

CHAPTER TWO

DERIVATIVES

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At the end of this unit students enable to:

- Define the derivative of a function at a point.
- Determine the slope of the tangent line to the graph of a function at a given point.
- State the basic properties of derivatives.
- Identify whether a given function is differentiable at a given point or not.
- Find the derivative of polynomial, exponential, logarithmic, trigonometric, hyperbolic functions, inverse functions, etc.
- Find the higher (or n^{th}) derivatives of a given function.

2.1. Definitions, Notions and Examples

Definition: Let a be a no in the domain of a function f . If the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists, we call this limit the derivative of f at a and we write it as $f'(a)$. So that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1)$$

- If the limit in Eq. (1) exists, we say that
- f has derivative at a
 - f is differentiable at a

or - $f'(a)$ exists.

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- If the limit in Eq. (1) does not exist we say that f is not differentiable at a or f has no derivative at a .
- If we put $x = a + h$ then, Eq. (1) can be written in form

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$

- This Eq. (2) is an alternative way of writing the derivative of function f at a .

Notations: The derivative of a function f at a may also be given by

$$\left. \frac{d}{dx} f(x) \right|_{x=a}, \quad Df(a) \quad \text{or} \quad \frac{dy}{dx}(a) \quad \text{where, } y = f(x)$$

2.2. Differentiability

Note that: The derivative of a function f with respect to x is a function whose domain is the set of numbers for which f is differentiable and whose value at any real number x is given by

$$\left. \begin{aligned} f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \end{aligned} \right\} \quad (3)$$

Definition: If f is differentiable at each number in its domain, then f is said to be a differentiable function.

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Example 1: Let $f(x) = 2x^2 - 3x + 6$, then find $f'(1)$

Solution: Using Eq. (1), we have

$$\begin{aligned} f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{2x^2 - 3x + 6 - 5}{x - 1} = \lim_{x \rightarrow 1} 2x - 1 = 1 \end{aligned}$$

Therefore f is differentiable at 1 and $f'(1) = 1$.

Example 2: Let $f(x) = c$, where c is a constant, show that $f'(x) = 0$ for all x .

Solution: By definition of the derivative we find that

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{c - c}{t - x} = \lim_{t \rightarrow x} 0 = 0, \quad \forall x$$

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Example 3: Show that $f(x) = x^n, n \in \mathbb{N}$ is a differentiable function and $f'(x) = nx^{n-1}$ for all x .

Solution:

Note that: $t^n - x^n = (t - x) (t^{n-2} + t^{n-3}x + \dots + tx^{n-2} + x^{n-1})$

$$\begin{aligned}
 \text{We have } f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{t^n - x^n}{t - x} \\
 &= \lim_{t \rightarrow x} (t^{n-1} + t^{n-2}x + t^{n-3}x^2 + \dots + tx^{n-2} + x^{n-1}) \\
 &= \lim_{t \rightarrow x} (\underbrace{x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \dots + xx^{n-2} + x^{n-1}}_{n \text{ terms}}) \\
 &= nx^{n-1}
 \end{aligned}$$

Since $f'(x) = nx^{n-1}$ is defined for all x , f is a differentiable function.

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Observe that: we have derived the basic formula of calculus

$$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{N}$$

For example, if $f(x) = x^{17}$, then $f'(x) = 17x^{16}$.

Example 4: Let $f(x) = \sqrt{x+1}$, then find the derivative of f at $x = 3$

Solution: Using Eq. (1) we have

$$\begin{aligned} f'(3) &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4} \end{aligned}$$

Therefore f is differentiable at 3 and $f'(3) = \frac{1}{4}$.

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Example 5: Let $f(x) = |x + 4|$, then f in $f'(-4)$.

Solution: Using Eq. (1) we have

$$f'(-4) = \lim_{x \rightarrow -4} \frac{f(x) - f(-4)}{x + 4} = \lim_{x \rightarrow -4} \frac{|x + 4|}{x + 4}$$

Observe that $|x + 4| = \begin{cases} x + 4, & \text{if } x \geq -4 \\ -(x + 4), & \text{if } x < -4 \end{cases}$

$$\text{Thus, } \lim_{x \rightarrow -4^+} \frac{|x + 4|}{x + 4} = 1 \quad \text{and} \quad \lim_{x \rightarrow -4^-} \frac{|x + 4|}{x + 4} = -1$$

Since $\lim_{x \rightarrow -4^+} \frac{|x + 4|}{x + 4} \neq \lim_{x \rightarrow -4^-} \frac{|x + 4|}{x + 4}$, it follows that f is not differentiable at $x = -4$.

Follows that f is not differentials at $x = -4$.

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Example 6: Let $f(x) = x^2$, show that $f'(x) = 2x$ for all x in $\mathbb{R}$.

Solution: Using Eq. (3),

$$\begin{aligned} f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{t^2 - x^2}{t - x} = \lim_{t \rightarrow x} \frac{(t - x)(t + x)}{t - x} = \lim_{t \rightarrow x} (t + x) = 2x \end{aligned}$$

Example 7: Let $f(x) = \cos x$, show that $f'(x) = -\sin x$, $\forall x$ in $\mathbb{R}$.

Solution: Using limit definition of derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos x}{h} \\ &= \cos(x) \lim_{h \rightarrow 0} \frac{(\cos(h) - 1)}{h} - \sin(x) \lim_{h \rightarrow 0} \sin(h) = 0 - \sin x = -\sin x \end{aligned}$$

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Activity 1:

I. Use the definition of derivative to determine whether the given function has derivative at a or not. If it is differentiable at a find $f'(a)$.

1. $f(x) = x^{4/3}$; $a = 0$

2. $f(x) = x^{3/4}$; $a = 0$

3. $f(x) = |2x - 5|$, $a = \frac{5}{2}$

4. $f(x) = \frac{1}{x^2}$, $a = 0$

5. $f(x) = \begin{cases} -x^2 + 4x & \text{for } x < 0 \\ x^2 - 1 & \text{for } x \geq 0, \end{cases} \quad a = 0$

