UNIT V

PROBABILITY

THEORY & DISTRIBUTION

5 Probability

- Basic Definitions: Events, Sample Space, and Probabilities
- Basic Rules for Probability
- Conditional Probability
- Independence of Events
- Combinatorial Concepts
- The Law of Total Probability and Bayes' Theorem
- Joint Probability Table

5-1 Probability is:

- A quantitative measure of **uncertainty**
- A measure of the **strength of belief** in the occurrence of an uncertain event
- A measure of the degree of **chance or likelihood of occurrence** of an uncertain event
- Measured by a number between 0 and 1 (or between 0% and 100%)

Types of Probability

1. **Objective or Classical Probability**
   - ✓ Based on equally-likely events
   - ✓ Based on long-run relative frequency of events
   - ✓ Not based on personal beliefs
   - ✓ Is the same for all observers (objective)
   - ✓ Examples: toss a coin, throw a die, pick a card

Types of Probability (Continued)

2. **Subjective Probability**
   - ✓ Based on personal beliefs, experiences, prejudices, intuition - personal judgment
   - ✓ Different for all observers (subjective)
   - ✓ Examples: elections, new product introduction…

5-2 Basic Definitions

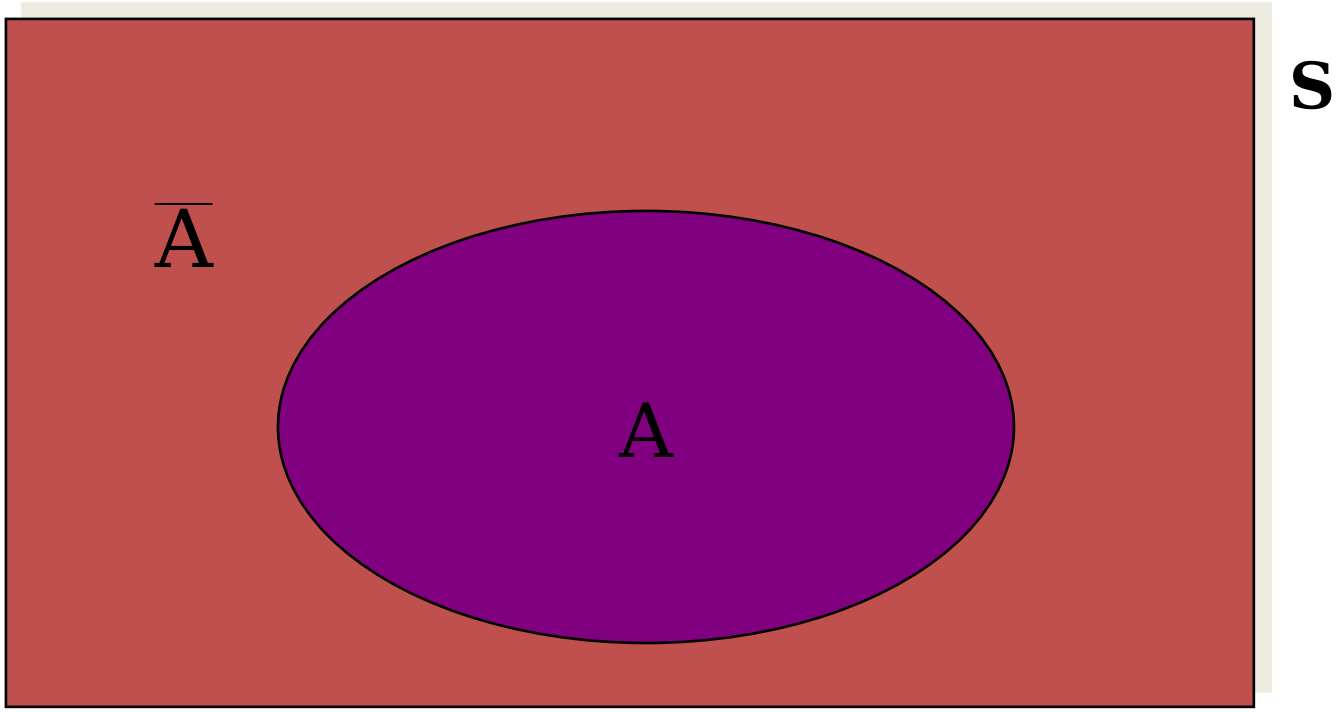
- **Set - a collection of elements or objects of interest**

✓ Empty set (denoted by ⊘)
  - a set containing no elements

✓ Universal set (denoted by S)
  - a set containing all possible elements

✓ Complement (**Not**). The complement of A is
  - a set containing all elements of S not in A

