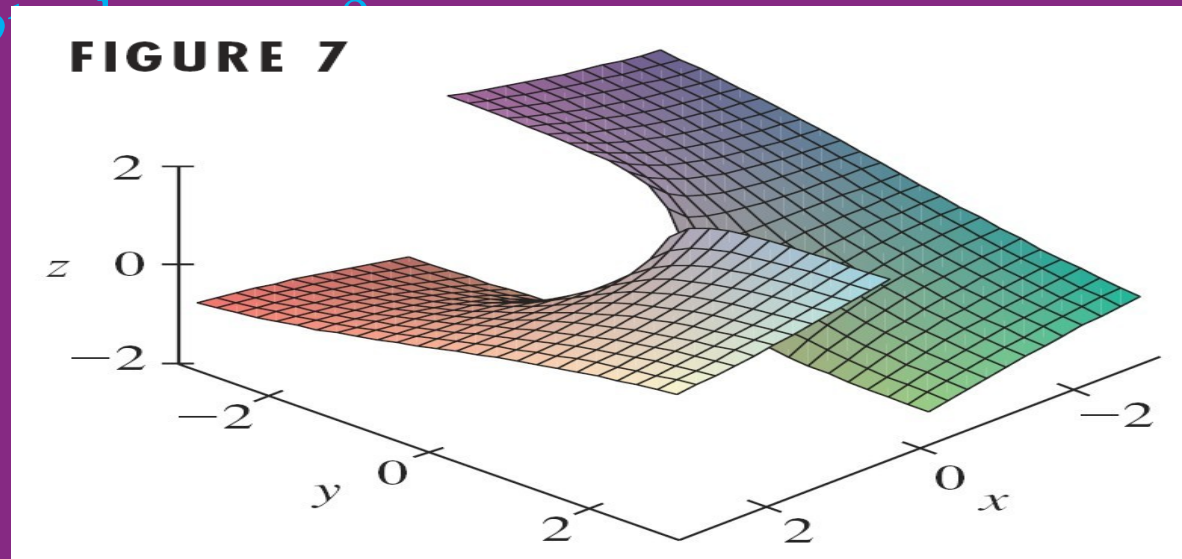
The function $f(x, y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line $x = 0$.

■ The function $g(t) = \arctan t$ is continuous everywhere.

■ So the composite function $g(f(x, y)) = \arctan(y/x) = h(x, y)$

is continuous except on the line $x = 0$.

■ Here is the graph of f :



EXERCISES

✖ If $f(x, y) = \frac{xy^2}{x^2 + y^4}$ does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

✖ Find the limit, if it exists, or show that the limit does not exist.

✖ a) $\lim_{(x, y) \rightarrow (0, 0)} \frac{4 - xy^2}{x^2 + y^4}$ Ans: $\frac{2}{3}$

c) $\lim_{(x, y) \rightarrow (0, 0)} \frac{xy \cos y}{y^2 + x^2}$

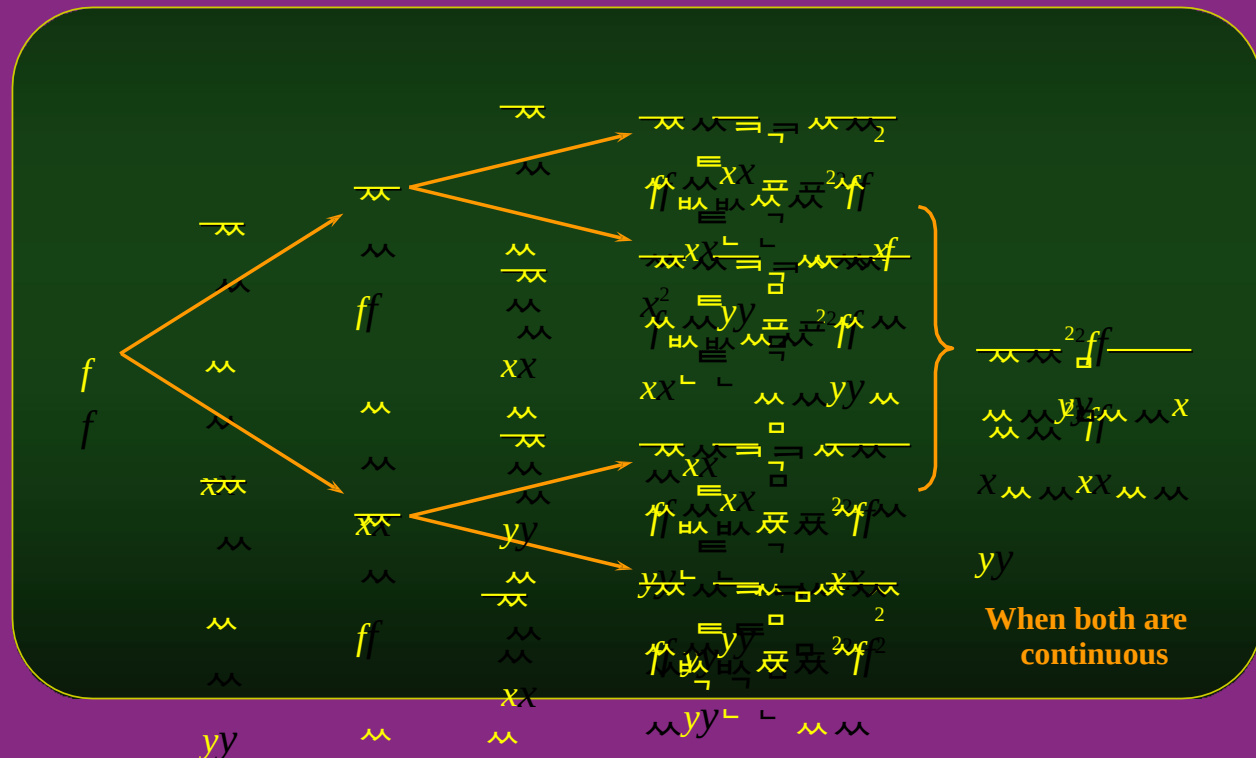
Ans: does not exist

b) $\lim_{(x, y) \rightarrow (0, 0)} \frac{y^4}{3y^4}$ Ans: does not exist

d) $\lim_{(x, y) \rightarrow (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}}$ Ans: 0

3.2

Partial Derivatives



FIRST PARTIAL DERIVATIVES

First Partial Derivatives of $f(x, y)$

- Suppose $f(x, y)$ is a function of two variables x and y .
- Then, the first partial derivative of f with respect to x at the point (x, y) is

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided the limit exists.

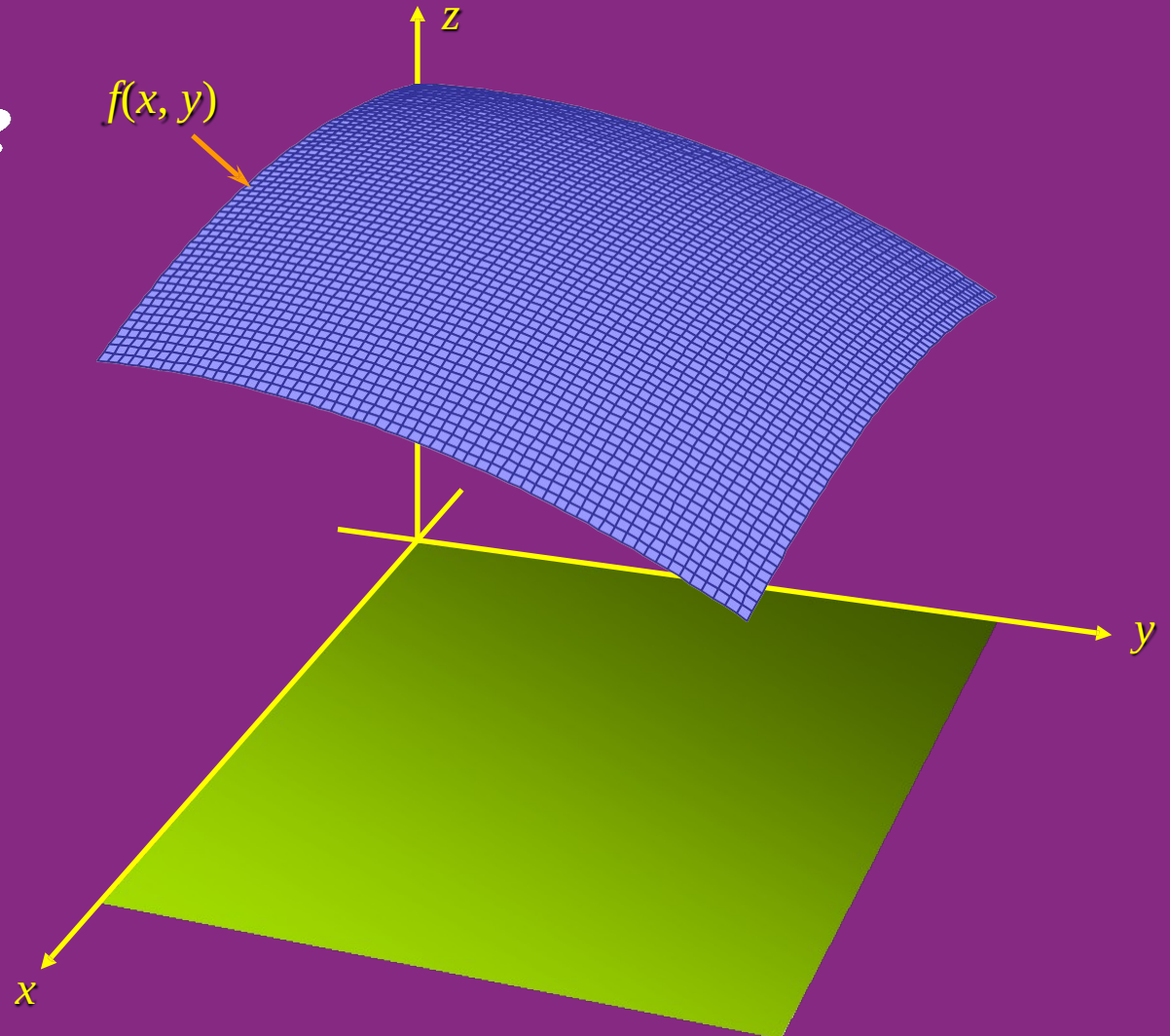
- The first partial derivative of f with respect to y at the point (x, y) is

$$\frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided the limit exists.

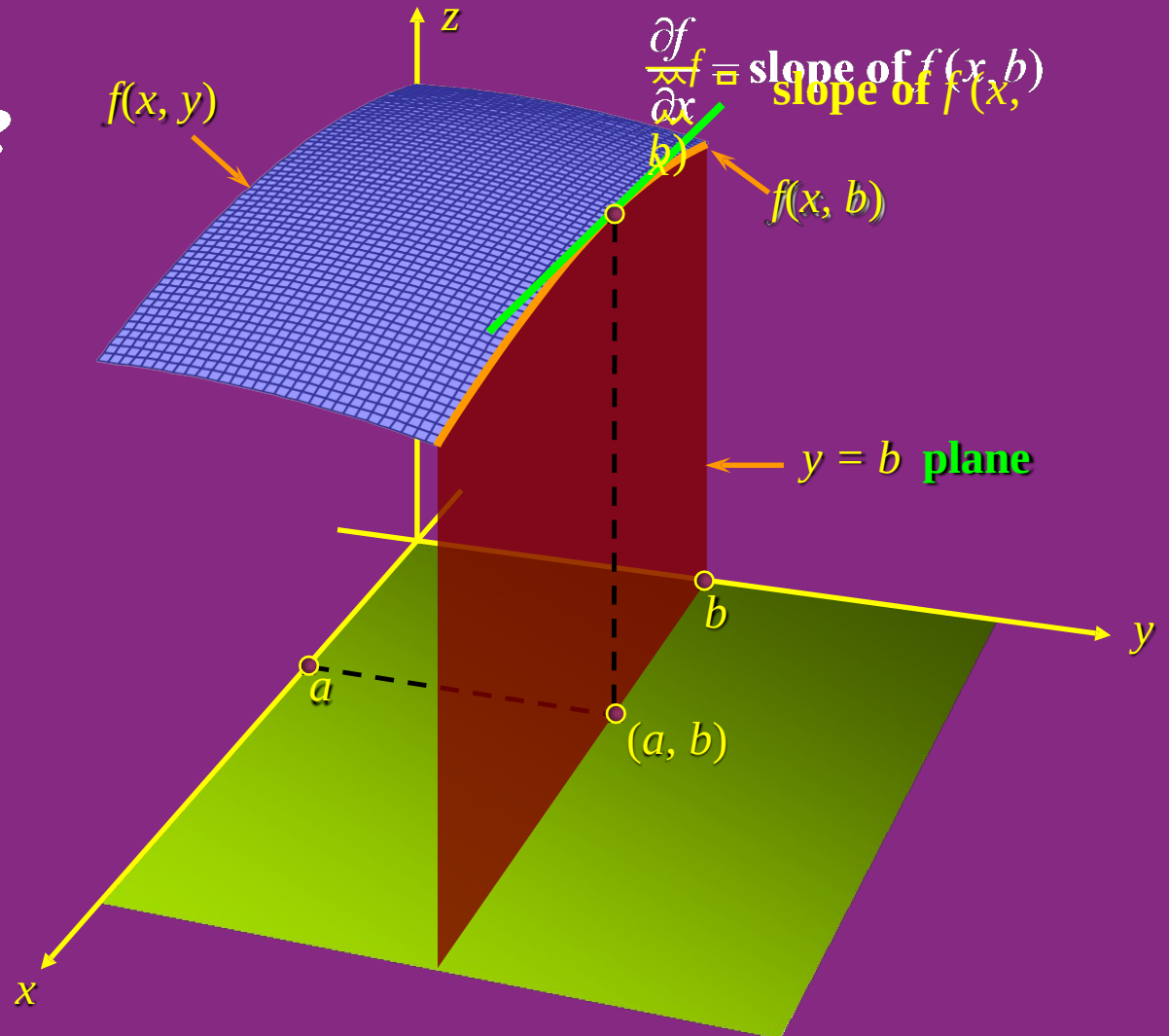
GEOMETRIC INTERPRETATION OF THE PARTIAL DERIVATIVE

What does $\frac{\partial f}{\partial x}$ mean?



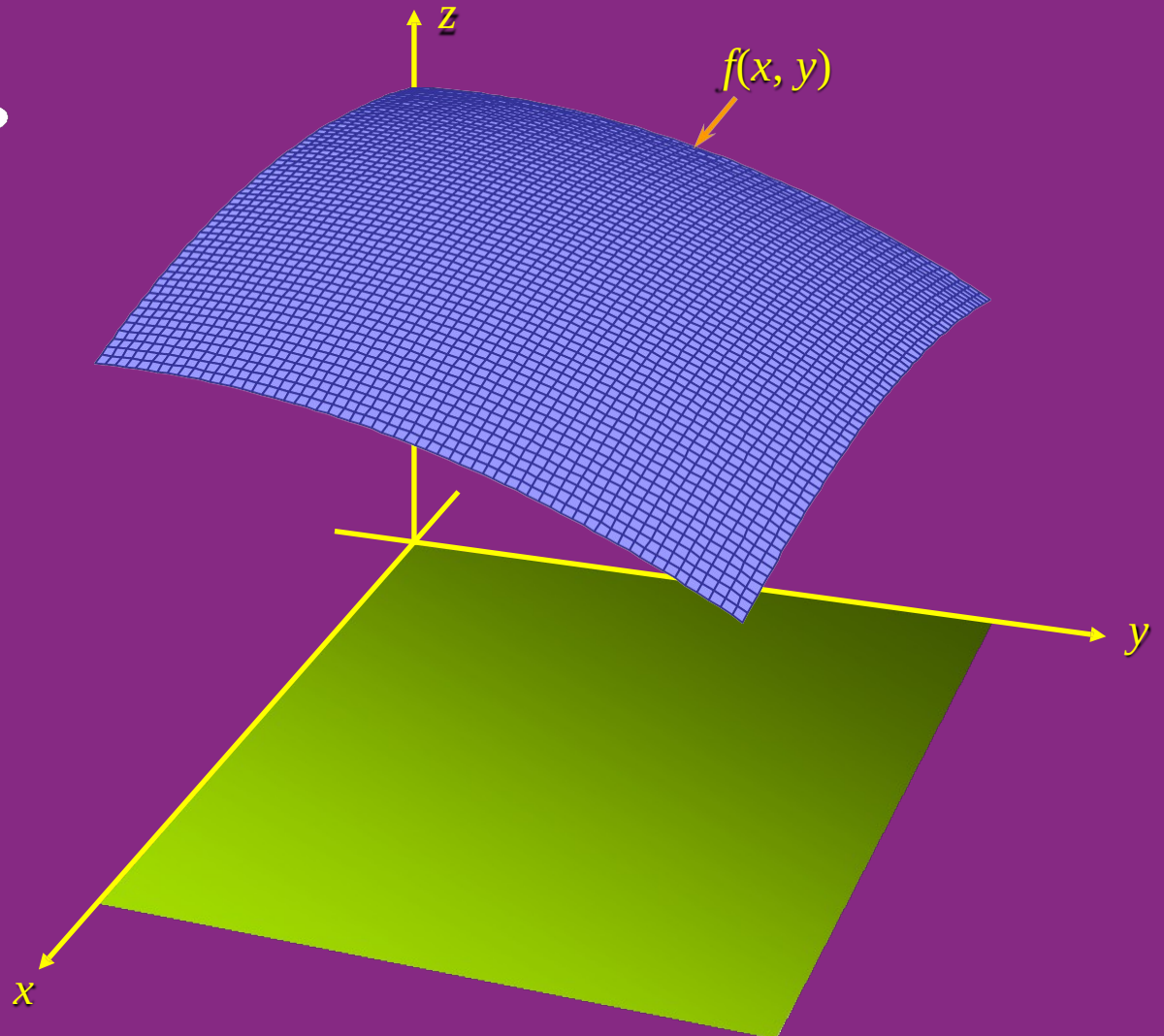
CONT'
D

What does $\frac{\partial f}{\partial x}$ mean?



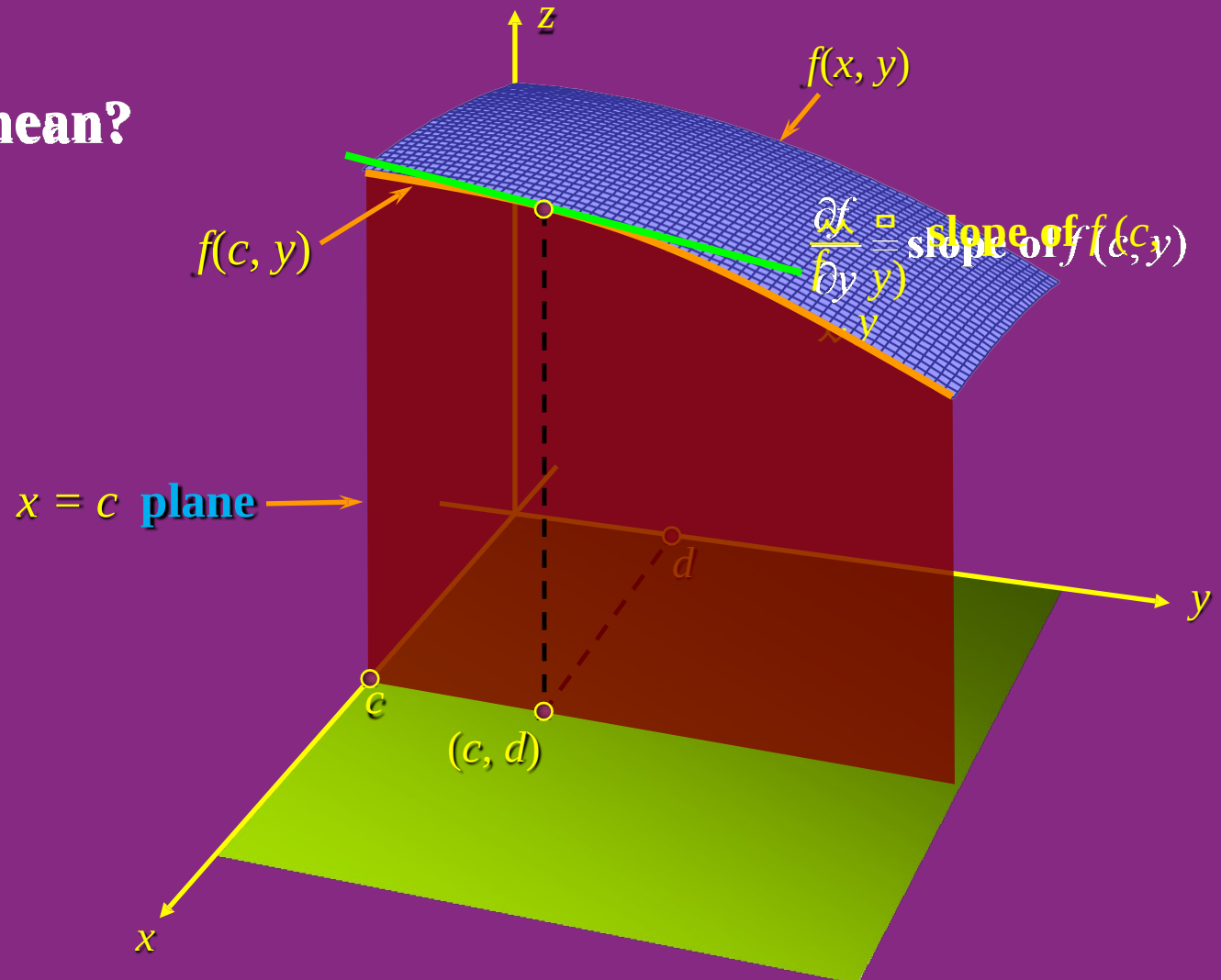
CONT'D

What does $\frac{\partial f}{\partial x}$ mean?