II. Determine whether the following function are differentiable or not

1. $f(x) = -2x^2 + x + 1$

2. $f(x) = \frac{x+1}{x-2}$

3. $f(x) = x^{1/3}$

4. $f(x) = \sin x$

5. $f(x) = \sqrt{x+1}$

6. $f(x) = |x| - x$

7. $f(x) = \begin{cases} 2x^2 - 1, & x \neq 1 \\ 4x, & x = 1 \end{cases}$

2.3. Geometrical Interpretation of the

Derivative

Let the graph of $y = f(x)$ be represented by the curve APQB shown in the figure below. The difference quotient

$$\frac{QR}{PR} = \frac{f(a+h) - f(a)}{h} = \tan \theta$$

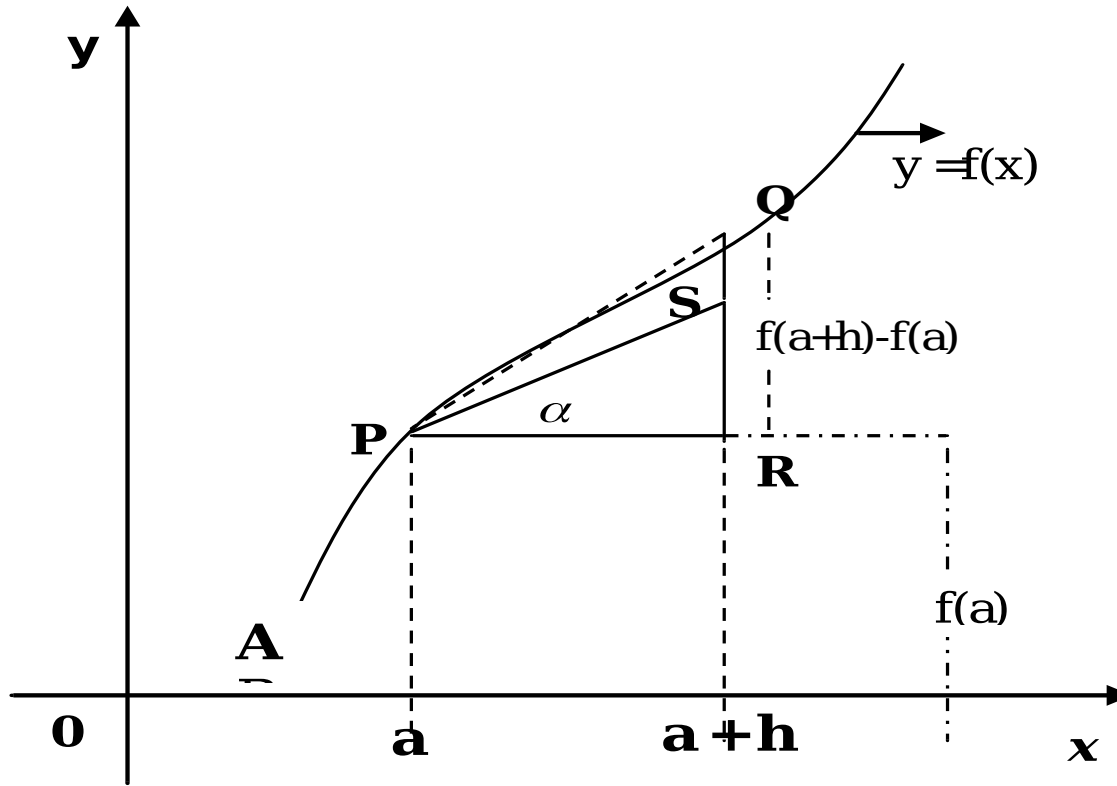
Is the slope of the secant line joining points P and Q of the curve.

As $h \rightarrow 0$, this secant line at the point P. Then

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \frac{SR}{PR} = \tan \alpha$$

is the slope of the tangent line to the curve at the point P.

Cont...



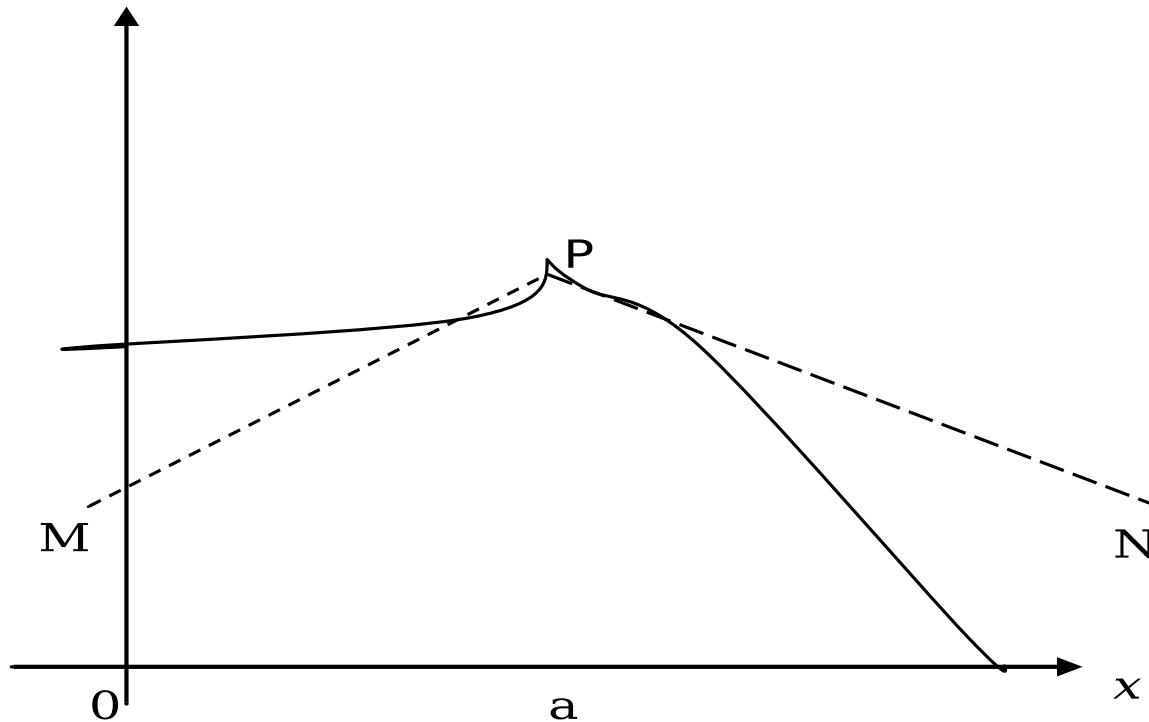
An equation for the tangent line to the curve $y = f(x)$ at a point where $x = a$ is given by

$$y - f(a) = f'(a) (x - a)$$

Cont....

The fact that a function can be continuous at a point and yet not be differentiable there is shown graphically in Fig. below. In this case there are two tangent lines at P represented by lines PM and PN and the slopes of these tangent lines are given by

$$f'(a^-) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad f'(a^+) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$$



Cont....

Definition: The tangent line to the graph of f at the point $P(a, f(a))$ is defined as:

- a) The line containing P with slope $f'(a)$ if $f'(a)$ exists
- b) The vertical line $x = a$ if $f'(a) = \infty$ or $f'(a) = -\infty$.

Remark:

- i. If neither (a) nor (b) holds the graph of f does not have a tangent line at P .
- ii. The normal line to the graph of f at the point $P(a, f(a))$ is the line continuing P perpendicular to the tangent line.

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Example 8: Find an equation of the tangent line and normal line to the graph of f at the indicated points.

a. $f(x) = x^2 + 2$, $P(1, 3)$

b. $f(x) = \sin x$, $P(\frac{\pi}{2}, 1)$

c. $f(x) = 3\sqrt{x}$, $P(0, 0)$

d. $f(x) = |x - 2|$, $P(2, 0)$

Solution

By the definition of the slope of the tangent line to the graph of the function f at the point $P(a, f(a))$ is given by

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

a) For $f(x) = x^2 + 2$ and $P = P(1, 3) \Rightarrow \text{slope} = f'(a) = 2$.

