

2. FLUID STATICS

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MOTION

2.1 PRESSURE SPECIFICATIONS

- Fluid at rest cannot support shear stress .
 - Mohr's circle reduces to a point.
- The normal stress on any plane through a fluid element at rest is equal to a unique value.
 - Fluid pressure-
 - positive for compression

2.1 PRESSURE SPECIFICATIONS...

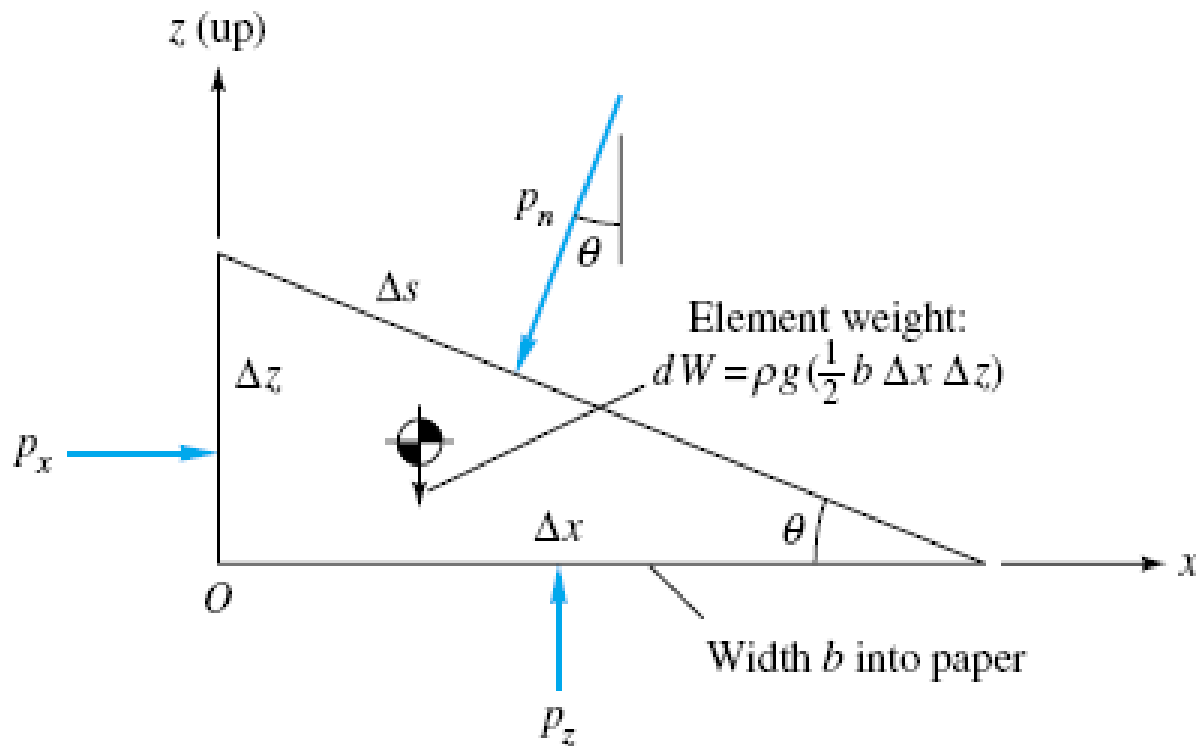


Fig. 2.1 Equilibrium of a Small Wedge of Fluid at Rest.

2.1 PRESSURE SPECIFICATIONS...

From equilibrium of forces,

$$\begin{aligned}\sum F_x = 0 &= P_x b \Delta z - P_n b \Delta s \sin \theta \\ \sum F_z = 0 &= P_z b \Delta x - P_n b \Delta s \cos \theta - \frac{1}{2} \rho g b \Delta x \Delta z\end{aligned}\quad (2.1)$$

From geometry,

$$\Delta s \sin \theta = \Delta z \qquad \Delta s \cos \theta = \Delta x \quad (2.2)$$

$$\Rightarrow P_x = P_n \qquad P_z = P_n + \frac{1}{2} \rho g \Delta z \quad (2.3)$$

2.1 PRESSURE SPECIFICATIONS...

Equation (2.3) illustrates two important principles of the hydrostatic, or shear-free, condition:

- There is no pressure change in the horizontal direction, and
- There is a vertical change in pressure proportional to the density, gravity, and depth change.

In general, there are three types of forces that act on a fluid element of size $dx dy dz$.

2.1 PRESSURE SPECIFICATIONS...

1. Pressure force

$$d\mathbf{F}_{\text{press}} = \left(-\mathbf{i} \frac{\partial P}{\partial x} - \mathbf{j} \frac{\partial P}{\partial y} - \mathbf{k} \frac{\partial P}{\partial z} \right) dx dy dz \quad (2.4)$$

2. Viscous force

$$d\mathbf{F}_{\text{vs}} = \mu \left(\frac{\partial^2 \mathbf{v}}{\partial x^2} + \frac{\partial^2 \mathbf{v}}{\partial y^2} + \frac{\partial^2 \mathbf{v}}{\partial z^2} \right) dx dy dz \quad (2.5)$$

3. Body force (mainly gravity force)

$$d\mathbf{F}_{\text{body}} = \rho \mathbf{g} dx dy dz \quad (2.6)$$

2.1 PRESSURE SPECIFICATIONS...

The resultant of these three forces will accelerate the element as

$$d\mathbf{F}_{\text{Press}} + d\mathbf{F}_{\text{VS}} + d\mathbf{F}_{\text{body}} = \rho dx dy dz \quad (2.7)$$

$$\Rightarrow -\nabla P + \mu \nabla^2 \mathbf{V} + \rho \mathbf{g} = \rho \mathbf{a} \quad (2.8)$$

2.1 PRESSURE SPECIFICATIONS...

GAGE PRESSURE AND VACUUM PRESSURE

Pressures are usually specified as

- The *absolute* or total magnitude or
- The value *relative* to the local ambient atmosphere.

The second case occurs because many pressure measuring instruments are of *differential* type and record, not an absolute magnitude, but the difference between the fluid pressure and the atmosphere.

- $p > p_a$ Gage pressure: $p(\text{gage}) = p - p_a$
- $p < p_a$ Vacuum pressure: $p(\text{vacuum}) = p_a - p$

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

- For a static fluid, the acceleration and the viscous force are zero and the general equation (2.8) reduces to

$$\nabla p = \rho \mathbf{g} \quad (2.9)$$

Recall from vector analysis that the vector ∇p expresses the magnitude and direction of the maximum spatial rate of increase of the scalar property p .

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

Equation (2.9) can be rewritten as

$$\mathbf{i} \frac{\partial P}{\partial x} + \mathbf{j} \frac{\partial P}{\partial y} + \mathbf{k} \frac{\partial P}{\partial z} = -\mathbf{k} \rho g \quad (2.10)$$

$$\Rightarrow \frac{\partial P}{\partial x} = 0; \quad \frac{\partial P}{\partial y} = 0; \quad \frac{\partial P}{\partial z} = -\rho g \quad (2.11)$$

$$\Rightarrow \frac{dP}{dz} = -\rho g \quad (2.12)$$

$$\Rightarrow P_2 - P_1 = - \int_1^2 \rho g dz$$

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

For liquids, which are usually considered as incompressible, equation (2.12) simplifies to

$$P_2 - P_1 = -\rho g \Delta z = \rho g(z_1 - z_2) = \gamma(z_1 - z_2)$$

or

$$(2.13)$$

$$z_2 - z_1 = \frac{P_1}{\gamma} - \frac{P_2}{\gamma}$$

From equation (2.11), it can be concluded that pressure in a continuously distributed uniform static fluid varies only with vertical distance and is independent of the shape of the container.