CONT'D

What does $\frac{\partial f}{\partial y}$ mean?



EXAMPLE 1

- Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function

$$f(x, y) = x^2 + xy^2 + y^3$$

- Use the partials to determine the rate of change of f in the x -direction and in the y -direction at the point $(1, 2)$.

Solution:

- To compute $\frac{\partial f}{\partial x}$, think of the variable y as a constant and differentiate the resulting function of x with respect to x :

$$f(x, y) = x^2 + y^2 x$$

$$\frac{\partial f}{\partial x} = 2x + y^2$$

SOLUTION(CONT'D)

- ✖ To compute $\frac{\partial f}{\partial y}$, think of the variable x as a constant and differentiate the resulting function of y with respect to y :

$$f(x, y) = x^2 + xy^2$$

$$\frac{\partial f}{\partial y} = 0 + 2xy = 2xy$$

- ✖ The rate of change of f in the x -direction at the point $(1, 2)$ is given by

$$\left. \frac{\partial f}{\partial x} \right|_{(1,2)} = 2(1) = 2$$

- ✖ The rate of change of f in the y -direction at the point $(1, 2)$ is given by

$$\left. \frac{\partial f}{\partial y} \right|_{(1,2)} = 2(1)(2) = 4$$

$$3(2)^2 = 12$$

EXAMPLE 2

✎ Find the first partial derivatives of the function

$$w(x, y) = \frac{x^2 + y^2}{xy}$$

Solution

✎ To compute $\frac{\partial w}{\partial x}$, think of the variable y as a constant and differentiate the resulting function of x with respect to x :

$$\begin{aligned} w(x, y) &= \frac{x^2 + y^2}{xy} \\ \frac{\partial w}{\partial x} &= \frac{(x^2 + y^2) \cdot y - xy(2x)}{(xy)^2} \\ &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \end{aligned}$$

SOLUTION (CONT'D)

- ✕ To compute $\frac{\partial w}{\partial y}$, think of the **variable** x as a **constant** and
 differentiate the resulting function of y with respect to y :

$$w(x, y) = \frac{xy}{x^2 + y^2}$$

$$\frac{\partial w}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

EXAMPLE 3

- Find the first partial derivatives of the function

HH

$$g(s, t) = (s^2 + st + t^2)^5$$

Solution

- To compute $\frac{\partial g}{\partial s}$, think of the variable t as a constant and differentiate the resulting function of s with respect to s :

$$g(s, t) = (s^2 + st + t^2)^5$$

$$\begin{aligned} \frac{\partial g}{\partial s} &= 5(s^2 + st + t^2)^4 \cdot (2s + t) \\ &= 5(2s + t)(s^2 + st + t^2)^4 \end{aligned}$$

SOLUTION (CONT'D)

- ✖ To compute $\frac{\partial g}{\partial t}$, think of the **variable** s as a **constant** and **differentiate** the resulting function of t with respect to t :

$$g(s, t) = (s^2 + st + t^2)^5$$

$$\begin{aligned}\frac{\partial g}{\partial t} &= 5(s^2 + st + t^2)^4 \cdot (-s + 2t) \\ &= 5(2t - s)(s^2 + st + t^2)^4\end{aligned}$$

Exercise

- ✖ Find the **first partial derivatives** of the function

$$h(u, v) = e^{u^2 v}$$

EXAMPLE 4

Find the first partial derivatives of the function

$$w = f(x, y, z) = xyz + xe^{yz} + x \ln y$$

Solution

Here we have a function of three variables, x , y , and z , and we are required to compute

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

$$, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$

For short, we can label these first partial derivatives respectively f_x , f_y , and f_z .

SOLUTION (CONT'D)

- To find f_x , think of the variables y and z as a constant and differentiate the resulting function of x with respect to x .

$$w = f(x, y, z) = xyz = xe^{yz} = x \ln y$$

$$f_x = yz = e^{yz} + \ln y = \ln y$$

- To find f_y , think of the variables x and z as a constant and differentiate the resulting function of y with respect to y .

$$w = f(x, y, z) = xyz = xe^{yz} = x \ln y$$

$$f_y = xz = xze^{yz} + \frac{x}{y} = \frac{x}{y}$$

SOLUTION (CONT'D)

- ✎ To find f_z , think of the variables x and y as a constant and differentiate the resulting function of z with respect to z :

$$w = f(x, y, z) = xyz + xe^{yz} =$$

$$f_z = xy + xye^{yz}$$

✎ Exercise:

Find $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$ if Z is defined implicitly as a function of x and y by the equation

$$x^3 = y^3 = z^3 =$$

$$6xyz = 1$$

SECOND ORDER PARTIAL DERIVATIVES

- ▀ The first partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of a function $f(x, y)$ of two variables x and y are also functions of x and y .
- ▀ As such, we may differentiate each of the functions f_x and f_y to obtain the second-order partial derivatives of f .
- ▀ Differentiating f_x with respect to x leads to the f_{xx} derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

CONT'D

- But the function f_x can also be differentiated with respect to y leading to a different second partial derivative

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (f_x)$$

- Similarly, differentiating the function f_y with respect to x leads to the second partial derivative

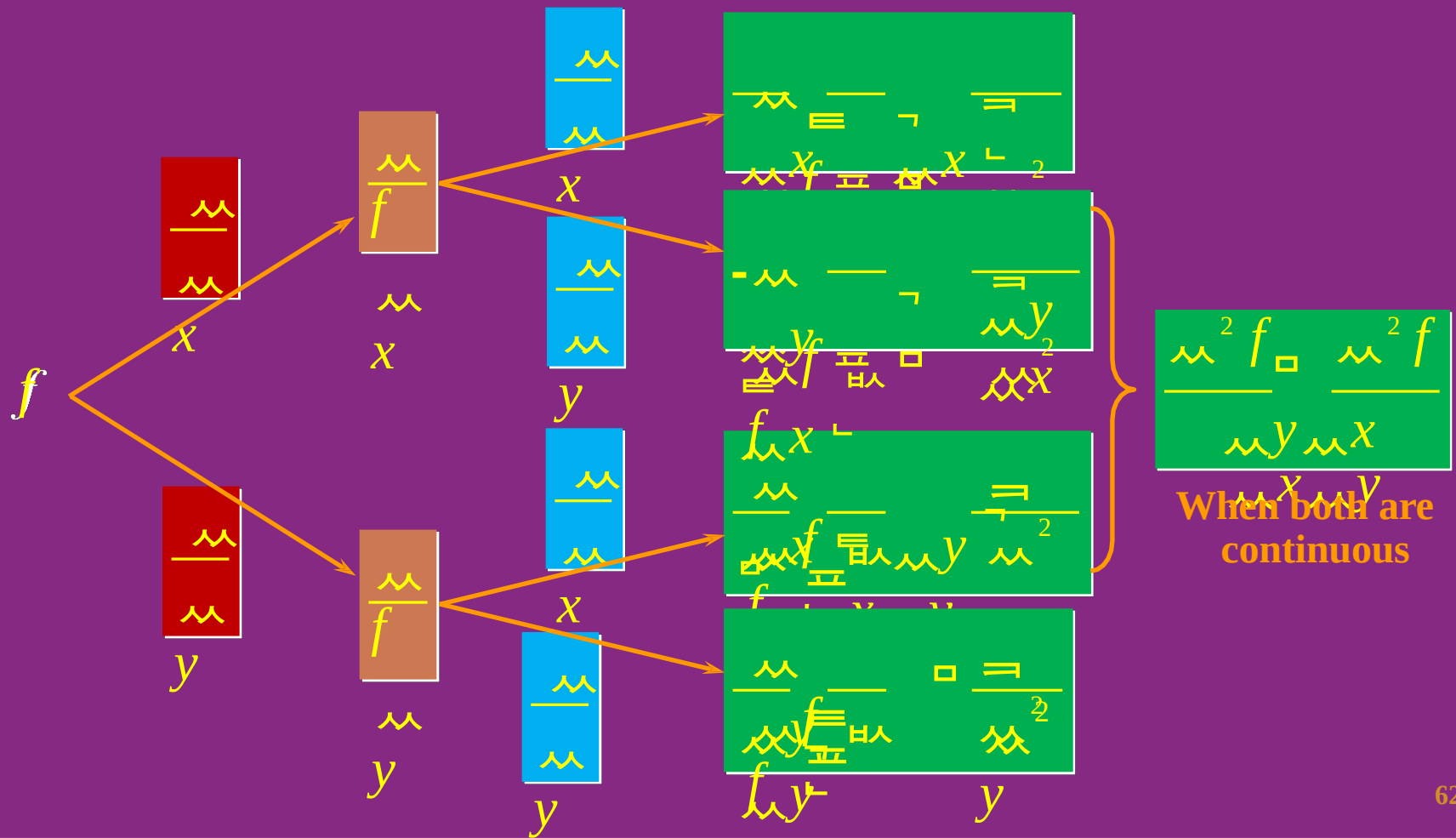
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (f_y)$$

- Finally, the function f_y can also be differentiated with respect to x leading to the second partial derivative

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (f_y)$$

CONT'D

- Thus, **four** second-order partial derivatives can be obtained of a function of **two** variables:



EXAMPLE 1

Find the **second-order partial derivatives** of the function

$$f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$$

Solution

First, calculate f_x and use it to find f_{xx} and f_{xy}

$$f_x = \frac{\partial}{\partial x}(x^3 + 3x^2y + 3xy^2 + y^3) = 3x^2 + 6xy + 3y^2$$

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(3x^2 + 6xy + 3y^2) \\ &= 6x + 6y \\ &= 6(x + y) \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(3x^2 + 6xy + 3y^2) \\ &= 6x + 6y \\ &= 6(y + x) \end{aligned}$$

CONT'D

- Then, calculate f_y and use it to find f_{yx} and f_{xy} .

$$f_y = \frac{\partial}{\partial y}(x^3 + 3x^2y + 3xy^2 + y^3) = 3x^2 + 6xy + 3y^2$$

$$f_{yx} = \frac{\partial}{\partial x}(-3x^2 + 6xy + 2y) = -6x + 6y = 6(y - x)$$

$$f_{yy} = \frac{\partial}{\partial y}(-3x^2 + 6xy + 2y) = 6x + 2 = 2(3x + 1)$$

- Exercise: Find the second-order partial derivatives of the function

$$f(x, y) = e^{xy}$$

DIRECTIONAL DERIVATIVE AND GRADIENT VECTOR

DIRECTIONAL DERIVATIVE

- In this section, we introduce a type of derivative, called a **directional derivative**, that enables us to find: The rate of change of a function of two or more variables in any direction.
- Recall that, if $z = f(x, y)$, then the partial derivatives f_x and f_y

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

DIRECTIONAL DERIVATIVE AND GRADIENT VECTOR

DIRECTIONAL DERIVATIVES

Definition: Let f be a function defined on a set centered at (x_0, y_0) , $u = a_1 i + a_2 j$ be a unit vector. Then the directional derivative of f in the direction of u at (x_0, y_0) is denoted by:

$$D_u f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha_1, y_0 + ha_2) - f(x_0, y_0)}{h}$$

Provided this limit exists.

Observe that if $u = i$, then

$$D_u f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$$

And if $u = j$, then

$$D_u f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0)$$

THEOREM

- Let f be differentiable function at (x_0, y_0) . Then f has a directional derivative at (x_0, y_0) in every

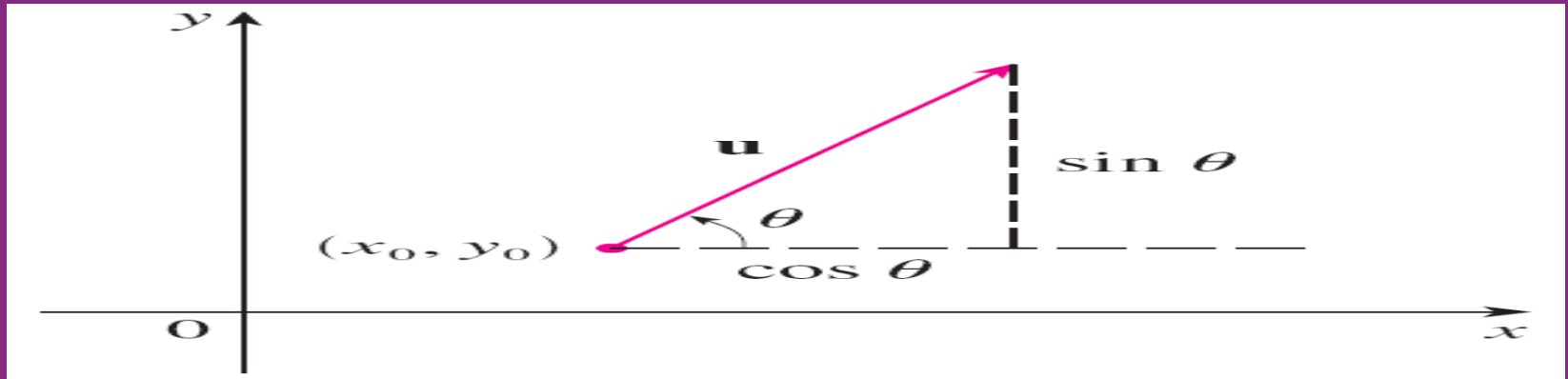
direction. If $u = a_1 i + a_2 j$ is a unit vector, then

$$D_u f(x_0, y_0) = f_x(x_0, y_0)a_1 + f_y(x_0, y_0)a_2$$

- Suppose the unit vector u makes an angle θ with the positive x -axis, as shown. Then, we can write $u = \cos \theta i + \sin \theta j$ and the formula in the above Theorem becomes

$$D_u f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

CONT'D



Example 1: $f(x, y) = 6x^2 + 3x^2 + y^2$ and $u = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$. Find $D_u f(1, 2)$.

Let

Solution: Notice that u is a unit vector, we will compute

$D_u f(1, 2)$ by the above equation. First we have to

calculate the partial derivative of f :

$$D_x f(x, y) = 6x \quad \text{and} \quad D_y f(x, y) = 2y$$

$$D_x f(1, 2) = 6 \quad \text{and} \quad D_y f(1, 2) = 4$$

CONT'D

Note: The directional derivative in the direction of a unit vector u is defined to be

$$u = \frac{a}{\|a\|}$$

$D_u f(x_0, y_0)$, where Example 2

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$

at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

ss

$$\nabla f(x, y) = 2xy^3 \mathbf{i} + (3x^2y^2 - 4) \mathbf{j}$$

$$\nabla f(2, -1) = -4 \mathbf{i} + 8 \mathbf{j}$$

CONT'D

- Note that $\mathbf{v}$ is not a unit vector. However, since unit vector in the direction of $\mathbf{v}$ is:

$$|\mathbf{v}| = \sqrt{29}, \text{ the}$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}} \mathbf{i} + \frac{5}{\sqrt{29}} \mathbf{j}$$

- Therefore, we have:

$$\begin{aligned} D_{\mathbf{u}} f(2, -1) &= \nabla f(2, -1) \cdot \mathbf{u} \\ &= (4\mathbf{i} - 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}} \mathbf{i} + \frac{5}{\sqrt{29}} \mathbf{j} \right) \\ &= \frac{4 \cdot 2}{\sqrt{29}} - \frac{8 \cdot 5}{\sqrt{29}} \\ &= \frac{8 - 40}{\sqrt{29}} \\ &= -\frac{32}{\sqrt{29}} \end{aligned}$$

EXAMPLE 3

Find the directional derivative $D_u f(x, y)$ if:

$f(x, y) = x^3 - 3xy + 4y^2$ and u is the unit vector
 angle $\theta = \pi/6$. What is $D_u f(1, 2)$? given

Solution: From the above formula we have:

$$\begin{aligned}
 D_u f(x, y) &= f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \\
 &= (3x^2 - 3y) \cos \frac{\pi}{6} + (3y) \sin \frac{\pi}{6} \\
 &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + 3y \frac{1}{2} \\
 &= \frac{\sqrt{3}}{2} (3x^2 - 3y) + \frac{3y}{2}
 \end{aligned}$$

CONT'

D

Therefore,

$$D_u f(1, 2) = \frac{1}{2} \sqrt{3} \cdot 3(1)^2 + 3(1) = \frac{\sqrt{3}}{2} + 3$$

- Notice from the above Theorem that the directional derivative can be written as the dot product of two vectors:

$$D_u f(x, y) = f_x(x, y) a + f_y(x, y) b$$

$$= \begin{pmatrix} f_x(x, y) & f_y(x, y) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

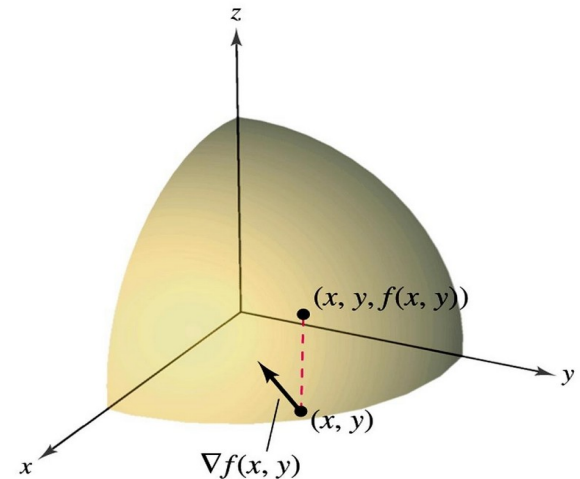
DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f** , denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as “del f .” Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y) , the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

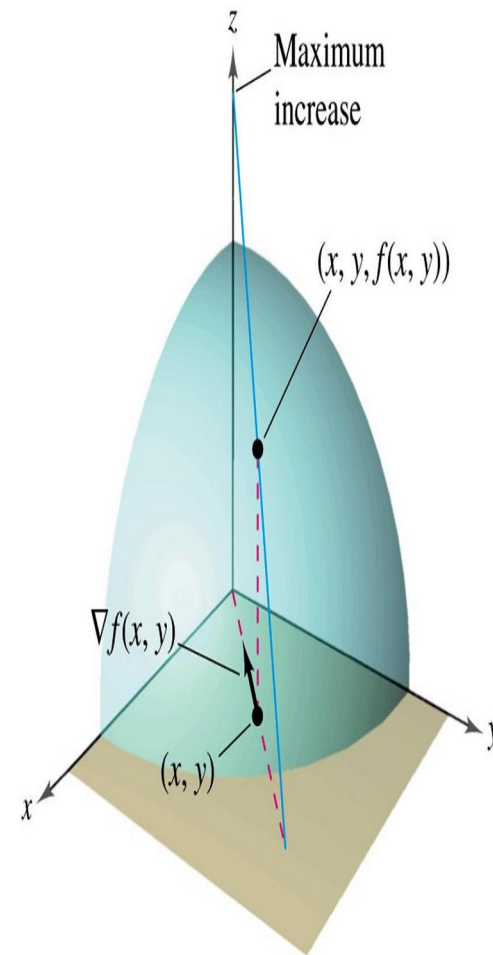


PROPERTIES OF THE GRADIENT

THEOREM 13.11 Properties of the Gradient

Let f be differentiable at the point (x, y) .

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}}f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}}f(x, y)$ is $-\|\nabla f(x, y)\|$.



GRADIENT IS NORMAL TO LEVEL CURVES

THEOREM 13.12 Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0) .

DIRECTIONAL DERIVATIVE AND GRADIENT FOR FUNCTIONS OF THREE VARIABLES

Directional Derivative and Gradient for Three Variables

Let f be a function of x , y , and z , with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined to be

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|. \quad \text{Maximum value of } D_{\mathbf{u}}f(x, y, z)$$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\|. \quad \text{Minimum value of } D_{\mathbf{u}}f(x, y, z)$$

CONT'D

Example 1: If $f(x, y) = \sin x + e^{xy}$, then find grad $(0,1)$.

Solution: First we have to evaluate the gradient of the function at x and y . Then

$$\text{grad } f(x, y) = f_x, f_y$$

$$= \cos x, ye^{xy}, xe^{xy}$$

$$\text{grad } f(0,1) = 2, 0$$

CONT'D

Example 2: If $f(x, y, z) = x \sin yz$, find:

- The gradient of f
- The directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution: The gradient of f is:

$$\begin{aligned} \nabla f(x, y, z) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= \left(\sin yz, x \cos yz, y \cos yz \right) \end{aligned}$$

CONT'D

⇒ At $(1, 3, 0)$, we have:

- The unit vector in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is:

$$f(1, 3, 0) = 0,$$

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$$

- Hence, the above Equation gives:

$$D_{\mathbf{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u} = 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \right) = 3 \cdot \frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

- ▮ Suppose we have a function f of **two or three variables** and we consider all possible directional derivatives of f at a given point.
 - These give the rates of change of f in all possible directions.
- We can then ask the questions:
 - In which of these directions does f change fastest?
 - What is the maximum rate of change?

THEOREM

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is :

- It occurs when $\mathbf{u}$ has the same direction as the gradient vector

- From the previous Equation, we

$$D_{\mathbf{u}} f = \|\nabla f\| \|\mathbf{u}\| \cos \theta$$

- where θ is the angle between ∇f and $\mathbf{u}$. Since $\|\mathbf{u}\| = 1$, we have

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

CONT'D

▀ The maximum value of $\cos \theta$ is 1. This occurs when $\theta = 0$.

▀ So, the maximum value of $\|D_u f\|$

$D_u f$ is:

▀ It occurs when $\theta = 0$, that is, when u has the same direction as ∇f .

EXAMPLE 6

a. If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

b. In what direction does f have the maximum rate of change?

c. What is this maximum rate of change?