Now using *point slope form* of equation of a line, the equation of the tangent line to the graph of f at $P(a, f(a))$ is given by

$$y = f'(a)(x - a) + f(a)$$

Cont....

Thus the required equation of the line is

$$y = 2(x - 1) + 3 = 2x + 1.$$

And an equation of the normal line to the graph f at the point $P(a, f(a))$ is given by

$$y = -\frac{1}{f'(a)}(x - a) + f(a) = -(x - 1) + 3 = -\frac{1}{2}x + \frac{7}{2}$$

b) For $f(x) = \sin x$ and $P = P\left(\frac{\pi}{2}, 1\right)$

$\Rightarrow$ slope $= f'\left(\frac{\pi}{2}\right) = 0$ and an equation of the tangent line is the line $y = 1$. Since the tangent is horizontal, the normal line is a vertical line through the point $P\left(\frac{\pi}{2}, 1\right)$.

Hence an equation of the normal line is $x = \frac{\pi}{2}$.

Cont....

c) Let $f(x) = \sqrt[3]{x}$ and $P(0, 0)$

slope = $f'(0)$, but $f'(0)$ is not finite .

Hence the graph of f has a vertical tangent line at $x = 0$, and the corresponding normal line to the graph of f of $(0, 0)$ is the line $y=0$.

d) Let $f(x) = |x - 2|$ and $P(2, 0)$

Since $f'(2)$ does not exist, the graph $f(x) = |x - 2|$ has no tangent line at the point $P(2, 0)$ and consequently there is no normal line.

Activity 2: For the following functions find an equation of the tangent line and the normal line at the given points if it exists.

1. $f(x) = \sqrt{x}$; $(4, 2)$

2. $f(x) = 4 - x^2$; $(2, 0)$

3. $f(x) = 2\cos x$; $(0, 2)$

4. $f(x) = |x + 3|$; $(-3, 0)$

2.4. Properties of Derivative

Since there are limit theorems of sums, products and quotients of functions, there also corresponding theorems for derivatives, which we will use them later as a tool for finding the derivatives of various functions:

Theorem: If f is differentiable at a point a , then f is continuous at a i.e. if $f'(a)$ exists, then $\lim_{x \rightarrow a} f(x) = f(a)$.

Proof: To show that f is continuous at a , it suffices to show that

$$\lim_{x \rightarrow a} f(x) = f(a), \text{ or equivalently, } \lim_{x \rightarrow a} [f(x) - f(a)] = 0$$

Notice that $f'(a)$ exists by hypothesis. Hence by the limit theorems, we have

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

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Remark: The converse of the above theorem is not true.

For example, $f(x) = |x|$ is continuous at $x=0$, but not differentiable at $x=0$.

Theorems: Let f and g be differentiable at a point a , then the $f \pm g$, the constant multiple kf , the product $f.g$ are all differentiable at a and respectively given by

$$1. (f \pm g)'(a) = f'(a) \pm g'(a)$$

$$2. (kf)'(a) = k.f'(a), \text{ when } k \text{ is a constant}$$

$$3. (f.g)'(a) = f'(a).g(a) + f(a).g'(a) \quad (\text{product rule})$$

$$4. \left(\frac{f}{g} \right)'(a) = \frac{f'(a).g(a) - f(a).g'(a)}{(g(a))^2}, \text{ for } g(a) \neq 0 \quad (\text{quotient rule})$$

Proof: left as an exercise for the reader!

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Example 9: Let $h(x) = x^2 + \cos x$, find a formula for $h'(x)$ and compute $h'\left(\frac{\pi}{2}\right)$.

Solution: Let $f(x) = x^2$ and $g(x) = \cos x$, then

$$h(x) = f(x) + g(x)$$

$$\Rightarrow h'(x) = f'(x) + g'(x)$$

$$= 2x - \sin x, \text{ for all } x$$

$$\text{and } h'\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = \pi - 1$$

Example 10:

a) Let $f(x) = 8\sqrt{x}$, find a formula for $f'(x)$

Solution: $f'(x) = (8\sqrt{x})' = 8(\sqrt{x})' = \frac{4}{\sqrt{x}}$. Why?

b) Find $\frac{d}{dx}\left(4x^7 - \sqrt{2}x^4 + \frac{1}{7}x^3 - 20x + 8\right)$

Cont....

Solution: we find that

$$\frac{d}{dx} \left(4x^7 - \sqrt{2}x^4 + \frac{1}{7}x^3 - 20x + 8 \right) = 28x^6 - 4\sqrt{2}x^3 + \frac{3}{7}x^2 - 20.$$

c) Let $f(x) = 5x^3 \sin x$. Find $f'(x)$.

If we put $h(x) = 5x^3$ and $g(x) = \sin x$, then $f(x) = h(x)g(x)$.

Since $h'(x) = 15x^2$ and $g'(x) = \cos x$, using product rule for derivative we have

$$\begin{aligned} f'(x) &= h'(x)g(x) + h(x)g'(x) \\ &= 15x^2 \sin x + 5x^3 \cos x \end{aligned}$$

d) Find $\frac{d}{dx}(\sqrt{x}(2x - 3) \cos x)$

Solution: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, $\frac{d}{dx}(2x - 3) = 2$, $\frac{d}{dx}(\cos x) = -\sin x$

Thus,

$$\begin{aligned}\frac{d}{dx}(\sqrt{x}(2x-3)\cos x) &= (2x-3)\cos x \frac{d}{dx}(\sqrt{x}) + \sqrt{x}\cos x \frac{d}{dx}(2x-3) \\ &\quad + \sqrt{x}(2x-3) \frac{d}{dx}(\cos x) \\ &= \frac{2x-3}{2\sqrt{x}}\cos x + 2\sqrt{x}\cos x - \sqrt{x}(2x-3)\sin x\end{aligned}$$

e) Let $F(x) = \frac{3x^7}{x^3 - 2x}$. Find $F'(x)$

Solution: If $f(x) = 3x^7$ and $g(x) = x^3 - 2x$, then

$$F(x) = \frac{f}{g}, \quad f'(x) = 21x^6 \text{ and } g'(x) = 3x^2 - 2$$

Therefore,

$$\begin{aligned}F'(x) &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} = \frac{21x^6(x^3 - 2x) - 3x^7(3x^2 - 2)}{(x^3 - 2x)^2} \\ &= \frac{21x^9 - 42x^7 - 9x^9 + 6x^7}{(x^3 - 2x)^2} = \frac{12x^9 - 36x^7}{(x^3 - 2x)^2}\end{aligned}$$

Cont....

f) Show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Solution: Since $\tan x = \frac{\sin x}{\cos x}$, we conclude that

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left(\frac{d}{dx}\sin x\right) \cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Example 11. Show that $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$, where $n \in \mathbf{Z}^+$.

