

CHAPTER ONE

VECTORS AND VECTOR SPACES (Tutorial)

1.1. Scalars and vectors in $R^2 \wedge R^3$

Activity 1.1

Describe the difference between scalar and vector quantities in $R^2 \wedge R^3$ and give some practical examples.

Example 1.1: Determine whether the following pairs vectors in R^3 is parallel not.

a. $A=2i-3j \wedge B=-3i+2j+k$

b. $A=2i+4j-k \wedge B=-i-2j+\frac{1}{2}k$

1.2. Vector addition and Scalar multiplication

Activity 1.2

Find the resultant sum of the pairs of vector given in example 1 above and show the diagram representation of the resultant vectors.

1.3. Dot (Scalar) product, Magnitude of a vector, Angle between two Vectors, Orthogonal Projection, Direction angles and direction cosines.

Example 1.2: Find any unit vectors that are perpendicular to the vector $A=(3,4)$.

Solution: Let $B=(a,b)$ be a unit vector perpendicular to A .

Then $\|B\|=1=a^2+b^2 \wedge A \cdot B=3a+4b=0$.

Then solving simultaneously we get $B=\left(\frac{-4}{5}, \frac{3}{5}\right) \vee B=\left(\frac{4}{5}, -\frac{3}{5}\right)$.

Example 1.3: If the angle between the vectors A and B is $\theta=\frac{\pi}{6}$ with each other and $\|A\|=\sqrt{3}$ and $\|B\|=1$, then calculate the cosine the angle between the vectors $A+B$ and $A-B$.

Solution: If φ is the angle between $A+B$ and $A-B$, then $\cos \varphi = \frac{(A+B) \cdot (A-B)}{\|A+B\| \|A-B\|}$.

$$\text{But } \|A+B\|^2 = \|A\|^2 + 2A \cdot B + \|B\|^2 = 3 + 2(\sqrt{3})(1)\cos(\pi/6) + 1 = 7.$$

$$\Rightarrow \|A+B\| = \sqrt{7}$$

$$\text{Similarly } \|A-B\| = 1 \wedge (A+B) \cdot (A-B) = \|A\|^2 - \|B\|^2 = 1$$

$$\text{Thus, } \cos \varphi = \frac{1}{\sqrt{7}} \Rightarrow \varphi = \arccos\left(\frac{1}{\sqrt{7}}\right).$$

Example 1.4: Let u and v be a pair orthogonal vectors such that $\|u\| = t$ and $\|v\| = r$.

Find the angle between the vector $p = \frac{tv+ru}{t+r}$ and the vector u .

Solution: Let θ be the angle between the vectors p and u .

$$\text{Then } \cos \theta = \frac{u \cdot p}{\|p\| \|u\|} = \frac{u \cdot (tv+ru)}{(t+r)\|p\| \|u\|} = \frac{tu \cdot v + r\|u\|^2}{(t+r)t\|p\|} = \frac{rt}{(t+r)\|p\|}, \text{ since } u \cdot v = 0 \wedge \|u\| = t.$$

$$\text{But } \|p\|^2 = \frac{\|tv+ru\|^2}{(t+r)^2} = \frac{t^2\|v\|^2 + 2tr(u \cdot v) + r^2\|u\|^2}{(t+r)^2} = 2$$

$$\Rightarrow \|p\| = \frac{\sqrt{2}rt}{t+r}$$

$$\text{Thus, } \cos \theta = \frac{rt}{(t+r)\|p\|} = \frac{rt}{(t+r)} \frac{(t+r)}{\sqrt{2}rt} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \pi/4.$$

Activity 1.3

1. If A and B are perpendicular unit vectors, show that $\|A-B\| = \sqrt{2}$.
2. Let $A = -i + 2j + 2k$. Then find the directional cosines & directional angles of A .

1.4. The Cross Product

Example 1.5: Let $u = (-1, \sqrt{3}, 0)$, $v = (1, -1, 1)$ and $w = (0, 1, -2)$ be vectors in space.