Complement of a Set

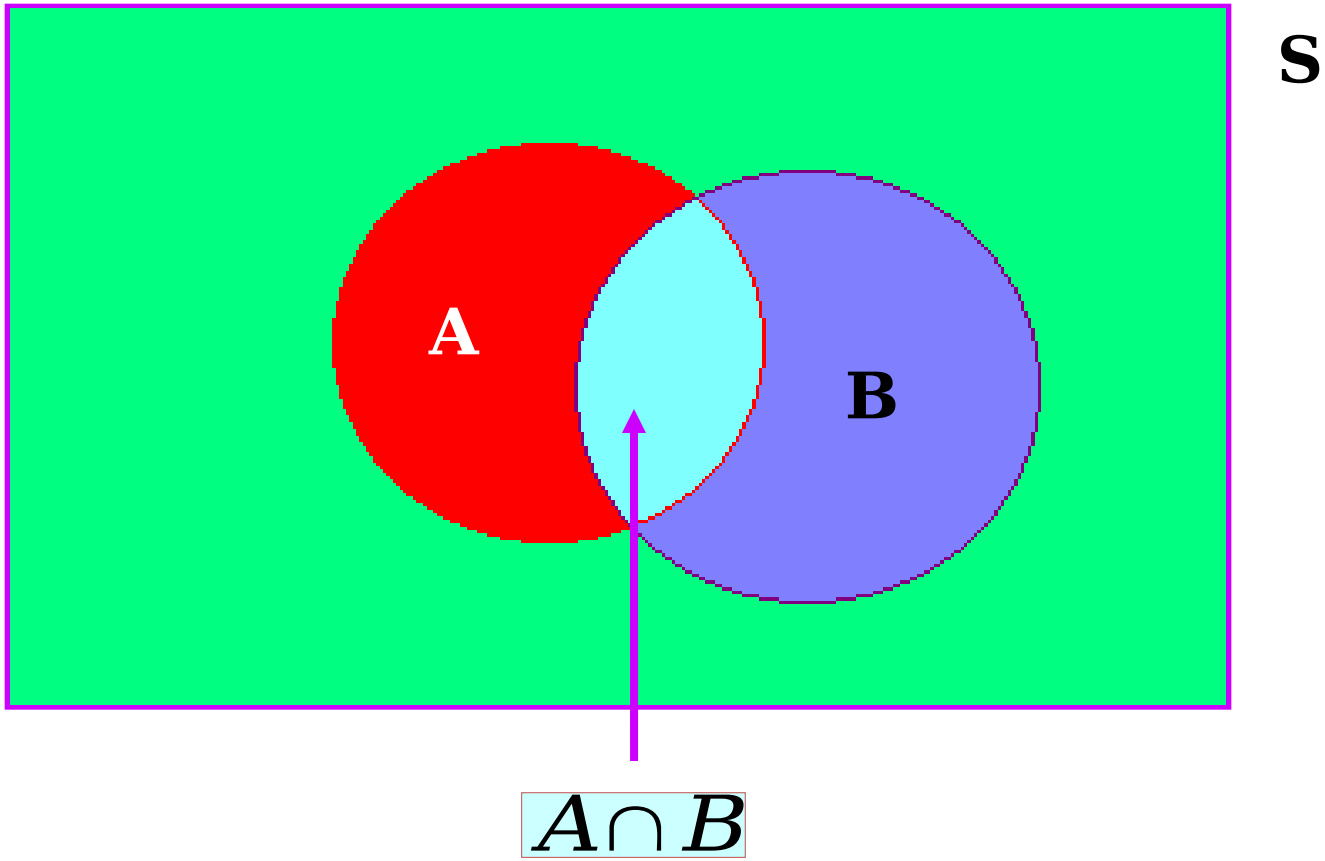


Venn Diagram illustrating the Complement of an event

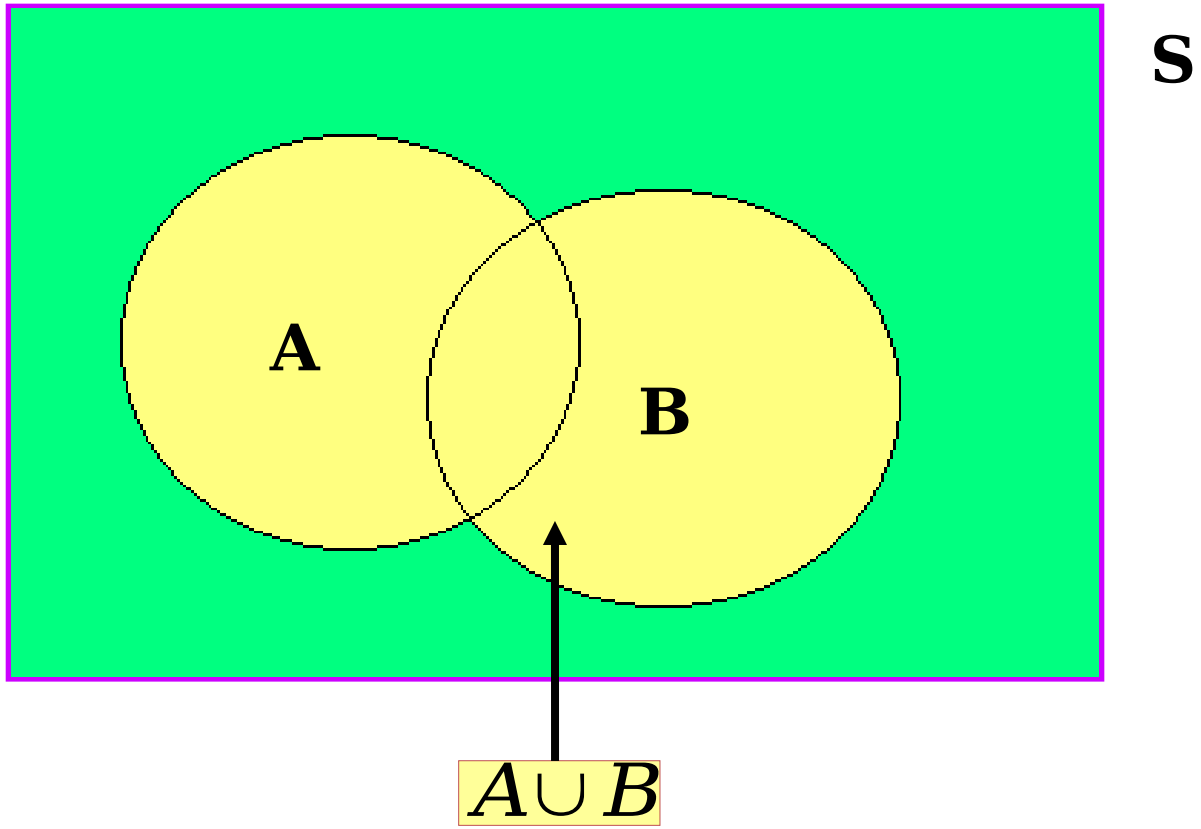
Basic Definitions (Continued)

- **Intersection** (And) $(A \cap B)$
  - a set containing all elements in both A and B
- **Union** (Or) $(A \cup B)$
  - a set containing all elements in A or B or both