Solution: We find that

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{1}{(x^n)^2} \cdot \frac{d}{dx}x^n = -\frac{1}{x^{2n}} \cdot nx^{n-1} = -nx^{-n-1}.$$

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Derivatives of the remaining trig functions can be determined the same way.

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cdot \cot x$$

Activity 5:

1. Find the derivative of the following functions

a) $f(x) = -2x^3 + 6x^2 - 5x + \sqrt{2}$ f) $f(x) = -4x^{-3} + 2 \cos x$

b) $f(x) = \left(x - \frac{1}{x} \right) (x^2 + 3 \sin x)$ g) $f(x) = \sqrt{x} (x^2 + 3 \sin x)$

c) $f(x) = \sin^3 x$

h) $f(x) = \frac{-x^3 + 9}{x^2 + 7}$

d) $f(x) = \csc x$

i) $f(x) = \cot x - \sqrt{x} \sec x$

e) $f(x) = \frac{x^2 + \sqrt{x}}{\sin x - \cos x}$

j) $f(x) = x^{-20} + \pi \tan x$

2. Use trigonometric identities to find $\frac{d}{dx}$ of the following functions

a) $f(x) = \sin 2x$ b) $f(x) = \cos^2 \left(\frac{x}{2} \right)$ c) $f(x) = \tan (-x)$

2.5. The Chain Rule

Consider the function $F(x) = \sqrt{x^2 + 1}$. If we put $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$ for all x then F is the composite of g and f . In this subtopic, we are going to develop a general rule for the derivative of a composite function in terms of the derivatives of the component function.

Theorem (The Chain Rule): Let f be differentiable at a , and g be differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) f'(a) \quad (1)$$

In general, $(g \circ f)'(x) = g'(f(x)) f'(x)$

for all x such that f is differentiable at x and g is differentiable at $f(x)$.

Proof:

Cont....

$$\text{Let } G(x) = \begin{cases} \frac{g(x) - g(f(a))}{x - f(a)} & \text{for } x \neq f(a) \\ g'(f(a)) & \text{for } x = f(a) \end{cases}$$

By the definition of the derivative of g at $f(a)$ and by the Substitution Theorem, we have:

$$\lim_{x \rightarrow a} G(f(x)) = \lim_{y \rightarrow f(a)} G(y) = \lim_{y \rightarrow f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a)) = G(f(a))$$

Hence G is differentiable at $f(a)$. On the other hand:

$$\begin{aligned} \frac{g(f(x)) - g(f(a))}{x - a} &= \frac{f(x) - f(a)}{f(x) - f(a)} \cdot \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} = G(f(x)) \left(\frac{f(x) - f(a)}{x - a} \right) \\ \therefore (g \circ f)'(a) &= \lim_{x \rightarrow a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \rightarrow a} G(f(x)) \cdot \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= G(f(a)) \cdot f'(a) = g'(f(a)) \cdot f'(a) \end{aligned}$$

Cont....

Example 12. Differentiate the following function

i) $h(x) = \cos(2x)$

ii) $h(x) = \sin \sqrt{1+x^8}$

iii) $h(x) = (x - x^3 + 4)^{-9}$

iv) $h(x) = \tan x^2$

Solution:

i) $h(x) = \cos(2x)$. Let $f(x) = 2x$ and $g(x) = \cos x$,
Then $h(x) = (g \circ f)(x)$, $f'(x) = 2$ and $g'(x) = -\sin x$.

$$\Rightarrow h'(x) = g'(f(x)) \cdot f'(x)$$

$$= g'(2x) \cdot 2 = -\sin(2x) \cdot 2 = -2 \sin(2x)$$

ii) $h(x) = \sin \sqrt{1+x^8}$

Let $f(x) = 1 + x^8$ and $g(x) = \sqrt{x}$, then $h(x) = (g \circ f)(x)$,

$$f'(x) = 8x^7, \quad g'(x) = \frac{1}{2\sqrt{x}}$$

$$\therefore h'(x) = g'(f(x)) \cdot f'(x) = \frac{1}{2\sqrt{1+x^8}} \cdot 8x^7 = \frac{1}{2\sqrt{1+x^8}} \cdot 8x^7 = \frac{4x^7}{\sqrt{1+x^8}}$$

Cont....

iii) $h(x) = (x - x^3 + 4)^{-9}$

Let $f(x) = x - x^3 + 4$ and $g(x) = x^{-9}$, then

$$h(x) = (g \circ f)(x), \quad f'(x) = 1 - 3x^2 \quad \text{and} \quad g'(x) = -9x^{-10}$$

Therefore,

$$h'(x) = g'(f(x)) \cdot f'(x)$$

$$= -9(f(x))^{-10} \cdot (1 - 3x^2) = -9(x - x^3 + 4)^{-10} (1 - 3x^2)$$

iv) $h(x) = \tan(x^2)$

Let $f(x) = x^2$ and $g(x) = \tan x$, then

$$h(x) = (g \circ f)(x), \quad f'(x) = 2x \quad \text{and} \quad g'(x) = \sec^2 x$$

Therefore, $h'(x) = g'(f(x)) f'(x) = \sec^2(f(x)) 2x = 2x \sec^2(x^2)$.

Cont....

Remark: If $k(x) = (hogof)(x)$ and f is differentiable at x , g is differentiable at $f(x)$ and h is differentiable at $(gof)(x)$, then the composition of h , g and f is differential at x and is given by:

$$k'(x) = h'(g(f(x)))g'(f(x))f'(x)$$

In Leibniz notation this becomes

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dy} \cdot \frac{dy}{dx}, \text{ where } y = f(x), v = g(y) \text{ and } u = h(v).$$

Example 13: Let $F(x) = \sqrt{\sec(\tan x)}$. Find the derivative of F .

Solution: If we take $f(x) = \tan x$, $g(x) = \sec x$ and $h(x) = \sqrt{x}$, then
 $F(x) = (hogof)(x)$

Now, $f'(x) = \sec^2 x$, $g'(x) = \sec x \tan x$ and $h'(x) = \frac{1}{2\sqrt{x}}$

$$\therefore F'(x) = h'(g(f(x)))g'(f(x))f'(x)$$

$$= \frac{1}{2\sqrt{g(f(x))}} \sec(f(x)) \tan(f(x)) \sec^2 x$$

$$= \frac{1}{2} \sqrt{\sec(\tan x)} \tan(\tan x) \sec^2 x$$

Cont....

