

Introduction

Mathematical logic started roughly towards the end of the 19th century with the English mathematician George Boole. It is the foundation on which proofs and arguments rest. In general, logic is about reasoning. It is about the validity of arguments, consistency among statements (or propositions, as they are called in logic) and matters of truth and falsehood. In a formal sense logic is concerned only with the *form* of arguments and the principles of valid inference.

As in most areas of study, the best way to get to know logic and to become skilled in its application is to do it, rather than philosophise it. In this course we are not going to be concerned about the wider philosophical aspects of logic. We will mainly engage in a solid study of so-called classical logic because it serves as a secure foundation for an investigation of all of logic and because of its application to problems and issues in computer science, in mathematics, in philosophy, indeed to problems of reasoning wherever they arise.

1.1 Statements and propositional logic

Propositional logic is the study of the logic of sentences. We know interrogative, declarative and exclamatory types of sentences. In propositional logic we deal with declarative or assertive types sentences. A proposition (or a statement) is classified as a declarative sentence to which one of the truth values, i.e. true or false, can be assigned.

Definition 1.1 A sentence is a group of words that really make sense.

Example 1.1 mars.

- | | |
|--|------------------------------|
| 1. 8 is the square root of 16. | 6. There is life in |
| 2. The capital city of Nigeria is Lagos. | 7. Take two aspirins. |
| 3. Take care! | 8. Blue birds are beautiful. |
| 4. Edgar Alan Poe was a French author. | 9. Get well soon. |
| 5. Jumping through the window. | 10. Do you speak Spanish? |

We can see that (1) and (2) are statements whose truth value is false. (3) is a simple exclamation. (4) is a statement but it may need some investigation. (5) is not even a sentence because it doesn't give sense. (6) is a statement but its truth value may

change depending on the time of investigation. (7) is a simple command or suggestion. (8) is not a statement since it may have both truth values. (9) is a simple wish and not a statement. (10) is a question, so it also is not a statement.

Definition 1.2 A sentence is said to be a statement or a proposition if and only if it is either true or false but not both.

Example 1.2 Sentences such as: Genet is beautiful, Abdi is handsome, Girma is rich are not considered as proposition throughout this chapter because their truth value can be true and false at the same time.

When a proposition is true, we say that its truth value is T. When it is false, we say that its truth value is F.

Notation. We usually denote propositions with lower cases of the English alphabet such as p, q, r, s . . .

1.2 Propositional connectives.

Just as we form new sentences using words like “and”, “but”, “or”, “if..., then”, we can get new propositions from the given propositions using connectives. Propositional connectives are operations used with propositions. But a new sentence obtained from the given propositions using connectives will be a proposition only when the new sentence has a truth value either T or F (but not both). The truth value of the new sentence depends on the logical connectives used and the truth value of the given propositions.

We now define the following five connectives.

- | | |
|------------------------|-------------------------------------|
| (i) Negation (NOT) | (iv) Implication (IF ... THEN) |
| (ii) Conjunction (AND) | (v) Bi-Implication (IF AND ONLY IF) |
| (iii) Disjunction (OR) | |

Negation (NOT)

If p is a proposition then the negation of p (read as “not p”) is a proposition (denoted by $\neg p$) whose truth value is T if p has the truth value F, and whose truth value is F if

p has the truth value T. Usually, the truth values of a proposition defined using a connective are listed in a table (Table 1.1) called the **truth table** for that connective.

Table 1.1 Truth Table for Negation

p	$\neg p$
T	F
F	T

Conjunction (AND)

If p and q are two propositions, then the conjunction of p and q (read as “ p and q ”) is a proposition (denoted by $p \wedge q$) whose truth value is T if and only if both p and q have truth values T; otherwise $p \wedge q$ is F. Consider Table 1.2

Table 1.2 Truth Table for Conjunction

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

If p and q are two propositions, then the disjunction of p and q (read as “ p or q ”) is a proposition (denoted by $p \vee q$) whose truth value is F if and only if both p and q have truth values F; otherwise $p \vee q$ is T. Consider Table 1.3

Table 1.3 Truth Table for Disjunction

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.3 Let p : This book is good and q : This book is cheap. Write the following sentences in symbolic form.

- (i) This book is good and cheap.
- (ii) This book is costly but good.
- (iii) This book is not good but cheap.
- (iv) This book is neither cheap nor good.
- (v) This book is either cheap or good.

Note: The truth table for $p \wedge q$ and $q \wedge p$ coincide. So we say that $p \wedge q$ and $q \wedge p$ are equivalent (see definition 1.4). But in natural language this need not happen. For example, the two sentences, namely “I went home and talked to my wife” and “I talked to my wife and went home”, have different meanings. Obviously we cannot write the second sentence in place of the first sentence.

Implication (IF p ... THEN q)

When p and q are two propositions joined by the connective implies, then the formed by proposition (denoted by $p \Rightarrow q$, and read as “**if p . . . , then q** ”) is called a logical implication. The truth value of $p \Rightarrow q$ is false if and only if p is T and q is F. The truth values of $p \Rightarrow q$ are given in Table 1.4.

Table 1.4 Truth Table for Implication

P	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Because implication plays such an essential role in mathematical reasoning, a variety of terminology is used to express $p \Rightarrow q$. You will encounter most if not all of the following ways to express implication.

- 1. “if p . . . , then q ”
- 2. “if p . . . , q ”
- 3. “ p is sufficient for q ”
- 4. “ p implies q ”
- 7. “ q when p ”
- 8. “ q follows from p ”
- 9. “ q is necessary for p ”
- 10. “ q is whenever p ”

5. “p only if q”

11. “q provided p”

6. “q if p”
is q”

12. “a necessary condition for p

Note that implication doesn't completely correspond to the ordinary understanding of the word “if ..., then...” or “it follows that”. The 3rd and 4th rows of the table of implication imply that if p is false, then whatever q is, the sentence $p \Rightarrow q$ is considered to be true. Thus, a false statement may imply anything.

Example 1.4 Find the truth values of the following propositions:

(i) If 5 is not an integer, then $\frac{1}{5}$ is an integer.