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

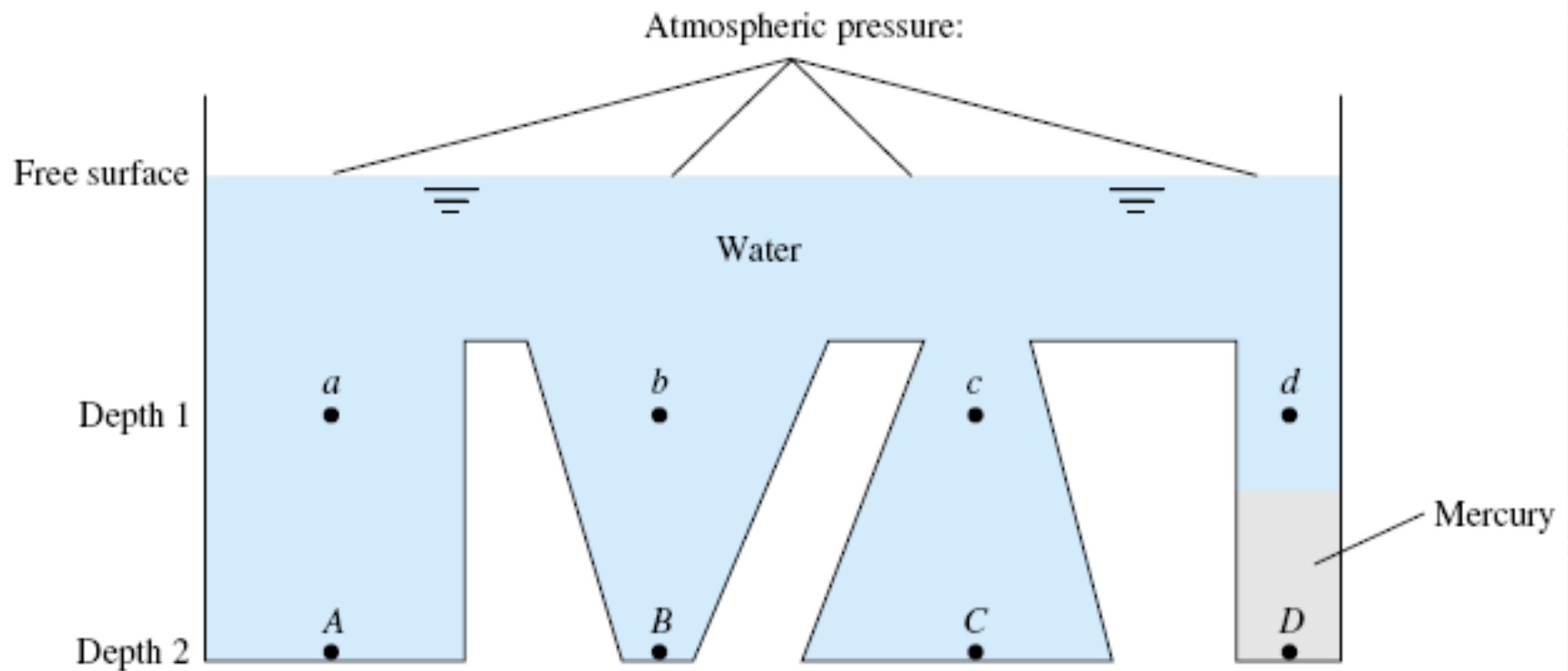


Fig. 2.2 Hydrostatic-Pressure Independent of Shape

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

THE MERCURY BAROMETER

Barometer (Fig. 2.3) is a simple instrument that measures the atmospheric pressure using the hydrostatic equation (2.13).

The height that the mercury column rises is given by

$$h = \frac{p_{\text{atm}}}{\gamma_{\text{Mercury}}}$$

(2.14)

2.2 HYDROSTATIC PRESSURE DISTRIBUTIONS ...

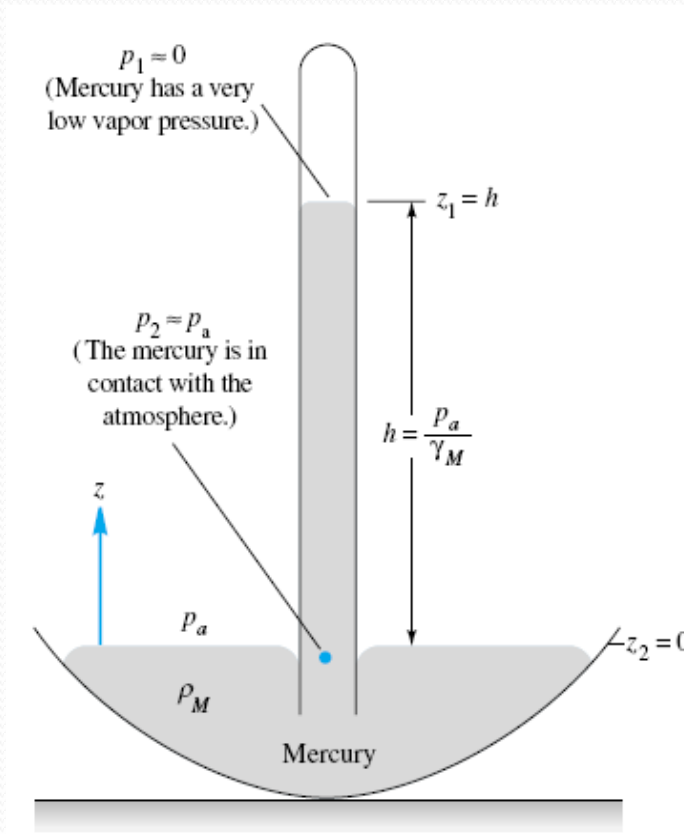


Fig. 2.3 Schematic of a Barometer

2.3 MANOMETRY

- From the hydrostatic formula (2.13), a change in elevation $z_2 - z_1$ of a liquid is equivalent to a change in pressure $(P_2 - P_1)/\rho g$.
- Thus, a static column of one or more liquids or gases can be used to measure pressure differences between two points. Such a device is called a manometer (Fig. 2.4).

2.3 MANOMETRY...

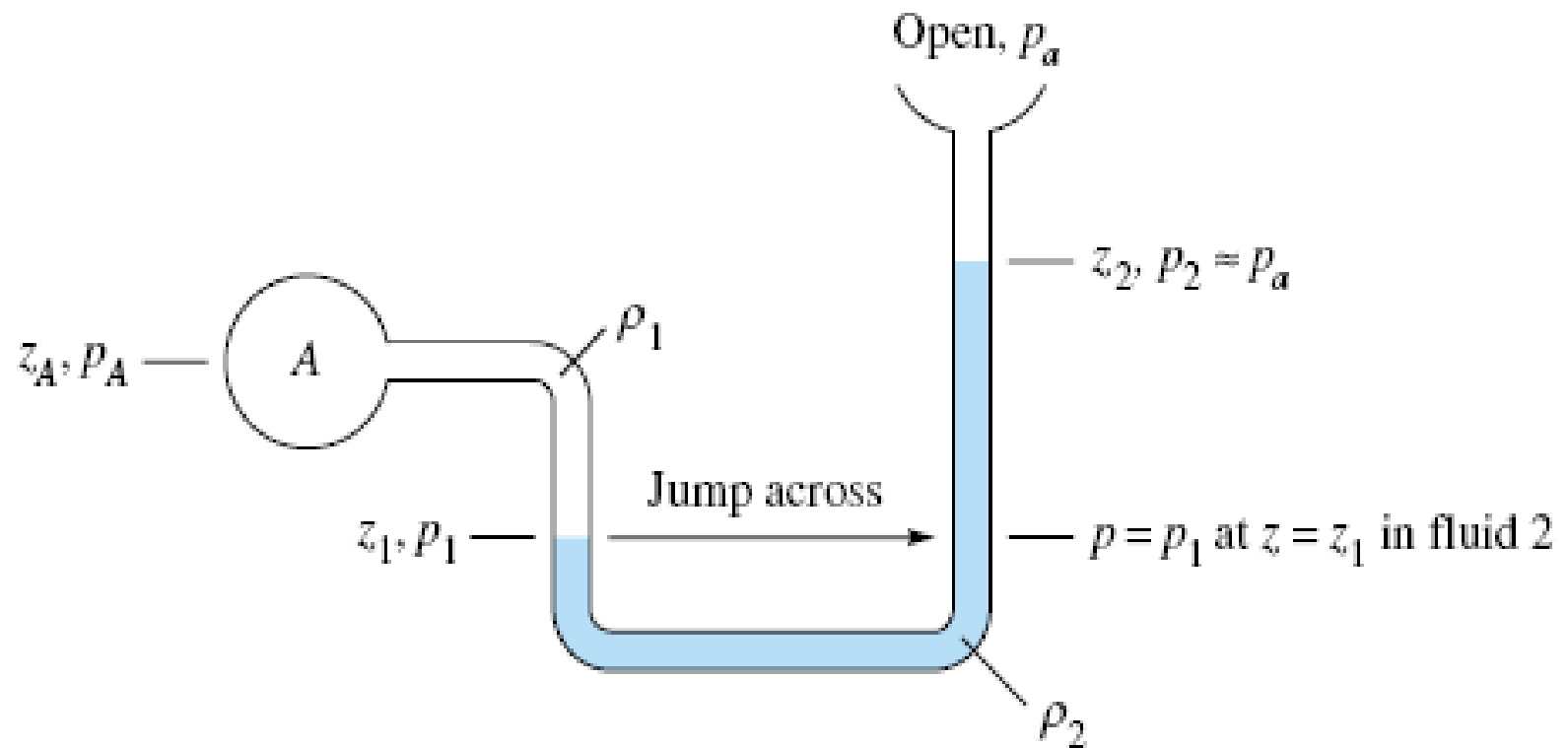


Fig. 2.4 Simple Open Manometer

2.3 MANOMETRY...

- The simple open manometer shown in Fig. 2.4 measures the pressure difference between point A and the atmosphere. The two pressures P_A and P_a can be related as

$$P_A + \rho \lambda_1 (z_A - z_1) - \rho \lambda_2 (z_2 - z_1) = P_2 \circ P_a \quad (2.15)$$