Example 14: Given $y = (\sqrt{x^5 - 6x})^{-3}$, find $\frac{dy}{dx}$

Solution: Using Leibniz notation we have

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{du} \frac{du}{dx}$$

Let $u = x^5 - 6x$, $v = \sqrt{u}$ and $y = v^{-3}$, then

$$\frac{du}{dx} = 5x^4 - 6, \quad \frac{dv}{du} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{dy}{dv} = -3v^{-4}$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= -3v^{-4} \cdot \frac{1}{2\sqrt{u}} \cdot (5x^4 - 6) \\ &= -3(\sqrt{x^5 - 6x})^{-4} \frac{1}{2\sqrt{x^5 - 6x}} (5x^4 - 6) \\ &= -\frac{3}{2}(x^5 - 6x)^{-\frac{9}{2}} (5x^4 - 6) \end{aligned}$$

Note: $\frac{d}{dx} x^{\frac{n}{m}} = \frac{n}{m} x^{\frac{n}{m}-1}$

Example 15: Let $f(x) = (x^3 - 1)^{7/5}$. Find $f'(x)$.

Solution: We have

$$f'(x) = \frac{7}{5} (x^3 - 1)^{(7/5 - 1)} \cdot 3x^2 = \frac{21}{5} x^2 (x^3 - 1)^{2/5}$$

Activity 6:

1. Prove that the derivative of an odd function is an even function.
2. Suppose f is an even function. If $f'(a) = k$, find $f'(-a)$, where a is an element of the domain of f' , and k is a constant.

2.6. Derivatives of Logarithmic and Exponential Functions

In this sub unit, we are going to learn about the derivatives of Logarithmic and Exponential functions.

Theorem: Let a be a positive real number such that $a \neq 1$, then

$$\text{i. } \frac{d}{dx}(\log_a^x) = \frac{1}{x} \log_a^e = \frac{1}{x \ln a}$$

$$\text{ii. } \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof: left as an exercise for the reader!

Examples 16: Find the derivatives of the following functions.

$$\text{a) } f(x) = \log_3^x \quad \text{b) } g(x) = \log_2(x^2 - 4x + 1) \quad \text{c) } h(x) = \ln(\sin x)$$

Solution:

$$\text{a) } \frac{d}{dx}(f(x)) = \frac{d}{dx}(\log_3^x) = \frac{1}{x \ln 3}.$$

b) Since $g(x) = \log_2(x^2 - 4x + 1)$ is a composite function of s and t where $s(x) = \log_2^x$ and $t(x) = x^2 - 4x + 1$, such that

$$g(x) = s(t(x)) = s(x^2 - 4x + 1) = \log_2(x^2 - 4x + 1).$$

Cont....

Applying the Chain rule we get

$$\frac{d}{dx} g(x) = s'(t(x))t'(x)$$

$$= \frac{1}{\ln 2(x^2 - 4x + 1)} (2x - 4) = \frac{2x - 4}{\ln 2(x^2 - 4x + 1)}$$

$$c) \frac{d}{dx} h(x) = \frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x.$$

$$\text{Therefore, } \frac{d}{dx} \ln(\sin x) = \cot x$$

Theorem: Let a be a positive real number ($a > 0$), such that $a \neq 1$,

$$\text{then } \frac{d}{dx} (a^x) = (\ln a) \cdot a^x \text{ for all } x \text{ in } \mathbb{R}.$$

Proof: Let $f(x) = a^x \Rightarrow \ln f(x) = \ln(a^x) = x \ln a$

Cont....

$$\text{Hence, } \frac{d}{dx}(\ln a^x) = \frac{d}{dx}(x \ln a) = \ln a$$

$$\Rightarrow \frac{1}{a^x} \frac{d}{dx}(a^x) = \ln a$$

$$\text{Therefore, } \frac{d}{dx}(a^x) = a^x \ln a.$$

$$\textbf{Remarks: a) } \frac{d}{dx}(\log_a^{f(x)}) = \frac{f'(x)}{f(x) \ln a}$$

$$\textbf{b) } \frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}, \text{ where } f \text{ is a differentiable functn.$$

$$\textbf{c) } \frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a, \text{ where } f \text{ is a differentiable fn.$$

$$\text{Moreover for any real number } x, \text{ it holds that } \frac{d}{dx}(e^x) = e^x.$$

Cont....

Examples 17: Find the derivative of each of the following functions.

a) $f(x) = 5^x$

c) $g(x) = e^{x^3}$

b) $h(x) = h(x) = a^{x^2} \quad (a > 0, a \neq 1)$

d) $k(x) = 2^{\cos x}$

Solution:

a) $\frac{d}{dx} f(x) = \frac{d}{dx} (5^x) = 5^x \ln 5.$

b) $\frac{d}{dx} g(x) = \frac{d}{dx} (e^{x^3}) = 3x^2 e^{x^3}.$

c) $\frac{d}{dx} h(x) = \frac{d}{dx} (a^{x^2}) = 2x \cdot a^{x^2} \cdot \ln a.$

d) $\frac{d}{dx} k(x) = \frac{d}{dx} (2^{\cos x}) = \ln 2 (-\sin x) 2^{\cos x} = -\sin x \cdot \ln 2 (2^{\cos x}).$

Activity 7: Find the derivative of the following functions

a) $f(x) = \log_4^{2x}$

h) $g(x) = \log_8^{\sqrt{x}}$

b) $h(x) = \log_{\sqrt{3}}^{(\ell n x^5)}$

i) $k(x) = \ln(1 + x^2)^3$

c) $l(x) = \ell n \left(\frac{x+1}{x-1} \right)$

j) $m(x) = \ell n (\tan x + \sqrt{x^2 + 2})$

Cont....

Examples 18: Find

a $y = \ln x^3$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{x^3} (3x^2)$
 $\therefore \frac{dy}{dx} = \frac{3}{x}$

OR

$\Rightarrow \frac{dy}{dx} = 3 \frac{1}{x} \therefore \frac{dy}{dx} = \frac{3}{x}$

b

$\Rightarrow \frac{dy}{dx} = 4^{4x^4} (\ln 4) (16x^3)$

c

$\Rightarrow \frac{dy}{dx} = \frac{1}{4x^5} \frac{1}{\ln 2} 20x^4 = \frac{5}{x \ln 2}$

d

$\Rightarrow \frac{dy}{dx} = -\sin(\ln(4x^3)) \frac{1}{4x^3} (12x^2)$
 $\therefore \frac{dy}{dx} = -\frac{3 \sin(\ln(4x^3))}{x}$

e)

$\Rightarrow \frac{dy}{dx} = \frac{1}{(3x^5+5)^5} \frac{1}{\ln 3} 5(3x^5+5)^4 (15x^4)$
 $\therefore \frac{dy}{dx} = \frac{75x^4}{(3x^5+5) \ln 3}$

f

$\Rightarrow \frac{dy}{dx} = 3^{\cos x^4} (\ln 3) (-\sin x^4) (4x^3)$
 $\therefore \frac{dy}{dx} = -4x^3 3^{\cos x^4} \sin x^4 \ln 3$

2.7. Higher Derivatives

If $f'(x)$ (the derivative of f at x) exists, then $f'(x)$ is called the first derivative of f at x , since f' is a function that maps x to $f'(x)$ we can speak about the derivative of f' at x denoted by $f''(x)$ which is given by

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}, \text{ provided this limit exists.}$$

If the above limit exists, we call it the second derivative of f at x .

