

CHAPTER 5

Dimensional Analysis

5.1 INTRODUCTION

- Dimensional analysis is a method for reducing the number and complexity of experimental variables.
- If a phenomenon depends upon n dimensional variables, dimensional analysis will reduce the problem to only k dimensionless variables, where the reduction $n-k=1, 2, 3$, or 4 , depending upon the problem complexity.

5.1 INTRODUCTION...

- Generally $n-k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions).
- In fluid mechanics, the four basic dimensions are
 - MLT $\rightarrow$ I system or
 - FLT $\rightarrow$ I system.

5.1 INTRODUCTION...

- Dimensional analysis has other benefits in addition to reducing number of variables.

a. Enormous savings in time and money.

Suppose one knew that the force F on a particular body immersed in a stream of fluid depended only on the body length L , stream velocity V , fluid density ρ , and fluid viscosity μ , that is,

$$F = f(L, V, \rho, \mu) \quad (5.1)$$

5.1 INTRODUCTION...

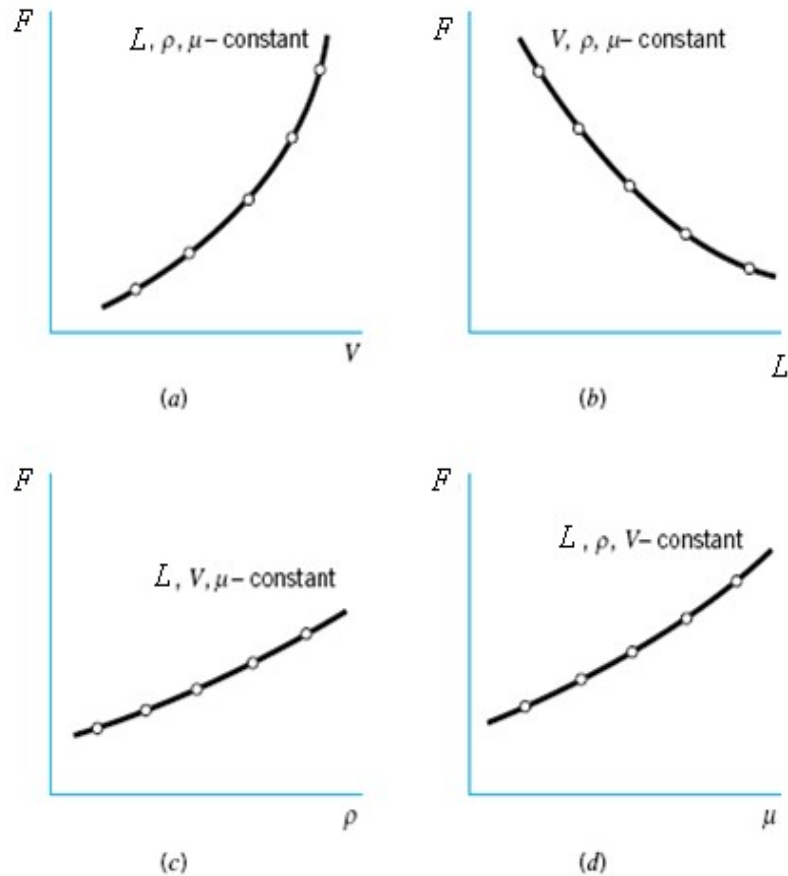


Fig. 5.1 Illustrative plots showing how the force is affected by different factors

5.1 INTRODUCTION...

However, with dimensional analysis, equation (5.1) can be reduced to

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \quad (5.2)$$

$$\Rightarrow C_F = g(\text{Re})$$

$F/(\rho V^2 L^2)$ is force coefficient and $\rho V L / \mu$ Reynolds number.

The function g is different mathematically from the original function f , but it contains all the same information.

Nothing is lost in a dimensional analysis.

5.1 INTRODUCTION...

b. It helps our thinking and planning for an experiment or theory.

It suggests dimensionless ways of writing equations before we waste money on computer time to find solutions.

It suggests variables which can be discarded; sometimes dimensional analysis will immediately reject variables, and at other times it groups them off to the side, where a few simple tests will show them to be unimportant.

Finally, dimensional analysis will often give a great deal of insight into the form of the physical relationship we are trying to study.

5.1 INTRODUCTION...

c. It provides scaling laws.