2.4 HYDROSTATIC FORCES ON PLANE SURFACES

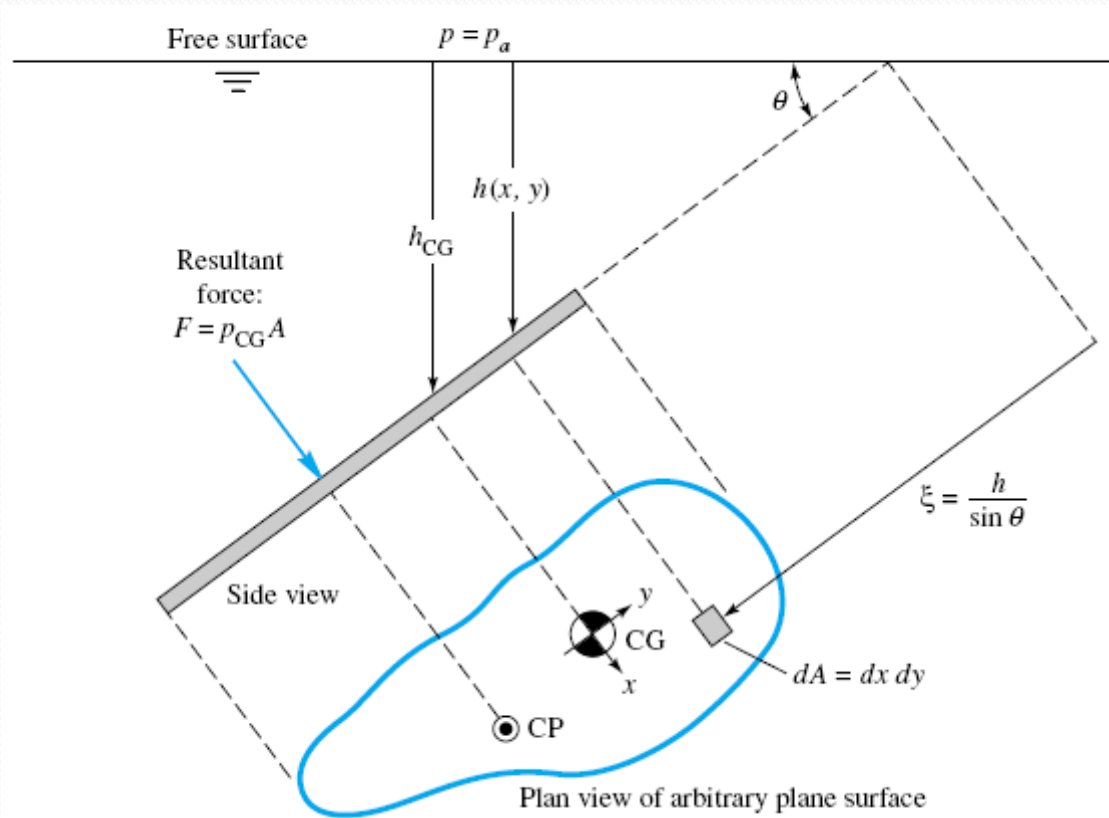


Fig. 2.5 Hydrostatic Force and Center of Pressure on an Arbitrary Plane Surface

2.4 HYDROSTATIC FORCES ON PLANE SURFACES ...

- The absolute pressure on one side of the element is

$$P = P_a + \rho g h \quad (2.16)$$

- The corresponding total hydrostatic force on one side of the element is
- $$F = \int P dA = \int (P_a + \rho g h) dA = P_a A + \rho g \int h dA \quad (2.17)$$

- From geometry,
- $$h = \frac{y}{\sin \theta} \quad (2.18)$$

- Then equation (2.17) can be rewritten as
- $$F = \int P dA = P_a A + \rho g \sin \theta \int y dA \quad (2.19)$$

2.4 HYDROSTATIC FORCES ON PLANE SURFACES ...

- But from the definition of a centroid from the free surface,

$$\xi_{CG} = \frac{1}{A} \int \xi dA \Rightarrow \int \xi dA = A \xi_{CG} \quad (2.20)$$

- Then

$$F = PaA + \gamma \sin \theta A \xi_{CG} \quad (2.21)$$

- But again from geometry

$$\sin \theta \xi_{CG} = h_{CG} \quad (2.22)$$

$$\Rightarrow F = PaA + \gamma A h_{CG} = (Pa + \gamma h_{CG}) A \quad (2.23)$$

2.4 HYDROSTATIC FORCES ON PLANE SURFACES ...

- The coordinates (x_{CP}, y_{CP}) of the center of pressure is obtained in the following way.

$$Fy_{CP} = \int y P dA = \int y (Pa + \gamma \xi \sin \theta) dA = Pa \int y dA + \gamma \sin \theta \int y \xi dA \quad (2.24)$$

- The term $\int y P dA = Pa \int y dA$ vanishes from the definition of a centroid.

$$\left. \begin{aligned} Fy_{CP} &= \gamma \sin \theta \int y \xi dA = \gamma \sin \theta \int y (\xi_{CG} - y) dA \\ &= \gamma \sin \theta \left(\int \xi_{CG} y dA - \int y^2 dA \right) \\ &= - \gamma \sin \theta I_{xx} \end{aligned} \right\} \quad (2.25)$$

$$y_{CP} = \frac{- \gamma \sin \theta I_{xx}}{(Pa + \gamma h_{CG}) A} \quad (2.26)$$

2.4 HYDROSTATIC FORCES ON PLANE SURFACES ...

- Similarly,

$$\begin{aligned} F_{x_{CP}} &= \int x P dA = \int x (P_a + \gamma (\xi_{CG} - y) \sin \theta) dA \\ &= - \gamma \sin \theta \int xy dA = - \gamma \sin \theta I_{xy} \end{aligned} \quad (2.27)$$

$$\Rightarrow x_{CP} = \frac{- \gamma \sin \theta I_{xy}}{(P_a + \gamma h_{CG})_A} \quad (2.28)$$

In most applications one side of the surfaces is exposed to ambient air while the other side is in contact with a liquid, Fig. 2.6. In such cases, the atmospheric pressure acts on both sides and balance each other.

2.4 HYDROSTATIC FORCES ON PLANE SURFACES ...

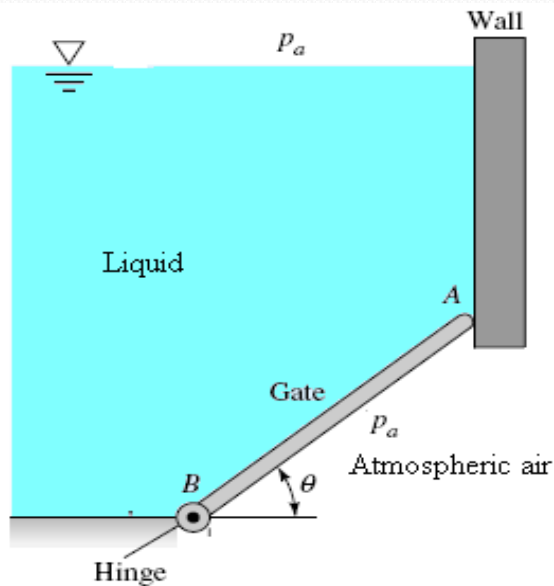


Fig. 2.6 Surface Exposed to Liquid on One Side and to Atmospheric Air on the Other Side

In such cases,

$$y_{CP} = \frac{-\sin \theta I_{xx}}{h_{CG} A}$$

$$x_{CP} = \frac{-\sin \theta I_{xy}}{h_{CG} A}$$

and

2.5 HYDROSTATIC FORCES ON CURVED SURFACES

- The resultant pressure force on a curved surface is most easily computed by separating it into horizontal and vertical components.
- There are two general rules when computing hydrostatic forces on curved surfaces.
 - The horizontal component of force on a curved surface equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component.
 - The vertical component of pressure force on a curved surface equals in magnitude and direction the weight of the entire column of fluid, both liquid