• **Solution:** We first compute the gradient vector:

$$\nabla f(x, y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$= \begin{pmatrix} e^y \\ x e^y \end{pmatrix}$$

SOLUTION

- The unit vector in the

direction

$$\mathbf{u} =$$

$$\frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\overrightarrow{PQ} =$$

$$\begin{pmatrix} 1.5 \\ 2 \end{pmatrix}$$

So, the rate of change of f in the direction from P to Q is:

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{5} (1 \cdot 3 + 2 \cdot 4) = 1$$

- According to the above **Theorem**, f increases fastest in the
- So, the maximum rate of change is.

$$|\nabla f(2, 0)| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

EXERCISE

1. Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$$

where:

- T is measured in degrees Celsius.
- x, y, z is measured in meters.

a. In which direction does the temperature increase fastest

at the point $(1, 1, -2)$?

b. What is the maximum rate of increase?

TANGENT PLANE AND NORMAL LINE TO A SURFACE

You have represented the surfaces in space primarily by equations of the form $z = f(x, y)$.

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$.

For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as $F(x, y, z) = f(x, y) - z$.

Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

$$F(x, y, z) = 0.$$

EXAMPLE

For the function $F(x, y, z) = x^2 + y^2 + z^2 - 4$ describe the level surface given by $F(x, y, z) = 0$. **Solution:**

The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

TANGENT PLANE AND NORMAL LINE TO A SURFACE

In the process of finding a **normal line to a surface**, you are also able to solve the problem of finding **a tangent plane to the surface**.

Let S be a surface given by $F(x, y, z) = 0$ and let $P(x_0, y_0, z_0)$ be a point on S .

Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, for all t ,

$$F(x(t), y(t), z(t)) = 0.$$

TANGENT PLANES

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1: Find the tangent plane to the elliptic paraboloid

$$z = 2x^2 + y^2 \text{ at the point } (1, 1, 3).$$

Solution: Let $f(x, y) = 2x^2 + y^2$. Then,

$$f_x(x, y) = 4x$$

$$f_y(x, y) = 2y$$

$$f_x(1, 1) = 4$$

$$f_y(1, 1) = 2$$

So, the above Equation gives the equation of the tangent plane at $(1, 1, 3)$ as: $z - 3 = 4(x - 1) + 2(y - 1)$

TANGENT PLANE AND NORMAL LINE TO A SURFACE

If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$0 = F'(t)$$

$$= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) +$$

$F_z(x, y, z)z'(t)$ At (x_0, y_0, z_0) , the equivalent

vector for

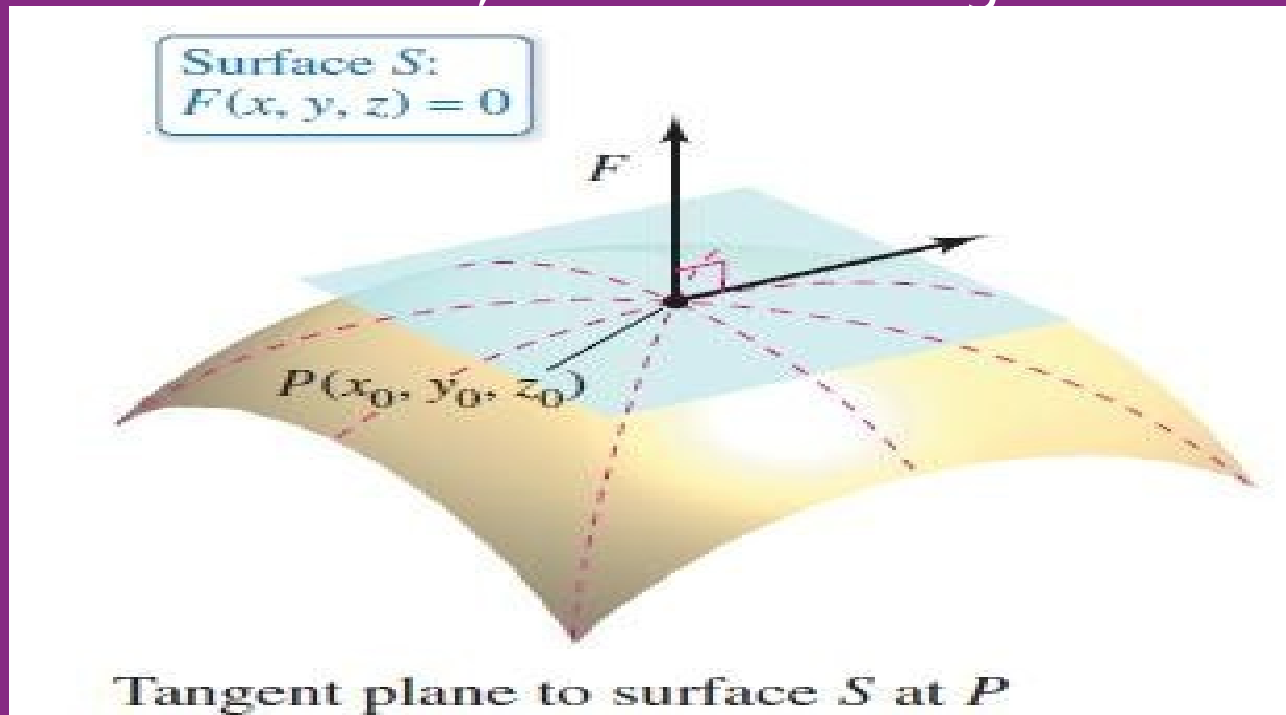
$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

Gradient

Tangent
vector

TANGENT PLANE AND NORMAL LINE TO A SURFACE

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P . So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P , as shown in Figure.



TANGENT PLANE AND NORMAL LINE TO A SURFACE

Definitions of Tangent Plane and Normal Line

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}.$$

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane** to S at P .
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line** to S at P .

To find an equation for the tangent plane to S at (x_0, y_0, z_0) , let (x, y, z) be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane.

TANGENT PLANE AND NORMAL LINE TO A SURFACE

Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0) , it must be orthogonal to every vector in the tangent plane, and you have

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$$

which leads to the next theorem.

THEOREM 13.13 Equation of Tangent Plane

If F is differentiable at (x_0, y_0, z_0) , then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

EXAMPLE 2

- Find an equation of the tangent plane to the hyperboloid

$$z^2 - 2x^2 - 2y^2 = 12 \quad \text{at the point } (1, -1, 4).$$

Solution:

Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

We have

$$F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y \quad \text{and} \\ F_z(x, y, z) = 2z.$$

SOLUTION(CON T'D)

cont'd

At the point $(1, -1, 4)$ the partial derivatives are

$$F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4,$$

and

$$F_z(1, -1, 4) = 8.$$

So, an equation of the tangent plane at $(1, -1, 4)$ is

$$\begin{aligned} -4(x - 1) + 4(y + 1) + 8(z - 4) &= 0 \\ -4x + 4 + 4y + 4 + 8z - 32 &= 0 \\ -4x + 4y + 8z - 24 &= 0 \\ x - y - 2z + 6 &= 0. \end{aligned}$$

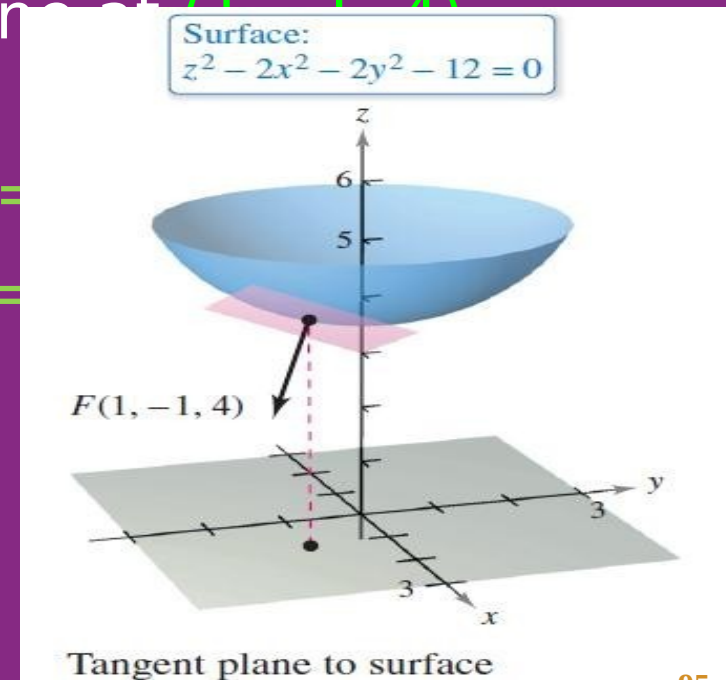


Figure shows a portion of the

TANGENT PLANE AND NORMAL LINE TO A SURFACE

To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

Then S is given by the level surface $F(x, y, z) = 0$, and by the above

Theorem, an equation of the tangent plane to S at

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

- **Exercise:** Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{3} = 1$$

Tangent Planes and Linear Approximations

In this section, we will learn how
to:

Approximate functions using
tangent planes and linear functions.

LINEAR APPROXIMATIONS

⇒ In general, an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

⇒ The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) +$$

$f_y(a, b)(y - b)$ is called the

linearization of f at (a, b) .

⇒ The approximation:

CONT'D

⇒ We have defined tangent planes for surfaces

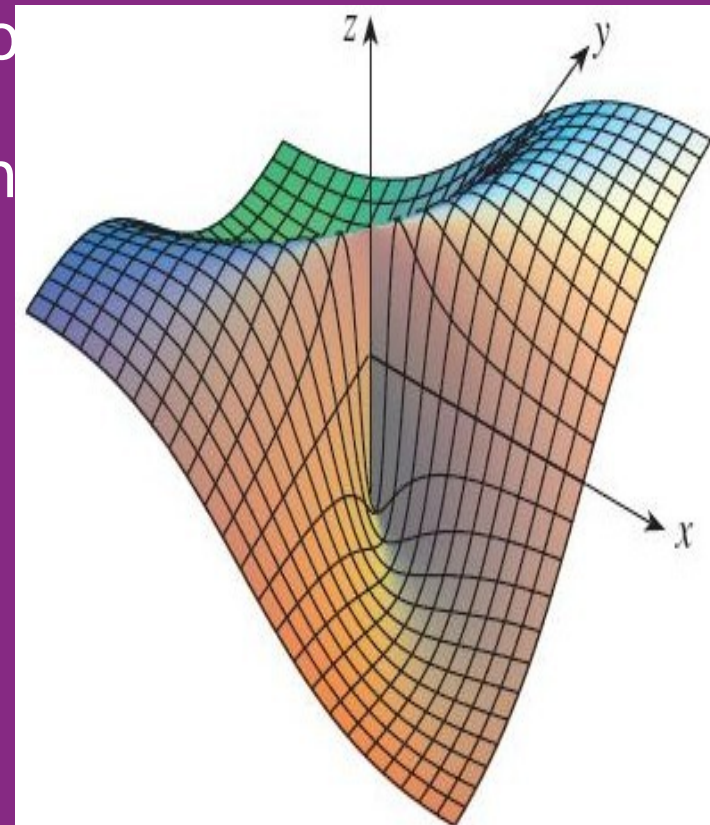
$z = f(x, y)$, where f has continuous first partial derivatives.

• What happens if f_x and f_y are not continuous?

⇒ The figure pictures such a function

• Its equation is:

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$



CONT'D

⇒ You can verify that its partial derivatives exist at the origin and, in

fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$.

⇒ However, f_x and f_y are not continuous.

⇒ Thus, the linear approximation would be $f(x, y) \approx 0$.

- However, $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

⇒ Thus, a function of two variables can behave badly even though

both of its partial derivatives exist.

CONT'D

⇒ Recall that, for a function of one variable, $y = f(x)$, if x

changes from a to $a + \Delta x$,

we defined the increment of y as:

$$\Delta y = f(a + \Delta x) - f(a)$$

⇒ Now, consider a function of two variables, $z = f(x, y)$.

⇒ Suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$.

⇒ Then, the corresponding increment of z is:

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

CONT'D

- Thus, the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Defn.: If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

- ◆ That is, the tangent plane approximates the graph of f well near the point of tangency.

CONT'D

⇒ It's sometimes hard to use the Definition directly to check the differentiability of a function.

- However, the next theorem provides a convenient sufficient condition for differentiability.

Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

EXAMPLE

a. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there.

b. Then, use it to approximate $f(1.1,$

$-0.1)$. **Solution:** The partial

$$f_x(x, y) = e^{xy} + xye^{xy} \quad f_y(x, y) = x^2e^{xy}$$

derivatives are:

$$f_x(1, 0) = 1 \quad f_y(1, 0) = 0$$

- Both f_x and f_y are continuous functions.
- So, f is differentiable by the above Theorem.

CONT'D

⇒ The linearization is:

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y$$

$$= x + y$$

⇒ The corresponding linear approximation is:

$$xe^{xy} \approx x + y$$

⇒ So, $f(1.1, -0.1) \approx 1.1 - 0.1 = 1$

- Compare this with the actual value of

DIFFERENTIALS

⇒ For a differentiable function of one variable, $y = f(x)$, we

define the differential dx to be an independent variable.

- That is, dx can be given the value of any real number.
- Then, the differential of y is defined as:

$$dy = f'(x) dx$$

⇒ For a differentiable function of two variables, $z = f(x, y)$, we

define the differentials dx and dy to be independent variables.

CONT'
D

$$\frac{dz}{dy} = \frac{f_x(x, y)}{f_y(x, y)} \frac{dx}{dy} = \frac{f_x(x, y)}{f_y(x, y)} \frac{dx}{dy}$$

◦ Sometimes, the notation df is used in place of dz

⇒ If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in the above Equation , then the differential of z is:

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

EXAMPLE

a. If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .

b. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz and dz .

Solution: From the above definition:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (2x + 3y) dx + (3x - 2y) dy$$

CONT'D

Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$,

$dy = \Delta y = -0.04$, we get:

$$\begin{aligned} dz &= [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) \\ &= 0.65 \end{aligned}$$

⇒ The increment of z is:

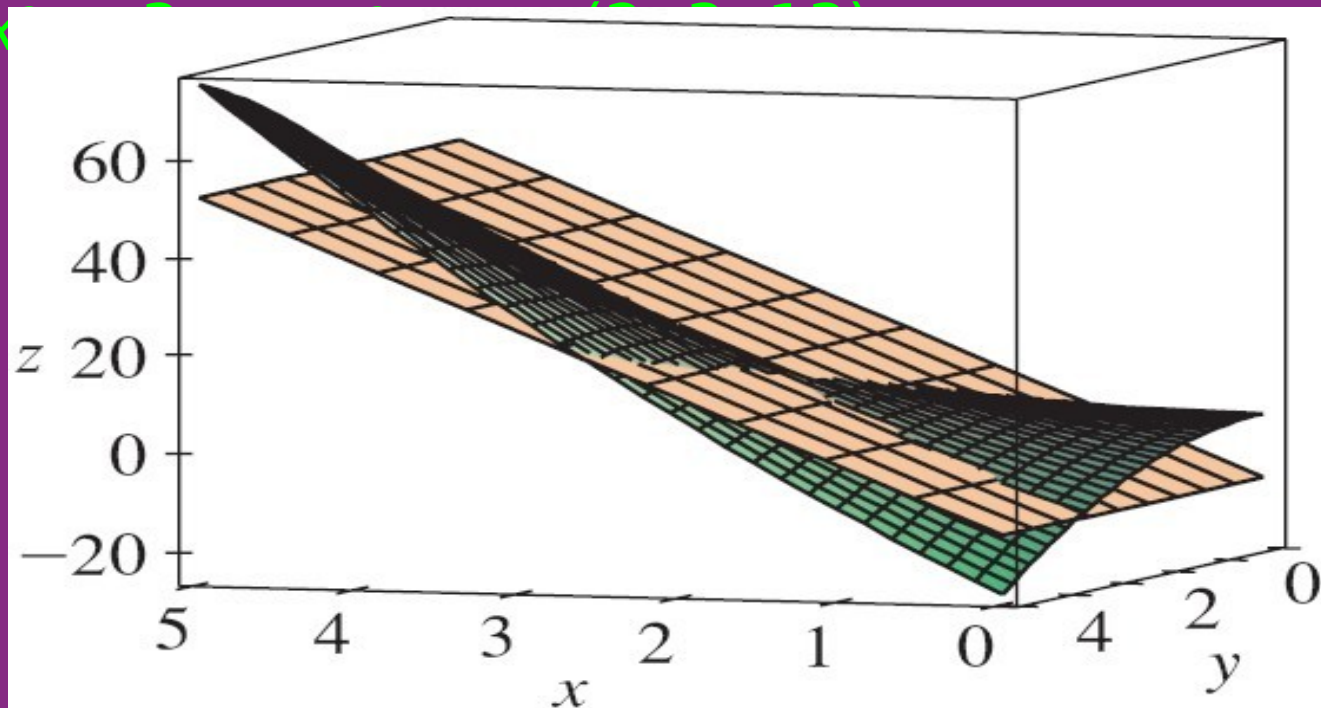
$$\begin{aligned} \Delta z &= f(2.05, 2.96) - f(2, 3) \\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - \\ &\quad [2^2 + 3(2)(3) - 3^2] \\ &= 0.6449 \end{aligned}$$

▪ Notice that Δz
but dz is

CONT'D

- ◆ In the above Example , dz is close to Δz because the tangent plane is a good approximation to the surface

$$z = x^2 + y^2$$



EXERCISE

1. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each.
 - Use differentials to estimate the maximum error in the calculated volume of the cone

FUNCTIONS OF THREE OR MORE VARIABLES

- Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.

- The linear approximation is:

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) +$$

$$f_z(a, b, c)(z - c)$$

- The linearization $L(x, y, z)$ is the right side of this

CONT'D

- If $w = f(x, y, z)$, the increment of w is:

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

- The differential dw is defined in terms of the differentials dx , dy , and dz of the independent variables by:

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

Example: The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2

SOLUTION

If the dimensions of the box are x , y , and z , its volume is

$$V = xyz.$$

- Thus, We are given that

$$|\Delta x| \leq 0.2, |\Delta y| \leq 0.2, |\Delta z| \leq 0.2$$

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ &= yz dx + xz dy + xy dz \end{aligned}$$

- To find the largest error in the volume, we use $dx = 0.2, dy =$

CONT'
D

$$\Rightarrow \text{Thus, } \Delta V \approx dV$$

$$= (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2)$$

$$= 1980$$

$\Rightarrow$ So, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm³ in the calculated volume.

- This may seem like a large error.
- However, it's only about 1% of the volume of the box

THE CHAIN RULE

In this section, we will learn about The Chain Rule and its

application in implicit differentiation.

- Recall that the Chain Rule for functions of a single variable gives the following rule for differentiating a composite function.
- If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t , and

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

function

$$\frac{dy}{dx} \frac{dx}{dt}$$

CONT'D

- The first version deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t .
 - This means that z is indirectly a function of t , $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t .
- We assume that f is differentiable
 - Recall that this is the case when f_x and f_y are continuous

THE CHAIN RULE (CASE 1)

Theorem 1: Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then, z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

EXAMPLE

- If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

- The Chain Rule gives:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= (2xy + 3y^4)(2 \cos 2t)$$

CONT'D

- It's not necessary to substitute the expressions for x and y in terms of t .

- We simply observe that, when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$

Thus,

$$\left. \frac{dz}{dt} \right|_t = (0 - 3)(2 \cos 0) - (0 - 0) \\ = (-3)(2) = -6$$

CONT'D

- If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y) , then The composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C .
- The derivative $\frac{dz}{dt}$ represents the rate at which the temperature changes along C .

CONT'D

HH

- We now consider the situation where $z = f(x, y)$, but each of x and y is a function of two variables s and t : $x = g(s, t)$, $y = h(s, t)$.
 - Then, z is indirectly a function of s and t , and we wish to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.
- In computing $\frac{\partial z}{\partial t}$, we hold s fixed and compute the derivative of z with respect to t . So, we can apply **Theorem 1** to obtain:

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

CONT'D

- A similar argument holds for $\frac{\partial z}{\partial s}$.
- So, we have proved the following version of the Chain Rule.

THE CHAIN RULE (CASE 2)

- Suppose $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

EXAMPLE

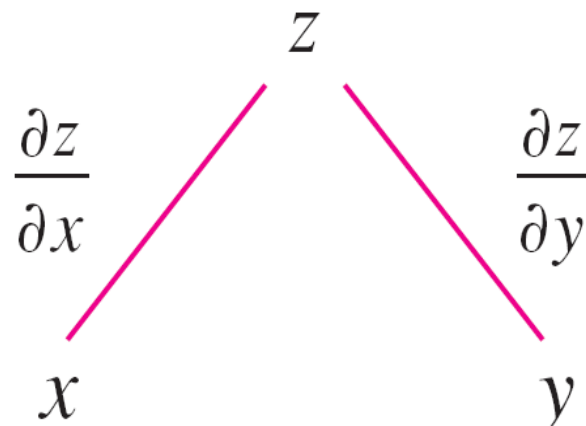
• If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

- Applying Case 2 of the Chain Rule, we get the following results.