Examples 19:

a) Let $f(x) = \sin x$. Find a formula for $f''(x)$.

Solution:

Since $f'(x) = \cos x$, it follows that $f''(x) = -\sin x$.

b) Let $f(x) = 3x^{\frac{1}{2}}$. Find a formula for $f''(x)$.

Solution: $f'(x) = 3 \left(\frac{1}{2} x^{\frac{1}{2}-1} \right) = 3 \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{3}{2} x^{-\frac{1}{2}}$

$$\text{and } f''(x) = \frac{3}{2} \left(-\frac{1}{2} x^{\frac{1}{2}-1} \right) = \frac{3}{2} \left(-\frac{1}{2} x^{-\frac{3}{2}} \right) = -\frac{3}{4} x^{-\frac{3}{2}}.$$

Cont....

For any positive integer $n \geq 3$ we can define the n^{th} derivative of f at x denoted by $f^{(n)}(x)$ or $D^n f(x)$. By denoting $f^{(n-1)}$ for the $(n-1)^{\text{th}}$ derivative of f , we can write

$$f^{(n)}(x) = \lim_{h \rightarrow 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

provided the limit exists.

The second derivative, the third derivative and so on are called higher derivatives. We say that f is n times differentiable if $f^{(n)}(x)$ exists for all x in the domain of f .

Example 20:

Let $f(x) = \cos x$. Find the n^{th} derivative of f , for $n \in \mathbf{Z}^+$.

Solution

$$\begin{aligned} f'(x) &= -\sin x, & f''(x) &= -\cos x, & f'''(x) &= \sin x, & f^{(4)}(x) &= \cos x, \\ f^{(5)}(x) &= -\sin x, & f^{(6)}(x) &= -\cos x, & \dots, \\ f^{(2n+1)}(x) &= (-1)^{n+1} \sin x, & f^{(2n)}(x) &= (-1)^n \cos x. \end{aligned}$$

Cont....

Example 21: Let $f(x) = x^5 - 3x^3 + 2x^2 + 1$, find all higher derivatives of f .

Solution: $f'(x) = 5x^4 - 9x^2 + 4x$, $f^{(4)}(x) = 120x$

$$f''(x) = 20x^3 - 18x + 4, \quad f^{(5)}(x) = 120$$

$$f^{(3)}(x) = 60x^2 - 18, \quad \text{and} \quad f^{(n)}(x) = 0, \quad \forall n \geq 6$$

Note: *Leibniz notation*, the second, third and fourth derivatives are:

$$\frac{d^2 y}{dx^2}, \quad \frac{d^3 y}{dx^3}, \quad \frac{d^4 y}{dx^4}$$

Example 22: Let $y = x \sin x$. Using the Leibniz notation find the first four derivatives of y .

Solution:

$$\frac{dy}{dx} = \sin x + x \cos x, \quad \frac{d^2 y}{dx^2} = \cos x + \cos x - x \sin x = 2 \cos x - x \sin x$$

$$\frac{d^3 y}{dx^3} = -2 \sin x - \sin x - x \cos x = -3 \sin x - x \cos x$$

$$\frac{d^4 y}{dx^4} = -3 \cos x - \cos x + x \sin x = -4 \cos x + x \sin x.$$

Activity 8:

1. Find the second derivative of the function

a) $f(x) = -12x^5 + \frac{1}{4}x^4 - \sqrt{1-x}$ b) $f(x) = \frac{x+1}{x-1}$ c) $f(x) = \sqrt{\sqrt{x} + \sin x}$

2. Find $\frac{d^2y}{dx^2}$ for the following functions

a) $y = x^{\frac{3}{2}}$

c) $y = (x^4 - \tan x)^5$

b) $y = \sec x$

d) $y = \frac{1}{x \ln x}$

e) $y = \frac{x + \sin x}{x + 2}$

3. Find the third derivative of the function

a) $f(x) = x^2 |x|$ b) $f(x) = \frac{1}{3x+1}$

c) $f(x) = \frac{3x}{4x+5}$

4. Find $\frac{d^3y}{dx^3}$ for the following functions

a) $y = 3x^2$

b) $y = x^{\frac{7}{2}} - 2x^3$

c) $y = \frac{2}{(1-x)^2}$

5. Find the n^{th} derivative of the function

a) $f(x) = \sin x - \cos x$ b) $f(x) = \frac{1}{x+1}$ c) $f(x) = \sin(nx)$ d) $f(x) = \ln x$

e) $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ ($a, b, c, d, e \in \mathbb{R}$ $a \neq 0$) g) $f(x) = e^{ax}$

2.8. Implicit Differentiation

The differentiable functions we have encountered so far have been defined by algebraic expressions such as

$$y = \frac{x^2 + 1}{x^3 - 4}, \quad y = \sin(\tan x^2), \text{ etc}$$

define functions f and g where

$$f(x) = \frac{x^2 + 1}{x^3 - 4} \text{ and } g(x) = \sin(\tan x^2).$$

However not all functions are defined in such an explicit way. If we consider the equation

$$x^2 - x = y^2 + 3y + 2,$$

it is not so easy to solved for y in terms of x . But there may exist a function f such that

$$x^2 - x = (f(x))^2 + 3f(x) + 2, \text{ holds for every } x \text{ in the domain of } f.$$

Such a function is said to be defined *implicitly* by the given equation and the process of finding the derivatives of such function is called *Implicit differentiation*.

Cont....

Example 23: Find the derivative of the implicit function given by
 $x^2 + y^2 = 25$ (*Equation of circle*)

Solution: Assume y is differentiable function of x and differentiating both sides with respect to x , we get:

$$2x + 2y \frac{dy}{dx} = 0 \quad \text{and solving for } \frac{dy}{dx}, \text{ we have:}$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{or} \quad f'(x) = -\frac{x}{y}.$$

Example 24: Find $\frac{dy}{dx}$ for the implicit functions

a) $xy^3 - 3x^2 = xy + 5x$

b) $e^{xy} + y \ln x = \cos 2x$

Solution :

a) Differentiating both sides with respect to x .

$$\text{i.e.} \quad \frac{d}{dx}(xy^3) - 3 \frac{d}{dx}(x^2) = \frac{d}{dx}(xy) + 5 \frac{dx}{dx}$$

$$\Rightarrow y^3 + 3xy^2 \frac{dy}{dx} - 6x = y + x \frac{dy}{dx} + 5$$

$$\Rightarrow (3xy^2 - x) \cdot \frac{dy}{dx} = y + 6x - y^3 + 5 \Rightarrow \frac{dy}{dx} = \frac{6x + y - y^3 + 5}{3xy^2 - x}.$$

Cont....

