

Addis Ababa Science & Technology University
Department of Electrical & Electronics Engineering

Probability and Random Process (EEEG-2114)

Chapter 2: Random Variables

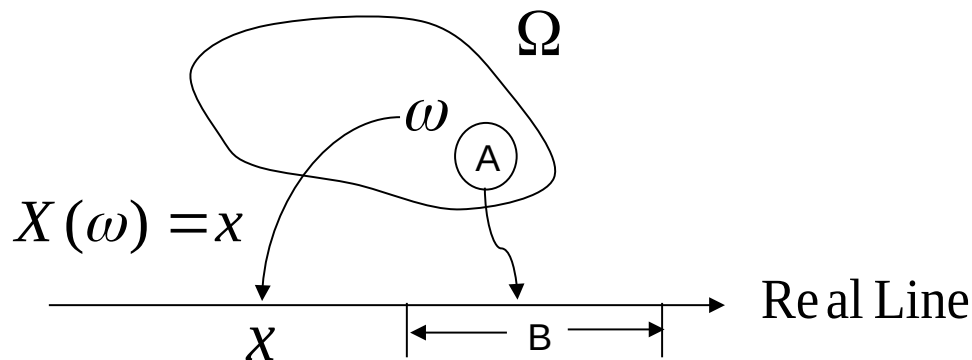
Random Variables

Outline

- Introduction
- The Cumulative Distribution Function
- Probability Density and Mass Functions
- Expected Value, Variance and Moments
- Some Special Distributions
- Functions of One Random Variable

Introduction

- A random variable X is a function that assigns a real number $X(\omega)$ to each outcome ω in the sample space Ω of a random experiment.
- The sample space Ω is the domain of the random variable and the set R_X of all values taken on by X is the range of the random variable.
- Thus, R_X is the subset of all real numbers.



Introduction Cont'd.....

- If X is a random variable, then $\{\omega: X(\omega) \leq x\} = \{X \leq x\}$ is an event for every X in R_X .

Example: Consider a random experiment of tossing a fair coin three times. The sequence of heads and tails is noted and the sample space Ω is given by:

$$\Omega = \{HHH, HHT, HTH, THH, THT, HTT, TTH, TTT\}$$

Let X be the number of heads in three coin tosses. X assigns each possible outcome ω in the sample space Ω a number from the set $R_X = \{0, 1, 2, 3\}$.

$\omega:$	<i>HHH</i>	<i>HHT</i>	<i>HTH</i>	<i>THH</i>	<i>THT</i>	<i>HTT</i>	<i>TTH</i>	<i>TTT</i>
$X(\omega):$	3	2	2	2	1	1	1	0

The Cumulative Distribution Function

- The cumulative distribution function (cdf) of a random variable X is defined as the probability of the event $\{X \leq x\}$.

$$F_X(x) = P(X \leq x)$$

Properties of the cdf, $F_X(x)$:

- The cdf has the following properties.
 - $F_X(x)$ is a non - negative function, i.e.,
$$0 \leq F_X(x) \leq 1$$
 - $\lim_{x \rightarrow \infty} F_X(x) = 1$
 - $\lim_{x \rightarrow -\infty} F_X(x) = 0$

The Cumulative Distribution Function Cont'd.....

iv. $F_X(x)$ is a non-decreasing function of X , i.e.,

If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$

v. $P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$

vi. $P(X > x) = 1 - F_X(x)$

Example:

Find the cdf of the random variable X which is defined as the number of heads in three tosses of a fair coin.

The Cumulative Distribution Function

Solution:

- ✓ We know that X takes on only the values 0, 1, 2 and 3 with probabilities $1/8$, $3/8$, $3/8$ and $1/8$ respectively.
- ✓ Thus, $F_X(x)$ is simply the sum of the probabilities of the outcomes from the set $\{0, 1, 2, 3\}$ that are less than or equal to x .

$$\therefore F_X(x) = \begin{cases} 0, & x < 0 \\ 1/8, & 0 \leq x < 1 \\ 1/2, & 1 \leq x < 2 \\ 7/8, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Types of Random Variables

- There are two basic types of random variables.

i. Continuous Random Variable

- ✓ A continuous random variable is defined as a random variable whose cdf, $F_X(x)$, is continuous every where and can be written as an integral of some non-negative function $f(x)$, i.e.,

$$F_X(x) = \int_{-\infty}^{\infty} f(u)du$$

ii. Discrete Random Variable

- ✓ A discrete random variable is defined as a random variable whose cdf, $F_X(x)$, is a right continuous, staircase function of X with jumps at a countable set of points $x_0, x_1, x_2, \dots$.