(ii) If 5 is an integer, then $\frac{1}{5}$ is an integer.

Bi - Implication (p IF AND ONLY IF q)

If p and q are two statements, then “p if and only if q” (usually written as “p iff q”) is a statement (denoted by $p \Leftrightarrow q$) whose truth value is T when the truth values of p and q are the same and whose truth value is F when the statements differ. The truth values of $p \Leftrightarrow q$ are given in Table 1.5.

Table 1.5 Truth Table for Bi-Implication

p	Q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6 summarizes the representation and meaning of the five logical connectives discussed above.

Table 1.6 Five Basic Logical Connectives

Connective	Symbol	Resulting proposition	Read as

Negation	$\neg$	$\neg p$	Not p
Conjunction	$\wedge$	$p \wedge q$	p and q
Disjunction	$\vee$	$p \vee q$	p or q
Implication	$\Rightarrow$	$p \Rightarrow q$	If p then q
Bi-Implication	$\Leftrightarrow$	$p \Leftrightarrow q$	p if and only if q

Example 1.5 Translate the following sentences into symbolic forms denoting the propositions as:

p: It is raining, q: I have the time, r: I will go to a movie.

- (i) It is not raining.
- (ii) It is not raining and I will not go to a movie.
- (iii) I will go to a movie only if it is not raining.
- (iv) If it is not raining and I have time, then I will go to a movie.

Example 1.6 If p, q, r are the propositions as given in Example 1.5, write the sentences in English corresponding to the following propositional forms:

- (i) $(\neg p \wedge q) \Leftrightarrow r$
- (ii) $(q \Rightarrow r) \wedge (r \Rightarrow q)$
- (iii) $\neg(q \vee r)$
- (iv) $r \Rightarrow (\neg p \wedge q)$

1.3 Compound propositions and their equivalences

1.3.1 Compound propositions

Suppose p, q, r and s are given propositions. Using the five connectives discussed we can form the following propositions.

$$\neg p \vee q, r \Rightarrow s, p \wedge s, r \Rightarrow q$$

Each of the above combination is a proposition. Hence we can form the following proposition again by using the logical connectives.

For example: (i) $(p \wedge s) \Leftrightarrow (\neg p \vee q)$ (ii) $(r \Rightarrow s) \wedge (r \Rightarrow q)$

Definition 1.3 Propositions that involve connectives are called compound propositions; otherwise they are elementary propositions.

Notation. We usually denote compound propositions with upper cases of the English alphabet such as P, Q, R, S . . .

Note that for 2 variables we have 4 rows are necessary; for 3 variables 8 rows are necessary. In general for n different variables the number of different logical possibilities is found by the formula 2^n .

Example 1.7 Construct truth table for each of the following compound propositions.

$$(i) \quad (p \vee q) \wedge (p \Leftrightarrow q)$$

$$(ii) \quad (p \vee q) \Rightarrow (p \vee r) \Rightarrow (r \vee q)$$

1.3.2 Equivalent propositions

Definition 1.4 Two compound propositions P and Q are said to be equivalent (denoted by $p \cong q$ or $p \equiv q$) iff they have identical corresponding truth values.

Example 1.8 Check the equivalence of each of the following compound propositions:

$$(i) \quad (p \Rightarrow q) \wedge (\neg p \vee q) ; (\neg q \wedge p) \vee (q \vee \neg p)$$

$$(ii) \quad (p \vee q) \wedge (p \Rightarrow r) ; (\neg p \vee r) \wedge (\neg p \Rightarrow q)$$

$$(iii) \quad p \Rightarrow (q \wedge r) ; (p \Rightarrow q) \wedge r$$

From (iii) you can see that $p \Rightarrow (q \wedge r)$ and $(p \Rightarrow q) \wedge r$ are not equivalent. This shows that putting brackets in their proper places is very important and the compound form of $p \Rightarrow q \wedge r$ will be meaningless.

1.4 Tautology and contradiction

Definition 1.5 A compound proposition is said to be a tautology iff it is true for all its logical possibilities.

Definition 1.6 A compound proposition is said to be a contradiction iff it is false for all its logical possibilities.

Definition 1.7 If a compound proposition is neither a tautology nor a contradiction, then it is known as a contingency.

Example 1.9 Check whether each of the following compound propositions is a tautology, a contradiction or a contingency.

- (i) $(\neg p \wedge q) \wedge (q \Rightarrow p)$
- (ii) $(p \vee q) \Rightarrow (r \wedge \neg q)$
- (iii) $(p \Rightarrow q) \Leftrightarrow (q \vee \neg p)$
- (iv) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow p)$

Note that when two equivalent propositions are joined by the connective bi-implies, then the result is always a tautology.

1.5 Converse, Inverse and Contrapositive of a Conditional Statement.

Suppose $p \Rightarrow q$ is the given conditional statement.

- $q \Rightarrow p$ is called the converse of $p \Rightarrow q$.
- $\neg p \Rightarrow \neg q$ is called the inverse of $p \Rightarrow q$.
- $\neg q \Rightarrow \neg p$ is called the contrapositive of $p \Rightarrow q$.

Example 1.9 Find the converse, the inverse and the contrapositive of each of the following propositions.

- a. $\neg p \Rightarrow q$
- b. $p \Rightarrow \neg q$
- c. Whenever I eat my lunch I drink Fanta.
- d. If the moon is made of gas, then 2 is a prime number.

1.6 The Laws of Algebra of Propositions

Let p , q and r be three arbitrary propositions.