Then find

- a) the unit vector in the opposite direction of u .
- b) the area of the parallelogram formed by v and w .

c) The angle between $u \times v$ and $u - v$.

Solutions:

a) The unit vector in the opposite direction of

u is given by $\frac{-u}{\|u\|} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right)$.

b) The area of the parallelogram is given by $Area = \|v \times w\| = \sqrt{6}$.

c) Since the $u \times v$ is orthogonal to both u and v , it is also orthogonal to $u - v$.

Example 1.6: Find the volume the parallelepiped with the vectors $u = i + k$, $v = 2i + j + 4k$ and $w = j + k$ as three of its edges.

Solution: The volume of the parallelepiped V is the absolute value of the triple scalar product of the three vectors u, v and w

$$\text{Thus, } V = |u \cdot (v \times w)| = \left| \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \right| = 1 \text{ cubic units}$$

Example 1.7: If the area of an equilateral triangle with adjacent sides u and v , is $25\sqrt{3} \text{ cm}^2$, then find $u \cdot v$.

Solution: Since it is an equilateral triangle, the angle between u and v is $\theta = \frac{\pi}{3}$.

The area A of the triangle is $A = \frac{1}{2} \|u \times v\| = \frac{1}{2} \|u\| \|v\| \sin\left(\frac{\pi}{3}\right) = 25\sqrt{3} \text{ cm}^2$

$$\Rightarrow \|u\| \|v\| = 100$$

Therefore, $u \cdot v = \|u\| \|v\| \cos\left(\frac{\pi}{3}\right) = 100 \left(\frac{1}{2}\right) = 50$.

Activity 1.4

Find the area of the triangle with vertices $A(1, -1, 0)$, $B(2, 1, -1)$ and $C(-1, 1, 2)$.

1.5. Lines and Planes in R^3

Example 1.8: Find the parametric and symmetric equations of the line passing through the point (2, 3, -4) and parallel to the vector, $\langle -1, 2, 5 \rangle$.

Solution: Simply use the parametric and symmetric equations for any line given a point on the line and the direction vector.

Parametric Equations:

$$x=2-t, y=3+2t, z=-4+5t$$

Symmetric equations:

$$\frac{x-2}{-1} = \frac{y-3}{2} = \frac{z+4}{5}$$

Example 1.9: Determine if the following two lines are parallel or identical.

$$l: x=3+t, y=2-2t, z=4+t \wedge m: x=5-2t, y=-2+4t, z=1-2t.$$

Solution: First look at the direction vectors:

$$v_1 = i - 2j + k \wedge v_2 = -2i + 4j + 2k$$

$\Rightarrow v_2 = -2v_1$, which shows the two lines are parallel.

Now we must determine if they are identical. So we need to determine if they pass through the same points. So we need to determine if the two sets of parametric equations produce the same points for different values of t .

Let $t=0$ for Line 1, the point produced is (3, 2, 4). Set the x from Line 2 equal to the x -coordinate produced by Line 1 and solve for t .

$$3 = 5 - 2t \Rightarrow t = 1.$$

Now let $t=1$ for Line 2 and the point (3, 2, -1) is produced. Since the z -coordinates are not equal, the lines are not identical.

Example 1.10: Determine if the lines intersect. If so, find the point of intersection and the cosine of the angle of intersection.

$$\begin{aligned} \text{Line 1: } x &= 3 + 2t \\ y &= -2t \\ z &= 4 - t \end{aligned}$$

$$\begin{aligned} \text{Line 2: } x &= 4 - t \\ y &= 3 + 5t \\ z &= 2 - t \end{aligned}$$

Solution: Direction vectors: $v_1 = \langle 2, -2, -1 \rangle$ $v_2 = \langle -1, 5, -1 \rangle$

Since $v_2 \neq k \cdot v_1$, the lines are not parallel. Thus they either intersect or they are skew lines.

Keep in mind that the lines may have a point of intersection or a common point, but not necessarily for the same value of t . So equate each coordinate, but replace the t in Line 2 with an s .

$$\left. \begin{array}{l} x: 3+2t=4-s \\ y: -2t=3+5s \\ z: 4-t=2-s \end{array} \right\} \text{system of 3 equations with 2 unknowns, solve the first 2}$$

with the third equation.