Dimensional analysis provides scaling laws which can convert data from a cheap, small model to design information for an expensive, large prototype.

As an example, we do not build a million-dollar airplane and see whether it has enough lift force.

We measure the lift on a small model and use a scaling law to predict the lift on the full-scale prototype airplane.

5.1 INTRODUCTION...

When the scaling law is valid, we say that a condition of similarity exists between the model and the prototype.

In the simple case of equation (5.1), similarity is achieved if the Reynolds number is the same for the model and prototype because the function g then requires the force coefficient to be the same also:

$$\text{Re}_m = \text{Re}_p \Rightarrow C_{Fm} = C_{Fp} \quad (5.3)$$

where subscripts m and p mean model and prototype, respectively.

5.2 THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY

- “If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; i.e., each of its additive terms will have the same dimensions.”
- Consider the displacement of a falling body
$$S = S_o + V_o t + \frac{1}{2} g t^2$$

(5.4)

Each term has dimension {L}.

5.2 THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY...

- Consider also the Bernoulli's equation for incompressible flow

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{const.} \quad (5.5)$$

Each term has dimensions $\{L^2T^{-2}\}$.

- The above equations illustrate some other factors that often enter into a dimensional analysis:

5.2 THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY...

1. Dimensional variables ; S and t (p , V , z).
2. Dimensional constants; S_0 , V_0 , and g , (ρ , g , and C).
3. Pure constants ($\frac{1}{2}$ and the exponent 2).
 - There are some physical variables that are naturally dimensionless by virtue of their definition as ratios of dimensional quantities. –strain, Poisson's ratio, specific gravity...
 - All angles are dimensionless.

5.3 BUCKINGHAM PI THEOREM

- Is one of the methods of reducing a number of dimensional variables to less number of dimensionless groups.
- The dimensionless groups found from the theorem are power products denoted by π_1, π_2, π_3 , etc.
- The method allows the π s to be found in sequential order without resorting to free exponents.

5.3 BUCKINGHAM PI THEOREM...

- If a physical process satisfies the principle of dimensional homogeneity and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or π 's.
- The reduction $j = n - k$ equals the maximum number of variables which do not form a π among themselves and is always less than or equal to the number of dimensions describing the variables.

5.3 BUCKINGHAM PI THEOREM...

- Once the reduction j is found, then j scaling variables which do not form a π among themselves will be selected.
- Each desired π group will be a power product of these j variables plus one additional variable which is assigned any convenient nonzero exponent.
- Each π group thus found is independent.

5.3 BUCKINGHAM PI THEOREM...

Quantity	Symbol	Dimensions	
		MLT^{-1}	FLT^{-1}
Length	L	L	L
Area	A	L^2	L^2
Volume	V	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	dm/dt	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$d\epsilon/dt$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	σ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	θ	θ
Specific heat	c_p, c_v	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\theta^{-1}$	$FT^{-1}\theta^{-1}$
Expansion coefficient	α	θ^{-1}	θ^{-1}

5.3 BUCKINGHAM PI THEOREM...

- There are in general six steps involved when converting dimensional relations in to non dimensional ones:
 1. List and count the n variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
 2. List the dimensions of each variable according to $\{MLT^{-1}\}$ or $\{FLT^{-1}\}$.
 3. Find the reduction j . Initially guess j equal to the number of different dimensions present, and look for j variables which do not form a pi product. If no luck, reduce j by 1 and look again. With practice, you will find j rapidly.

5.3 BUCKINGHAM PI THEOREM...

4. Select j scaling parameters which do not form a π product. Make sure they please you and have some generality if possible, because they will then appear in every one of your π groups. Pick density or velocity or length.
5. Add one additional variable to your j repeating variables, and form a power product. Algebraically find the exponents which make the product dimensionless.
6. Write the final dimensionless function, and check your work to make sure all π groups are dimensionless.

5.4 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS

- The concept of dimensional analysis can be used to reveal basic dimensionless parameters, e.g., Reynolds number, in their proper form and proper position, giving clues to when they are negligible in the differential equations governing fluid flow.
- Let us briefly apply this technique to the incompressible-flow continuity and momentum equations with constant viscosity:

5.4 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS...