(1) Idempotent Laws

a) $p \wedge p \equiv p$

b) $p \vee p \equiv p$

(2) Commutative Laws

a) $p \wedge q \equiv q \wedge p$

b) $p \vee q \equiv q \vee p$

(3) Associative Laws

a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(4) Distributive Laws

a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(5) De Morgan's Laws

a) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

b) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

(6) The Law of Double Negation

$$\neg(\neg p) \equiv p$$

(7) Identity Laws

a) $p \wedge T \equiv p$ b) $p \wedge F \equiv F$ c) $p \vee T \equiv T$ d) $p \vee F \equiv p$

(8) Complement Laws

a) $p \wedge \neg p \equiv F$

b) $p \vee \neg p \equiv T$

(9) Contrapositive Law

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

(10) Definitions

a) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

b) $p \Rightarrow q \equiv \neg p \vee q$

Example 1.10: Using the Laws of Algebra of Propositions simplify $(\neg p \Rightarrow q) \wedge (\neg p \vee q)$

Solution: $(\neg p \Rightarrow q) \wedge (\neg p \vee q)$

$$\equiv (p \vee q) \wedge (\neg p \vee q) \dots \dots \dots \text{Definition } [\neg p \Rightarrow q \equiv p \vee q]$$

$$\equiv q \vee (p \wedge \neg p) \dots \dots \dots \text{Distributive Laws}$$

$$\equiv q \vee F \dots \dots \dots \text{Complement Laws}$$

$$\equiv q \dots \dots \dots \text{Identity Laws}$$

Example 1.11: Using the Important Properties of Logical Connectives Check whether each of the following compound proposition is a tautology, a contradiction or a contingency.

a) $p \Rightarrow (q \Rightarrow p)$
 $\Rightarrow \neg p$

b) $[\neg p \vee (q \Rightarrow p)] \Rightarrow p$

c) $[(p \Rightarrow q) \wedge \neg p]$

Solution:

a. $p \Rightarrow (q \Rightarrow p)$

$$\equiv \neg p \vee (q \Rightarrow p) \dots \dots \dots \text{Definition}$$

$$\equiv \neg p \vee (\neg q \vee p) \dots \dots \dots \text{Definition}$$

$$\equiv \neg p \vee (p \vee \neg q) \dots \dots \dots \text{Commutative Laws}$$

$$\equiv (\neg p \vee p) \vee \neg q \dots \dots \dots \text{Associative Laws}$$

$$\equiv T \vee \neg q \dots \dots \dots \text{Complement Laws}$$

$$\equiv T \dots \dots \dots \text{Identity Laws}$$

$\therefore$ The compound proposition $p \Rightarrow (q \Rightarrow p)$ is a tautology.

b. $[\neg p \vee (q \Rightarrow p)] \Rightarrow p$

$$\equiv [\neg p \vee (\neg q \vee p)] \Rightarrow p \dots \dots \dots \text{Definition } [q \Rightarrow p \equiv \neg q \vee p]$$

]

$$\equiv [\neg p \vee (p \vee \neg q)] \Rightarrow p \dots \dots \dots \text{Commutative Laws}$$

$$\equiv [(\neg p \vee p) \vee \neg q] \Rightarrow p \dots \dots \dots \text{Associative Laws}$$

$$\equiv [T \vee \neg q] \Rightarrow p \dots \dots \dots \text{Complement Laws}$$

$$\equiv T \Rightarrow p \dots \dots \dots \text{Identity Laws}$$

$$\equiv \neg T \vee p \dots\dots\dots \text{Definition } [T \Rightarrow p \equiv \neg T \vee p]$$

$$\equiv F \vee p \dots\dots\dots \text{The Law of Negation}$$

$$\equiv p \dots\dots\dots \text{Identity Laws}$$

$\therefore$ The compound proposition $[\neg p \vee (q \Rightarrow p)] \Rightarrow p$ is a contingency .

$$\text{c. } [(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg p$$

$$\equiv [(\neg p \vee q) \wedge \neg p] \Rightarrow \neg p \dots\dots\dots \text{Definition}$$

$$\equiv [\neg p \wedge (\neg p \vee q)] \Rightarrow \neg p \dots\dots\dots \text{Commutative Laws}$$

$$\equiv \neg p \Rightarrow \neg p \dots\dots\dots \text{Absorption Laws}$$

$$\equiv p \vee \neg p \dots\dots\dots \text{Definition } [\neg p \Rightarrow \neg p \equiv p \vee \neg p]$$

$$\equiv F \dots\dots\dots \text{Complement Laws}$$

$\therefore$ The compound proposition $[\neg p \vee (q \Rightarrow p)] \Rightarrow p$ is a contradiction.

1.7 Predicate Logic and Open Propositions

1.7.1 Predicate Logic

So far we have considered only propositional logic. In this section we are going to learn predicate logic at elementary level.

Consider two propositions “*Amin is a student*”, and “*Azeb is a student*”. As propositions, there is no relation between them, but we know they have something in common. Both Amin and Azeb share the property of being *a student*. By replacing x by Amin or Azeb (or any other name), we get many propositions. The common feature expressed by “is a student” is called a *predicate*.

Definition 1.8 A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate. For example, “*is a student*” is a predicate. Another predicate that can be attributed to numbers is “greater than one”.

- i) 7 is “greater than one” ii) 0 is “greater than 1” iii) Fasil is “greater than one”

In ordinary language given a predicate, there is more or less a determined class of individuals to which it applies. In our case in (i) and (ii), the predicate “greater than 1” would be agreed to have been used appropriately, though of course (i) is true and (ii) is false, But if we ask the truth value of (iii) some people might refuse to pass judgment on the ground that it is nonsensical because the predicate “greater than 1” applies to numbers, However, we take it that any predicate can be attributed to any individual and hence (iii) is perfectly a good proposition like the first, which happens to be false.

1.7.2 Open Propositions

Definition 1.9 An open proposition is a sentence which contains one or more variables and becomes a proposition when a particular individual is/are substituted in it.

Notation: Sentences involving predicates describing the property of objects are denoted by $p(x)$, where p denotes the predicate and x the variable denoting any object.

Example 1.12: $p(x)$: x is the capital city of Bulgaria.

$q(x, y)$: x is the prime minister of y .

$r(t)$: t is less than 4.

In the case of $p(x)$, if we replace x by Nairobi, we obtain

$p(\text{Nairobi})$: Nairobi is the capital city of Bulgaria which is false

and if we replace Sofia in $p(x)$, we obtain a statement

$p(\text{Sofia})$: Sofia is the capital city of Bulgaria which is a true statement.