2.5 HYDROSTATIC FORCES ON CURVED SURFACES...

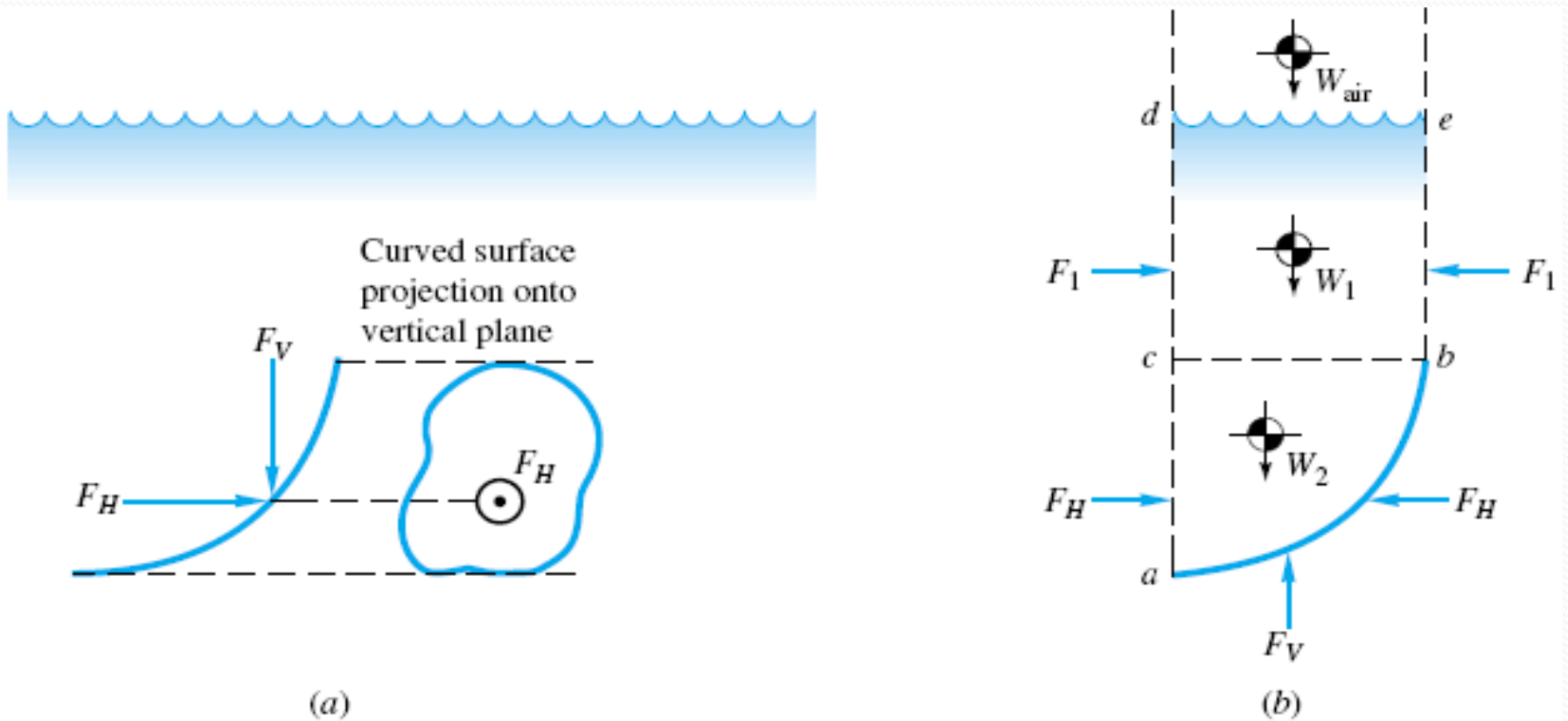


Fig. 2.7 Computation of Hydrostatic Force on a Curved Surface

2.5 HYDROSTATIC FORCES ON CURVED SURFACES...

$$F_H = (Pa + \gamma h_{CG}) A_{projected}$$

$$y_{CP} = \frac{-\gamma I_{xx}}{(Pa + \gamma h_{CG}) A_{projected}}$$

$$x_{CP} = \frac{-\gamma I_{xy}}{(Pa + \gamma h_{CG}) A_{projected}}$$

$$F_V = W_{air} + W_1 + W_2$$

The weight of air is usually negligible.

2.6 BUOYANCY AND STABILITY

- For a completely immersed or floating body, the following two laws of buoyancy discovered by Archimedes are applied.
 - A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces.
 - A floating body displaces its own weight in the fluid in which it floats.

2.6 BUOYANCY AND STABILITY ...

STABILITY OF FLOATING OBJECTS

- If little disturbance is applied to a floating body, it may overturn (unstable body) or restore (stable body) to its original position.
- Unstability of floating bodies should be avoided during design.

2.6 BUOYANCY AND STABILITY ...

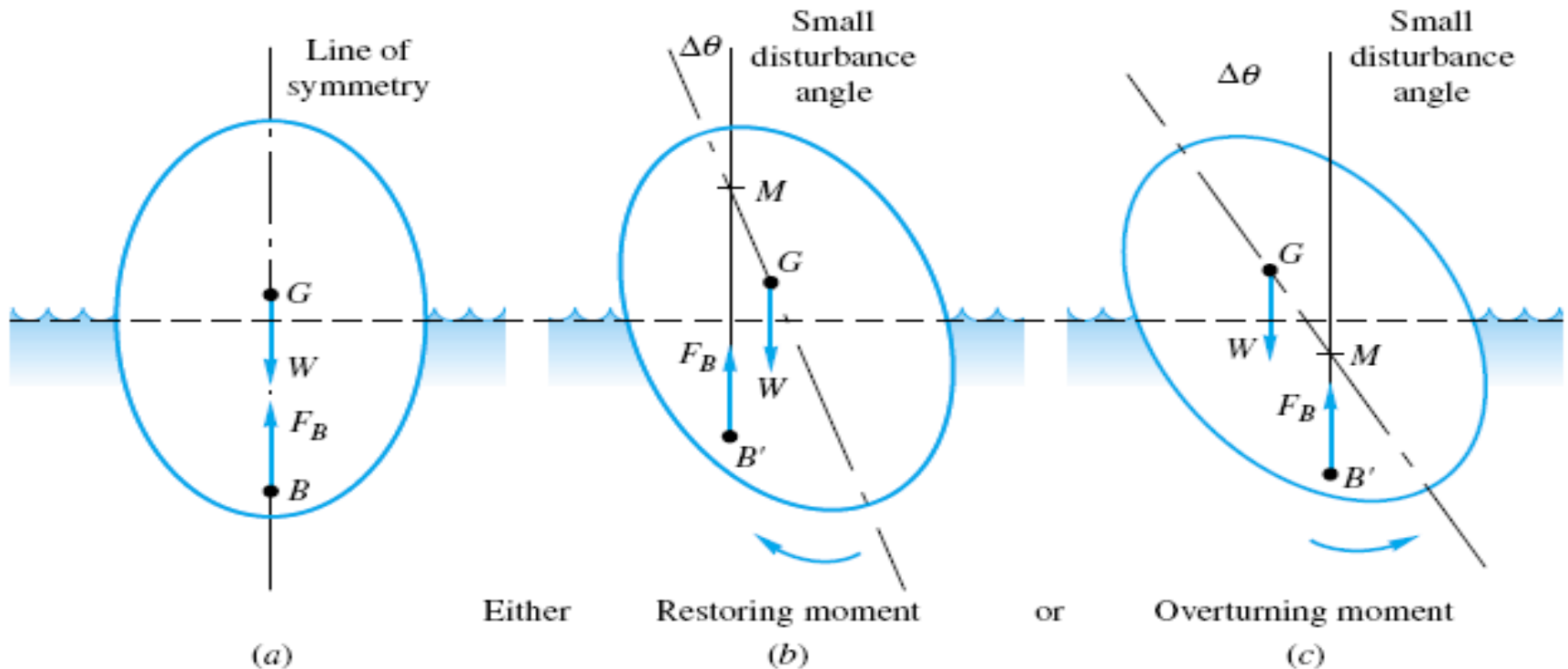


Fig. 2.8 Calculation of the Metacenter M of a Floating Body

2.6 BUOYANCY AND STABILITY ...

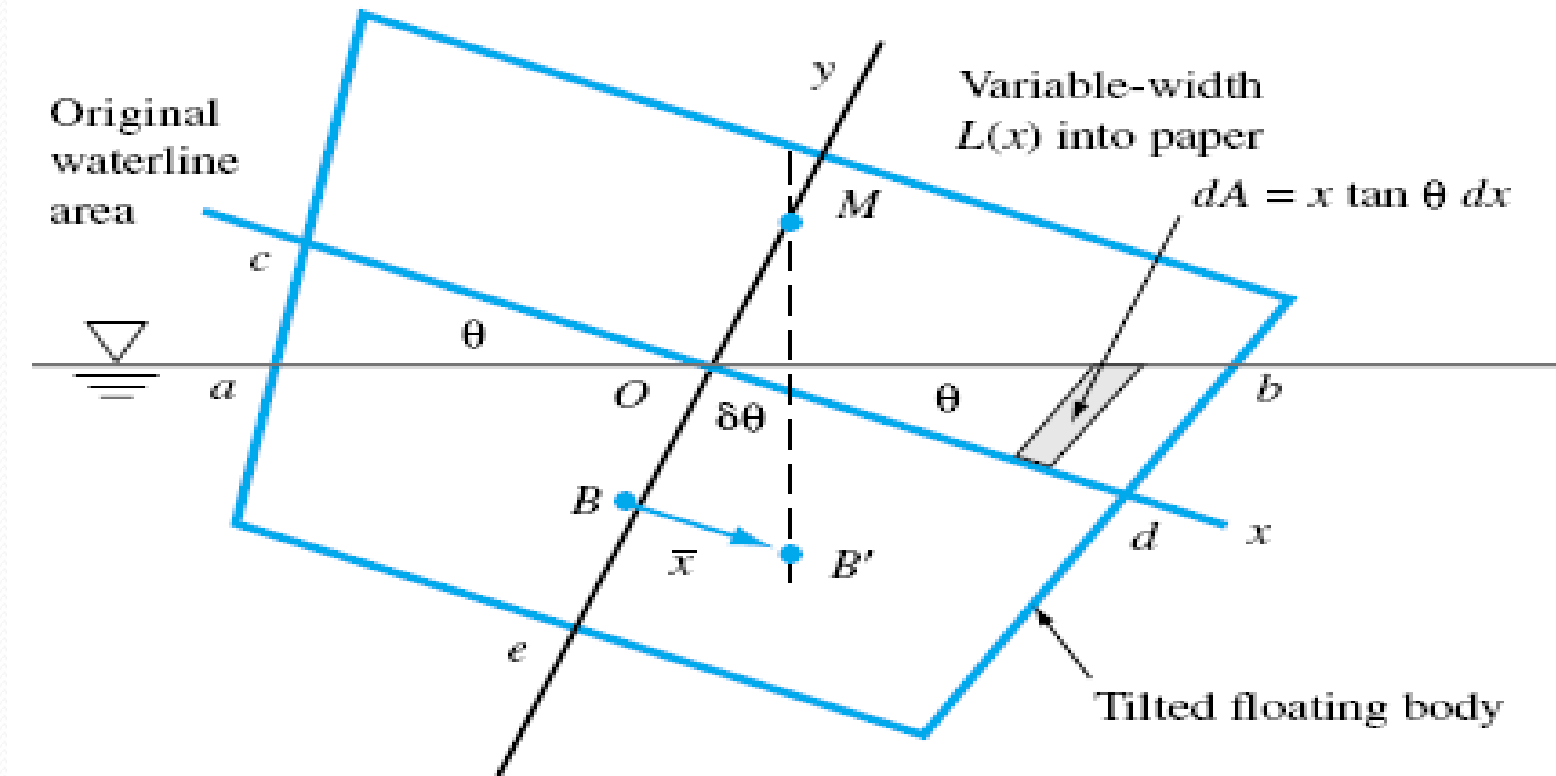


Fig. 2.9 Calculation of Metacentric Height

2.6 BUOYANCY AND STABILITY ...

- The y -axis of the body is assumed to be a line of symmetry.
- Tilting the body a small angle φ then submerges small wedge Obd and uncovers an equal wedge cOa , as shown.
- The new position B' of the center of buoyancy is calculated as the centroid of the submerged portion $aObde$ of the body:

2.6 BUOYANCY AND STABILITY ...

$$\begin{aligned}
 \bar{x}V_{aObxde} &= \int_{cOdea} xdv + \int_{Obd} xdv - \int_{cOa} xdv = 0 + \int_{Obd} x(LdA) - \int_{cOa} x(LdA) \\
 &= \int_{Obd} xL(x \tan \theta dx) - \int_{cOa} xL(-x \tan \theta dx) \\
 &= \tan \theta \int_{waterline} x^2 L(dx) \\
 &= \tan \theta \int_{waterline} x^2 (dA) \\
 &= I_o \tan \theta
 \end{aligned}
 \tag{2.29}$$

Then

$$MB = \frac{\bar{x}}{\tan \theta} = \frac{I_o}{V_{submerged}} \tag{2.30}$$

2.6 BUOYANCY AND STABILITY ...

The engineer would determine the distance from G to B from the basic shape and design of the floating body and then make the calculation of I_o and the submerged volume

$V_{\text{submerged}}$.