Solving the system, we get $t = 1$ and $s = -1$.

Line 1: $t = 1$ produces the point $(5, -2, 3)$

Line 2: $s = -1$ produces the point $(5, -2, 3)$. So the lines intersect at this point.

To find the angle of intersection of the two lines,

$$\cos \theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}, \text{ where } \theta \text{ is the angle between } u \text{ and } v$$

The angle θ between two intersecting lines should be less than 90° , so we use absolute value in the numerator:

$$\begin{aligned} \cos \theta &= \frac{|\langle 2, -2, -1 \rangle \cdot \langle -1, 5, -1 \rangle|}{\sqrt{2^2 + (-2)^2 + (-1)^2} \cdot \sqrt{(-1)^2 + 5^2 + (-1)^2}} \\ \cos \theta &= \frac{|-2 - 10 + 1|}{\sqrt{9} \cdot \sqrt{27}} = \frac{|-11|}{9\sqrt{3}} \approx 0.706 \\ \Rightarrow \theta &= \arccos(0.706) \end{aligned}$$

Example 1.12: Given the points $(1, 2, -1)$, $(4, 0, 3)$ and $(2, -1, 5)$ in a plane, find the equation of the plane in general form.

Solution: To write the equation of the plane we need a point (we have three) and a vector normal to the plane. So we need to find a vector normal to the plane. First find two vectors in the plane, and then recall

that their cross product will be a vector normal to both those vectors and thus normal to the plane.

Two vectors: From (1, 2, -1) to (4, 0, 3): $\langle 4-1, 0-2, 3+1 \rangle = \langle 3, -2, 4 \rangle$

From (1, 2, -1) to (2, -1, 5): $\langle 2-1, -1-2, 5+1 \rangle = \langle 1, -3, 6 \rangle$

Their cross product:

$$\begin{vmatrix} \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & -3 & 6 \end{vmatrix} = \vec{i} \begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} = -14\vec{j} - 7\vec{k} = -14\vec{j} - 7\vec{k}$$

Equation of the plane:

$$0(x-1) - 14(y-2) - 7(z+1) = 0$$

$$-14y - 7z + 21 = 0$$

or

$$2y + z - 3 = 0$$

Example 1.13: Find the line of intersection for the planes

$$x + 3y + 4z = 0 \wedge x - 3y + 2z = 0.$$

Solution: To find the common intersection, solve the equations simultaneously.

Multiply the first equation by -1 and add the two to eliminate x.

$$\begin{aligned} -1 \cdot (x + 3y + 4z = 0) &\rightarrow -x - 3y - 4z = 0 \\ x - 3y + 2z = 0 &\rightarrow \underline{x - 3y + 2z = 0} \\ -6y - 2z &= 0 \text{ or } y = -\frac{1}{3}z \end{aligned}$$

Back substitute y into one of the first equations and solve for x.

$$\begin{aligned} x + 3 \cdot \left(-\frac{1}{3}z\right) + 4z &= 0 \\ x - z + 4z &= 0 \\ x &= -3z \end{aligned}$$

Finally if you let $z = t$, the parametric equations for the line are

$$x = -3t, \quad y = -\frac{1}{3}t \quad \text{and} \quad z = t$$

Distance in space

Example 1.14: Find the distance between the point $Q(3, 1, -5)$ to the plane

$$4x + 2y - z = 8.$$

Solution: We know the normal to the plane is $\langle 4, 2, -1 \rangle$ from the general form of a plane. We can find a point in the plane simply by letting x and y equal 0 and solving for z : $P(0, 0, -8)$ is a point in the plane.