Sets: A Intersecting with B



Sets: A Union B

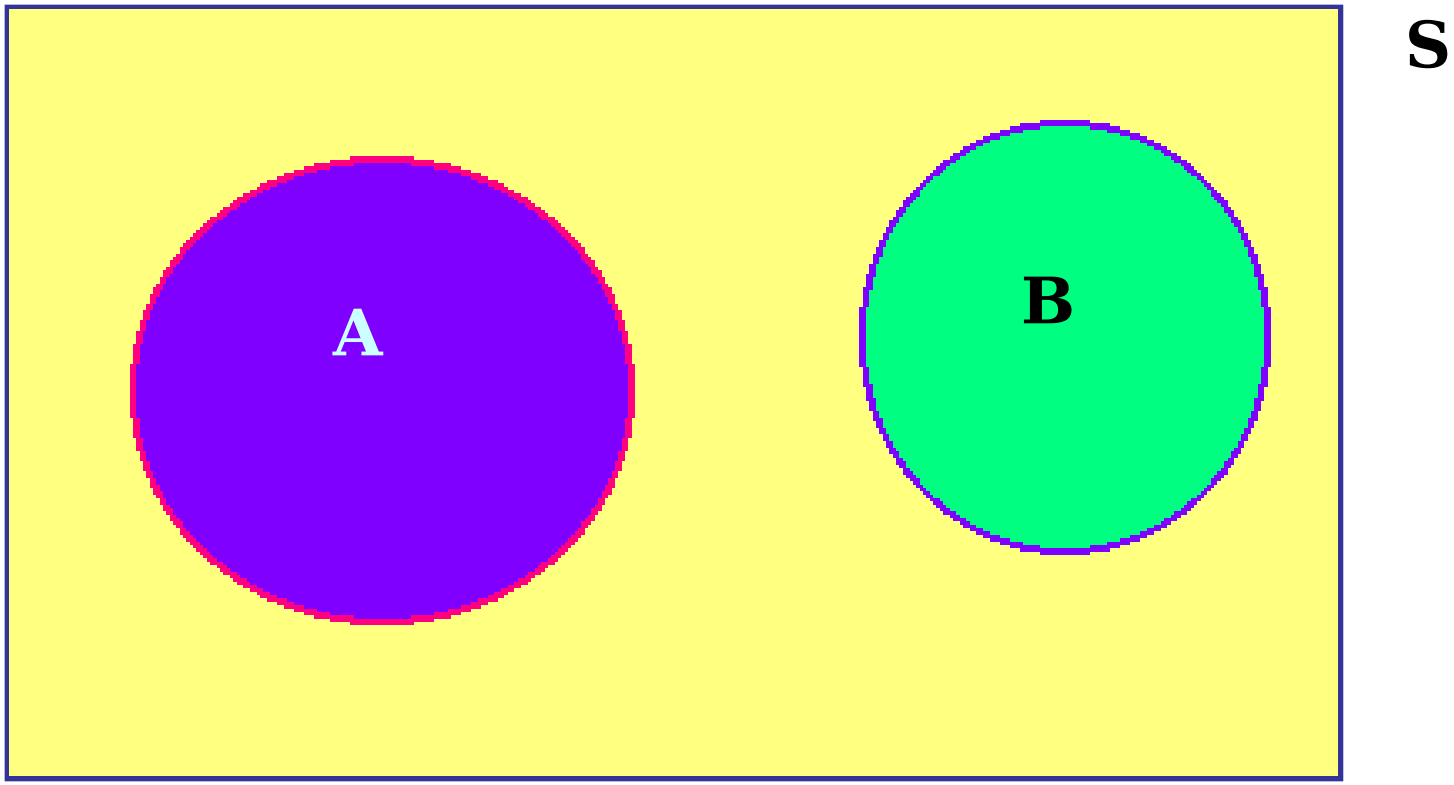


Basic Definitions (Continued)

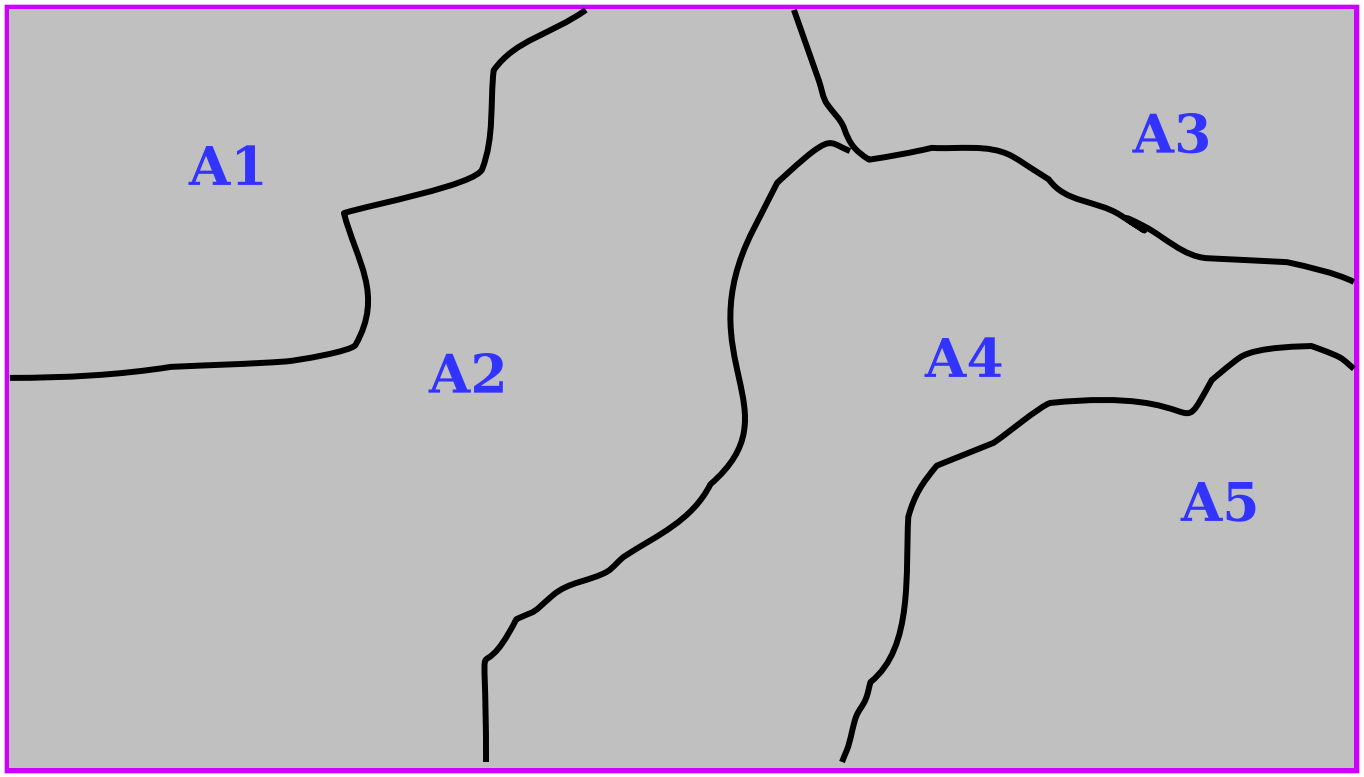
- **Mutually exclusive or disjoint sets**
  - sets having no elements in common, having no intersection, whose intersection is the empty set
- **Partition**
  - a collection of mutually exclusive sets which together include all possible elements, whose union is the universal set

Mutually Exclusive or Disjoint Sets

have nothing in common



Sets: Partition



Experiment

- Process that leads to one of several possible outcomes *, e.g.:
  - ✓ Coin toss
    - Heads, Tails
  - ✓ Throw die
    - 1, 2, 3, 4, 5, 6
  - ✓ Pick a card
    - AH, KH, QH, ...
  - ✓ Introduce a new product
- Each trial of an experiment has a single observed outcome.
- The precise outcome of a random experiment is unknown before a trial.

* Also called a basic outcome, elementary event, or simple eve

Events : Definition

- Sample Space or Event Set
  - ✓ Set of all possible outcomes (universal set) for a given experiment
    - E.g.: Roll a regular six-sided die
      - S = {1,2,3,4,5,6}
- Event
  - ✓ Collection of outcomes having a common characteristic
    - E.g.: Even number
      - A = {2,4,6}
  - Event A occurs if an outcome in the set A occurs
- Probability of an event
  - ✓ Sum of the probabilities of the outcomes of which it consists
    - P(A) = P(2) + P(4) + P(6)

Equally-likely Probabilities

- For example:
  - ✓ Throw a die
    - Six possible outcomes {1,2,3,4,5,6}
    - If each is equally-likely, the probability of each is 1/6 = 0.1667 = 16.67%
    - $P(e) = \frac{1}{n(S)}$
    - Probability of each equally-likely outcome is 1 divided by the number of possible outcomes
  - ✓ Event A (even number)
    - P(A) = P(2) + P(4) + P(6) = 1/6 + 1/6 + 1/6 = 1/2
    - $P(A) = \sum P(e)$ for e in A

$$= \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

5-3 Basic Rules for Probability

- **Range of Values for P(A):** $0 \le P(A) \le 1$
- **Complements** - Probability of ***not*** A

$$P(\overline{A}) = 1 - P(A)$$

- **Intersection** - Probability of both A ***and*** B

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

$$P(A \cap C) = 0$$

✓ **Mutually exclusive events** (A and C) :

Basic Rules for Probability (Continued)

- **Union** - Probability of A *or* B or both (rule of unions)

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = P(A) + P(B) - P(A \cap B)$$

  - ✓ Mutually exclusive events: If A and B are mutually exclusive, then $P(A \cap B) = 0$ so $P(A \cup B) = P(A) + P(B)$

5-4 Conditional Probability

- **Conditional Probability** - Probability of A *given* B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ where } P(B) \neq 0$$

✓ Independent events: $P(A|B) = P(A)$

$P(B|A) = P(B)$

Conditional Probability (continued)

Rules of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{so} \quad P(A \cap B) = P(A|B)P(B)$$
$$= P(B|A)P(A)$$

If events A and D are **independent events**:

$$P(A|D) = P(A)$$

$$P(D|A) = P(D)$$

so $P(A \cap D) = P(A)P(D)$

Example1: Suppose there are 30 applicants for a job in a certain organization, which are cross- classified by their sex and color.