In the same manner if we replace x by Benjamin Netanyahu and y by Israel, we form a proposition

$q(\text{B. Netanyahu, Israel})$: B. Netanyahu is the prime minister of Israel.

Also if we substitute t by 4, we obtain a statement

$r(4)$: 4 is less than 4.

1.8 Compound Open Propositions

The formation of compound open propositions is the same as the formation of compound propositions. If for example we have two open propositions $p(x)$ and $q(x)$, we can form compound open propositions by using logical connectives.

1. $p(x) \wedge q(x)$ – Logical conjunction
2. $p(x) \vee q(x)$ – Logical disjunction
3. $p(x) \Rightarrow q(x)$ – Logical implication
4. $p(x) \Leftrightarrow q(x)$ – Logical bi-implication
5. $\neg p(x)$ – Logical negation

If we want to construct more complicated ones, we have to follow the same method simply as we used in the case of propositions.

Example 1.13: Let $p(x)$: x is a composite number

$q(x)$: x is an odd number

$r(x)$: x is a multiple of 3

Then a) $p(x) \wedge q(x)$: x is a composite number and x is an odd number

b) $p(x) \Rightarrow (q(x) \vee r(x))$: If x is a composite number, then either it is odd or a multiple of 3.

c) $\neg p(x) \Leftrightarrow q(x)$: x is not a composite number iff it is odd.

Definition 1.10 Two open propositions $p(x)$ and $q(x)$ are said to be equivalent iff $p(a) \equiv q(a)$ for all individual a . Note that if the universe U is specified, then $p(x)$ and $q(x)$ are equivalent iff $p(a) \equiv q(a)$ for all $a \in U$.

Example 1.14: Let $U = \{-2, -1, 0, 1, 2\}$, $p(x) : x^2 - 4 = 0$ and
 $q(x) : |x| \geq 2$.

Then for all $a \in U$ we can show that $p(x)$ and $q(x)$ have the same truth values.

$$p(-2): (-2)^2 - 4 = 0 \quad \textbf{(T)} \quad ; \quad q(-2): |-2| \geq 2 \quad \textbf{(T)}$$

$$p(-1): (-1)^2 - 4 = 0 \quad \textbf{(F)} \quad ; \quad q(-1): |-1| \geq 2 \quad \textbf{(F)}$$

$$p(0): (0)^2 - 4 = 0 \quad (\mathbf{F}) \quad ; \quad q(0): |0| \geq 2 \quad (\mathbf{F})$$

$$p(1): (1)^2 - 4 = 0 \quad (\mathbf{F}) \quad ; \quad q(1): |1| \geq 2 \quad (\mathbf{F})$$

$$p(2): (2)^2 - 4 = 0 \quad (\mathbf{T}) \quad ; \quad q(2): |2| \geq 2 \quad (\mathbf{T})$$

$$\therefore p(x) \equiv q(x)$$

Example 1.15: Let $p(x): |x| \leq 1$ and $q(x): x^2 - 1 = 0$, where $U = \{-1, 0, 1\}$.

Check whether $p(x)$ and $q(x)$ are equivalent or not.

$$p(-1): |-1| \leq 1 \quad (\mathbf{T}) \quad ; \quad q(-1): (-1)^2 - 1 = 0 \quad (\mathbf{T})$$

$$p(0): |0| \leq 1 \quad (\mathbf{T}) \quad ; \quad q(0): (0)^2 - 1 = 0 \quad (\mathbf{F})$$

$$p(1): |1| \leq 1 \quad (\mathbf{T}) \quad ; \quad q(1): (1)^2 - 1 = 0 \quad (\mathbf{T})$$

Since $p(x)$ and $q(x)$ have different corresponding truth values (in the second row), we can conclude by definition that $p(x)$ and $q(x)$ are not equivalent.

Definition 1.11 Let U be the universal set. An open proposition $p(x)$ is said to be a tautology iff $p(a)$ is true for all its logical possibilities (i.e. iff $p(a)$ is true for all $a \in U$).

Example 1.16: Let $U = \{-2, -1, 0, 1, 2\}$, $p(x): |x| \leq 3$ and $q(x): \sqrt{1-x} \leq 2$.

Check whether each of the above open proposition is a tautology or not.

Solution: $p(-2): |-2| \leq 3 \quad (\mathbf{T}) \quad ; \quad q(-2): \sqrt{1-(-2)} = \sqrt{3} \leq 2.$
 $(\mathbf{T})$

$p(-1): |-1| \leq 3 \quad (\mathbf{T}) \quad ; \quad q(-1): \sqrt{1-(-1)} = \sqrt{2} \leq 2.$
 $(\mathbf{T})$

$p(0): |0| \leq 3 \quad (\mathbf{T}) \quad ; \quad q(0): \sqrt{1-(0)} = \sqrt{1} \leq 2. \quad (\mathbf{T})$

$p(1): |1| \leq 3 \quad (\mathbf{T}) \quad ; \quad q(1): \sqrt{1-(-1)} = \sqrt{0} \leq 2. \quad (\mathbf{T})$

$$p(2): |2| \leq 3 \quad \textbf{(T)} \quad ; \quad \sqrt{1-(2)} = \sqrt{-1} \leq 2. \quad \textbf{(F)}$$

As we can see $p(x)$ is a tautology but $q(x)$ is not.

1.9 Quantified Propositions

Let's consider the following open propositions with universe $\mathbf{R}$.

- i) $p(x) : x^2 + 3 \geq 0$
- ii) $q(x) : x^2 - 9 = 0$
- iii) $r(x) : x^2 + x + 2 = 0$

We can see that

$p(x)$ is true for all $x \in \mathbf{R}$.

$q(x)$ is true for some $x \in \mathbf{R}$.

$r(x)$ is false for all $x \in \mathbf{R}$.

Hence for a given open proposition $p(x)$ with the universe U , we observe that there are three different possibilities:

$p(x)$ is always true for all $x \in \mathbf{R}$.

$q(x)$ is true only for some $x \in \mathbf{R}$.

$r(x)$ is always false for all $x \in \mathbf{R}$.