- If the metacentric height MG is positive, the body is stable for small disturbances.
- Note that if GB is negative, that is, B is *above* G , the body is always stable.

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION

- In rigid-body motion, all particles are in combined translation and rotation, and there is no relative motion between particles.
- With no relative motion, there are no strains or strain rates, so that the viscous term $\mu \nabla^2 \mathbf{V}$ in equation (2.8) vanishes, leaving a balance between pressure, gravity, and particle acceleration

$$\left. \begin{aligned} - \nabla P + \rho \mathbf{g} &= \rho \mathbf{a} \\ \Rightarrow \nabla P &= \rho(\mathbf{g} - \mathbf{a}) \end{aligned} \right\} \quad (2.31)$$

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

- For a liquid container under uniform linear motion

$$\nabla p = \rho(\mathbf{k}(g + a_z) + \mathbf{i}a_x)$$

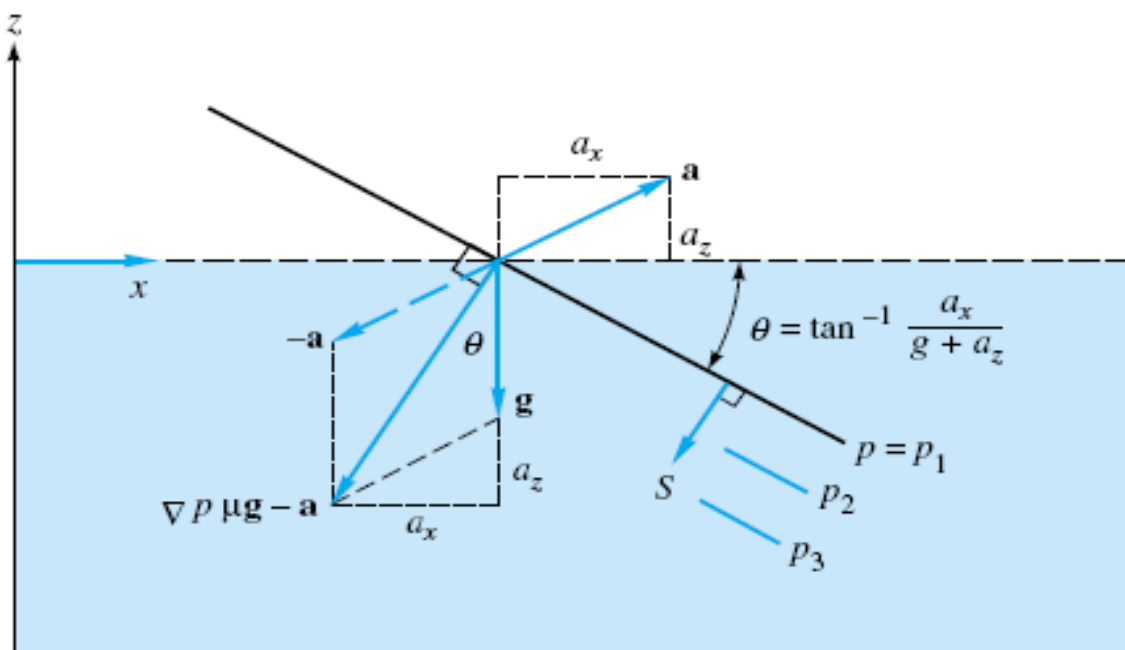


Fig. 2.10 Tank of Liquid in Rigid-Body Acceleration

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

$$\theta = \tan^{-1} \left(\frac{a_x}{g + a_z} \right) \quad (2.33)$$

- For liquid under rigid body rotation, the acceleration $\mathbf{a}$ in equation (2.31) becomes

Then

$$\left. \begin{aligned} \nabla P &= -\rho(\mathbf{k}g - \mathbf{i}_r\Omega^2 r) \\ \mathbf{i}_r \frac{\partial P}{\partial r} + \mathbf{k} \frac{\partial P}{\partial z} &= \rho(-\mathbf{k}g + \mathbf{i}_r\Omega^2 r) \end{aligned} \right\} (2.35)$$

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

$$\Rightarrow \frac{\partial P}{\partial r} = \rho \Omega^2 r \quad \frac{\partial P}{\partial z} = -\rho g \quad (2.36)$$

$$\left. \begin{aligned} \frac{\partial P}{\partial r} = \rho \Omega^2 r &\Rightarrow P = \rho \Omega^2 \frac{r^2}{2} + f(z) \\ \Rightarrow \frac{\partial P}{\partial z} = f'(z) &= -\rho g \\ \Rightarrow f(z) &= -\rho g z + \text{Const} \end{aligned} \right\} (2.37)$$

$$\Rightarrow P(r, z) = \rho \Omega^2 \frac{r^2}{2} - \rho g z + \text{Const} \quad (2.38)$$

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

- If $P = P_0$ at $(r, z) = (0, 0)$, then $Const = P_0$. The final desired distribution is