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= e^{st^2} t^2 \sin(s^2 t) + 2st e^{st^2} \cos(s^2 t) \end{aligned}$$

CONT'D

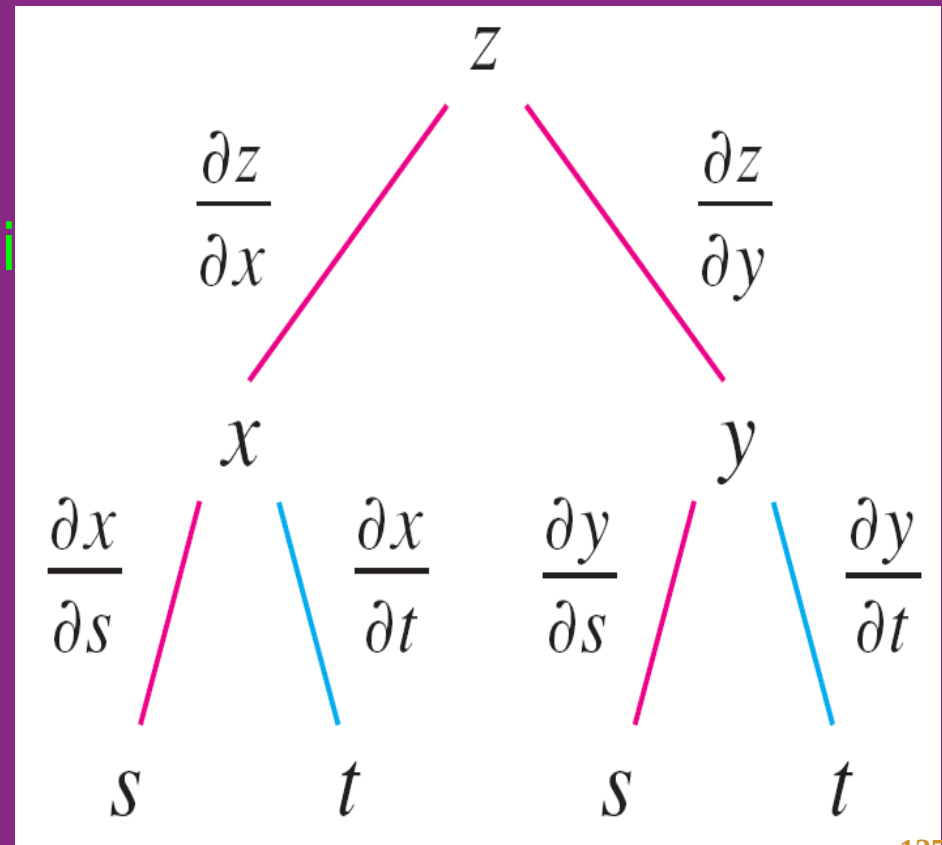
- Case 2 of the Chain Rule contains three types of variables:
 - s and t are independent variables.
 - x and y are called intermediate variables.
 - z is the dependent variable.
- To remember the Chain Rule, it's helpful to draw a tree diagram, as follows.
- We draw branches (the dependent variable z and the independent variables x and y to indicate



CONT'D

Then, we draw branches from x and y to the independent variables s and t .

- On each branch, we write the corresponding partial derivative

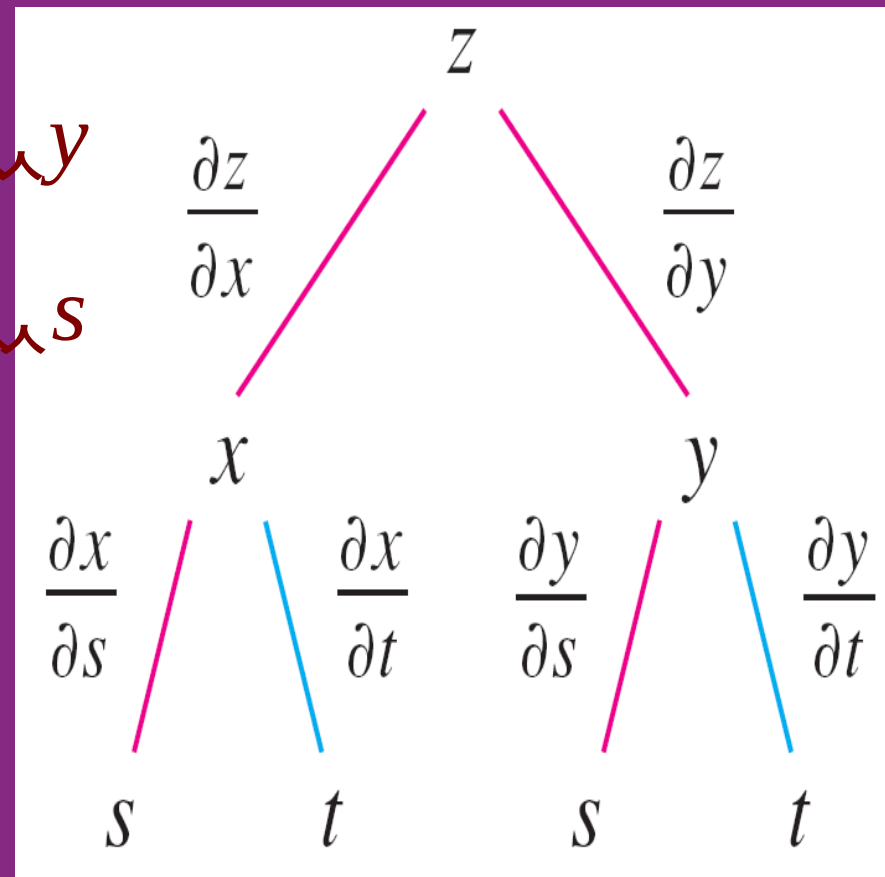


CONT'D

- ▀ To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

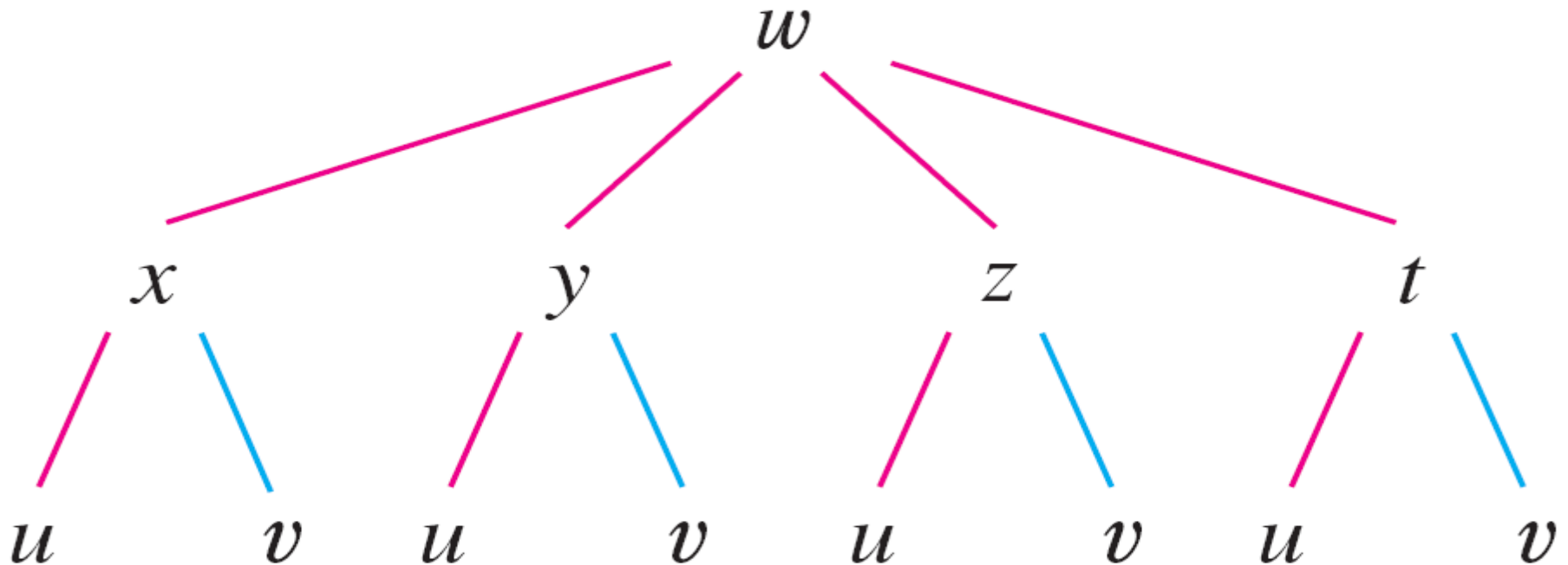
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

- ▀ Similarly, we find $\partial z/\partial t$ by using the paths from z to t .



THE CHAIN RULE (GEN. VERSION)

- ✘ **Example:** Write out the Chain Rule for $\frac{\partial y}{\partial u}$ the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $t = t(u, v)$



CONT'D

- With the aid of the tree diagram, we can now write the required expressions:

$$\mu W \square \quad \mu W \mu X \text{---} \mu W \mu Y \text{---} \mu W \text{---}$$

$\bar{0}$ $\bar{0}$

$$\mu Z \text{---} \mu W \mu t$$

$\bar{0}$

$$\mu u \text{---} \text{---} \mu X \text{---} \text{---} \mu u \text{---} \text{---} \mu y \text{---} \text{---}$$

μu μZ μu

$\mu t \mu u$

EXAMPLE

✘ If $u = x^4y + y^2z^3$, where

$$x = rse^t, y = rs^2e^{-t}, z = r^2s \sin t$$

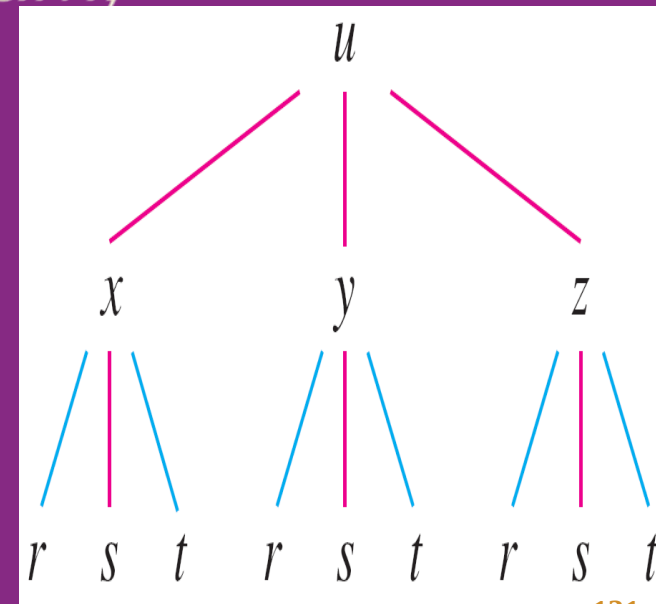
find the value of $\frac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$

✘ With the help of this tree diagram, we have:

$$\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) +$$



CONT'D

When $r = 2$, $s = 1$, and $t = 0$,
we have:

$$x = 2, y = 2, z$$

Thus,

$$\frac{\mu^U}{\mu^S} \square \frac{(64)(2)}{(0)(0)} \equiv \frac{(16)(4)}{(0)(0)} \equiv 192$$

EXERCISES

1. If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t \frac{\partial g}{\partial s} = s \frac{\partial g}{\partial t}$$

2. If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find:

a) $\frac{\partial z}{\partial r}$

b) $\frac{\partial^2 z}{\partial r^2}$

IMPLICIT DIFFERENTIATION

- We suppose that an equation of the form
 $F(x, y) = 0$ defines y implicitly as a differentiable function of x .
 - That is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f .
- If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x .

CONT'D

- ◆ Since both x and y are functions of x , we obtain:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx}$$

- ✕ However, $\frac{dx}{dx} = 1$. So, if $\frac{\partial F}{\partial y} \neq 0$, we solve for $\frac{dy}{dx}$ and obtain:

$$\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = - \frac{F_x}{F_y}$$

CONT'D

- To get the equation, we assumed $F(x, y) = 0$ defines y implicitly as a function of x .
- The Implicit Function Theorem, proved in advanced calculus, gives conditions under which this assumption is valid.
- The theorem states the

CONT'D

- Then, the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by the above equation.
- Example: Find y' if $x^3 + y^3 = 6xy$.

Solution: The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

- So,
:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = \frac{x^2 - 2}{y^2 - 2x}$$

CONT'D

- Now, we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.
- This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f .
- If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, f(x, y)) = 0$ as follows:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

CONT'D

- However,

$$\frac{\partial \mathcal{L}}{\partial x} = 1 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = 0$$

- So, that equation becomes:

$$\frac{\partial \mathcal{L}}{\partial z} = 0$$

- If $\frac{\partial \mathcal{L}}{\partial z} \neq 0$, we solve for $\frac{\partial \mathcal{L}}{\partial z}$ and obtain the first formula in these equations.

CONT'D

☐

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad \square$$

◆ The formula for $\frac{\partial z}{\partial y}$ is obtained in a similar manner.

- Again, a version of the Implicit Function Theorem gives conditions under which our assumption is valid.

CONT'D

- This version states the following.
- Suppose F is defined within a sphere containing (a, b, c) , where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x , F_y , and F_z are continuous inside the sphere.
- Then, the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) ,

EXAMPLE

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$

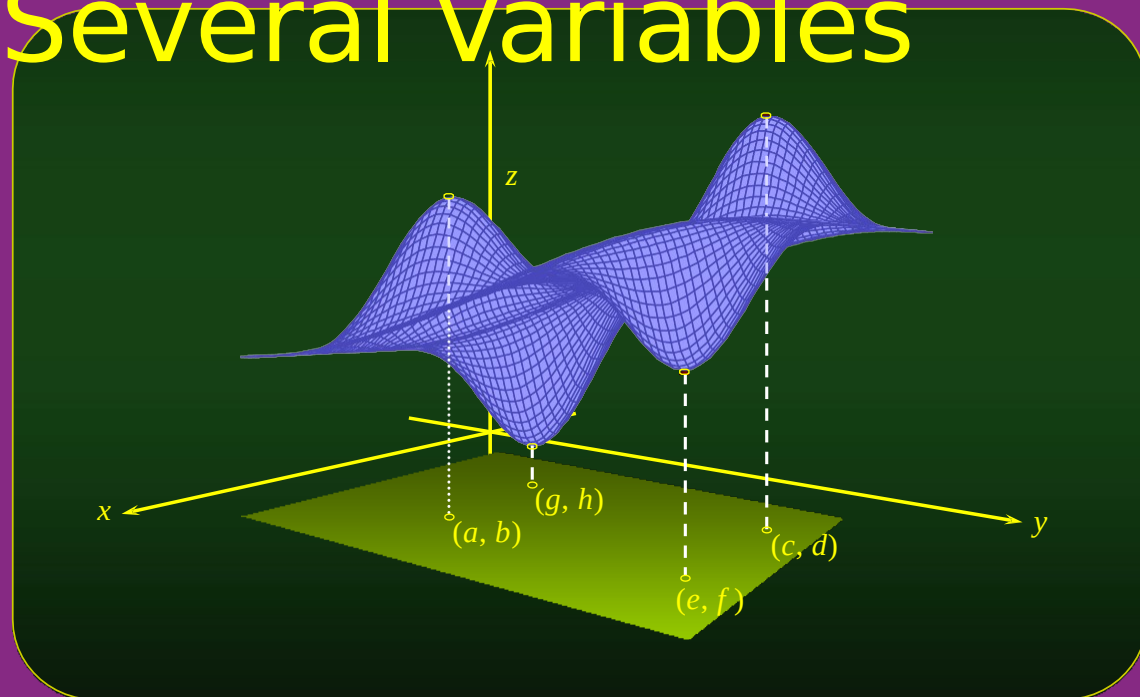
Solutions: Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$

Then, from the above Equations, we have:

$$\begin{aligned} \frac{\partial F}{\partial x} &= 3x^2 + 6yz = 0 & \frac{\partial F}{\partial y} &= 3y^2 + 6xz = 0 \\ \frac{\partial F}{\partial z} &= 3z^2 + 6xy = 0 \end{aligned}$$

3.3

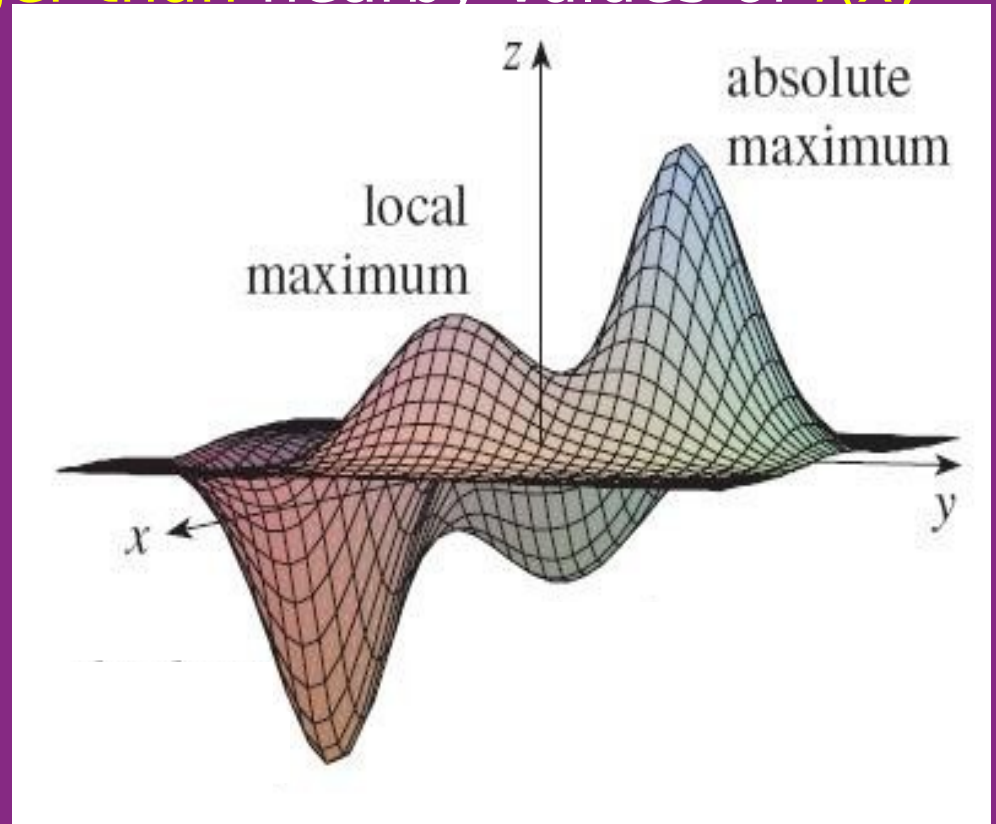
Maxima and Minima of Functions of Several Variables



ABSOLUTE MAXIMUM

▀ There are two points (a, b) where f has a **local maximum** —that is, where $f(a, b)$ is **larger than** nearby values of $f(x, y)$.

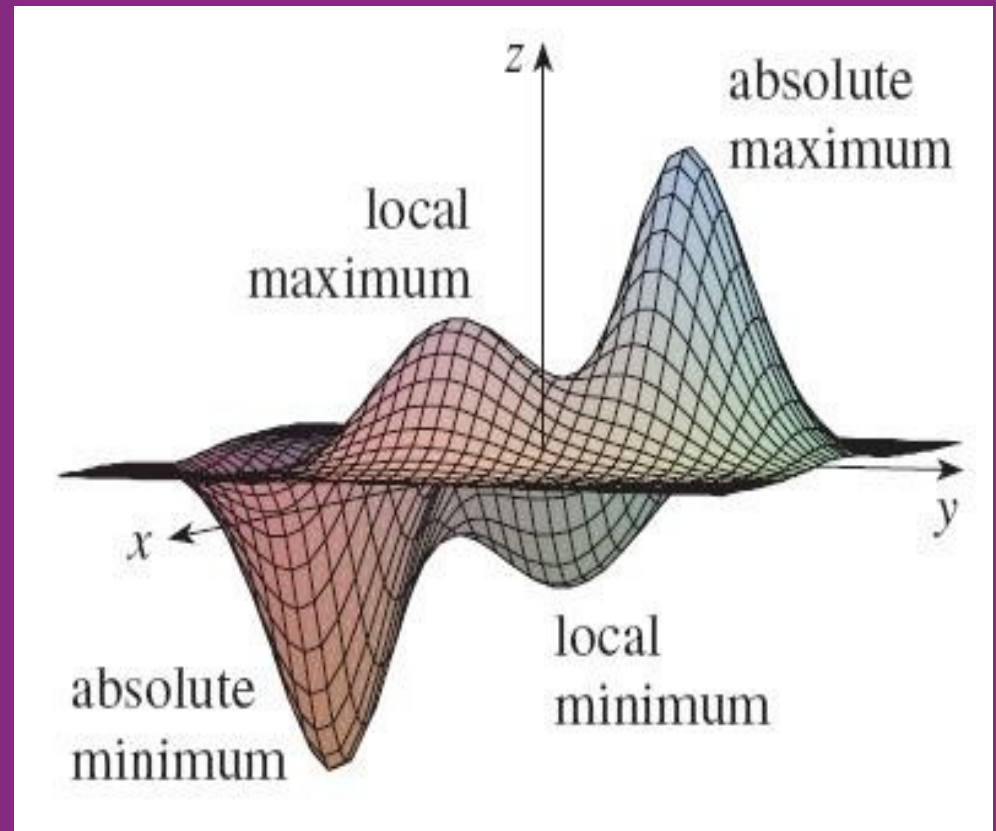
- The larger of these two values is the **absolute maximum**



ABSOLUTE MINIMUM

▮ Likewise, f has **two local minima** —where $f(a, b)$ is smaller than nearby values.

- The smaller of these two values is the **absolute minimum**.



RELATIVE EXTREMA OF A FUNCTION OF TWO VARIABLES

- ▮ Let f be a function defined on a region R containing the point (a, b) .
- ▮ Then, f has a relative maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b) .
 - The number $f(a, b)$ is called a relative maximum value.
- ▮ Similarly, f has a relative minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b) .

DEFN. OF LOCAL MAX. & LOCAL MIN. VALUE

▮ A function of two variables has a **local maximum** at (a, b) if

$$f(x, y) \leq f(a, b) \text{ when } (x, y) \text{ is near } (a, b).$$

▮ This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .

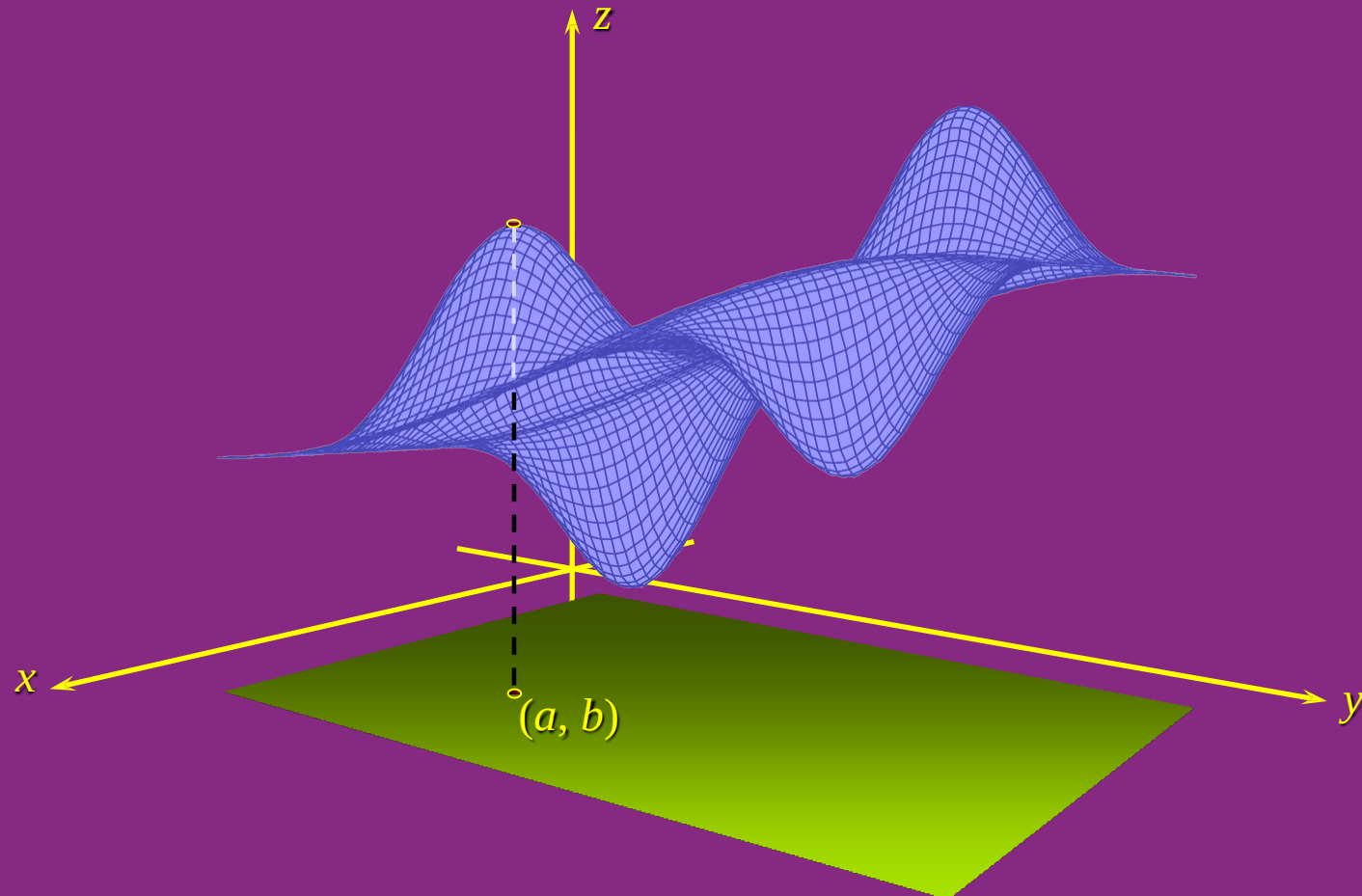
◦ The number $f(a, b)$ is called a **local maximum value**.

▮ If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a local minimum at (a, b) . $f(a, b)$ is a **local minimum value**.

Note: If the inequalities in the above Definition hold for all points (x, y) in the domain of f , then f has

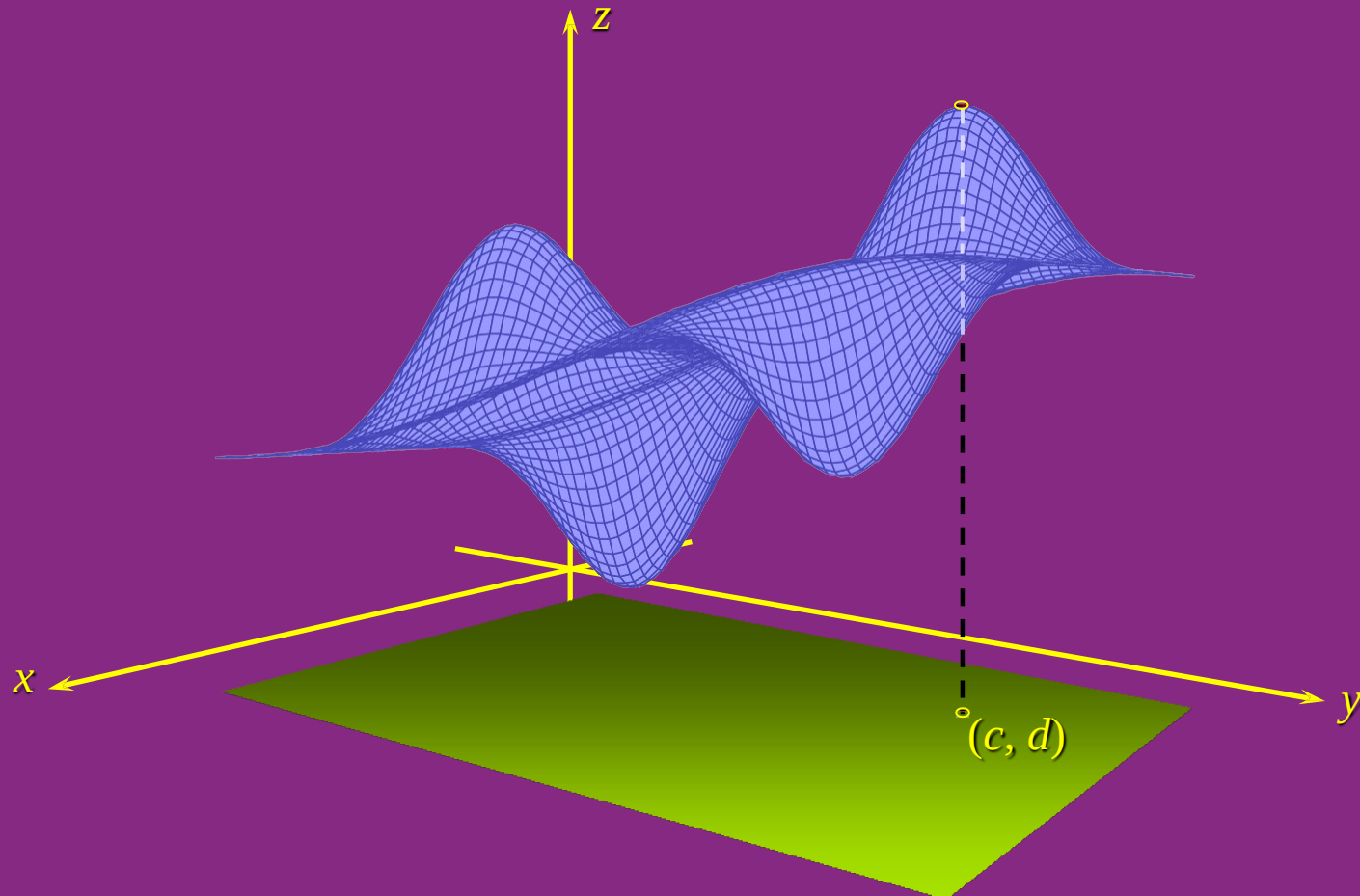
Graphic Example

- ◆ There is a **relative maximum** at (a, b) .



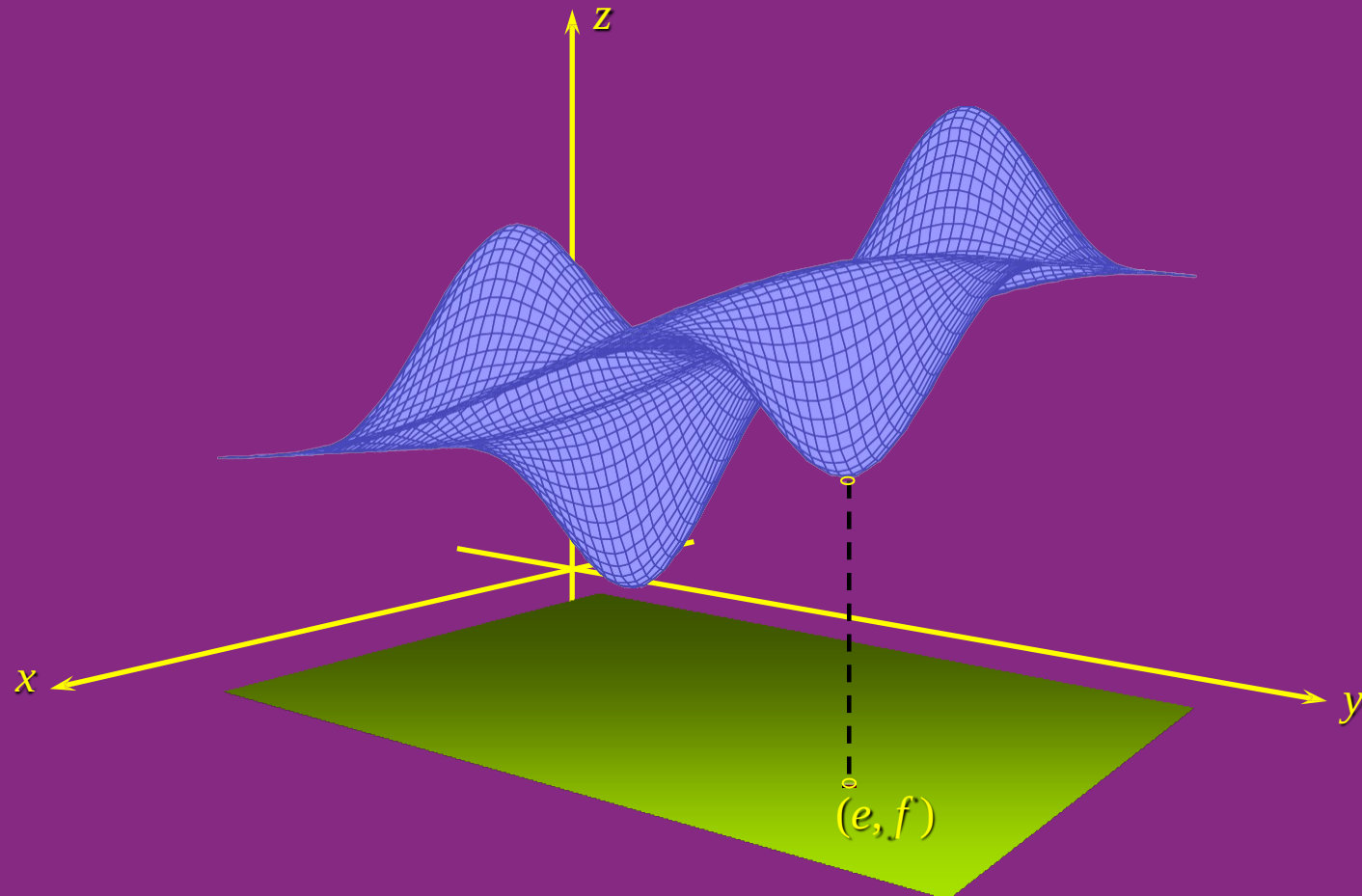
Graphic Example

- ◆ There is an **absolute maximum** at (c, d) . (It is also a **relative maximum**)



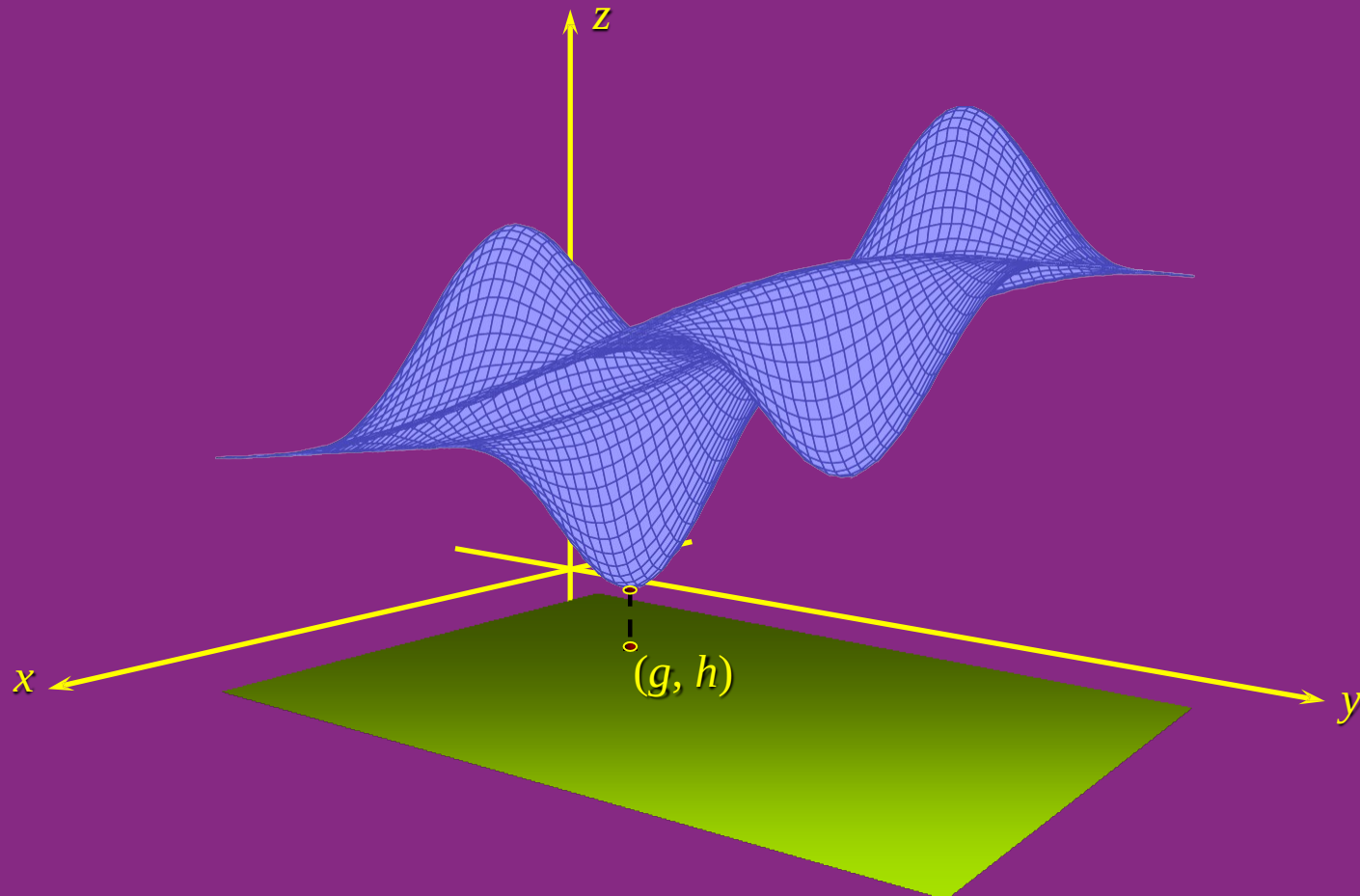
Graphic Example

◆ There is a **relative minimum** at (e, f) .



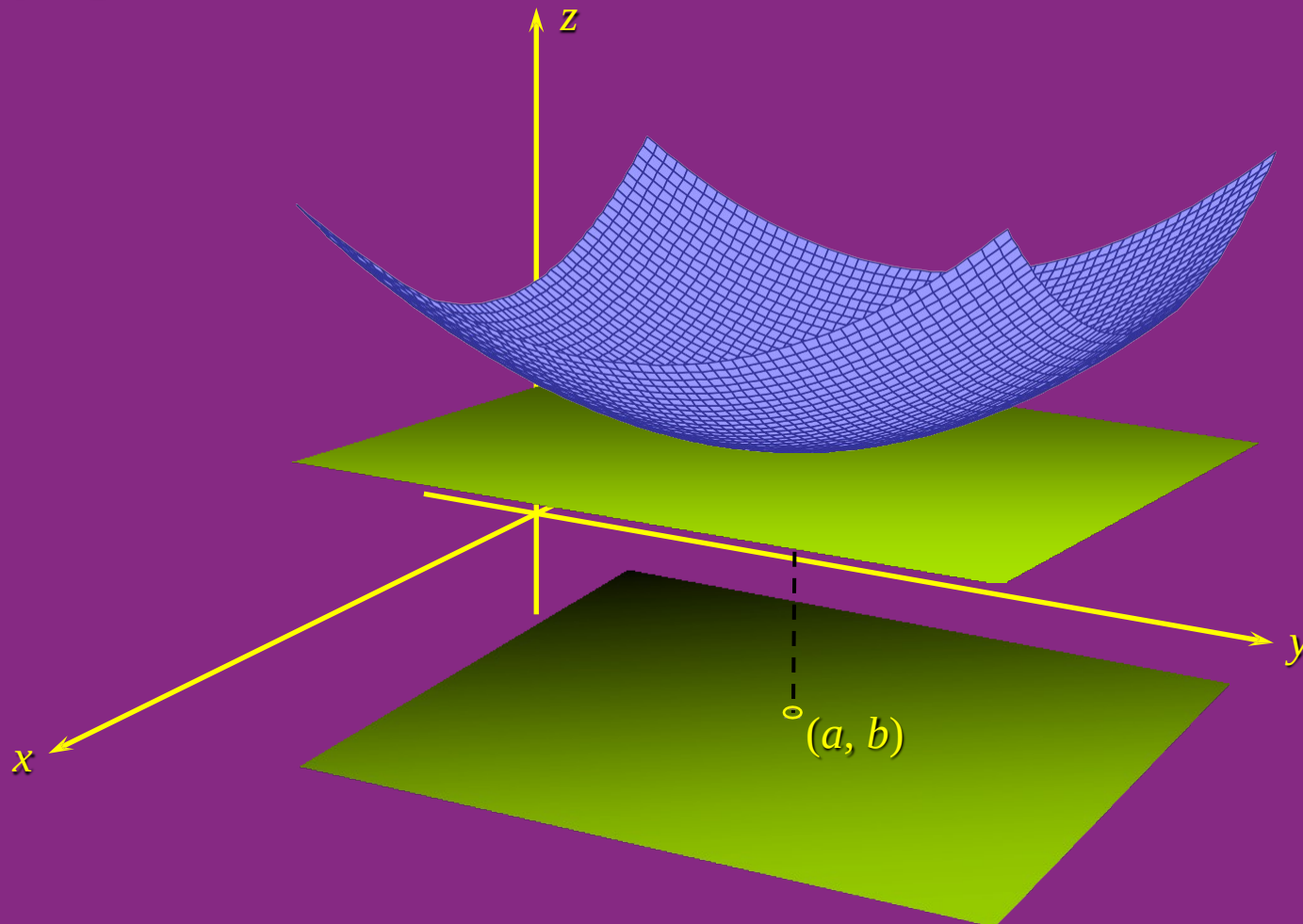
Graphic Example

- ◆ There is an **absolute minimum** at (g, h) . (It is also a **relative minimum**)



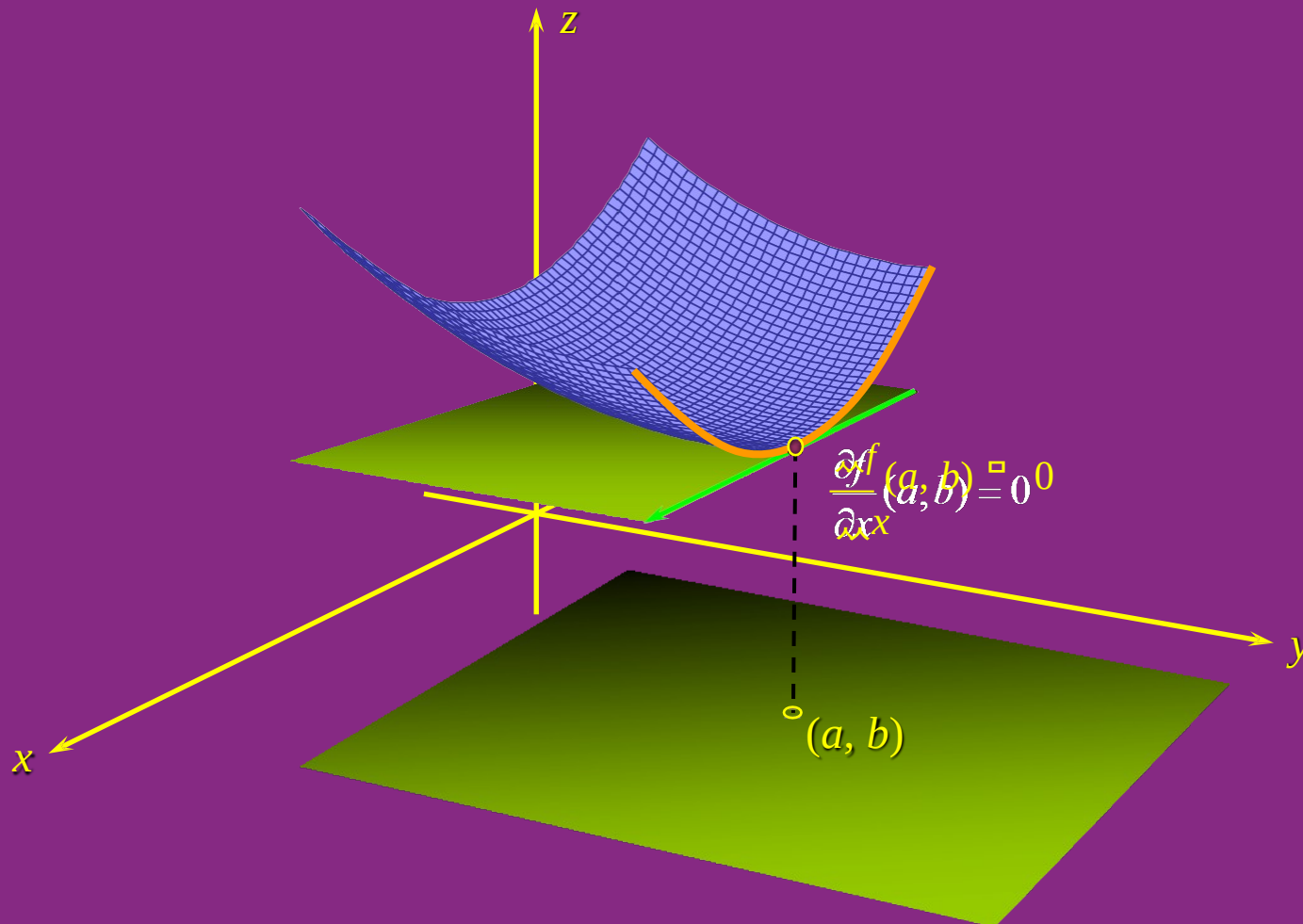
Relative Minima

- ◆ At a **minimum point** of the graph of a function of two variables, such as point (a, b) below, **the plane tangent to the graph of the function is horizontal** (assuming the surface of the graph is smooth):