Definition 1.12 The phrase “for all x ” is called a Universal Quantifier and is denoted by $\forall x$.

“for all x ”, “for each x ”, “for every x ” have the same meaning.

If $p(x)$ is an open proposition with universe U , then $(\forall x) p(x)$ is a quantified proposition and read as “every x in the universe has the property p ”.

To show that $(\forall x)p(x)$ is false, it is enough to find at least one $a \in U$ such that $p(a)$ is false. Such an element $a \in U$ is called a counter example.

Example 1.17: Let $U = \mathbf{R}$ and define open propositions $p(x)$, $q(x)$ and $r(x)$ by

$$p(x): \frac{x}{x} = 1, \quad q(x): (x - 3)^2 = x^2 - 6x + 9 \quad r(x): \frac{x^2 - 25}{x - 5} =$$

$$x + 5$$

Find the truth value for

$$\text{i) } (\forall x) p(x) \quad \text{ii) } (\forall x) q(x) \quad \text{iii) } (\forall x) r(x)$$

Solution: i) $(\forall x) p(x) \equiv (\forall x \in \mathbf{R}) \left[\frac{x}{x} = 1 \right]$

This quantified proposition is false. Counter example $x = 0$.

$$\text{ii) } (\forall x) q(x) \equiv (\forall x \in \mathbf{R}) \left[(x - 3)^2 = x^2 - 6x + 9 \right]$$

This quantified proposition is true because it works for any value of x in $\mathbf{R}$.

$$\text{iii) } (\forall x) r(x) \equiv (\forall x \in \mathbf{R}) \left[\frac{x^2 - 25}{x - 5} = x + 5 \right]$$

This quantified proposition is false. Counter example $x = 5$.

Definition 1.13 The phrase “for some x ” is called an Existential Quantifier and is denoted by $\exists x$.

“for some x ”, “for at least one x ”, “there exists an x ” have the same meaning.

If $p(x)$ is an open proposition with universe U , then $(\exists x) p(x)$ is a quantified proposition and read as “there is at least one x in the universe which has the property p ”.

To show that $(\exists x)p(x)$ is true, it is enough to find at least one $a \in U$ such that $p(a)$ is true.

Example 1.18: Let $U = \mathbf{R}$ and define open propositions $p(x)$, $q(x)$ and $r(x)$ by

$$p(x): x \times 0 = x, \quad q(x): x^2 - x + 7 = 0 \quad r(x): \frac{x^3 - 8}{x - 2} \neq x^2 +$$

$$2x + 4$$

Find the truth value for

$$\text{i) } (\exists x) p(x) \quad \text{ii) } (\exists x) q(x) \quad \text{iii) } (\exists x) r(x)$$

Solution: i) $(\exists x) p(x) \equiv (\exists x \in \mathbf{R}) [x \times 0 = x]$

This quantified proposition is true. Example $x = 0$.

$$\text{iv) } (\exists x) q(x) \equiv (\exists x \in \mathbf{R}) [x^2 - x + 7 = 0]$$

This quantified proposition is false because we can't find any x in $\mathbf{R}$ satisfying the property of q .

$$\text{v) } (\exists x) r(x) \equiv (\exists x \in \mathbf{R}) \left[\frac{x^3 - 8}{x - 2} \neq x^2 + 2x + 4 \right]$$

This quantified proposition is true. Example $x = 2$.

1.10 Relationship between quantified propositions

Suppose we have an open proposition. We know that $(\forall x) p(x)$ is a quantified proposition. The question follows is what will be the truth value for the negation of $(\forall x) p(x)$ (which is $\neg (\forall x) p(x)$)

We have discussed that $(\forall x) p(x)$ is false if we can find an individual " a " such that $p(x)$ is false. If we succeed in getting such an individual, then $(\exists x) \neg p(x)$ is true. Hence $(\forall x) p(x)$ will be false, if $(\exists x) \neg p(x)$ is true. Thus we can conclude that the negation of $(\forall x) p(x)$ is $(\exists x) \neg p(x)$.

$$\text{i.e } \neg (\forall x) p(x) \equiv (\exists x) \neg p(x).$$

For example If we let $U = \mathbf{R}$ and $p(x) : \frac{x^2 - 36}{x - 6} = x + 6$, then one can easily see that

$$\neg (\forall x) p(x) \equiv \neg (\forall x) \left[\frac{x^2 - 36}{x - 6} = x + 6 \right] \text{ is the same as } (\exists x) \neg p(x) \equiv (\exists x) \neg \left[\frac{x^2 - 36}{x - 6} = x + 6 \right]$$

$$\equiv (\exists x \in \mathbf{R}) \left[\frac{x^2-36}{x-6} \neq x+6 \right] \text{ and that } x \text{ is } x-6.$$

In the same manner one can show that $\neg ((\exists x) p(x)) \equiv (\forall x) \neg p(x)$.

As an example we can take $U = \mathbf{R}$ and $p(x) : \sqrt{x} < 0$.

Here $\neg ((\exists x) p(x)) \equiv \neg ((\exists x) [\sqrt{x} < 0])$, which means $(\forall x) \neg p(x) \equiv (\forall x) \neg [\sqrt{x} < 0]$

$\equiv (\forall x) [\sqrt{x} \geq 0]$ which is a true statement.

1.11 Quantifiers in combination

1. $(\forall x) (\forall y) p(x, y) :-$ Every x and every y in the universe satisfy the property p .
2. $(\exists x) (\exists y) p(x, y) :-$ There exists at least one x and at least one y which satisfy the property p .
3. $(\forall x) (\exists y) p(x, y) :-$ For each x in the universe there exists at least one y satisfying the property p .
4. $(\exists y) (\forall x) p(x, y) :-$ There exists y in the universe which works for all x satisfying the property p .