$$P(r, z) = \rho\Omega^2 \frac{r^2}{2} - \rho gz + P_0 \quad (2.39)$$

Lines (surfaces) of constant pressure

$$P_1 = \rho\Omega^2 \frac{r^2}{2} - \rho gz + P_0 = \text{constant}$$

$$\Rightarrow z = \frac{P_0 - P_1}{\rho g} + \frac{\Omega^2 r^2}{2g}$$

$$\Rightarrow z = a + br^2 \quad (2.40)$$

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

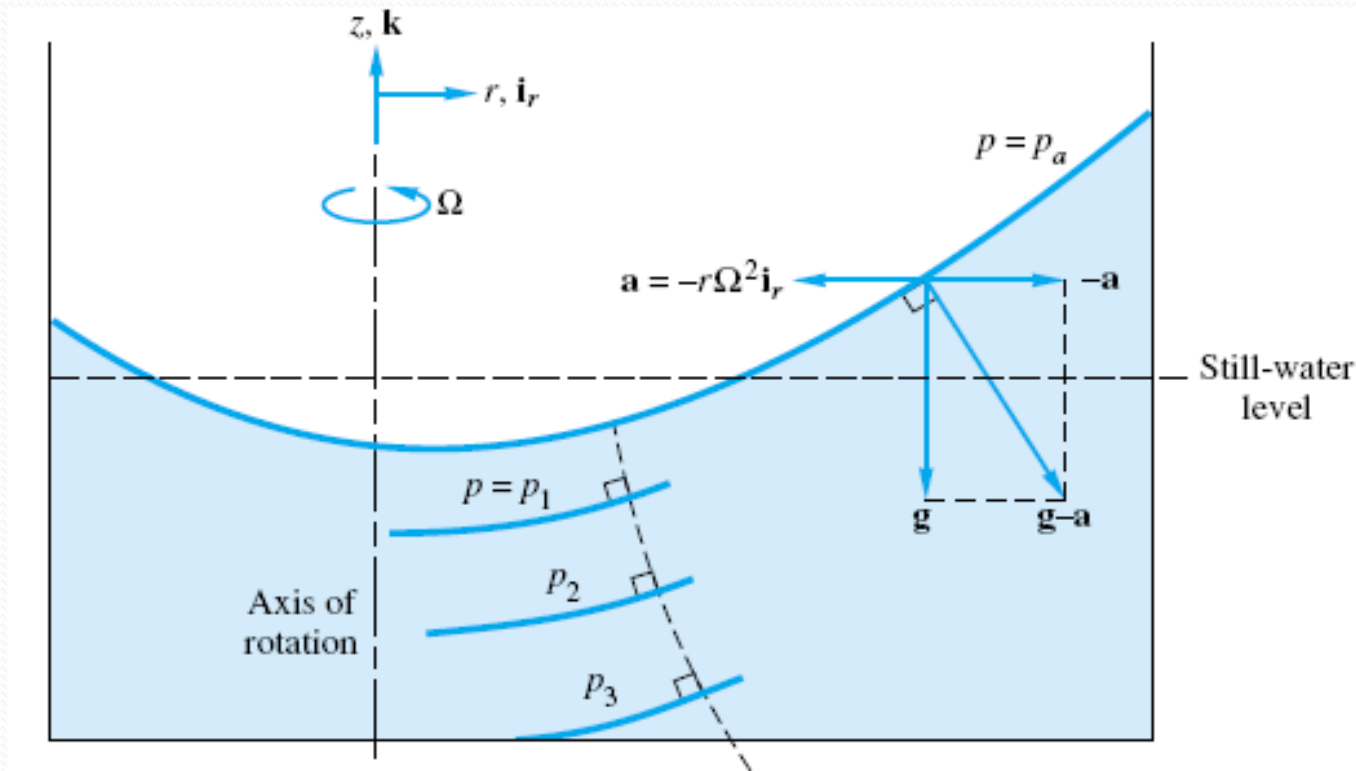


Fig. 2.11 Liquid under Rigid Body Rotation

2.7 PRESSURE VARIATION WITH RIGID BODY MOTION...

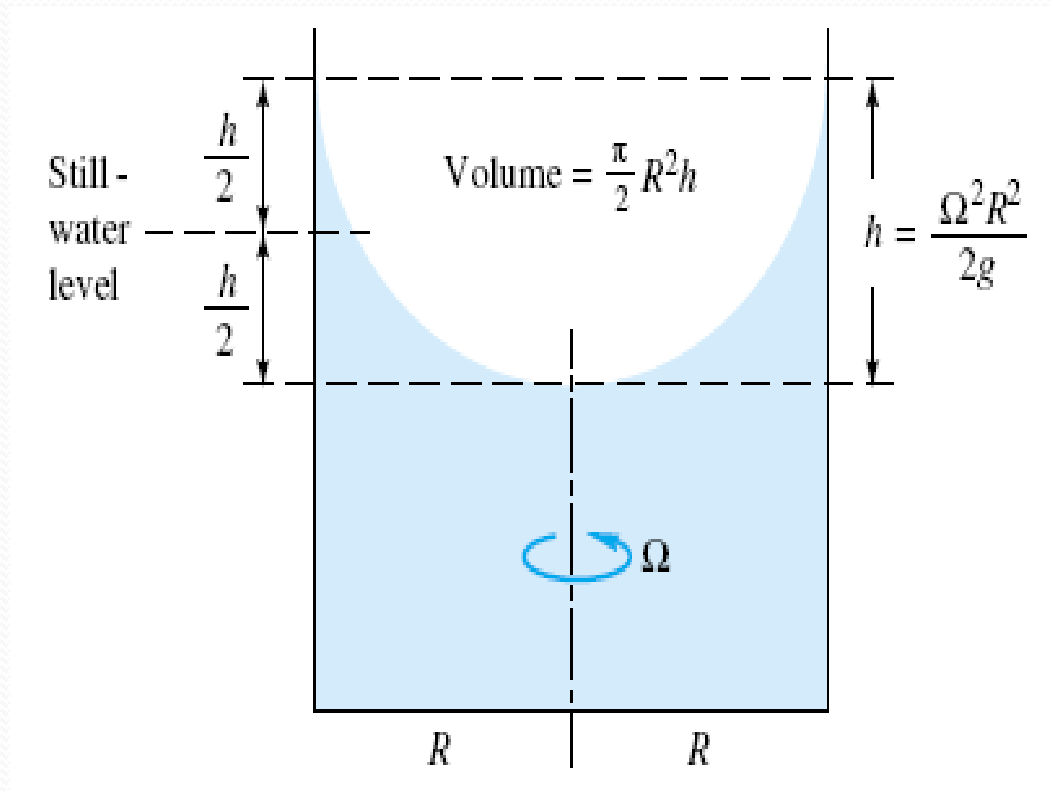


Fig. 2.12 Determining the Free Surface Position for Rotation of a Cylinder of Fluid about its Central Axis.

Example 2.1

- At 20°C gage *A* reads 350 kPa absolute. What is the height h of the water in cm? What should gage *B* read in kPa absolute? See Fig.

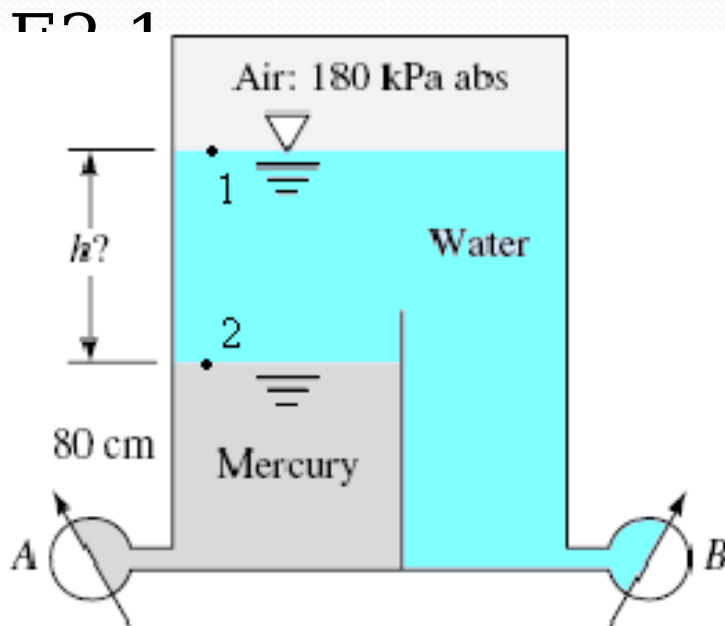


Fig. E2.1

Solution

Write hydrostatic equations between points 1 and 2, and between points 2 and A. Eliminate pressure of point 2.

$$P_2 - P_1 = \rho_{\text{water}}(z_1 - z_2) = \rho_{\text{water}} h \text{ and } P_A -$$

$$P_2 = \rho_{\text{mercury}}(z_2 - z_A) = \rho_{\text{mercury}} * 0.8$$

$$\Rightarrow h = \frac{P_A - P_1}{\rho_{\text{water}}} = \frac{133100 - 180000}{9790} + \frac{\rho_{\text{mercury}} * 0.8}{\rho_{\text{water}}} = \underline{6.488\text{m}}$$

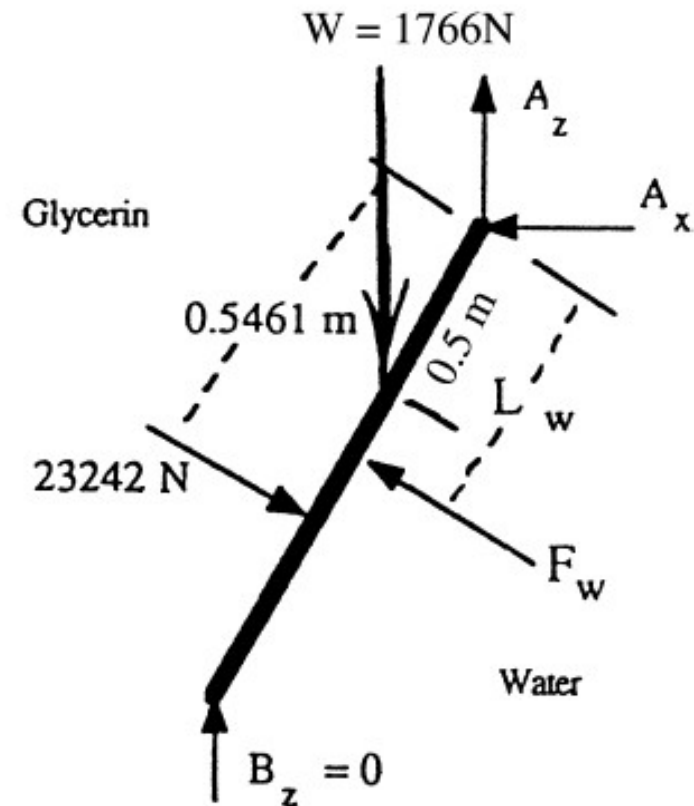
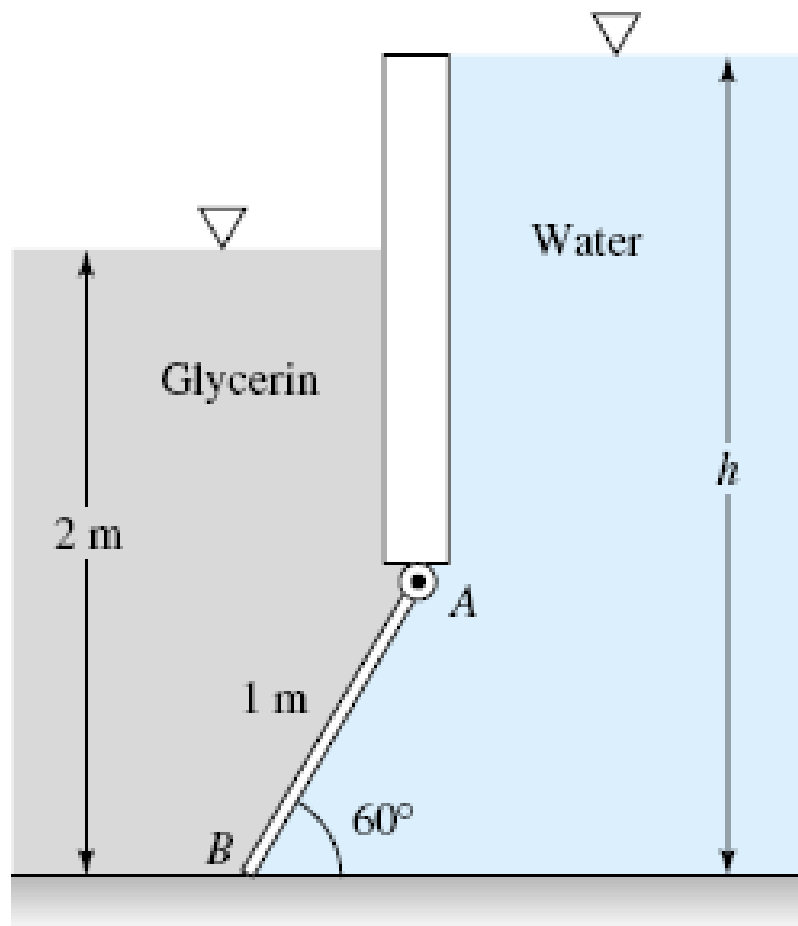
Pressure at point B can be obtained as

$$P_B = P_1 + \rho_{\text{water}}(h + 0.8) = 180000 + 9790 * (6.488 + 0.8) = 251349.52 \text{ pa}$$

Example 2.2

Gate AB is a homogeneous mass of 180 kg, 1.2 m wide into the paper, hinged at A , and resting on a smooth bottom at B . All fluids are at 20°C. For what water depth h will the force at point B be zero?