The Probability Density Function

- The probability density function (pdf) of a continuous random variable X is defined as the derivative of the cdf, $F_X(x)$, i.e.,

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Properties of the pdf, $f_X(x)$:

i. For all values of X , $f_X(x) \geq 0$

ii. $\int_{-\infty}^{\infty} f_X(x) dx = 1$

iii. $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x) dx$

The Probability Mass Function

- The probability mass function (pmf) of a discrete random variable X is defined as:

$$P_X(X = x_i) = P_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

Properties of the pmf, $P_X(x_i)$:

i. $0 \leq P_X(x_i) \leq 1, k = 1, 2, \dots$

ii. $P_X(x) = 0$, if $x \neq x_k, k = 1, 2, \dots$

iii. $\sum_k P_X(x_k) = 1$

Calculating the Cumulative Distribution Function

- The cdf of a continuous random variable X can be obtained by integrating the pdf, i.e.,

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$

- Similarly, the cdf of a discrete random variable X can be obtained by using the formula:

$$F_X(x) = \sum_{x_k \leq x} P_X(x_k) U(x - x_k)$$

Expected Value, Variance and Moments

i. Expected Value (Mean)

- The expected value (mean) of a continuous random variable X , denoted by μ_X or $E(X)$, is defined as:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} xf_X(x)dx$$

- Similarly, the expected value of a discrete random variable X is given by:

$$\mu_X = E(X) = \sum_k x_k P_X(x_k)$$

- Mean represents the average value of the random variable in a very large number of trials.

Expected Value, Variance and Moments Cont'd.....

ii. Variance

- The variance of a continuous random variable X , denoted by σ^2_X or $\text{VAR}(X)$, is defined as:

$$\sigma^2_X = \text{Var}(X) = E[(X - \mu_X)^2]$$

$$\Rightarrow \sigma^2_X = \text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

- Expanding $(x - \mu_X)^2$ in the above equation and simplifying the resulting equation, we will get:

$$\sigma^2_X = \text{Var}(X) = E(X^2) - [E(X)]^2$$

Expected Value, Variance and Moments Cont'd.....

- The variance of a discrete random variable X is given by:

$$\sigma^2_X = \text{Var}(X) = \sum_k (x_k - \mu_X)^2 P_X(x_k)$$

- The standard deviation of a random variable X , denoted by σ_X , is simply the square root of the variance, i.e.,

$$\sigma_X = \sqrt{E(X - \mu_X)^2} = \sqrt{\text{Var}(X)}$$

iii. Moments

- The n^{th} moment of a continuous random variable X is defined as:

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x) dx, \quad n \geq 1$$

Expected Value, Variance and Moments Cont'd.....

- Similarly, the n^{th} moment of a discrete random variable X is given by:

$$E(X^n) = \sum_k x_k^n P_X(x_k), \quad n \geq 1$$

- Mean of X is the first moment of the random variable X .

Some Special Distributions

i. Continuous Probability Distributions

1. Normal (Gaussian) Distribution

- The random variable X is said to be normal or Gaussian random variable if its pdf is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2}.$$

- The corresponding distribution function is given by:

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2 / 2\sigma^2} dy = G\left(\frac{x-\mu}{\sigma}\right)$$

$$\text{where } G(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Some Special Distributions Cont'd.....

- The normal or Gaussian distribution is the most common continuous probability distribution.

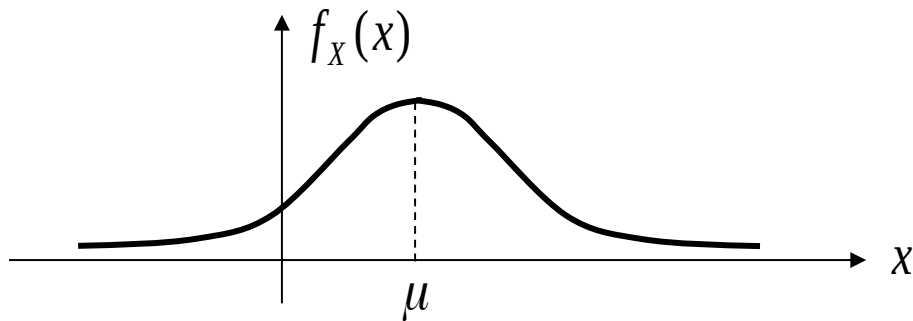


Fig. *Normal or Gaussian Distribution*

Some Special Distributions Cont'd.....

2. Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

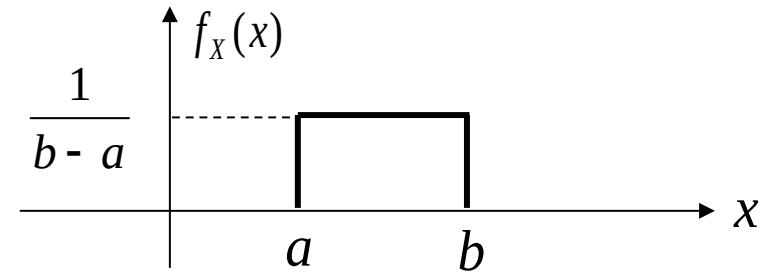


Fig. *Uniform Distribution*

$$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

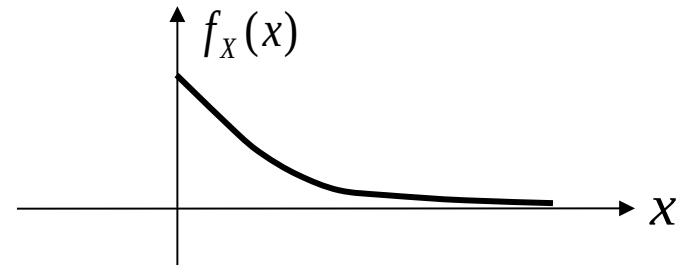


Fig. *Exponential Distribution*

Some Special Distributions Cont'd.....

4. Gamma Distribution

5. Beta Distribution

where

$$f_X(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1} (1-x)^{b-1}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta(a,b) = \int_0^1 u^{a-1} (1-u)^{b-1} du.$$

Some Special Distributions Cont'd.....

6. Rayleigh Distribution

7. Cauchy Distribution

8. Laplace Distribution

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \frac{\alpha / \pi}{\alpha^2 + (x - \mu)^2}, \quad -\infty < x < +\infty.$$

$$f_X(x) = \frac{1}{2\lambda} e^{-|x|/\lambda}, \quad -\infty < x < +\infty.$$

Some Special Distributions Cont'd....

i. Discrete Probability Distributions

1. Bernoulli Distribution

$$P(X = 0) = q, \quad P(X = 1) = p.$$

2. Binomial Distribution

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

3. Poisson Distribution

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots, \infty.$$

Some Special Distributions Cont'd....

4. Hypergeometric Distribution

$$P(X = k) = \frac{\binom{m}{k} \binom{N-m}{n-k}}{\binom{N}{n}}, \quad \max(0, m + n - N) \leq k \leq \min(m, n)$$

5. Geometric Distribution

$$P(X = k) = pq^k, \quad k = 0, 1, 2, \dots, \infty, \quad q = 1 - p.$$

6. Negative Binomial Distribution

$$P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \dots.$$

Random Variable Examples

Example-1:

The pdf of a continuous random variable is given by:

$$f_X(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- a. Determine the value of k .
- b. Find the corresponding cdf of X .
- c. Find $P(1/4 \leq X \leq 1)$
- d. Evaluate the mean and variance of X .

Random Variable Examples Cont'd.....

Solution:

$$\begin{aligned} a. \quad \int_{-\infty}^{\infty} f_X(x) dx &= 1 \Rightarrow \int_0^1 kx dx = 1 \\ &\Rightarrow k \left(\frac{x^2}{2} \right) \Big|_0^1 = 1 \\ &\Rightarrow \frac{k}{2} = 1 \\ &\therefore k = 2 \end{aligned}$$

$$\Rightarrow f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Random Variable Examples Cont'd.....

Solution:

b. The cdf of X is given by :

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$

Case 1: for $x < 0$

$$F_X(x) = 0, \text{ since } f_X(x) = 0, \text{ for } x < 0$$

Case 2: for $0 \leq x < 1$

$$F_X(x) = \int_{-\infty}^x f_X(u) du = \int_0^x 2u du = u^2 \Big|_0^x = x^2$$

Random Variable Examples Cont'd.....

Solution:

Case 3: for $x \geq 1$

$$F_X(x) = \int_0^1 f_X(u) du = \int_0^1 2u du = u^2 \Big|_0^1 = 1$$

$\therefore$ The cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Random Variable Examples Cont'd.....

Solution:

c. $P(1/4 \leq X \leq 1)$

i. Using the pdf

$$P(1/4 \leq X \leq 1) = \int_{1/4}^1 f_X(x) dx = \int_{1/4}^1 2x dx$$

$$\Rightarrow P(1/4 \leq X \leq 1) = x^2 \Big|_{1/4}^1 = 15/16$$

$$\therefore P(1/4 \leq X \leq 1) = 15/16$$

ii. Using the cdf

$$P(1/4 \leq X \leq 1) = F_X(1) - F_X(1/4)$$

$$\Rightarrow P(1/4 \leq X \leq 1) = 1 - (1/4)^2 = 15/16$$

$$\therefore P(1/4 \leq X \leq 1) = 15/16$$

Random Variable Examples Cont'd.....