Thus the vector, $PQ = \langle 3-0, 1-0, -5-(-8) \rangle = \langle 3, 1, 3 \rangle$

Now that we have the vector PQ and the normal, we simply use the formula for the distance between a point and a plane.

$$D = \left\| \text{proj}_{\vec{n}} PQ \right\| = \frac{|PQ \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\langle 3, 1, 3 \rangle \cdot \langle 4, 2, -1 \rangle|}{\sqrt{4^2 + 2^2 + (-1)^2}}$$

$$D = \frac{|12 + 2 - 3|}{\sqrt{16 + 4 + 1}} = \frac{11}{\sqrt{21}} \approx 2.4$$

Example 1.15: Find the distance between the planes $\pi_1: x + 2y - 2z = 3$

$\pi_2: 2x + 4y - 4z = 7$

Solution: We first rewrite the equations $\pi_1 \wedge \pi_2$ so that they have the same normal vector $\vec{n} = \langle a, b, c \rangle$.

That is: $\pi_1: x + 2y - 2z = 3$

$$\pi_2: x + 2y - 2z = \frac{7}{2} \implies d_1 = 3 \wedge d_2 = \frac{7}{2}$$

Thus, $D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$, where $a=1, b=2, c=-2$

$$\frac{\left| \frac{7}{2} - 3 \right|}{\sqrt{1 + 4 + 4}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

Activity 1.5

1. Determine the equation of the plane passing through the point

$P(5,3,1)$ if $N=(-1,3,-2)$ is its normal vector.

2. Find the distance from the given point P to the given plane π .

a) $P=(2,-1,4), \pi: 3x-y+7z=2$.

b) $P=(7,-2,-10), \pi: -2x+6y+8z+z=0$.

3. Find the line of intersection for the planes $x+3y+4z=0 \wedge x-3y+2z=0$.

1.6. Vector Spaces and Subspaces

Example 1.16: Show that the set of points $\{(x,y,z): 3x+4y-z=0\}$ is a subspace of $\mathbb{R}^3$.

Activity 1.6

1. Determine whether the following set of vectors in the vector space $V=\mathbb{R}^3$ are linearly dependent or independent.

a. $\{(1,0,0), (0,1,0), (0,0,3)\}$.

b. $\{(2,6,0), (2,4,1), (1,1,1)\}$.

2. Show that the set of vectors $\{(1,1), (-1,2)\}$ forms a basis of $\mathbb{R}^2$.

Example 1.17: If $v=(1,k,5)$ is the linear combination of $u=(1,-3,2) \wedge w=(2,-1,1)$, then find the value(s) of k .

Solution: $v=\alpha u+\beta w=(\alpha+2\beta, -3\alpha-2\beta, 2\alpha+\beta)$

$\Rightarrow (1,k,5)=(\alpha+2\beta, -3\alpha-2\beta, 2\alpha+\beta)$

Then solving simultaneously for $\alpha \wedge \beta$, and equating the entries in the coordinates, we get

$$k=-3\alpha-2\beta=-7.$$

Example 1.18: Let $V=\mathbb{R}^3$ be a vector space with the usual addition and scalar multiplication over $\mathbb{R}$. Then:

a. Show that $W=\{(x,y,z): x+2y+3z=0\}$ is a subspace of V .

b. $B=\{(1,1,-2), (1,0,1), (1,2,0)\}$ forms a basis of V .

Solutions:

a) i) Since $(0,0,0) \in W$, clearly $W \neq \emptyset$.

ii) Let $u = (x, y, z) \wedge v = (r, s, t)$ are elements of W . Then $x + 2y + 3z = 0 \wedge r + 2s + 3t = 0$.

Then $u + v = (x + r, y + s, t + z)$ and

$$(x + r) + 2(y + s) + 3(t + z) = (x + 2y + 3z) + (r + 2s + 3t) = 0 + 0 = 0.$$

Thus, $u + v \in W$.

iii) For any scalar $c \in \mathbb{R}$ and any

$u = (x, y, z) \in W$, we have $cu \in W$ because $cx + 2cy + 3cz = c(x + 2y + 3z) = 0$.

Therefore, W is a subspace of V .

b) Exercise.

Example 1.19: Let V be the vector space of all real valued functions of real variable t . Then which of the following set of functions are LD/LI? Justify!

a. $\{t, t^2, \sin t\}$

b. $\{\cos^2 t, \sin^2 t, 1\}$

Solution: Class work