Example 2.2...



Solution

Let $\rho_A = 12360 \text{ N/m}^3$ for glycerin and 9790 N/m^3 for water. The centroid of AB is 0.433m vertically below A, so $h_{CG} = 2.0 - 0.433 = 1.567 \text{ m}$, and we may compute the glycerin force and its line of action:

$$F_g = \rho_{CG} A h_{CG} = 12360 \times 1.567 \times 1.2 = 23242 \text{ N}$$

$$y_{CP,g} = \frac{-I_{xx} \sin \theta}{h_{CG} A} = - \frac{(1/12) \times 1.2 \times 1^3 \times \sin 60^\circ}{1.567 \times 1.2} = -0.0461 \text{ m}$$

Solution...

The water force and its line of action are

$$F_w = \gamma h_{CG} A = 9790 \times h_{CG} \times 1.2 = 23242 N$$

$$y_{CP,w} = \frac{-I_{xx} \sin \theta}{h_{CG} A} = - \frac{(1/12) \times 1.2 \times 1^3 \times \sin 60^\circ}{h_{CG} \times 1.2} = - \frac{0.0722}{h_{CG}}$$

The weight of the gate, $W = 180 \times (9.81) = 1766N$, acts at the centroid. Since the force at B equals zero, we may sum moments counterclockwise about A to find the water

$$\sum M_A = 0 \Rightarrow 23242 \times 0.5461 + 1766 \times 0.5 \cos 60^\circ - 9790 \times h_{CG} \times 1.2 \times (0.5 + 0.0722/h_{CG}) = 0$$

$$\Rightarrow h_{CG,water} = 2.09m$$

$$\Rightarrow h = h_{CG,water} + 0.433 = 2.52m$$

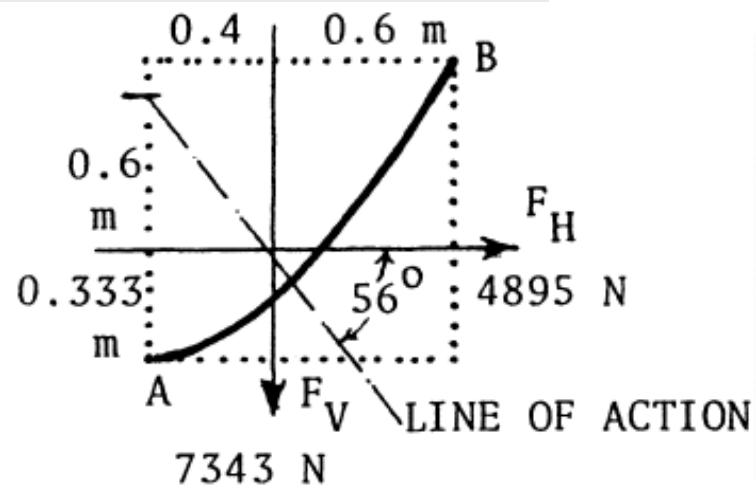
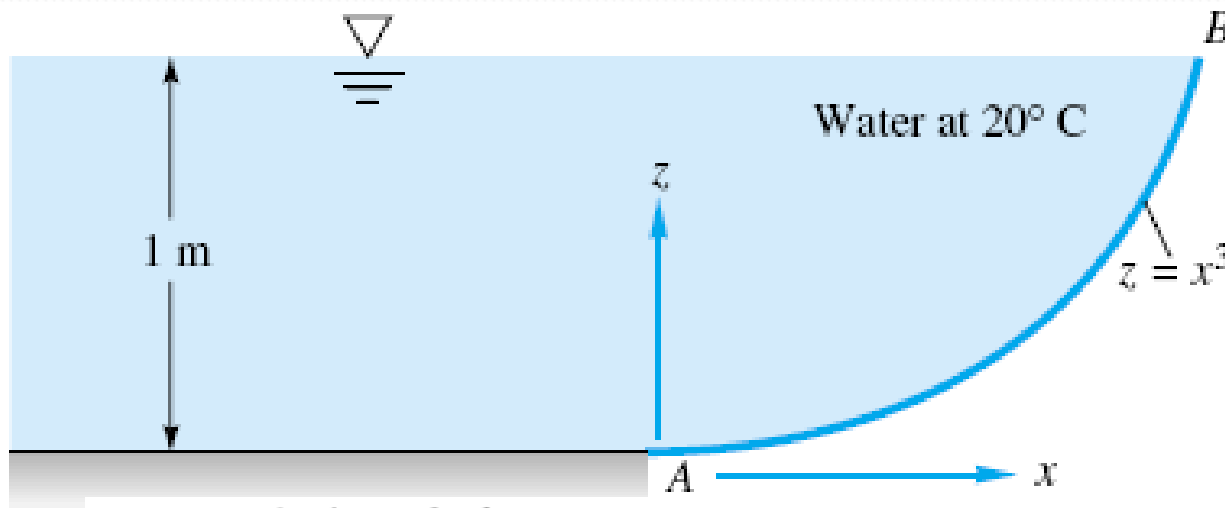
Example 2.3

Example 2.3

Determine

- a. The total hydrostatic force on the curved surface AB and
- b. Its line of action. Neglect atmospheric pressure, and let the surface have unit width.

Example 2.3...



Solution

The horizontal force is

$$F_H = (\gamma h_{CG}) A_{\text{projected}} = 9790 \times 0.5 \times 1 \times 1 = 4895 \text{ N at } 0.667 \text{ m below B}$$

For the cubic-shaped surface AB, the weight of water above is computed by integration:

$$\begin{aligned} F_V &= \gamma b \int_0^1 (1 - x^3) dx = \frac{3}{4} \gamma b \\ &= \frac{3}{4} \times 9790 \times 1 = 7343 \text{ N} \end{aligned}$$

Solution...

The line of action (water centroid) of the vertical force also has to be found by integration:

$$\bar{x} = \frac{\int x dA}{\int dA} = \frac{\int_0^1 x(1-x^3) dx}{\int_0^1 (1-x^3) dx} = \frac{3/10}{3/4} = 0.4 \text{ m}$$

The vertical force of 7343 N thus acts at 0.4 m to the right of point A, or 0.6 m to the left of B, as shown in the sketch above. The resultant hydrostatic force then is

$$F_{\text{total}} = \sqrt{(4895)^2 + (7343)^2} = 8825 \text{ N}$$

acting at **56.31°** down and to

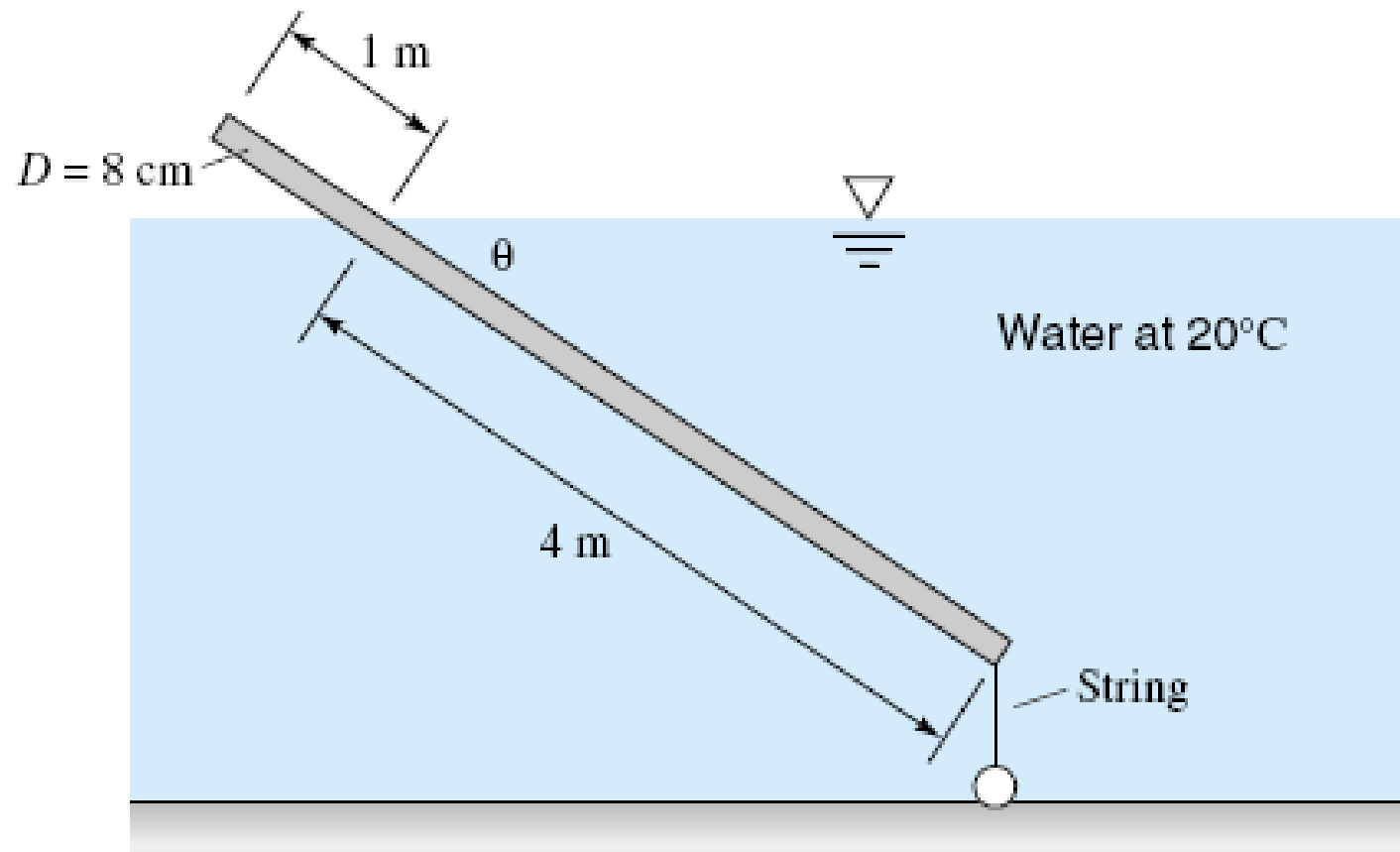
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Example 2.4