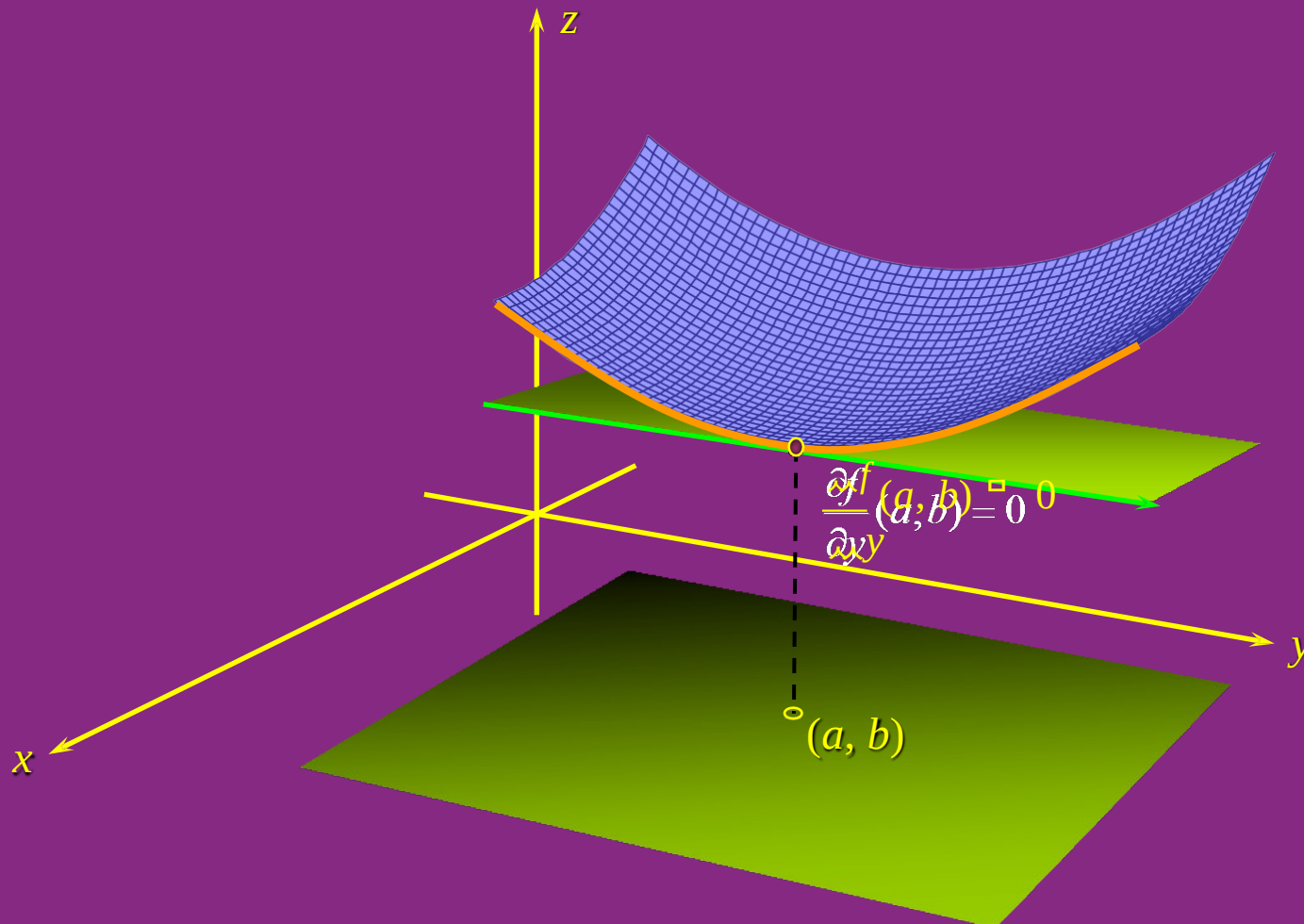
Relative Minima

- ◆ Thus, at a **minimum point**, the graph of the function has a **slope of zero** along a direction **parallel** to the **x-axis**:



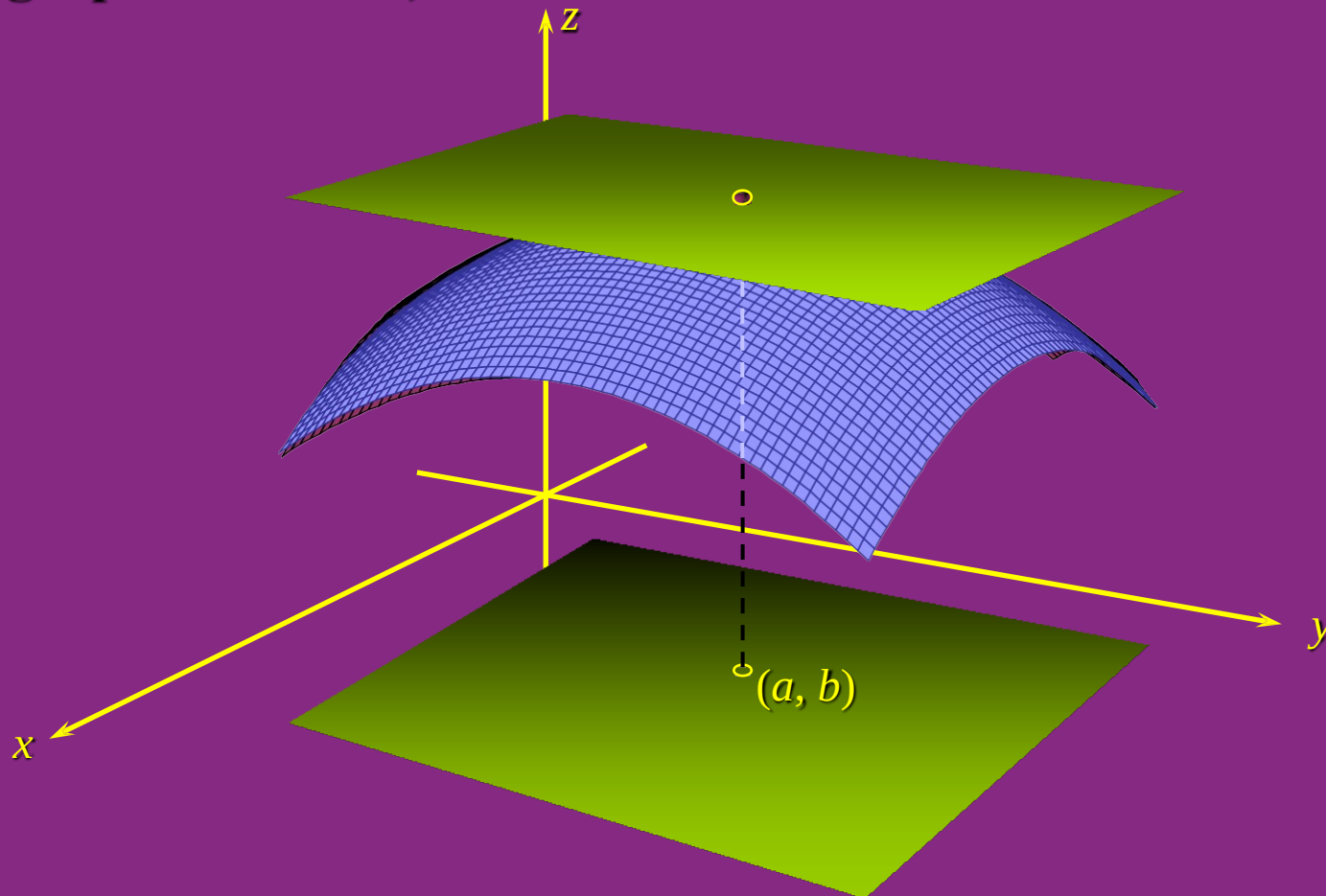
Relative Minima

- ◆ Similarly, at a **minimum point**, the graph of the function has a **slope of zero** along a direction **parallel to the y-axis**:



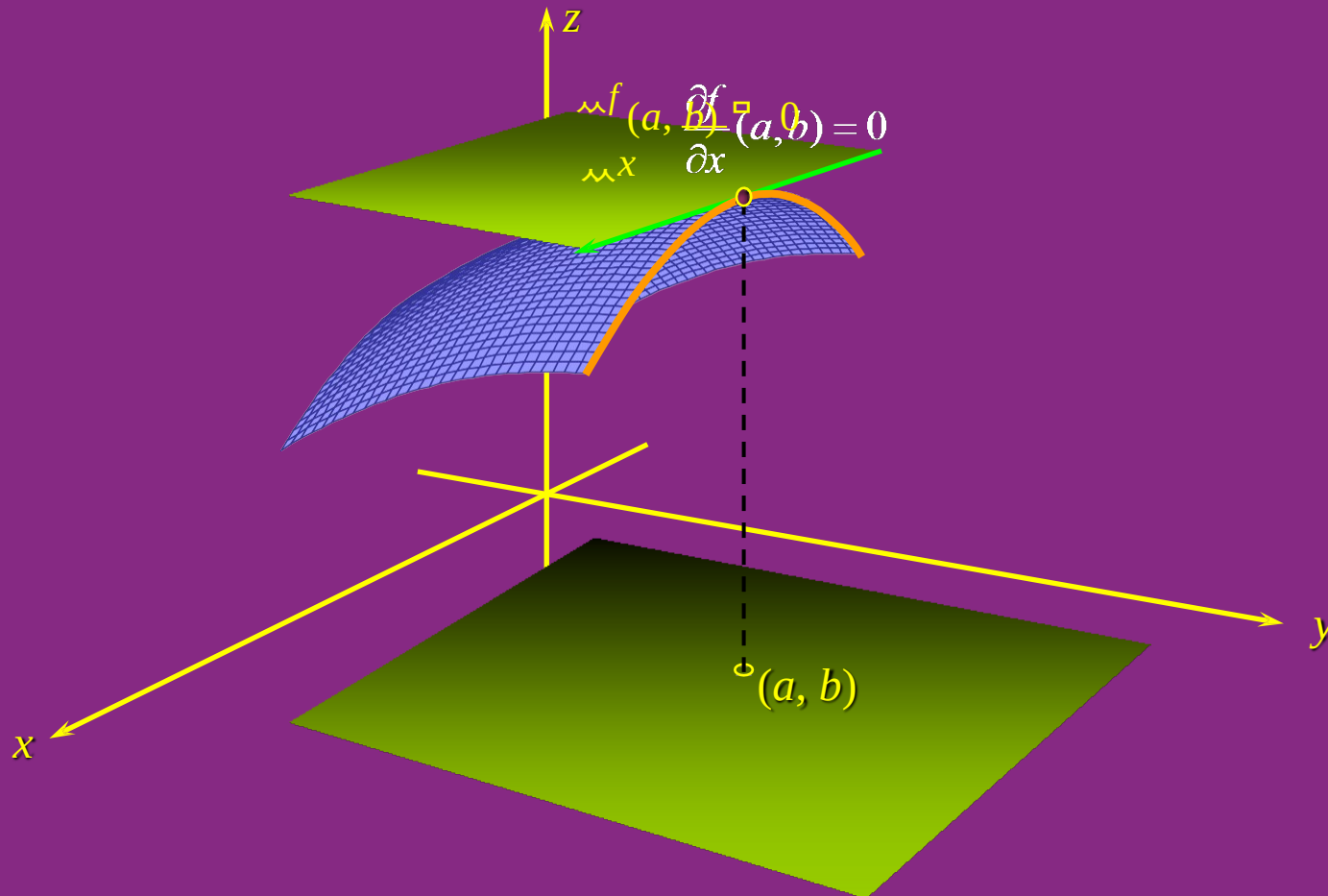
Relative Maxima

- ◆ At a **maximum point** of the graph of a function of two variables, such as point (a, b) below, **the plane tangent to the graph of the function is horizontal** (assuming the surface of the graph is smooth):



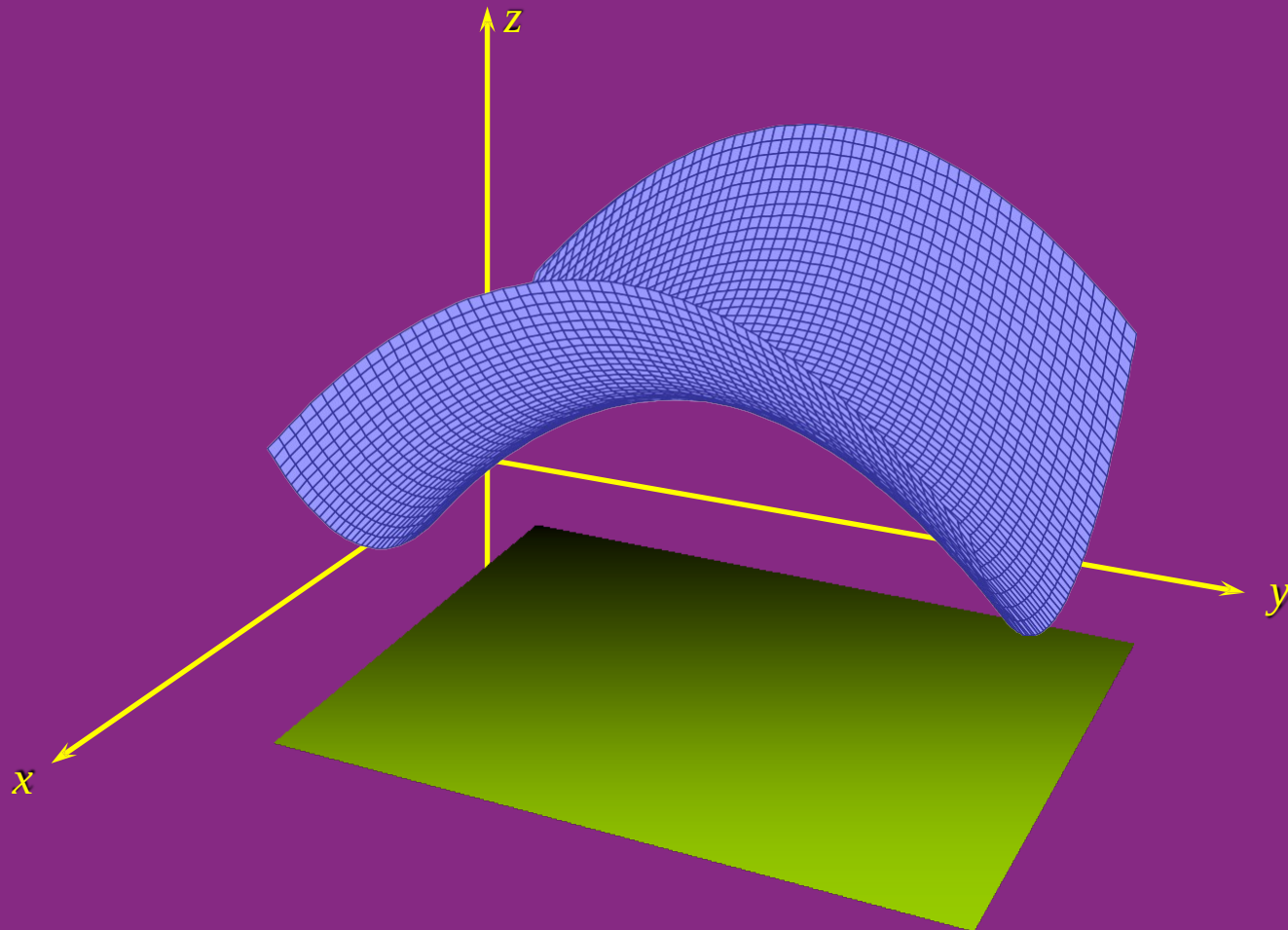
Relative Maxima

- ◆ Thus, at a **maximum point**, the graph of the function has a **slope of zero** along a direction **parallel** to the **x-axis**:



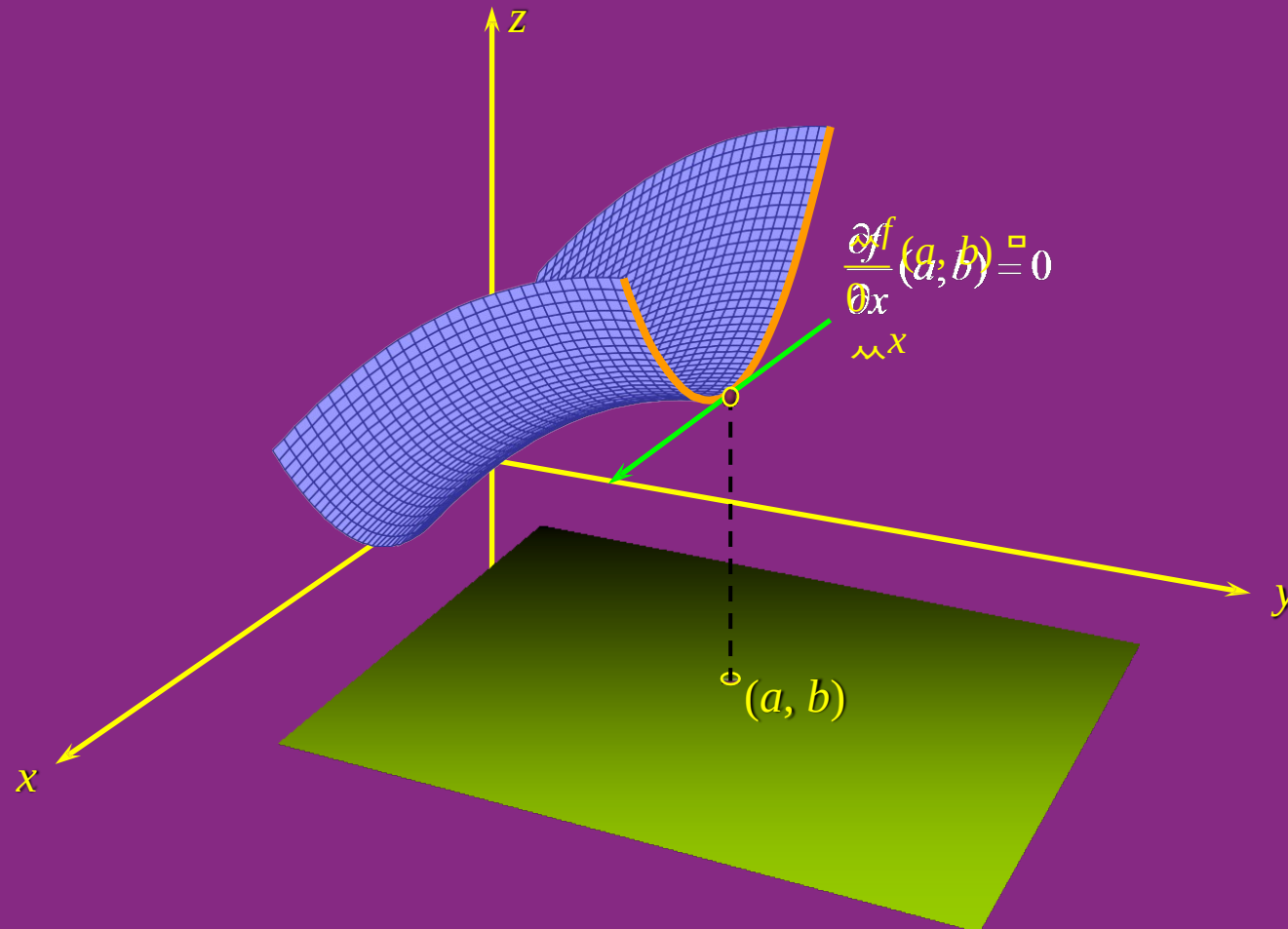
Saddle Point

- ◆ In the case of a saddle point, both partials are equal to zero, but the point is neither a maximum nor a minimum.