	Black	White
Male	12	8
Female	4	6

Assume that each applicant is equally likely to be chosen for a job. What is the probability that the applicant chosen is

a) black b) white c)male d) female e) male and black

f) female and black g) male and white h) female and white

Solution: Let B stands for the set of black applicant W stands for white applicant M stands for male applicant and F stands for female applicant

a) $P(B) = \frac{12 + 4}{30} = \frac{8}{15}$

b) $P(W) = \frac{8 + 6}{30} = \frac{7}{15}$

c) $P(M) = \frac{12 + 8}{30} = \frac{2}{3}$

d) $P(F) = \frac{4 + 6}{30} = \frac{1}{3}$

e) $P(M \cap B) = \frac{12}{30}$

g) $P(M \cap W) = \frac{8}{30}$

f) $P(F \cap B) = \frac{4}{30}$

h) $P(F \cap W) = \frac{6}{30}$

Example2: In **Example** 1 above find the probability that an applicant chosen is

a) male given that black is chosen b) male given that white is chosen

c) female given that black is chosen d) female given that white is chosen

Solution: a) $P(M/B) = \dfrac{P(M \cap B)}{P(B)} = \dfrac{12}{30} \Big/ \dfrac{8}{15} = \dfrac{12}{16} = \underline{\underline{\dfrac{3}{4}}}$

b) $P(M/W) = \dfrac{P(M \cap W)}{P(W)} = \dfrac{8}{30} \cdot \dfrac{15}{7} = \underline{\underline{\dfrac{4}{7}}}$

c) $P(F/B) = \dfrac{P(F \cap B)}{P(B)} = \dfrac{4}{30} \cdot \dfrac{15}{8} = \underline{\underline{\dfrac{1}{4}}}$

d) $P(F/W) = \dfrac{P(F \cap W)}{P(W)} = \dfrac{6}{30} \cdot \dfrac{15}{7} = \underline{\underline{\dfrac{3}{7}}}$

2-5 Independence of Events

onditions for the statistical independence of events A and B

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$
$$\text{and}$$
$$P(A \cap B) = P(A)P(B)$$

$$P(Ace|Heart) = \frac{P(Ace \cap Heart)}{P(Heart)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13} = P(Ace)$$

$$P(Heart|Ace) = \frac{P(Heart \cap Ace)}{P(Ace)} = \frac{\frac{1}{52}}{\frac{4}{52}} = \frac{1}{4} = P(Heart)$$

$$P(Ace \cap Heart) = \frac{4}{52} * \frac{13}{52} = \frac{1}{52} = P(Ace)P(Heart)$$

Independence of Events

Events *Television* (T) and *Billboard* (B) are assumed to be **independent.**

Product Rules for Independent Events

e probability of the **intersection** of several independent ev
he product of their separate individual probabilities:

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1)P(A_2)P(A_3)\cdots P(A_n)$$

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = 1 - P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3)\cdots P(\bar{A}_n)$$

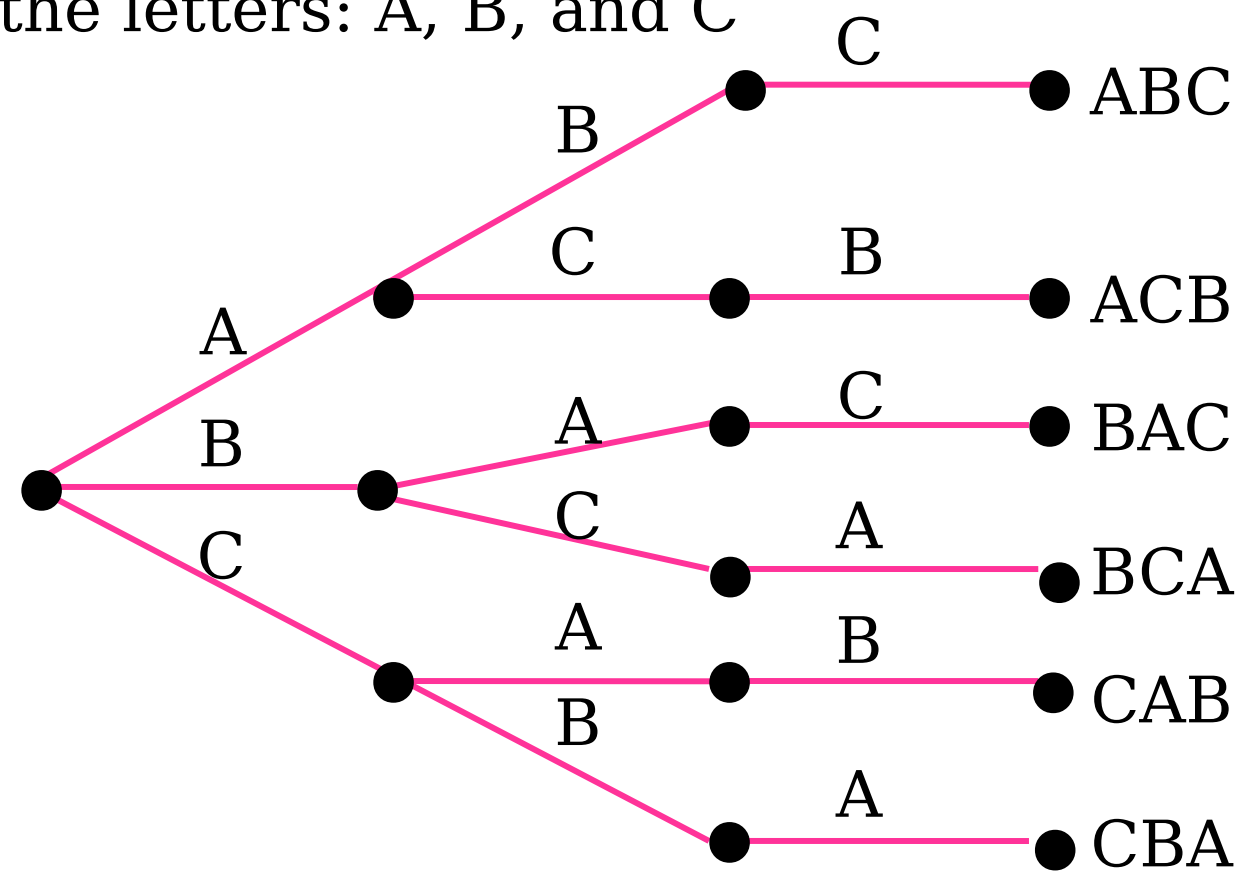
he probability of the **union** of several independent events
1 minus the product of probabilities of their complements:

5-6 Combinatorial Concepts

- Consider a pair of six-sided dice. There are six possible outcomes from throwing the first die {1,2,3,4,5,6} and six possible outcomes from throwing the second die {1,2,3,4,5,6}. Altogether, there are 6*6 = 36 possible outcomes from throwing the two dice.
- In general, **if there are *n* events and the event *i* can happen in *Ni* possible ways, then the number of ways in which the sequence of *n* events may occur is *N1N2…Nn***.
- Pick 5 cards from a deck of 52 - **with replacement**
  - ✓ $52*52*52*52*52=52^5$ 380,204,032 different possible outcomes
- Pick 5 cards from a deck of 52 - **without replacement**
  - ✓ 52*51*50*49*48 = 311,875,200 different possible outcomes

More on Combinatorial Concepts (Tree Diagram)

Order the letters: A, B, and C



Factorial

How many ways can you order the 3 letters A, B, and C?

There are 3 choices for the first letter, 2 for the second, and 1 for the last, so there are 3*2*1 = 6 possible ways to order the three letters A, B, and C.

How many ways are there to order the 6 letters A, B, C, D, E, and F?
(6*5*4*3*2*1 = 720)

Factorial: For any positive integer *n*, we define ***n factorial*** as:

- *n(n-1)(n-2)…(1).* We denote n factorial as ***n!***.
- The number *n!* is the number of ways in which *n*

Permutations (Order is important)

What if we chose only 3 out of the 6 letters A, B, C, D, E, and F?

There are 6 ways to choose the first letter, 5 ways to choose the second letter, and 4 ways to choose the third letter (leaving 3 letters unchosen). That makes 6*5*4=120 possible orderings or **permutations**.

Permutations are the possible **ordered** selections of r objects out of a total of n objects. The number of permutations of n objects ta

$$_nP_r = \frac{n!}{(n-r)!}$$

For example

$$_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6*5*4*3*2*1}{3*2*1} = 6*5*4 = 120$$

Combinations (Order is not Important)

Suppose that when we pick 3 letters out of the 6 letters A, B, C, D, E, and F we chose BCD, or BDC, or CBD, or CDB, or DBC, or DCB. (These are the 6 (3!) permutations or orderings of the 3 letters B, C, and D.) But these are orderings of the same combination of 3 letters. How many combinations of 6 different letters, taking 3 at a time, are there?

Combinations are the possible selections of r items from a group of n items $\binom{n}{r}$ regardless of the order of selection. The number of combinations is denoted $\binom{n}{r}$ is read as ***n choose r***. An alternative notation is ***nCr.*** We define the number of combinations of r out of n elements as:

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$$

For example

$$\binom{n}{r} = {}_6C_3 = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6*5*4*3*2*1}{(3*2*1)(3*2*1)} = \frac{6*5*4}{3*2*1} = \frac{120}{6} = 20$$

5-7 The Law of Total Probability and Bayes' Theorem

The law of total probability:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

In terms of conditional probabilities:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$= P(A|B)P(B) + P(A|\overline{B})P(\overline{B})$$

Bayes' Theorem

- **Bayes'** theorem enables you, knowing just a little more than the probability of A given B, to find the probability of B given A.
- Based on the definition of conditional probability and the law of total probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{P(A \cap B)}{P(A \cap B) + P(A \cap \bar{B})}$$

Applying the law of total probability to the denominator

$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

Applying the definition of conditional probability throughou

Bayes' Theorem Extended

- Given a partition of events B1,B2 ,...,Bn:

$$P(B_1|A) = \frac{P(A \cap B_1)}{P(A)}$$

Applying the law of total probability to the denominator

$$= \frac{P(A \cap B_1)}{\sum P(A \cap B_i)}$$

Applying the definition of conditional probability throughout

$$= \frac{P(A|B_1)P(B_1)}{\sum P(A|B_i)P(B_i)}$$

Example

- An economist believes that during periods of high economic growth, the U.S. dollar appreciates with probability 0.70; in periods of moderate economic growth, the dollar appreciates with probability 0.40; and during periods of low economic growth, the dollar appreciates with probability 0.20.
- During any period of time, the probability of high economic growth is 0.30, the probability of moderate economic growth is 0.50, and the probability of low economic growth is 0.50.
- Suppose the dollar has been appreciating during the present period. What is the probability we are experiencing a period of high economic growth?

Partition:

H - High growth P(H) = 0.30

M - Moderate growth P(M) = 0.50

L - Low growth P(L) = 0.20

Event A - Appreciation

$P(A|H) = 0.70$

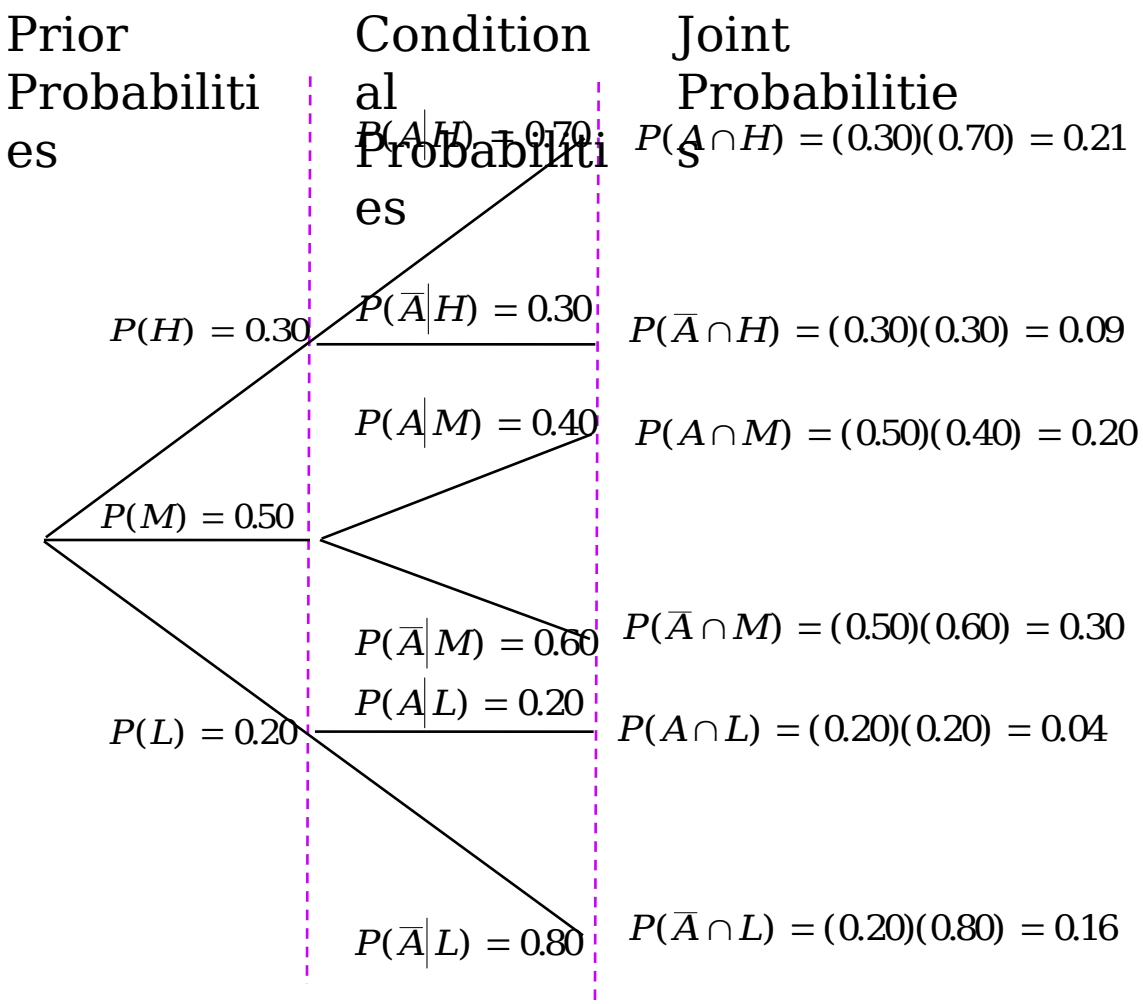
$P(A|M) = 0.40$

$P(A|L) = 0.20$

Cont'd

$$
\begin{aligned}
P(H|A) &= \frac{P(H \cap A)}{P(A)} \\
&= \frac{P(H \cap A)}{P(H \cap A) + P(M \cap A) + P(L \cap A)} \\
&= \frac{P(A|H)P(H)}{P(A|H)P(H) + P(A|M)P(M) + P(A|L)P(L)} \\
&= \frac{(0.70)(0.30)}{(0.70)(0.30) + (0.40)(0.50) + (0.20)(0.20)} \\
&= \frac{0.21}{0.21 + 0.20 + 0.04} = \frac{0.21}{0.45} \\
&= 0.467
\end{aligned}
$$

Tree Diagram



5-8 The Joint Probability Table

- A joint probability table is similar to a contingency table , except that it has probabilities in place of frequencies.
- The joint probability for Example 2-11 is shown below.
- The row totals and column totals are called marginal probabilities.

The Joint Probability Table

- A joint probability table is similar to a contingency table , except that it has probabilities in place of frequencies.
- The joint probability for Example 2-11 is shown on the next slide.
- The row totals and column totals are called marginal probabilities.

The Joint Probability Table: Example

- The joint probability table for the previous Example is summarized below

	High	Medium	Low	Total
$ Appreciates	0.21	0.2	0.04	0.45
$Depreciates	0.09	0.3	0.16	0.55

Marginal probabilities are the **row totals** and the **column totals**.

Example2: five cards bearing numerals 1, 3, 5, 7, and 9 are placed in a box and two are with drawn at random. What is the probability that the sum of the numbers shown on the cards drawn is

a) 4 b) 8 c) 16 d) an even number e) an odd number

Solution: U = {(1, 3), (1, 5), (1, 7), (1, 9), (3, 5) (3, 7), (3, 9), (5, 7), (5, 9), (7, 9)}

n(u) = 10

a) $P(E_1) = \frac{1}{10}$ b) $P(E_2) = \frac{2}{10}$ c) $P(E_3) = \frac{1}{10}$

d) $P(E_4) = 1$ e) $P(E_5) = 0$

Where $E_1 = \{(1, 3)\}$ $E_2 = \{(1, 7), (3, 5)\}$ $E_3 = \{(7, 9)\}$ $E_4 = u$ and $E_5 = \{\}$

Example5: How many five-digit numerals can be written using the digits 1, 3, 5, 7 and 9 if no digit is repeated in each numeral? If each numeral is equally likely to be chosen, what is the probability that the number chosen: -

a) is odd b) is even c) has unit digit is 9

d) is greater than 50,000 e) less than 40,000

Solution: The number of five digit numerals that could be written is P (5, 5) = 5! = 120

a) 1, since all the digit used to write the numerals are odd the unit digit is certainly odd and thus all the five digit numerals are odd.

b) 0, since the unit digit can never be even, the first digit numeral can never be even, thus it is an impossible event.

c) $\frac{1}{5}$, since the number of the five digit numerals whose unit digit is 9 is 4! = 24 and the probability of this event is $\frac{24}{120} = \frac{1}{5}$.

d) $\frac{3}{5}$, since for the number to be greater than 50,000 the 10,000th digit has to be selected only from 6, 7 or 9 and there after any of the numbers 1, 3, 5, 7 or 9 which was not already selected can be selected once. Hence there are $3 \times 4 \times 3 \times 2 \times 1 = 72$ different numbers greater than 50,000. So the probability of his event is $\frac{72}{120} = \frac{3}{5}$.

e) $\frac{2}{5}$, since there are $2 \times 4 \times 3 \times 2 \times 1 = 48$ different five digit numerals less than 40,000. So the probability of this event is $\frac{48}{120} = \frac{2}{5}$.

Example6: From a jar containing 4 white, 3 red and 2 black balls all identical except color, three balls are drawn at random. How many different out comes are there? What is the probability that an out come consists of

a. 3 white balls

b. 3 red balls

c. 2 white and 1 red balls

d. 2 red and 1 white balls

e. 1 white and 1 red balls

f. 2 red and 1 black

g. 1 red and 2 black

h. 1 one from each color

i. 2 white and 1 black balls

Solution: Totally there are 9 balls. Hence the number of possible outcomes of drawing 3 balls randomly is $c(9, 3) = 84$. Thus

a. $P(3W) = \dfrac{c(4,3)}{c(9,3)} = \dfrac{4}{84} = \underline{\underline{\dfrac{1}{21}}}$

b. $P(3R) = \dfrac{c(3,3)}{c(9,3)} = \underline{\underline{\dfrac{1}{84}}}$

c. $P(2W, 1R) = \dfrac{c(4,2) \times c(3,1)}{c(9,3)} = \dfrac{6 \times 3}{84} = \dfrac{18}{84} = \dfrac{3}{14}$

d. $P(1W, 2R) = \dfrac{c(4,1) \times c(3,2)}{c(9,3)} = \dfrac{4 \times 3}{84} = \underline{\underline{\dfrac{1}{7}}}$

e. $P(1W, 2B) = \dfrac{c(4,1) \times c(2,2)}{c(9,3)} = \dfrac{4}{84} = \underline{\underline{\dfrac{1}{21}}}$

f. $P(2R, 1B) = \dfrac{c(3,2) \times c(2,1)}{c(9,3)} = \dfrac{6}{84} = \underline{\underline{\dfrac{1}{14}}}$

g. $P(1R, 2B) = \dfrac{c(3,1) \times c(2,2)}{c(9,3)} = \dfrac{3}{84} = \underline{\underline{\dfrac{1}{28}}}$

h. $P(1W, 1R, 1B) = \dfrac{c(4,1) \times c(3,1) \times c(2,1)}{c(9,3)} = \dfrac{4 \times 3 \times 2}{84} = \underline{\underline{\dfrac{2}{7}}}$

i. $P(2W, 1B) = \dfrac{c(4,2) \times c(2,1)}{c(9,3)} = \dfrac{6 \times 2}{84} = \dfrac{12}{84} = \underline{\underline{\dfrac{1}{7}}}$

Rule of complementary events: If E and E′ are two complementary events of an experiment then $P(E) + P(E') = 1$

Example3: In rolling a regular die, what is the probability that the face appears up shows not composite number?