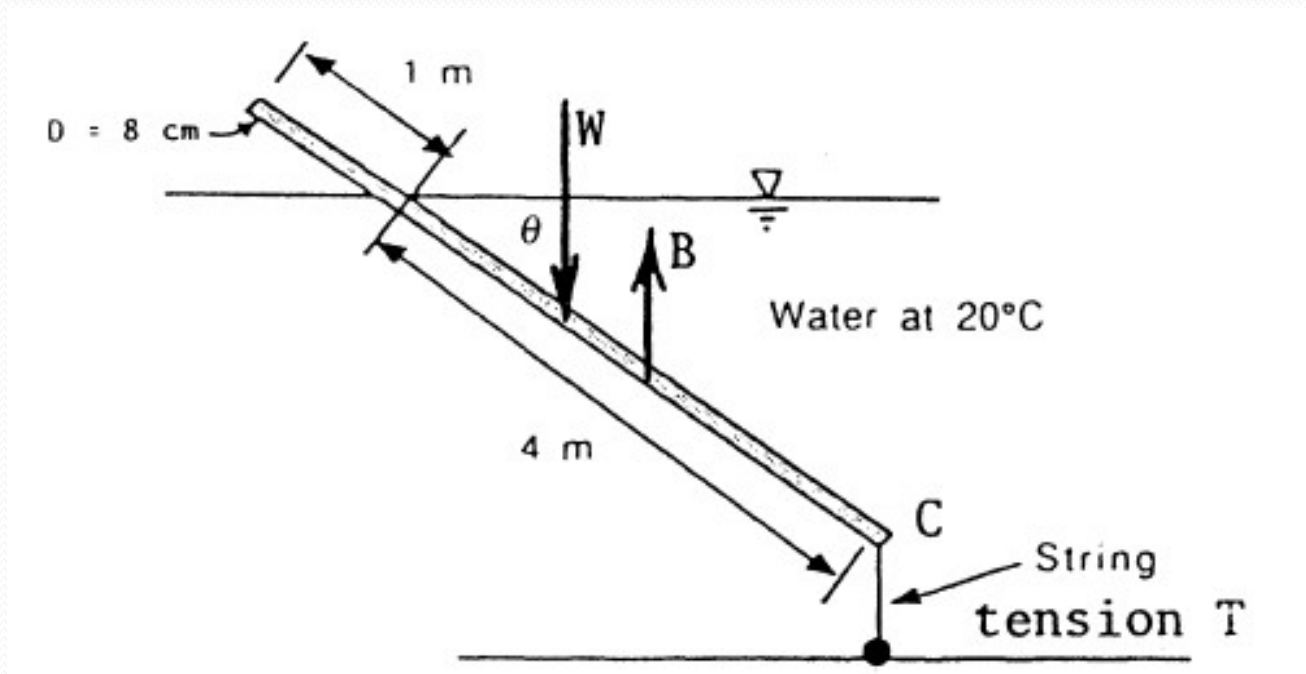
The uniform 5m long round wooden rod is tied to the bottom by a string. Determine

- a. The tension in the string and
- b. The specific gravity of the wood. Is it possible for the given information to determine the inclination angle θ ? Explain.

Example 2.4...



Example 2.4...



Solution

The rod weight acts at the middle, 2.5 m from point C, while the buoyancy is 2 m from C. Summing moments about C gives

$$\sum M_C = 0 = W \times 2.5 \sin \theta - B \times 2 \sin \theta$$

$$\Rightarrow W = 0.8B$$

$$\text{But } B = 9790 \times \pi/4 \times 0.08 \times 4 = 196.8 \text{ N}$$

$$\text{Thus } W = 0.8B = 157.5 = SG \times 9790 \times \pi/4 \times 0.08 \times 5$$

$$\Rightarrow \mathbf{SG = 0.64}$$

Solution...

Summation of vertical forces yields

$$\text{String tension } T = B - W = 196.8 - 157.5 \approx 39\text{N}$$

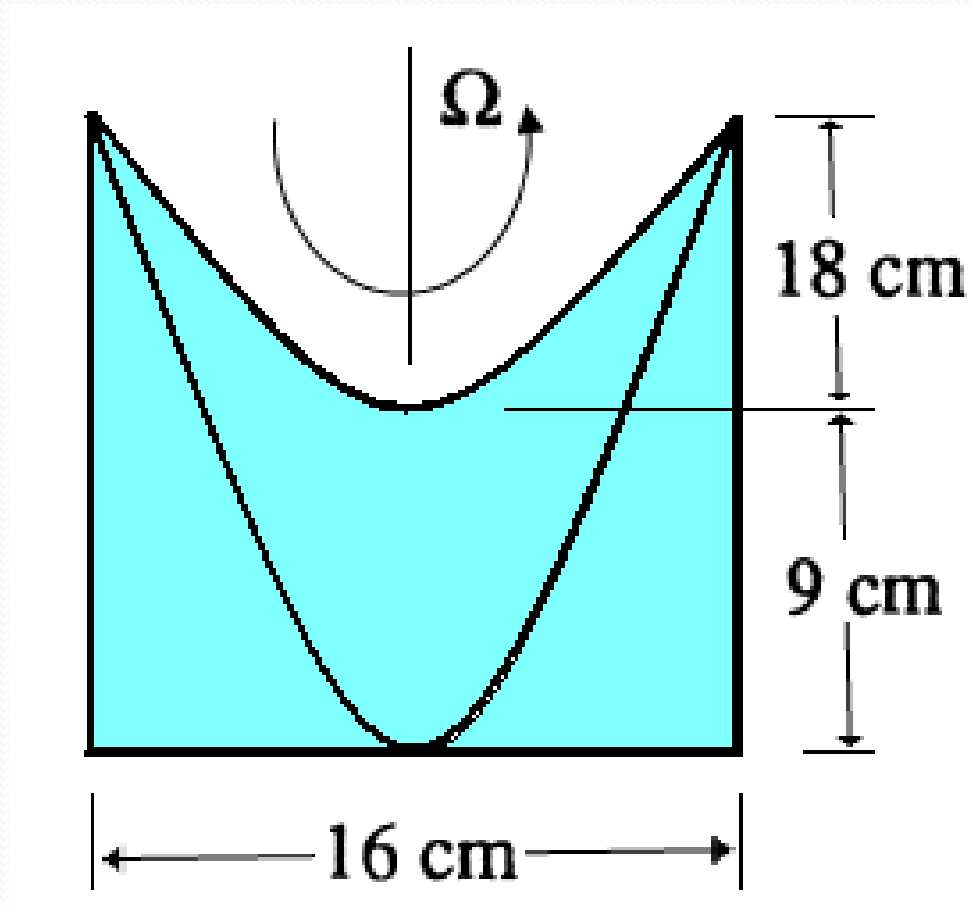
These results are independent of the angle θ ,
which cancels out of the moment balance.

Example 2.5

A 16cm diameter open cylinder 27 cm high is full of water. Compute the rigid-body rotation rate about its central axis, in r/min,

- a. For which one third of the water will spill out and
- b. For which the bottom will be barely exposed.

Solution



Solution...

- a. One-third will spill out if the resulting paraboloid surface is 18 cm deep:

$$h = 0.18 = \frac{\Omega^2 R^2}{2g} = \frac{\Omega^2 \times 0.08^2}{2 \times 9.81}$$

$$\Rightarrow \Omega = 23.5 \text{ rad/s} = \mathbf{224 \text{ r/min}}$$

- b. The bottom is barely exposed if the paraboloid surface is 27 cm deep:

$$h = 0.27 = \frac{\Omega^2 R^2}{2g} = \frac{\Omega^2 \times 0.08^2}{2 \times 9.81}$$

$$\Rightarrow \Omega = 28.8 \text{ rad/s} = \mathbf{275 \text{ r/min}}$$