Example 1.19: Let $U = \mathbf{R}$ be the universal set. Define $p(x, y)$ and $q(x, y)$ by

$$p(x, y) : (x - y)^2 = x^2 - 2xy + y^2, \quad q(x, y) : \frac{(x-y)^2}{x-y} = x + y$$

Find the truth value for

- i) $(\forall x) (\forall y) p(x, y)$ ii) $(\forall x) (\forall y) q(x, y)$

Solution: i) $(\forall x) (\forall y) p(x, y) \equiv (\forall x \in \mathbf{R}) (\forall y \in \mathbf{R}) [(x - y)^2 = x^2 - 2xy + y^2]$

This quantified proposition is true because $(x - y)^2 = x^2 - 2xy + y^2$ is an identity which is true for all x and for all y in the given universe.

$$\text{ii) } (\forall x) (\forall y) p(x, y) \equiv (\forall x \in \mathbf{R}) (\forall y \in \mathbf{R}) \left[\frac{(x-y)^2}{x-y} = x + y \right]$$

This quantified proposition is false. Counter example $x = 4$ and $y = 4$. Or in general $x = y$.

Example 1.20: Let $U = \mathbf{R}$ be the universal set. Define $p(x, y)$ and $q(x, y)$ by

$$p(x, y) : x + y = 7, \quad q(x, y) : \frac{1}{x-y} = 0.$$

Find the truth value for

$$\text{i) } (\exists x) (\exists y) p(x, y) \quad \text{ii) } (\exists x) (\exists y) q(x, y)$$

Solution: i) $(\exists x) (\exists y) p(x, y) \equiv (\exists x \in \mathbf{R}) (\exists y \in \mathbf{R}) [x + y = 7]$

This quantified proposition is true. Example $x = 1$ and $y = 6$.

$$\text{ii) } (\exists x) (\exists y) q(x, y) \equiv (\exists x \in \mathbf{R}) (\exists y \in \mathbf{R}) \left[\frac{1}{x-y} = 0 \right].$$

This quantified proposition is false because it is impossible to find x and y which satisfy the property of q .

Example 1.21: Let $U = \mathbf{R}$ be the universal set. Define $p(x, y)$ and $q(x, y)$ by

$$p(x, y) : x^2 - 1 = y, \quad q(x, y) : xy = 1$$

Find the truth value for

$$\text{i) } (\forall x) (\exists y) p(x, y) \quad \text{ii) } (\forall x) (\exists y) q(x, y)$$

Solution: i) $(\forall x) (\exists y) p(x, y) \equiv (\forall x \in \mathbf{R}) (\exists y \in \mathbf{R}) [x^2 - 1 = y]$

This quantified proposition is true. Because for each x in the universe we always can get y which satisfy the property p .

$$\text{ii) } (\forall x) (\exists y) q(x, y) \equiv (\forall x \in \mathbf{R}) (\exists y \in \mathbf{R}) [xy = 1]$$

This quantified proposition is false. Counter example $x = 0$.

Example 1.22: Let $U = \mathbf{R}$ be the universal set. Define $p(x, y)$ and $q(x, y)$ by

$$p(x, y) : xy = 0, \quad q(x, y) : y > x.$$

Find the truth value for

$$\text{i) } (\exists y) (\forall x) p(x, y) \qquad \text{ii) } (\exists y) (\forall x) q(x, y)$$

Solution: i) $(\exists y) (\forall x) p(x, y) \equiv (\exists y \in \mathbf{R}) (\forall x \in \mathbf{R}) [x y = 0]$

This quantified proposition is true. Example $y = 0$.

$$\text{ii) } (\exists y) (\forall x) q(x, y) \equiv (\exists y \in \mathbf{R}) (\forall x \in \mathbf{R}) [y > 1]$$

This quantified proposition is false because its meaning is: “there is a fixed number y , which is greater than all real number.

1.12 Translation of ordinary statements containing “all” and “some” to symbolic form

Let $p(x)$: x is an Ethiopian and $q(x)$: x is wise.

Now $p(x) \wedge q(x)$ represents an open proposition “ x is an Ethiopian and x is wise”,

And a quantified proposition $(\exists x)[p(x) \wedge q(x)]$ represents “there is someone who is both Ethiopian and wise”

and this statement is the same as saying “some Ethiopian is wise.”

Hence all sentences of the forms “some $p(x)$ are $q(x)$ ” should be symbolized by $(\exists x)[p(x) \wedge q(x)]$

Some examples are:

1. Some prime numbers are even
2. Some baby are cute.
3. Some students play cards all night

Similarly, sentences of the forms “some $p(x)$ is not / are not $q(x)$ ” are symbolized as $(\exists x)[p(x) \wedge \neg q(x)]$

Example 1.23: Let $p(x)$: x is an odd integer and $q(x)$: x is a prime number.

Write $(\exists x)[p(x) \wedge \neg q(x)]$ in sentence form.

Solution: The above quantified proposition can be represented by a sentence “There exists an odd integer which is not prime”. And this is the same thing to say “some odd integers are not prime”.

On the other hand, for the open propositions $p(x)$: x is an Ethiopian and $q(x)$: x is wise consider a quantified proposition $(\forall x)[p(x) \wedge q(x)]$.

$(\forall x) [p(x) \wedge q(x)]$ means “every person is both Ethiopian and wise”

Note that this is entirely different from the statement “all Ethiopians are wise”. This last statement is symbolized by $(\forall x) [p(x) \Rightarrow q(x)]$ and this statement is false if there is an Ethiopian who is not wise ; otherwise it is true. Some examples are:

	Statement form	Symbolic form
1	Every rational number is a real number.	$(\forall x) [Q(x) \Rightarrow R(x)]$
2	All fish swim.	$(\forall x) [F(x) \Rightarrow S(x)]$
3	All integers divisible by 2 are even.	$(\forall x) [(Z(x) \wedge D(x)) \Rightarrow E(x)]$
4	All integers divisible by 3 are odd.	$(\forall x) [(Z(x) \wedge D(x)) \Rightarrow O(x)]$

Again a sentence of the form “no $p(x)$ are $q(x)$ ” is the same as “all $p(x)$ are not $p(x)$ ”. Symbolically it is written as $(\forall x) [p(x) \Rightarrow \neg q(x)]$

For example “no Ethiopian is greedy” means “all Ethiopians are not greedy” and this statement (i.e. “no Ethiopian is greedy”) is different from the sentence “not all Ethiopians are greedy”. The second statement means “there are some Ethiopians who are not greedy” or “there exists at least one Ethiopian who is not greedy” and it can also be written as “there exists a person who is an Ethiopian and he is not greedy”. Symbolically it is given by $(\exists x)[p(x) \wedge \neg q(x)]$.