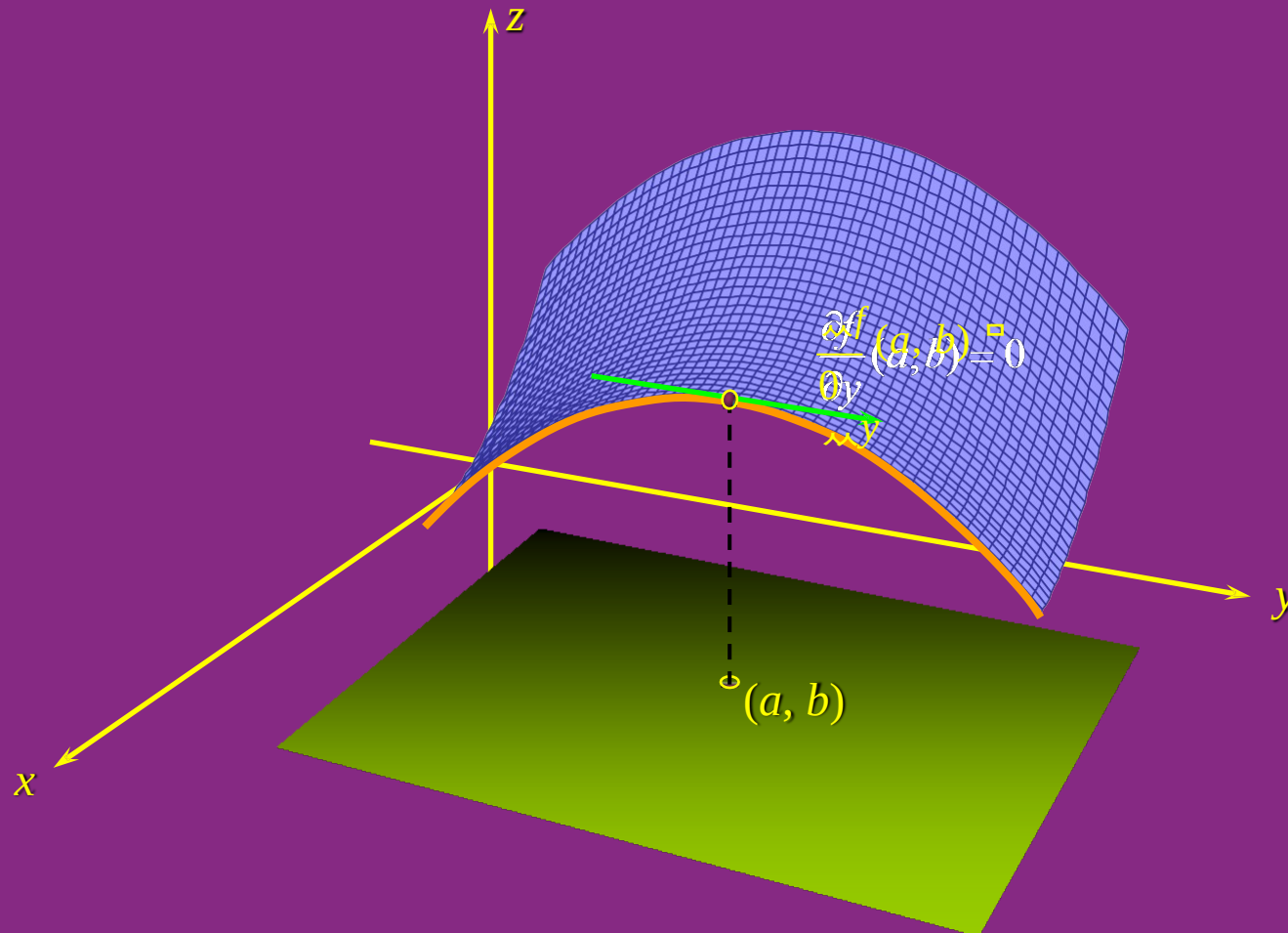
Saddle Point

In the case of a saddle point, the function is at a **minimum** along one vertical plane...



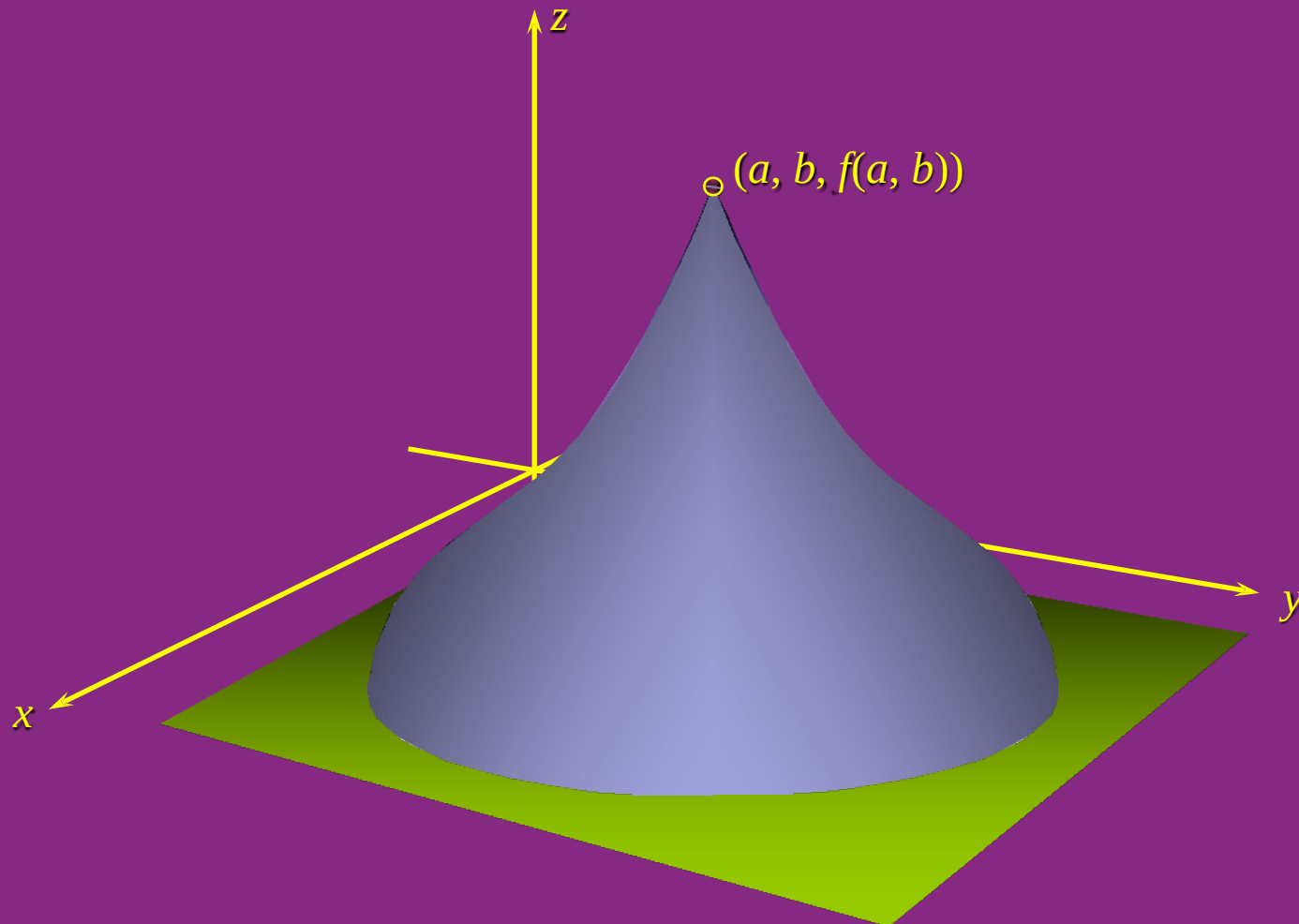
Saddle Point

... but at a **maximum** along the perpendicular vertical plane.



Extrema When Partial Derivatives are Not Defined

- ◆ A maximum (or minimum) may also occur when both partial derivatives are not defined, such as point (a, b) in the graph below:



LOCAL MAXIMUM & MINIMUM THEOREM

- ▮ If f has a local maximum or minimum at (a, b) and the first- order partial derivatives of f exist there, then

$$f_x(a, b) = 0$$

$$\text{and } f_y(a, b) = 0$$

Critical Point of a Function

- ▮ A critical point of a function f is a point in the domain of f where $f_x(a, b) = 0$ and $f_y(a, b) = 0$

or at least one of the partial derivatives does not exist.

EXAMPLE 1

▮ Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the local extreme values.

Solution:

$$f_x(x, y) = 2x$$

$$- 2$$

$$f_y(x, y) = 2y$$

$$- 6$$

▮ These partial derivatives are equal to 0 when $x = 1$ and $y = 3$.

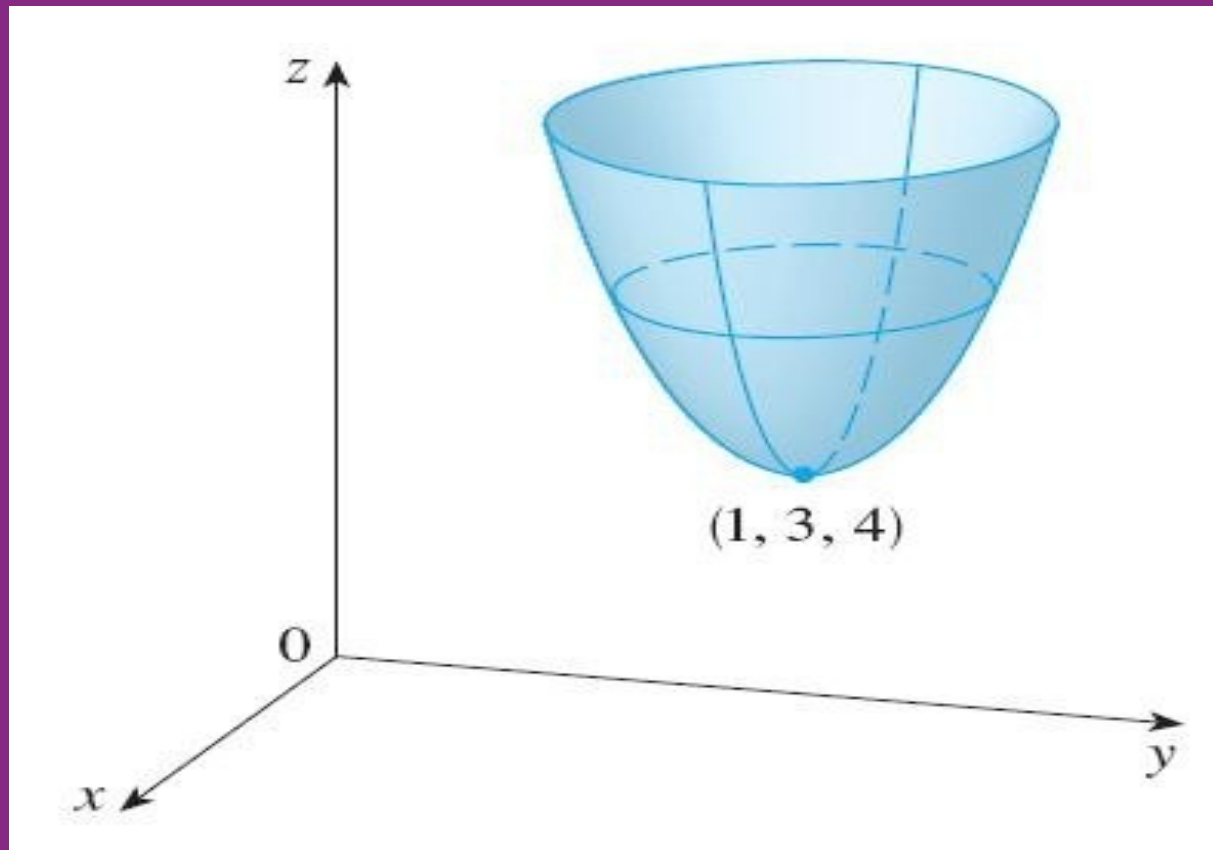
So, the only critical point is $(1, 3)$.

▮ By completing the square, we find: $f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$

▮ Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y .

CONT'D

RR This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$.



EXAMPLE 2

- Find the extreme values of $f(x, y)$
 $= y^2 - x^2$

Solution: Since $f_x = -2x$ and $f_y = -2y$, the only critical point is $(0, 0)$.

- Notice that, for points on the x -axis, we have $y = 0$.

So, $f(x, y) = -x^2 < 0$ (if $x \neq 0$).

- For points on the y -axis, we have $x = 0$.

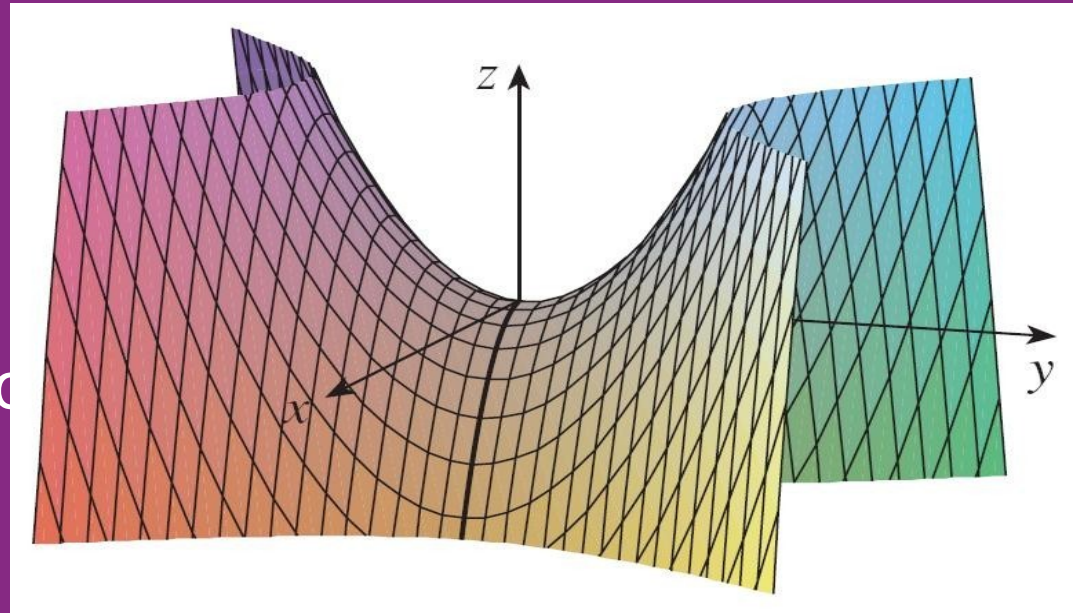
So, $f(x, y) = y^2 > 0$ (if $y \neq 0$).

- Thus, every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values.

- So, $f(0, 0) = 0$ can't be an extreme value for f .

CONT'D

- Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point.
- The figure shows how this is possible. The graph of f is the hyperbolic paraboloid
 - It has a horizontal tangent plane ($z = 0$) at the origin



THE SECOND DERIVATIVE TEST

Suppose that the second partial derivatives of f are continuous on a disk with center (a, b) .

$f_x(a, b) = 0$ and $f_y(a, b) = 0$ (i.e. (a, b) is a critical point of f).

Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$

Then,

a. $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ implies that $f(x, y)$ has a relative maximum at the point (a, b) .

b. $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ implies that $f(x, y)$ has a relative minimum at the point (a, b) .

c. $D(a, b) < 0$ implies that $f(x, y)$ has neither a relative maximum nor a relative minimum at

CONT'D

- ▮ To remember the formula for D , it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (f_{xx}f_{yy} - (f_{xy})^2)$$

- ▮ **Example 1 :** Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

SOLUTION

- ▮ We first locate the critical points:

$$f_x = 4x^3 - 4y$$

$$f_y = 4y^3 - 4x$$

- ▮ Setting these partial derivatives equal to 0, we obtain:

$$x^3 - y = 0$$

$$y^3 - x = 0$$

- ▮ To solve these equations, we substitute $y = x^3$ from the first equation into the second. This gives:

$$0 = x^9 - x$$

$$= x(x^8 - 1)$$

$$= x(x^4 - 1)(x^4 + 1)$$

$$= x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

CONT'D

- So, there are three real roots: $x = 0, 1, -1$
 - The three critical points are: $(0, 0), (1, 1), (-1, -1)$

- Next, we calculate the second partial

derivatives and $D(x, y)$: $f_{xx} = 12x^2$ $f_{xy} = -4$ $f_{yy} = 12y^2$

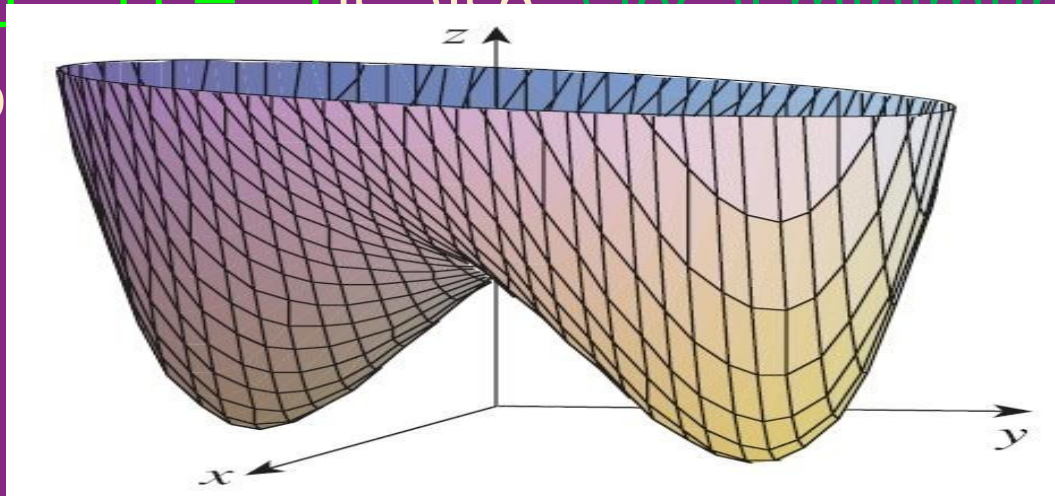
$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

- As $D(0, 0) = -16 < 0$, it follows from case c of the Second Derivatives Test that the origin is a saddle point.

CONT'D

- ▮ As $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case a of the test that $f(1, 1) = -1$ is a local minimum.
- ▮ Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$.
 - So $f(-1, -1) = -1$ is also a local minimum.

- ▮ The graph



EXAMPLE 1

Find the **relative extrema** of the

$$f(x, y) = x^2 + y^2$$

Solution: We

$$f_x = 2x$$

$$\text{and } f_y = 2y$$

have

To find the **critical points**, we set $f_x = 0$ and $f_y = 0$ and solve the

resulting system of **simultaneous equations**

$$2x = 0$$

$$\text{and } 2y = 0$$

obtaining $x = 0$, $y = 0$, or $(0, 0)$, as the **sole critical point**.

Next, apply the **second derivative test** to determine the nature of the critical point $(0, 0)$.

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = (2)(2) -$$

Thus,

We have $D(\bar{x}, y) = 4$, and in particular,

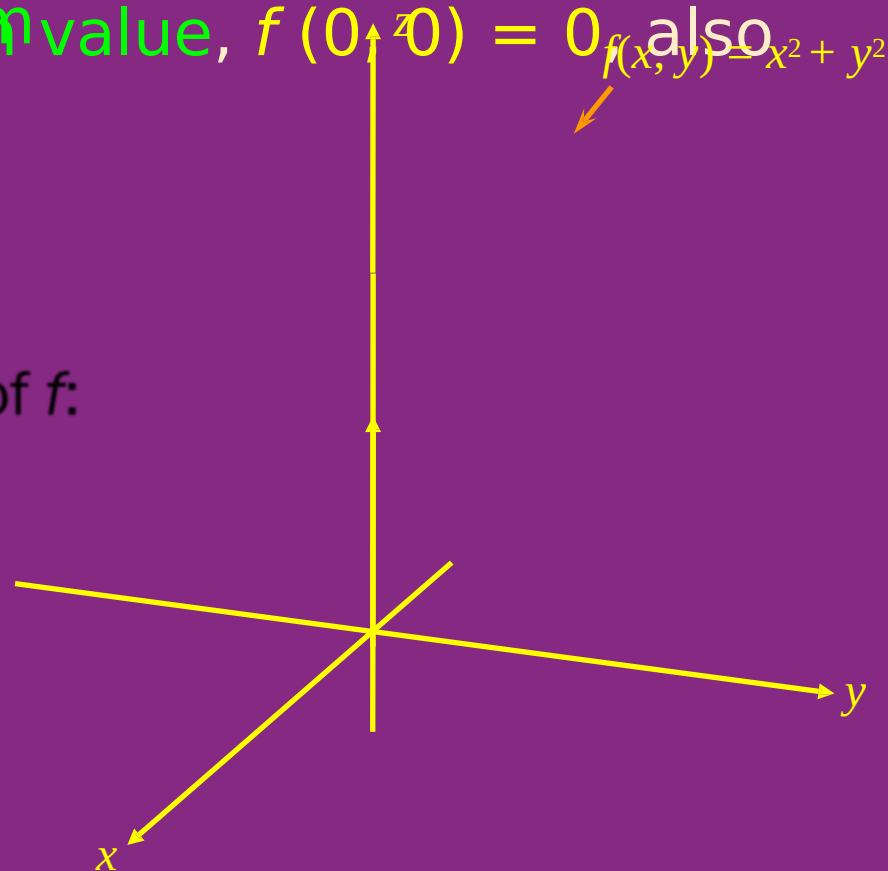
$$D(0, 0) = 4.$$

CONT'D

Since $D(0, 0) > 0$ and $f_{xx} = 2 > 0$, we conclude that f has a **relative minimum** at the point $(0, 0)$.

The **relative minimum value**, $f(0, 0) = 0$, also happens to be the **absolute minimum**

The **relative minimum value**, $f(0, 0) = 0$, also happens to be the **absolute minimum** of f :



EXAMPLE 2

Find the **relative extrema** of the function

$$f(x, y) = 3x^2 - 4xy + 4y^2 - 4x + 8y + 4$$

Solution:

We have $f_x = 6x - 4y - 4 = 0$ and $f_y = -4x + 8y + 8 = 0$ To find the **critical points**, we set $f_x = 0$ and $f_y = 0$ and **solve** the resulting system of **simultaneous equations**

$$6x - 4y - 4 = 0 \quad \text{and} \quad -4x + 8y + 8 = 0$$

obtaining $x = 0$, $y = -1$, or $(0, -1)$, as the **sole critical point**.