- Continuity: $\nabla \cdot \mathbf{V} = 0$ (5.6a)
- Momentum: $\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V}$ (5.6b)
- Equations (5.6) contain the three basic dimensions M, L, and T. All variables p , $\mathbf{V}$, x , y , z , and t can be nondimensionalized by using density and two reference constants which might be characteristic of the particular fluid flow:
- Reference velocity = U , Reference length = L

5.4 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS...

- Now define all relevant dimensionless variables, denoting them by an asterisk:

$$V^* = \frac{V}{U}$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{L} \quad (5.7)$$

$$z^* = \frac{z}{L}$$

$$t^* = \frac{tU}{L}$$

$$p^* = \frac{p + \rho g z}{\rho U^2}$$

- Since ρ , U , and L are all constants, the derivatives in equations (5.6) can all be handled in dimensionless form with dimensional coefficients.

5.4 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS...

- For example,

$$\frac{\partial u}{\partial x} = \frac{\partial(Uu^*)}{\partial(Lx^*)} = \frac{U}{L} \frac{\partial u^*}{\partial x^*}$$

- Substitute the variables from equations (5.7) into equations (5.6) and divide through by the leading dimensional coefficient.

- The resulting dimensionless equations of motion are:

$$\nabla^* \cdot \mathbf{v}^* = 0$$

- Continuity: $\frac{d\mathbf{v}^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L} \nabla^{*2} \mathbf{v}^*$ (5.8a)

- Momentum: $= -\nabla^* p^* + \frac{1}{\text{Re}} \nabla^{*2} \mathbf{v}^*$ (5.8b)

5.4 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS...

- The nondimensional momentum equation (5.8a) contains the famous Reynolds number, which is defined as the ratio of inertia force to viscous force.
- As it can be seen from equation (5.8b), the viscous force becomes negligible for high Reynolds number.

EXAMPLE 5.1

- The pressure difference $\equiv p$ across an explosion or blast wave is a function of the distance r from the blast center, time t , speed of sound a of the medium, and total energy E in the blast. Rewrite this relation in dimensionless form. How does $\equiv p$ change if E is doubled?

SOLUTION

Step 1: Write the function and count variables (find n).

$$\Xi \quad p = f(r, t, a, E) \quad n = 5$$

Step 2: List dimensions of each variable.

$\Xi \quad p$	r	t	a	E
$ML^{-1}T^{-2}$	L	T	LT^{-1}	ML^2T^{-2}

- Step 3: Find j.

There are three basic dimensions (MLT) and j is less or equal to 3. The variables r, t and E do not form pi group and j=3. Then $k = n - j = 5 - 3 = 2$.

SOLUTION...

- Step 5: find the pi groups.
 - The first pi group can be obtained by adding the variable **a** with power of 1 to the repeating variables selected in step 4.

$$\pi_1 = r^x t^y E^z a$$

- For π_1 to be dimensionless, the following should be fulfilled.

$$(L)^x (T)^y (ML^2T^{-2})^z (LT^{-1}) = M^0 L^0 T^0$$

- Collecting powers of similar terms

Mass: $z=0$; Length: $x+2z+1=0 \Rightarrow x=-1$

Time: $y-2z-1=0 \Rightarrow y=1 \Rightarrow \pi_1 = r^{-1} t a = \frac{ta}{r}$

SOLUTION...

- The second pi group can be obtained by adding the variable $\equiv p$ with power of 1 to the repeating variables.

$$\pi_2 = r^x t^y E^z \equiv p$$

- For π_2 to be dimensionless,

$$(L)^x (T)^y (ML^2T^{-2})^z (ML^{-1}T^{-2}) = M^0 L^0 T^0$$

- Collecting powers of similar terms

$$\text{Mass: } z + 1 = 0 \quad \Rightarrow z = -1$$

$$\text{Length: } x + 2z - 1 = 0 \quad \Rightarrow x = -2z + 1 = 3$$

$$\begin{aligned} \text{Time: } y - 2z - 2 &= 0 \\ \Rightarrow \pi_2 &= r^3 E^{-1} \Delta p = \frac{\Delta p r^3}{E} \end{aligned}$$

SOLUTION...

- Step 6: Final dimensionless function
 - Then one pi group is a function of the other,

$$\pi_2 = f(\pi_1)$$

$$\Rightarrow \frac{\Delta p r^3}{E} = f\left(\frac{ta}{r}\right)$$

$$\Rightarrow \Delta p = \frac{E}{r^3} f\left(\frac{ta}{r}\right)$$