1.13 Arguments and validity

Definition 1.14: A logical deduction (in argument form) is an assertion that a given set of statements $p_1, p_2, p_3, \dots, p_n$ (called premises) yield another statement Q (called conclusion).

Notation:

p_1

p_2

p_3

or $p_1, p_2, p_3, \dots, p_n \vdash Q$

•

•

•

 p_n

Example 1.23: If the moon is made of bananas, there are monkeys on the moon.
The moon is not made of bananas. Therefore there is no monkey on the moon.

Example 1.24: If the moon is made of bananas, there are monkeys on the moon.
There is no monkey on the moon. Therefore the moon is not made of bananas.

Solution: Let p: the moon is made of bananas.

q: there are monkeys on the moon.

Solution 1.23: The symbolic form of the given argument is $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$.

Now let's construct a truth table to see the truth values of the compound $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(p \Rightarrow q) \wedge \neg p$	$[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

As we can see from the above table a compound proposition $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$ is a contingency.

Solution 1.24: The symbolic form of the given argument is $[(p \Rightarrow q) \wedge \neg q] \Rightarrow \neg p$.

Constructing a truth table for the compound $[(p \Rightarrow q) \wedge \neg q] \Rightarrow \neg p$ we get:

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(p \Rightarrow q) \wedge \neg q$	$[(p \Rightarrow q) \wedge \neg q] \Rightarrow \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

in which a compound proposition $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$ is a tautology.

Definition 1.15: If the given argument is a tautology, then it is said to be a valid argument ; otherwise it is called an invalid argument.

Thus we can conclude that the argument in *example 1.23* is an invalid argument, where as the argument in *example 1.24* is a valid argument.

Example 1.25: Check the validity of the following argument.

*“If the rain does not come, the crops are going to be ruined and people will starve.
Either the crops are not ruined or people will not starve. Therefore the rain comes”.*

Solution: Let p : the rain comes

q : the crops are going to be ruined

r: people will starve

The symbolic form of the given argument is $[[\neg p \Rightarrow (q \wedge r)] \wedge (\neg q \vee \neg r)] \Rightarrow p$.

Let' denote the above compound proposition by Q because of the space limitation of the table.

p	q	r	$\neg p$	$\neg q$	$\neg r$	$q \wedge r$	$\neg p \Rightarrow (q \wedge r)$	$\neg q \vee \neg r$	$[\neg p \Rightarrow (q \wedge r)] \wedge (\neg q \vee \neg r)$	Q
T	T	T	F	F	F	T	T	F	F	T
T	T	F	F	F	T	F	T	T	T	T
T	F	T	F	T	F	F	T	T	T	T

T	F	F	F	T	T	F	T	T	T	T
F	T	T	T	F	F	T	T	F	F	T
F	T	F	T	F	T	F	F	T	F	T
F	F	T	T	T	F	F	F	T	F	T
F	F	F	T	T	T	F	F	T	F	T

Since the compound proposition $[[\neg p \Rightarrow (q \wedge r)] \wedge (\neg q \vee \neg r)] \Rightarrow p$ is a tautology, it follows by definition that the given argument is a valid argument.

Validity of an argument can also be defined in the following way:

Definition 1.16: An argument $p_1, p_2, p_3, \dots, p_n \vdash Q$ is said to be valid if Q is true whenever $p_1, p_2, p_3, \dots, p_n$ are accepted as true ; otherwise it is called a fallacy or an invalid argument.

Note: In checking the validity of an argument that involves propositions without quantifiers, we use truth tables as follows:

1. Construct a truth table that contains $p_1, p_2, p_3, \dots, p_n$ and Q.
2. Then check all the rows for which $p_1, p_2, p_3, \dots, p_n$ are true at the same time. If there is at least one possibility for Q to be false while all $p_1, p_2, p_3, \dots, p_n$ are true, then the argument is a fallacy or invalid ; otherwise the argument is a valid argument.

Example 1.25: Check the validity of the argument. $[[\neg p \Rightarrow (q \wedge r)] \wedge (\neg q \vee \neg r)] \Rightarrow p$ using the above two steps.

Solution: The compound proposition $[[\neg p \Rightarrow (q \wedge r)] \wedge (\neg q \vee \neg r)] \Rightarrow p$ can be written in argument form as: $\neg p \Rightarrow (q \wedge r), \neg q \vee \neg r \vdash q$.

p	q	R	$\neg$ p	$\neg$ q	$\neg$ r	$q \wedge$ r	$\neg p \Rightarrow (q \wedge$ r)	$\neg q \vee \neg r$
---	---	---	-------------	----------	-------------	-----------------	--------------------------------------	----------------------

T	T	T	F	F	F	T	T	F
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>

Since all the truth values of the conclusion are true when the truth values of the two premises are true at the same time, it follows that the given argument is a valid argument.

Example 1.26: Check the validity of the following argument using the above two steps.

“For the candidate to win, it is sufficient that he gets the support of the farmers. He gets the support of the farmers only if he has a good plan for agricultural development. He has no plan for agricultural development. Therefore the candidate will not win.”

Solution: Let p : the candidate wins

q : the candidate has the support of the farmers

r : the candidate has a good plan for agricultural development.

Thus the above argument can be expressed as:

$$q \Rightarrow p$$

$$q \Rightarrow r$$

$$\neg r$$

$$\text{or } q \Rightarrow p, q \Rightarrow r, \neg r \vdash \neg p$$

$$\text{or } [(q \Rightarrow p) \wedge (q \Rightarrow r) \wedge \neg r] \Rightarrow \neg p$$

$$\neg p$$

p	q	r	$\neg$ r	$q \Rightarrow$ p	$q \Rightarrow r$	$\neg p$
T	T	T	F	T	T	F
T	T	F	T	T	F	F
T	F	T	F	T	T	F
T	F	F	T	T	T	F
F	T	T	F	F	T	T
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	T	T	T	T

As we can see the three premises are true at the same time on the fourth and the eighth rows. But since the conclusion is false on the fourth row, it follows that the given argument is an invalid argument or is a fallacy.