b) $e^{xy} + y \ln x = \cos 2x$ is also an implicit function. We have

$$\frac{d}{dx}(e^{xy} + y \ln x) = \frac{d}{dx}(\cos 2x)$$

$$\Rightarrow ye^{xy} + xe^{xy} \frac{dy}{dx} + \frac{y}{x} + \ln x \frac{dy}{dx} = -2 \sin 2x$$

$$\Rightarrow (xe^{xy} + \ln x) \frac{dy}{dx} = - \left[2 \sin 2x + ye^{xy} + \frac{y}{x} \right] \Rightarrow \frac{dy}{dx} = - \frac{2x \sin 2x + xye^{xy+y}}{x^2 e^{xy} + x \ln x}$$

Example 25: Find an equation of the line that is tangent to the circle

$$x^2 + y^2 = 25 \text{ at the point } (3, -4).$$

Solution: The point $(3, -4)$ lies on the circle, which is the graph of a differentiable function y .

$$\frac{dy}{dx} = -\frac{x}{y}. \text{ So, that } \frac{dy}{dx} \Big|_{(3, -4)} = \frac{3}{4}.$$

Therefore the slope of the tangent line we seek is $\frac{3}{4}$, so an equation of the required line is

$$y - (-4) = \frac{3}{4}(x - 3) \text{ or } y = \frac{3}{4}x - \frac{25}{4}.$$

Cont....

Note: Differentiating an implicit function will create the derivative () needed to calculate the slope of any tangent on the curve or the rate of change at any value of the independent variable.

Example 26: Find

$$a \quad 3x^2 + y^2 = 14$$

$$\Rightarrow 6x \frac{dx}{dx} + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 6x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} = -6x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{6x}{2y}$$

$$\therefore \frac{dy}{dx} = -\frac{3x}{y}$$

$$b \quad x^3 y^5 + 3x = 8y^3 + 1$$

$$\Rightarrow x^3 \cdot 5y^4 \frac{dy}{dx} + 3x^2 \frac{dx}{dx} y^5 + 3 \frac{dx}{dx} = 24y^2 \frac{dy}{dx}$$

$$\Rightarrow 5x^3 y^4 \frac{dy}{dx} + 3x^2 y^5 + 3 = 24y^2 \frac{dy}{dx}$$

$$\Rightarrow 5x^3 y^4 \frac{dy}{dx} - 24y^2 \frac{dy}{dx} = -3x^2 y^5 - 3$$

$$\Rightarrow \frac{dy}{dx} (5x^3 y^4 - 24y^2) = -3x^2 y^5 - 3$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2 y^5 - 3}{5x^3 y^4 - 24y^2}$$

Cont....

Example 27: Find the equation of the tangent and normal lines for the following implicit function at the given point.

$$3x^2 + y^2 = 14 \quad (2, \sqrt{2})$$

Solution

$$\text{Here, } \frac{dy}{dx} = -\frac{3x}{y}$$

$$\Rightarrow m_{tan} = \frac{dy}{dx} = -\frac{3(2)}{\sqrt{2}}$$

and

$$m_{normal} = \frac{\sqrt{2}}{6}$$

and

$$\text{Equation of the normal line: } y - \sqrt{2} = \frac{\sqrt{2}}{6}(x - 2)$$

Cont....

Note: If y is a function defined implicitly in terms of x , and if $\frac{d^2y}{dx^2}$ exists,

we can differentiate $\frac{dy}{dx}$ with respect to x to obtain $\frac{d^2y}{dx^2}$.

Example 28: Let $x^2 + y = 25$. Find $\frac{d^2y}{dx^2}$ implicitly.

Solution: We already know that:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{x}{y} \right) = -\frac{(y^2 + x^2)}{y^3} = -\frac{25}{y^3}.$$

Activity 9:

1. Use implicit differentiation to find the derivative of y with respect to x .

a) $3y^2 = -2x^4$

c) $x = \frac{1 - \sqrt{y}}{1 + \sqrt{y}}$

d) $\frac{\sin y}{y^2 + 1} = 3x$

b) $y^2 = \frac{x^3}{2 - x}$

e) $x^2 + y^2 = \frac{y^2}{x^2}$

Cont....

2. Use implicit differentiation to find the derivative of y with respect to x at the given point.

a) $x^2 + y^2 = y$; $(0, 1)$

b) $xy = 2$; $(-2, -1)$

c) $y^2 \sin 2x = -2y$; $\left(\frac{\pi}{4}, -2\right)$

d) $\sin x = \cos y$; $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

e) $\frac{x}{x+2y} = 1 - y$; $(1, 0)$

3. Find an equation of the line tangent to the graph of the equation at the given point

a) $xy^2 = 18$; $(2, -3)$

b) $x^2 + y^2 = 3y$; $(-\sqrt{2}, 2)$

c) $\sin(x+y) = 2x$; $(0, \pi)$

4. Use implicitly differentiation to find $\frac{d^2y}{dx^2}$

a) $x^2 - y^4 = 6$

c) $x^2 \sin 2y = 1$

b) $x^x \ln y = y$

d) $2xy^2 = x^2y$

Alternative techniques of differentiating some functions

➤ This technique is useful in cases where it is easier to differentiate the logarithm of a function rather than the function itself.

Example 29: Find

a $y = 2x^{2x}$

b $y = x^{\sin x}$

c $y = (1-x)^2(x+1)^4$

Solution:

)

$$\Rightarrow \ln y = \ln (2x^{2x})$$

$$\Rightarrow \ln y = \ln 2 + \ln (x^{2x})$$

$$\Rightarrow \ln y = \ln 2 + 2x \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 0 + 2x \frac{1}{x} + 2 \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2 + 2 \ln x$$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = 2x^{2x}$$

$$\therefore \frac{dy}{dx} = 4x^{2x}$$

Cont....

$$\textbf{b} \text{ } \textcolor{red}{!} \ y = x^{\sin x} \Rightarrow \ln y = \ln (x^{\sin x}) \Rightarrow \ln y = \sin x \ln x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin x \frac{1}{x} + \cos x \ln x \Rightarrow \frac{dy}{dx} = y \textcolor{red}{!}$$

$$\therefore \frac{dy}{dx} = x^{\sin x} \textcolor{red}{!}$$

$$\textbf{c} \text{ } \textcolor{red}{!} \ y = (1-x)^2 (x+1)^4 \Rightarrow \ln y = \ln ((1-x)^2 (x+1)^4)$$

$$\Rightarrow \ln y = \ln (1-x)^2 + \ln (x+1)^4$$

$$\Rightarrow \ln y = 2 \ln (1-x) + 4 \ln (x+1)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2 \frac{1}{1-x} (-1) + 4 \frac{1}{x+1} \Rightarrow \frac{dy}{dx} = y \left(\frac{-2}{1-x} + \frac{4}{x+1} \right)$$

$$\therefore \frac{dy}{dx} = (1-x)^2 (x+1)^4 \left(\frac{-2}{1-x} + \frac{4}{x+1} \right)$$

2.11. Hyperbolic Functions

- Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names.
- In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle.