Next, apply the **second derivative test** to compute the **nature of the critical point** $(0, -1)$.
Thus, $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (6)(8) - (4)^2 = 32$

CONT'D

- ▮ We have $D(x, y) = 32$, and in particular, $D(0, -1) = 32$.
- ▮ Since $D(0, -1) > 0$ and $f_{xx} = 6 > 0$, we conclude that f has a **relative minimum** at the point $(0, -1)$.
- ▮ We also have $f_x = 2x$ and $f_y = 12y^2 - 24y + 36$. To find the **critical points**, we set $f_x = 0$ and $f_y = 0$ and solve the resulting equations:

$$2x = 0 \quad \text{and} \quad 12y^2 - 24y + 36 = 0$$
- ▮ The first equation implies that $x = 0$, while the second equation implies that $y = -1$ or $y = 3$.
- ▮ Thus, there are **two critical points** of f : $(0, -1)$ and $(0, 3)$.

CONT'D

- ▮ To apply the **second derivative test**, we calculate

$$f_{xx} = 2 \qquad f_{yy} = 24(y - 1) \qquad f_{xy} = 0$$

$$D(x, y) = f_{xx} \cdot f_{yy} - f_{xy}^2 = (2) \cdot 24(y - 1) - (0)^2 = 48(y - 1)$$

- ▮ Apply the **second derivative test** to the **critical point** $(0, -1)$:

- ▮ We have $D(x, y) = 48(y - 1)$.

- ▮ In particular, $D(0, -1) = 48[(-1) - 1] = -96$.

- ▮ Since $D(0, -1) = -96 < 0$ we conclude that f has a **saddle point** at $(0, -1)$.

- ▮ The **saddle point value** is $f(0, -1) = 22$, so there is a saddle

CONT'D

- ▮ Apply the **second derivative test** to the **critical point** $(0, 3)$:
- ▮ We have $D(x, y) = 48(y - 1)$.
- ▮ In particular, $D(0, 3) = 48[(3) - 1] = 96$.
- ▮ Since $D(0, -1) = 96 > 0$ and $f_{xx}(0, 3) = 2 > 0$, we conclude that f has a **relative minimum** at the point $(0, 3)$.
- ▮ The **relative minimum value**, $f(0, 3) = -106$, so there is a relative minimum at $(0, 3, -106)$.

EXAMPLE 2

- Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Solution:

The distance $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ from the point $(1, 0, -2)$ is:

- However, if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$.

Thus, we have: We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

CONT'D

- By solving the equations

$$\begin{aligned} f_x &= 2(x-1) - 2(6-x-2y) = 4x - 4y - 14 = 0 \\ f_y &= 2y - 4(6-x-2y) = 4x - 10y + 24 = 0 \end{aligned}$$

- we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$.
- Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have:

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0 \text{ and } f_{xx} > 0$$

- So, by the Second Derivatives Test, f has a local minimum at

$$\left(\frac{11}{6}, \frac{5}{3}\right)$$

- There must be a

CONT'D

Ex If $x = \frac{11}{6}$ and $y = \frac{5}{3}$ then

$$d = \sqrt{(x-1)^2 + y^2} = \sqrt{\left(6 - \frac{11}{6}\right)^2 + \left(\frac{5}{3}\right)^2} = \sqrt{\frac{5}{6}} = \frac{\sqrt{6}}{6}$$

Ex The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + 3z = 6$ is $\frac{5}{6}\sqrt{6}$.

Exerci

5. A rectangular box without a lid is to be made from 12 m² of cardboard. Find the maximum volume of such a box.

EXERCISE

1. Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also, find the highest point on the graph of f .

2. The total weekly revenue that Acrosonic realizes in producing and selling its loudspeaker system is given by

$$R(x, y) = \frac{1}{4}x^2 - \frac{3}{8}y^2 + \frac{1}{4}xy + 300x - 240y$$

where x denotes the number of fully assembled units and y denotes the number of loudspeakers.

Determine how many fully assembled units and how many kits should be produced per week to maximize production of these

$$C(x, y) = 180x + 140y + 5000$$

The total weekly cost attributable to the production of these

EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains absolute maximum value $f(x_1, y_1)$ and absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

BB To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest value from steps 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

EXAMPLE

- Find the absolute maximum and minimum values of the

function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle

$$D = \{(x, y) \mid 0 \leq x \leq 3, 0$$

$\leq y \leq 2\}$ Solution:

- As f is a polynomial, it is continuous on the closed, bounded rectangle D .

So, the above Theorem tells us there is both an absolute maximum and an absolute minimum.

- According to step 1 in the above Method, we first find the critical points. These occur when

$$f_x = 2x - 2y = 0$$

$$f_y = -2x + 2 = 0$$

- So, the only critical point is $(1, 1)$.

CONT'D

▀ In step 2, we look at the values of f on the boundary of D .

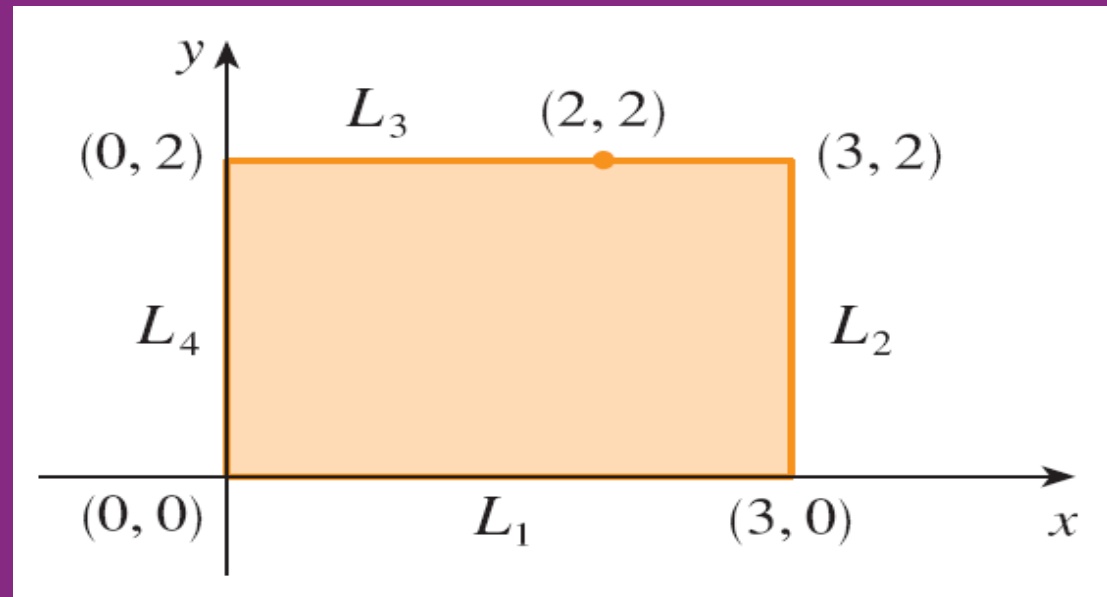
▀ This consists of the four line segments

$L_1, L_2, L_3,$
 L_4

▀ On L_1 , we have $y = 0$ and $0 \leq x \leq 3$.
This is an increasing function of x . So,

$0 \leq x$
 ≤ 3

- ▀ Its minimum value is:
 $f(0, 0) = 0$
- ▀ Its maximum value is:
 $f(3, 0) = 9$



CONT'
D

- On L_2 , we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad \begin{matrix} 0 \leq y \\ \leq 2 \end{matrix}$$

- This is a decreasing function. Its maximum value is: $f(3, 0) = 9$ and

- Its minimum value is: $f(3, 2) = 1$

- On L_3 , we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad \begin{matrix} 0 \leq \\ x \leq 3 \end{matrix}$$

- simply by observing that $f(x, 2) = (x - 2)^2$, we see that:

- The minimum value of the function is $f(2, 2) = 0$.

CONT'
D

Finally, on L_4 , we have $x = 0$ and $0 \leq y \leq 2$
 $f(0, y) = 2y$
 with:

- Maximum value $f(0, 2) = 4$
- Minimum value $f(0, 0) = 0$

Thus, on the boundary,

- The minimum value of f is 0.

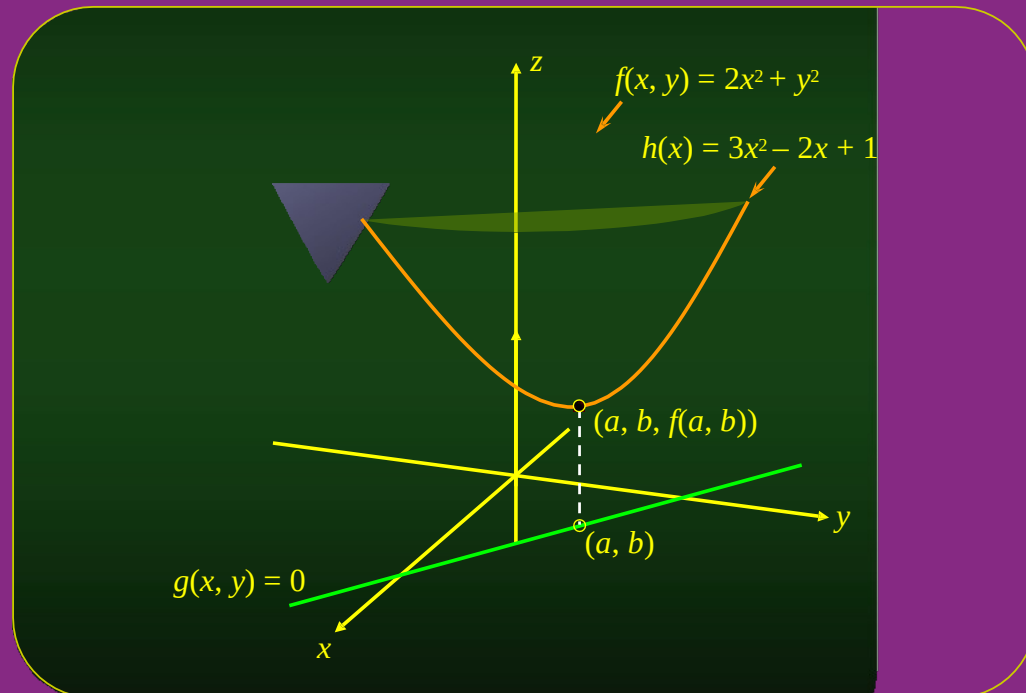
In step 3, we compare these values with the value $f(1, 1) = 1$ at the critical point.

We conclude that:

- The absolute maximum value of f on D is $f(3, 0) = 9$.
- The absolute minimum value of f on D is $f(0, 0) = 0$.

3.4

Constrained Maxima and Minima and the Method of Lagrange Multipliers



CONSTRAINED MAXIMA AND MINIMA

- ▄ In many practical optimization problems, we must maximize or minimize a function in which the independent variables are subjected to certain further constraints.
- ▄ We shall discuss a powerful method for determining relative extrema of a function $f(x, y)$ whose independent variables x and y are required to satisfy one or more constraints of the form $g(x, y) = 0$.

THE METHOD OF LAGRANGE MULTIPLIERS

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ and $\nabla f = \lambda \nabla g$ on the surface $g(x, y, z) = k$,
~~Extrema values exist~~

a. we proceed as follows:
 Find all values of x, y, z , and λ such that

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow x, y, z =$$

b. Evaluate $g = x, y, z = k$ that result from step a.

- The largest of these values is the maximum value of f .
- The smallest is the minimum value of f .

The number λ in the equation is called a Lagrange multiplier

LAGRANGE'S METHOD

▮ If we write the vector equation in terms of its components, then the equations in step a become:

$$f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g(x, y, z) = k$$

▮ This is a system of four equations in the four unknowns x, y, z , and λ . However, it is not necessary to find explicit values for λ .

Note: For functions of two variables, the method of Lagrange multipliers is similar to the method just described

▮ To find the extreme values of $f(x, y)$ subject to the

constraint $g(x, y) = k$, we find the extreme values of $f(x, y)$ and $g(x, y)$ subject to the constraint $g(x, y) = k$.

that: $f_x = \lambda g_x, f_y = \lambda g_y, g(x, y) = k$

EXAMPLE 1

▮ A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution: we let x , y , and z be the length, width, and height, respectively, of the box in meters.

▮ Then, we wish to maximize $V = xyz$ subject to the constraint

$$a(x, y, z) = 2xz + 2yz + xy = 12$$

▮ Using

look for

▮ This

λg_y

$$\begin{aligned} \nabla V &= \nabla g \quad \text{and} \quad g(x, y, z) = 12 \\ \lambda g_x &= \lambda g_y = \lambda g_z \end{aligned}$$

x, y, z , and λ such that:

$$V_z = \lambda g_z$$

(1)

$$2xz + 2yz + xy = 12$$

CONT'
D

▀ The equations become:

$$yz = \lambda(2z + y) \quad (2)$$

$$xz = \lambda(2z + x) \quad (3)$$

$$xy = \lambda(2x + 2y) \quad (4)$$

$$2xz + 2yz + xy = 12 \quad (5)$$

▀ There are no general rules for solving systems of equations.

▀ In this example, you might notice that if we multiply Equation 2 by x , Equation 3 by y , and Equation 4 by z , then left sides of the equations will be identical.

CONT'D

- Doing so, we have:

$$xyz = \lambda(2xz + \quad) \quad (6)$$

$$xy) \quad xyz =$$

$$\lambda(2yz + xy) \quad (7)$$

- We observe that $xyz \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from Equations 2, 3, and 4. This would contradict Equation 5. (8)

- Therefore, from Equations 6 and 7, we have:

$$2xz + xy = 2yz + xy$$

- which gives $xz = yz$. However, $z \neq 0$ (since $z = 0$ would give $V = 0$).

- Thus, $x = y$.

- From Equations 7 and 8, we have

$$2yz + xy = 2xz + 2yz$$

CONT'D

▮ If we now put $x = y = 2z$ in Equation 5, we get:

$$4z^2 + 4z^2 + 4z^2 = 12$$

▮ Since x , y , and z are all positive, we therefore have $z = 1$, and so $x = 2$ and $y = 2$.

▮ **Example 2:** Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution: We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$

▮ Using Lagrange multipliers, we solve the equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ as: $f_x = \lambda g_x$ and $f_y = \lambda g_y$ and $g(x, y) = 1$.

λg_y

CONT'D

▮ They can also be written as: $2x = 2x\lambda$ (9)

$$4y = 2y\lambda \quad (10)$$

$$x^2 + y^2 = 1 \quad (11)$$

- ▮ From Equation 9 , we have $x = 0$ or $\lambda = 1$
- If $x = 0$, then Equation 11 gives $y = \pm 1$.
 - If $\lambda = 1$, then $y = 0$ from Equation 10; so, we have $x = \pm 1$.

▮ Therefore, f has possible extreme values at the points

$$(0, 1), (0, -1), (1, 0), (-1, 0)$$

- Evaluating f at these four points, we find that:

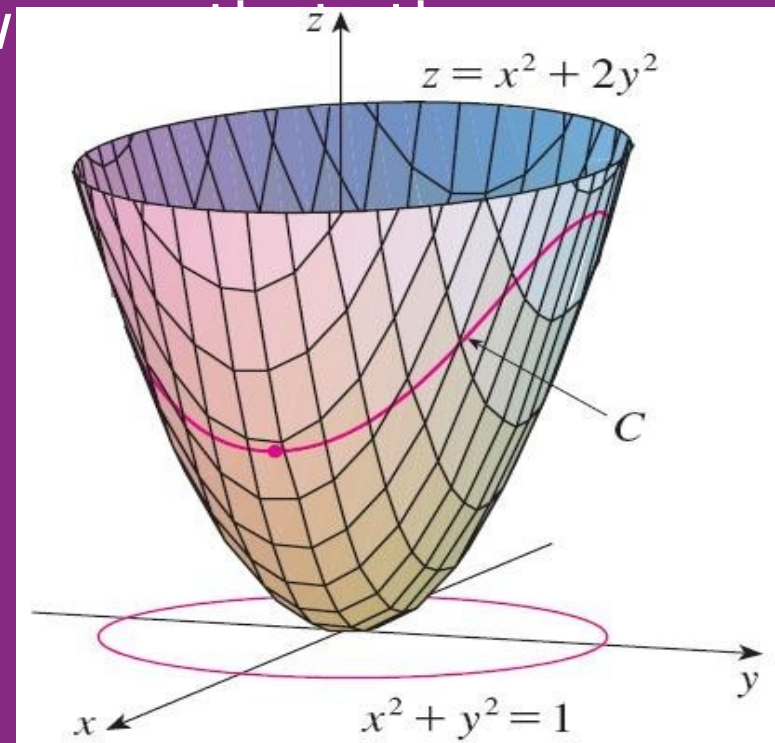
CONT'D

Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ is:

$$f(0, \pm 1) = 2$$

The minimum value is: $f(\pm 1, 0) = 1$

Checking with the figure, we see that these values look reasonable.



EXAMPLE 3

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$

Solution: According to the procedure in evaluating extreme value in the previous Section, we compare the values of f at the critical points with values at the points on the boundary.

▮ Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$.

- We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0$$

$$f(\pm 1, 0) = 1$$

$$f(0, \pm 1) = 2$$

▮ Therefore, the maximum value of f on the disk $x^2 + y^2 \leq 1$ is:

EXAMPLE 4

- Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Solution:

- The distance d from a point (x, y, z) on the sphere to the point $(3, 1, -1)$ is:

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

However, the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z)$$

$$= (x-3)^2 + (y-1)^2 + (z+1)^2$$

CONT'D

- ▀ The constraint is that the point (x, y, z) lies on the sphere, that is, $g(x, y, z) = x^2 + y^2 + z^2 = 4$
- ▀ According to the method of Lagrange multipliers, we solve:

$$\nabla f = \lambda \nabla g, g = 4$$

- ▀ That gives:

$$2(x - 3) = 2x\lambda \quad (12)$$

$$2(y - 1) = 2y\lambda \quad (13)$$

$$2(z + 1) = 2z\lambda \quad (14)$$

$$x^2 + y^2 + z^2 = 4 \quad (15)$$

CONT'D

- The simplest way to solve these equations is to solve for x , y , and z in terms of λ from Equations 12, 13, and 14, and then substitute these values into Equation 15.

- From Equation 12, we have:

$$x - 3 = x\lambda \text{ or } x(1 - \lambda) = 3$$

or

$$x = \frac{3}{1 - \lambda}$$

- Note that $1 - \lambda \neq 0$ because $\lambda \neq 1$ is impossible from Equation 12.

- Similarly, Equations 13 and 14 give:

$$y = \frac{1}{1 - \lambda} \quad z = \frac{1}{1 - \lambda}$$

CONT'D

So, from Equation 15, we have:

$$\frac{3^2}{(1-\lambda)^2} = \frac{1^2}{(1-\lambda)^2} \Rightarrow \frac{3^2}{(1-\lambda)^2} = \frac{1^2}{(1-\lambda)^2} \Rightarrow \frac{3^2}{(1-\lambda)^2} = \frac{1^2}{(1-\lambda)^2}$$

This gives $(1 - \lambda)^2 = 11/4$, $1 - \lambda = \pm \frac{\sqrt{11}}{2}$.

o

Thus,

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z) :

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right) \text{ and } \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right)$$

CONT'D

It's easy to see that f has a smaller value at the first of these points.

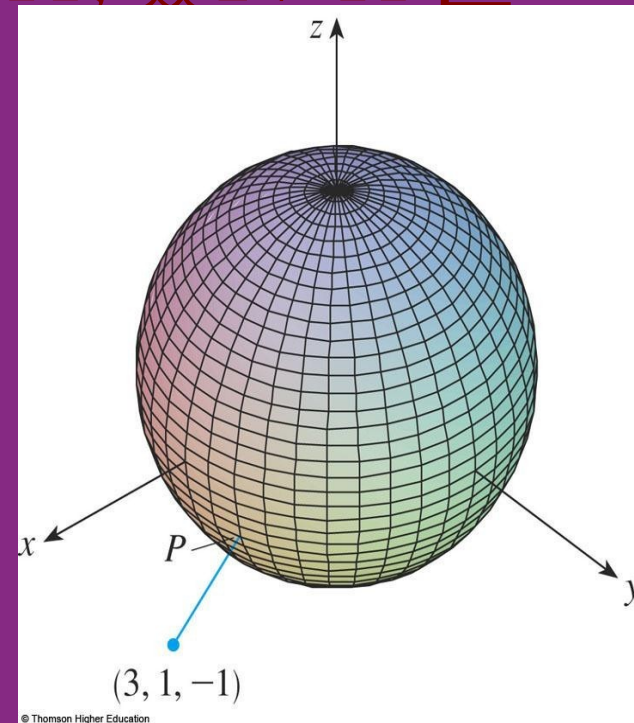
Thus, the closest point is:

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right)$$

The farthest
is:

$$\left(\frac{\sqrt{11}}{6}, \frac{\sqrt{11}}{2}, \frac{\sqrt{11}}{2} \right)$$

The figure shows the sphere and the nearest point in Example 4.



EXERCISE

1. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$

2. The Manager of the Central Jimma Hotel, is planning the addition of another swimming pool to the Jimma Community.

The chief engineer has suggested that an area of the form of an ellipse located in the rear of the path floor would be suitable for this purpose. It has been determined that the shape of the ellipse may be described by the equation

$$x^2 + 4y^2 = 3600$$

where x and y are measured in meter.

Find the dimensions of the rectangular pool with

**End of
Chapter
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