Exercises: Check the validity of each of the following arguments:

1. If I drive to work, then I will arrive tired. I drive to work. Therefore I will arrive tired.
2. If I drive to work, then I will arrive tired. I don't drive to work. Therefore I will not arrive tired.
3. If you invest in the stock market, then you will get rich. If you get rich, then you will be happy. Therefore If you invest in the stock market, then you will be happy.
4. If I try hard and have talent, then I will become a musician. If I become a musician, then I will be happy. Therefore if I am not happy, then I didn't try hard or I do not have talent.

There are rules known as “*Rules of Inferences*” which enable us to check or determine the validity of the given argument using deductive method.

1.14 Rules of Inferences

1. Modes Ponenes

p

2. Modes Tollens

$\neg q$

	$p \Rightarrow q$		$p \Rightarrow q$
	$\neg p$		q
3.	Principle of syllogism	4.	Principle of Adjunction
	$p \Rightarrow q$	a.	p
	$q \Rightarrow r$		q
			$p \vee q$
	$p \Rightarrow q$		$p \wedge q$
5.	Principle of detachment	6.	Modes Tollendo Ponnens
	$p \wedge q$		$\neg p$
			$p \vee q$
	p, q		
7.	Modes Ponnendo Tollens	8.	Principle of Equivalence
	$\neg (p \wedge q)$		$p \Rightarrow q$
	p		p
	$\neg q$		q
9.	Principle of Substitution		
	p		
	$q \Rightarrow p$		

Example 1.27: Check the validity of each of the following arguments.

- | | | | |
|-----------|--|-----------|----------------------------|
| a. | $\neg p \vee q, r \Rightarrow p, r$ | e. | $\neg p \wedge \neg q$ |
| b. | $\neg q \Rightarrow \neg p, r \Rightarrow p, \neg q$ | | $(q \vee r) \Rightarrow p$ |
| c. | $p \Rightarrow (q \vee r), \neg r, p$ | | |
| d. | $q \Rightarrow p, \neg t \Rightarrow \neg p, \neg r \vee s, r \vee \neg t$ | | $\neg r$ |

- f. "If 6 is even, then 2 does not divide 5. Either 7 is not prime or 2 divides 5. But 7 is prime. Therefore 6 is not even".

Note: Always remember that all the premises given are assumed to be true.

Solution: a. $\neg p \vee q, r \Rightarrow p, r \vdash p$

r is true premise

$r \Rightarrow p$ is true premise

p is true Modes Ponens (or Rule of Implication)

$\neg p$ is false Rule of Negation

$\neg p \vee q$ is true premise

q is true Modes Tollendo Ponens (or Rule of Disjunction)

$\therefore$ The given argument is a valid argument.

b. $\neg q \Rightarrow \neg p, r \Rightarrow p, \neg q \vdash r$

$\neg q$ is true premise

$\neg q \Rightarrow \neg p$ is true premise

$\neg p$ is true Modes Ponens (or Rule of Implication)

p is false Rule of Negation

$r \Rightarrow p$ is true premise

p is false Modes Tollens (or Rule of Implication)

$\therefore$ The given argument is an invalid argument.

c. $p \Rightarrow (q \vee r), \neg r, p \vdash q$

p is true premise

$\neg r$ is true premise

r is false Rule of Negation

$p \Rightarrow (q \vee r)$ is true premise

$q \vee r$ is true Principle of equivalence (or Rule of Implication)

q is true Modes Tollendo Ponens (or Rule of Disjunction)

$\therefore$ The given argument is a valid argument.

d. $q \Rightarrow p, \neg t \Rightarrow \neg p, \neg r \vee s, r \vee \neg t \vdash \neg s \Rightarrow \neg q.$

$q \Rightarrow p$ is true premise

$\neg t \Rightarrow \neg p$ is true premise

$p \Rightarrow t$ is true Law of Contrapositive

$q \Rightarrow t$ is true. Principle of Syllogism

$r \vee \neg t$ is true premise

$\neg r \Rightarrow \neg t$ is true Definition

$t \Rightarrow r$ is true Law of Contrapositive

$q \Rightarrow r$ is true Principle of Syllogism

$\neg r \vee s$ is true premise

$r \Rightarrow s$ is true Definition

$q \Rightarrow s$ is true Principle of Syllogism

$\neg s \Rightarrow \neg q$ Law of Contrapositive

$\therefore$ The given argument is a valid argument.

e. $\neg p \wedge \neg q$

$(q \vee r) \Rightarrow p$

or $\neg p \wedge \neg q, (q \vee r) \Rightarrow p \vdash \neg r$

$\neg r$

$\neg p \wedge \neg q$ is true premise

$\neg p, \neg q$ are true Principle of detachment (or Rule of $\wedge$)

p, q are false Rule of Negation

$(q \vee r) \Rightarrow p$ is true premise

$q \vee r$ is false Modes Tollens (or Rule of $\vee$)
 $\Rightarrow$

r is false or Rule of $\vee$

$\neg r$ is true Rule of Negation

$\therefore$ The given argument is a valid argument.

f. “If 6 is even, then 2 does not divide 5. Either 7 is not prime or 2 divides 5. But 7 is prime. Therefore 6 is not even”.

Let p : 6 is even
 q : 2 divides 5
 r : 7 is prime

Then the symbolic form of the above argument is

$p \Rightarrow \neg q, \neg r \vee q, r \vdash \neg p$
 r is true premise

$\neg r$ is false Rule of Negation

$\neg r \vee q$ is true premise

q is true Rule of $\vee$

$\neg q$ is false Rule of Negation

$p \Rightarrow \neg q$ is true premise

p is false Rule of $\Rightarrow$

$\neg p$ is true Rule of Negation

$\therefore$ The given argument is a valid argument.