Solution: $U = \{1,2,3,4,5,6\}$ Let $E = \{4,6\}$ the $\bar{E} = \{1,2,3,5\}$

$$\therefore P(E) = \frac{2}{6} \quad \text{and} \quad P(\bar{E}) = 1 - P(E) = 1 - \frac{2}{6} = \frac{4}{6}$$

Example4: In tossing a fair coin three times, what is the probability of achieving at least one head in the three tosses?

Solution: $U = (HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Let E be an event consisting of no head i.e. $E = \{TTT\}$ then E′ is an event consisting of at least one head.

$$\text{Since } P(E) = \frac{1}{8} \quad \text{and} \quad P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{8} = \frac{7}{8}$$

Example5: Suppose a family plan to have four children. What is the probability that not all the children have the same sex if it is equally likely for a son or daughter to be born?

Solution: n (u) = 16, let E be an event that the children are all sons or all daughters i.e.

$E = \{SSSS , DDDD\}$ then $P(E) = \frac{2}{16}$ and $P(\overline{E}) = 1 - P(E) = 1 - \frac{2}{16} = \frac{14}{16} = \frac{7}{8}$

Definition:

Two events are said to be independent if the occurrence of one does not affect the probability of the occurrence of the other. Several events are similarly independent if the occurrence of any one does not affect the probabilities of the occurrence of the other. If two events are not independent then they are said to be dependent. Similarly several events are not independent then they are said to be dependent.

Example9: Suppose from a box containing 7 white and 3 black balls, we draw 2 balls turn by turn with out replacement. What is the probability of drawing 1 white and 1 black ball?

Solution: The probability of drawing 1st white and then 2nd black is $\frac{7}{10} \times \frac{3}{9} = \frac{21}{90}$. The probability of drawing 1st black and 2nd white is $\frac{3}{10} \times \frac{7}{9} = \frac{21}{90}$. Hence the total probability of drawing 1 white and 1 black is

$$\frac{21}{90} + \frac{21}{90} = \frac{42}{90} = \frac{7}{15} \quad \text{or} \quad P(1w,1b) = \frac{c(7,1) \times c(3,1)}{c(10,2)} = \frac{7 \times 3}{45} = \underline{\underline{\frac{7}{15}}}$$

PROBABILITY DISTRIBUTION

- **A *probability distribution* is a correspondence, which assigns probabilities to the values of a random variable.**
- **<u>Example 1:</u>** Construct a probability distribution for the number of heads in tossing two fair coins simultaneously once.

  Solution: ***The sample space of the experiment contains the following:***

  $$S = \{HH, HT, TH, TT\}$$

- Let the random variable X denotes the ‘number of heads’. We then use the probability function P(X) to assign probability to each outcome consequently; the probability distribution is given below:

Outcome, X	0	1	2
Probability, P(X)	¼	½	¼

The probability distribution shows that the probability that the random variable can assume the value 0 is ¼, the value 1 is ½ and the value 2 is ¼. Note that the sum of these probabilities is 1.

Example 2: The number of mistakes a typist made in ten days of assessment is shown in the following table.

Nº of mistakes	2	3	4	5
Nº of days	1	4	3	2

a) Construct a probability distribution for the number of mistakes she committed.

b) Represent graphically the probability distribution in part (a).

THE BINOMIAL PROBABILITY DISTRIBUTION

- **Properties of the Binomial Experiment**
  - **Each trial can have only two outcomes or outcomes that can be reduced to two outcomes (success or failure).**
  - **There must be a fixed number of trials.**
  - **The outcomes of each trial must be independent.**
  - **The probability of a success must remain the same for each trial.**

In a Binomial experiment, the probability of exactly x successes in n trials is given by:

$$P(x)=\frac{n!}{(n-x)!x!}\cdot p^{x}\cdot q^{n-x}$$

Where x is the number of successes

$P(x)$ is the probability of success

n is the number of trials

P is the numerical probability of success

q is the numerical probability of failure

Example 1: Consider the experiment of tossing a coin three times. Show that it is a binomial experiment and find the probability of getting exactly two heads.

Solution:

This is a binomial experiment since

i) There are only two outcomes, head and tail.

ii) The number of trials is fixed (three)

iii) The probability of success, getting a head, does not change from trial to trial.

iv) The trials or tosses are independent, since the outcome of any trial is not affected by the outcome of any other trial

Then $p = 1/2, \quad q = 1\text{-}1/2 = 1/2$

$n = 3, \qquad x = 2$

$$P(x) = \frac{n!}{(n-x)!x!} \cdot p^x \cdot q^{n-x}$$

$$P(2) = \frac{3!}{(3-2)!2!} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{3-2} = \frac{3!}{1!2!} \cdot \left(\frac{1}{4}\right) \cdot \left(\frac{1}{2}\right)$$

$$= \frac{3}{8} \qquad = 0.375$$

Example 2: A new drug is effective 60% of the time. What is the probability that in a random sample of 4 patients, it will be effective on two of them?

Solution:

This is a Binomial experiment as the points of the experiment are satisfied. Define 'effective' as 'success' and 'non effective' as 'failure'. Then,

$p = 0.6, \quad q = 1 - 0.6 = 0.4, \quad n = 4, \quad x = 2$

Required $p(2) = ?$

$$P(2) = \frac{4!}{(4-2)!2!} . (0.6)^2 . (0.4)^2 = (6)(0.0576) = 0.3456$$

Hence, the drug will be effective on two of a random sample of 4 patients with a probability of 0.3456 (or 34.56%).

Mean, and Variance of a Probability Distribution

Definition

The mean, variance and standard deviation of a variable that has the Binomial distribution is found as:

Mean $\mu = n \cdot p$

Variance $\sigma^2 = n \cdot p \cdot q$

Standard deviation $\sigma = \sqrt{npq}$

Poisson Distribution

- A discrete probability distribution that is useful when n is large and p is small and when the independent variables occur over a period of time is called the *Poisson probability distribution.*
- The Poisson probability distribution assumes the following two conditions:
  - The probability of an occurrence is the same for any two intervals of equal length.
  - The occurrence or non-occurrence in any interval is independent of the occurrence or non-occurrence in any other interval.

The Poisson probability function

$$P(x;\lambda)=\frac{e^{-\lambda}.\lambda^{x}}{x!}$$

Where P(x, λ) is the probability of x occurrences in an interval of time, volume, area etc for a variable, λ denotes the mean number of occurrences and e ≅ 2.7183

- **Example1:** Past police records indicate a mean of five accidents per month while investigating the safety of a dangerous intersection. The number of accidents is distributed according to the probability in any month of
  a) Exactly 3 accidents.
  b) Fewer than 2 accidents.
- **Solution**: By assumption the given distribution is a Poisson probability distribution.

Solution: By assumption the given distribution is a Poisson probability distribution.