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

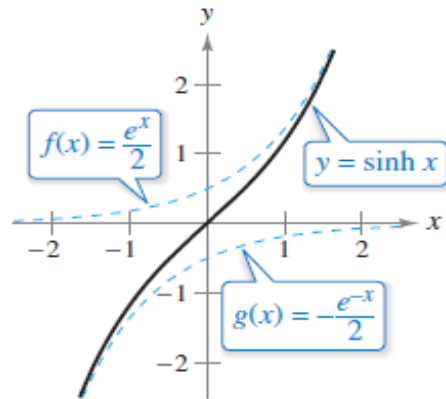
$$\operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

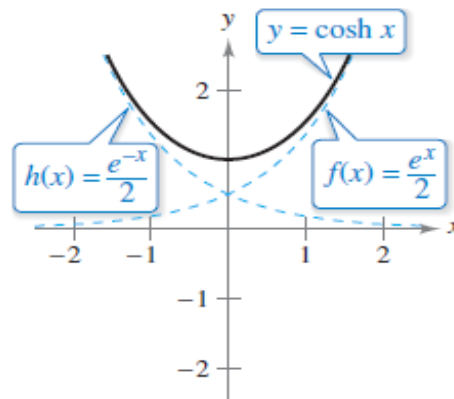
$$\operatorname{coth} x = \frac{1}{\tanh x}, \quad x \neq 0$$

Cont....

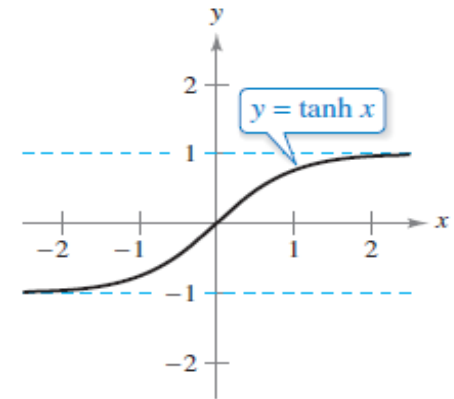
The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure



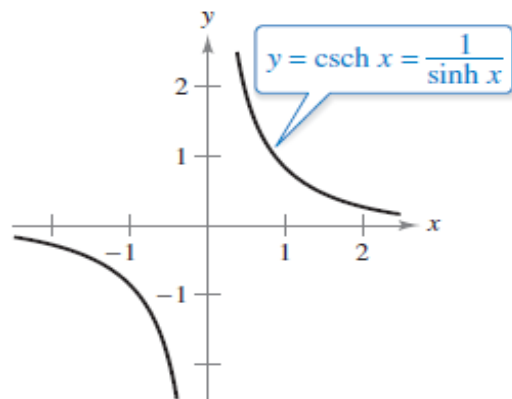
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



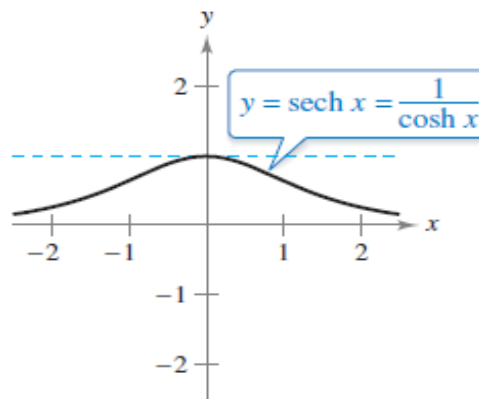
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



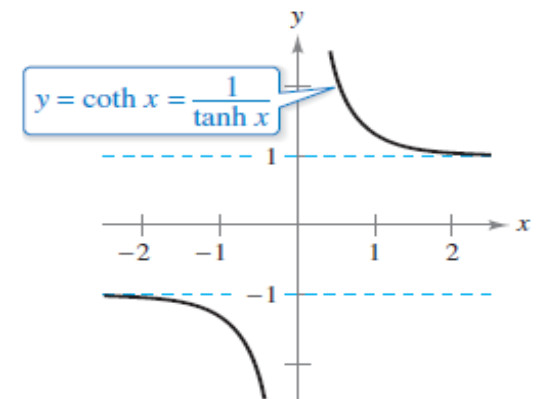
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

- **Note that** the graph of $\sinh x$ can be obtained by adding the corresponding y-coordinates of $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$.

and

- Likewise, the graph of $\cosh x$ can be obtained by adding the corresponding y-coordinates of $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.

and

HYPERBOLIC IDENTITIES

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Proof:

Exercise!

Derivative of Hyperbolic Functions

Derivatives can be found relatively easily using the definitions.

$$\frac{d}{dx} \sinh(x) = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx} \cosh(x) = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Cont....

$$\begin{aligned}
 \frac{d}{dx} \tanh(x) &= \frac{d}{dx} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
 &\text{(quotient rule)} \\
 &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} \\
 &= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}} \right)^2 = \operatorname{sech}^2(x)
 \end{aligned}$$

Furthermore,

$$\frac{d}{dx} \coth(x) = -\operatorname{csch}^2(x), \quad \frac{d}{dx} \operatorname{sech}(x) = -\operatorname{sech}(x) \tanh(x)$$

$$\frac{d}{dx} \operatorname{csch}(x) = -\operatorname{csch}(x) \coth(x)$$

Remarks:

1. All of the derivatives are similar to trig functions except for some of the signs. *sinh*, *cosh* and *tanh* are positive. The others are negative.
2. Any of the differentiation rules in Hyperbolic function can be combined with the product and Chain Rule.

Example 48:

1. and Chain Rule.

3.

$$\begin{aligned}
 4. \frac{d}{dx} (\ln(\cosh x)) &= \frac{\sinh x}{\cosh x} - \frac{\cancel{2} \sinh x}{\cancel{2} \cosh^3 x} = \tanh x - \tanh x \cdot \frac{1}{\cosh^2 x} = \tanh x \left(1 - \frac{1}{\cosh^2 x} \right) \\
 &= \tanh x \cdot \tanh^2 x = \tanh^3 x.
 \end{aligned}$$

Activity 11:

I. Find the derivatives of each of the functions in Exercises 1 - 12. In some cases it may be convenient to do some preliminary algebra. These problems involve hyperbolic functions and their inverses.

1. $f(x) = x \sinh x^3$

2. $f(x) = x \sinh^3 x$

3. $f(x) = \cosh(\ln(x^2 + 1))$

4. $f(x) = 3 \tanh^2 e^x$

5. $f(x) = \sqrt{\cosh^2 x + 1}$

6. $f(x) = \frac{\tanh \sqrt{x}}{\sqrt{\sinh x}}$

II. Use logarithmic differentiation to find the derivatives of each of the functions in Exercises 1 - 3.

1. $(\sin x)^x$

2. $(\sec x)^x$

3. $(\sin x)^{\cos x}$

End of Chapter two