Given that $\lambda=5$

a) $P(x)=\dfrac{\lambda^x \cdot e^{-\lambda}}{x!}$

x = 3

$$P(3)=\frac{(5^3).(2.7183)^{-5}}{3!}=\frac{(125)(0.00674)}{6}$$

$$= \underline{\underline{0.1404}}$$

b) Fewer than 2 accidents comprise 0 and 1 accident during any month.

$$\therefore \quad P(0)+P(1)=\frac{(5^0).(2.7183)^{-5}}{0!}+\frac{(5^1)(2.7183)^{-5}}{1!}$$

$$\cong 0.0674+0.3370$$

$$\cong \underline{\underline{0.4044}}$$

- **Example 2:** If there are 200 typographical errors randomly distributed in a 500-page manuscript, find the probability that a given page contains exactly 3 errors.
-

$$\lambda = \frac{200}{500} = 0.4$$

Or, 0.4 error per page.

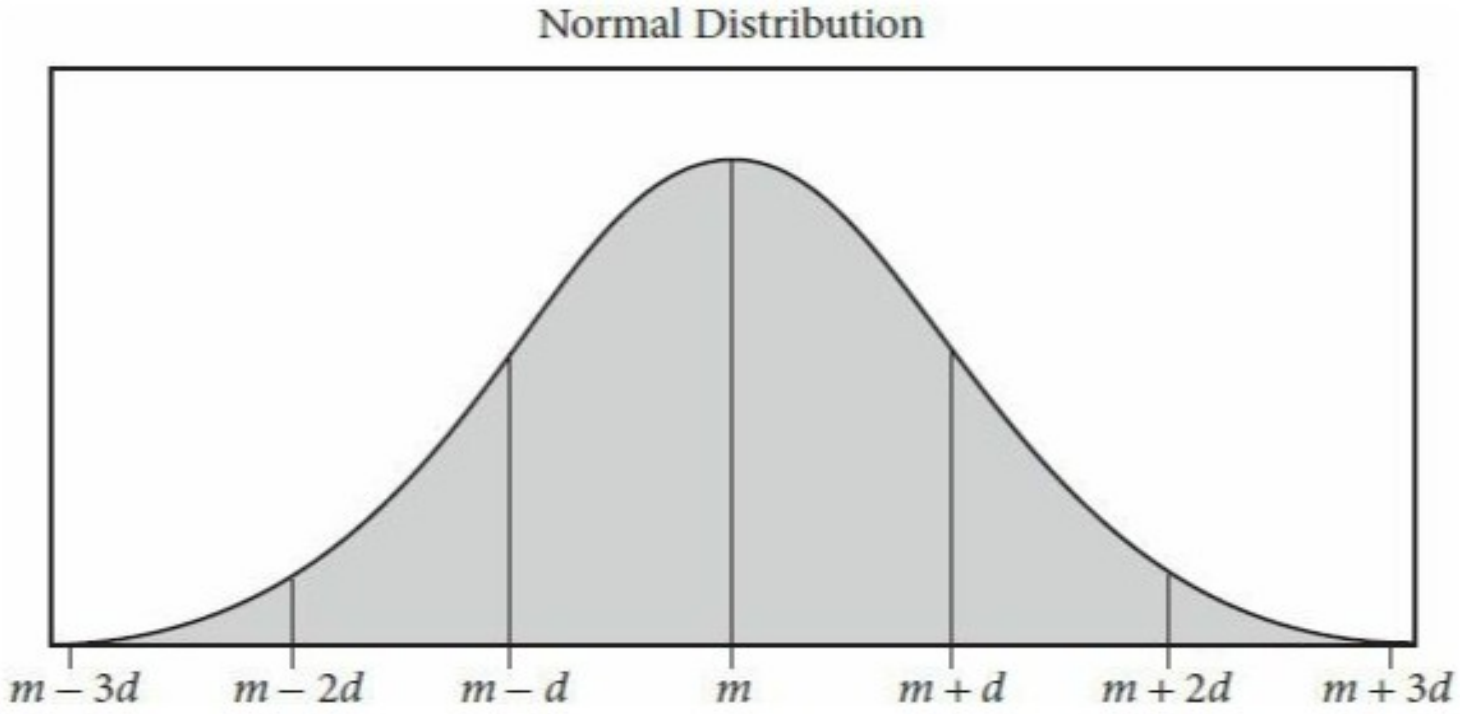
Since x = 3,

$$P(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{(2.7183)^{-0.4}\cdot(0.4)^{3}}{3!} \cong 0.00715$$

Thus, there is less than a 1% probability that a give page contains less than 3 errors.

CONTINUOUS PROBABILITY DISTRIBUTION

- **THE NORMAL PROBABILITY DISTRIBUTION**
- **The normal probability distribution is a**
  c **ed**
  c



Properties of the normal probability distribution

- Bell -shaped.
- The mean, median and mode are equal and are located at the center of the distribution.
- The curve is Uni-modal
- The curve is symmetrical about the mean.
- The curve is continuous and never touches the x-axis
- The total area under the normal distribution curve is equal to 1, or 100%
- The area under the normal curve that lies within one SD of the mean is approximately 0.68 or 68%, within two SD’s about 0.95 or 95% and within three SD’s about 0.997 or 99.7%

The standard normal probability distribution

- A random variable that has a normal distribution with a mean of 0 and a standard deviation of 1 is said to have a *standard normal probability distribution.*
- Recall that the standard score (z-score) of a value is the number of st that value is fr

$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard Deviation}}$$

Or, $$z = \frac{X - \mu}{\delta}$$

Example 1: Find the area under the normal curve between z=0 and z=2.34

Example 2: Find the area under the normal distribution curve between z = -1.93 and z = 2.35

Example 3: Find the probability that the z-value of a normally distributed variable lies to the left of 1.65 i,.e P(z < 1.91)

Example 4: Find P(z > 1.91)

Example 5: The scores for an IQ test are normally distributed with a mean of 100 and a standard deviation of 15. Find the percentage of IQ scores that will fall below 112.

$$Z = \frac{x - \mu}{\sigma} = \frac{112 - 100}{115} = 0.8$$

$$P(z < 0.8) = P(z < 0) + P(0 < z < 0.8) = 0.5000 + 0.2881 = 0.7881$$

Hence, 78.81% of the IQ scores fall below 112.

Example2: The monthly salaries of workers are normally distributed with a mean of Br 550 and standard deviation of Br 150 of workers whose monthly salaries are;

a) Between birr 600 and 700
b) Less than birr 700.

Example: The scores for Statistics are approximately normally distributed with mean of 70 marks and standard deviation 12 marks.

A. What score would a student need to score in the top 16.6% of student's scores?

B. What is the probability that a randomly selected group of 25 students will have a mean score of more than 75 marks?

- **Binomial probability distribution:**
- $$P(X) = \left(\frac{n!}{x!(n-x)!}\right)(p^x)(q^{(n-x)})$$

- **Poison probability distribution:**
- $$P(X) = \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)$$

- **Normal distribution:**

$$P(X) = \left(\frac{(\bar{x}-\mu)}{\delta}\right)$$


Thanks!