Solution:

d. Mean and Variance

i. Mean

$$\mu_X = E(X) = \int_0^1 x f_X(x) dx = \int_0^1 2x^2 dx$$

$$\Rightarrow \mu_X = \left. \frac{2x^3}{3} \right|_0^1 = 2/3$$

ii. Variance

$$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^1 x^2 f_X(x) dx = \int_0^1 2x^3 dx = 1/2$$

$$\Rightarrow \sigma_X^2 = \text{Var}(x) = 1/2 - (2/3)^2 = 1/18$$

Random Variable Examples Cont'd.....

Example-2:

Consider a discrete random variable X whose pmf is given by:

$$P_X(x_k) = \begin{cases} 1/3, & x_k = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance of X .

Random Variable Examples Cont'd.....

Solution:

i. Mean

$$\mu_X = E(X) = \sum_{k=1}^1 x_k P_X(x_k) = 1/3(-1 + 0 + 1) = 0$$

ii. Variance

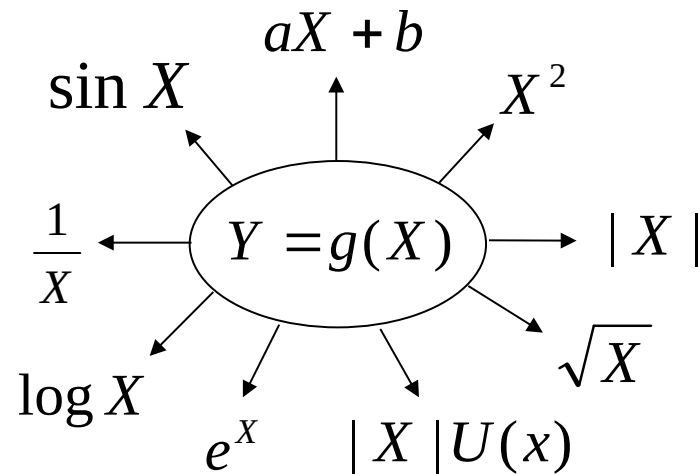
$$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{k=1}^1 x_k^2 P_X(x_k) = 1/3[(-1)^2 + (0)^2 + (1)^2] = 2/3$$

$$\Rightarrow \sigma_X^2 = \text{Var}(x) = 2/3 - (0)^2 = 2/3$$

Functions of One Random Variable

- Let X be a continuous random variable with pdf $f_X(x)$ and suppose $g(x)$ is a function of the random variable X defined as:
- We can determine the cdf and pdf of $Y = g(X)$ in terms of that of X .
- Consider some of the following functions.



Functions of a Random Variable Cont'd....

- Steps to determine $f_Y(y)$ from $f_X(x)$:

Method I:

1. Sketch the graph of $Y=g(X)$ and determine the range space of Y .
2. Determine the cdf of Y using the following basic approach.
3. Obtain $f_Y(y)$ from $F_Y(y)$ by using direct differentiation, i.e.,

$$F_Y(y) = P(g(X) \leq y) = P(Y \leq y)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Functions of a Random Variable Cont'd.....

Method II:

1. Sketch the graph of $Y=g(X)$ and determine the range space of Y .
2. If $Y=g(X)$ is one to one function and has an inverse transformation $x=g^{-1}(y)=h(y)$, then the pdf of Y is given by:
3. Obtain $Y=g(x)$ is not one-to-one function, then the pdf of Y can be obtained as follows.
 - i. Find the real roots of the function $Y=g(x)$ and denote them by x_i

$$f_Y(y) = \left| \frac{dx}{dy} \right| f_X(x) = \left| \frac{dh(y)}{dy} \right| f_X[h(y)]$$

Functions of a Random Variable Cont'd.....

ii. Determine the derivative of function $g(x_i)$ at every real root x_i , i.e. ,

iii. Find the pdf of Y by using the following formula.

$$g(x_i) = \frac{dx_i}{dy}$$

$$f_Y(y) = \sum_i \left| \frac{dx_i}{dy} \right| f_X(x_i) = \sum_i g(x_i) f_X(x_i)$$

Examples on Functions of One Random Variable

Examples:

a. Let $Y = aX + b$. Find $f_Y(y)$.

b. Let $Y = X^2$. Find $f_Y(y)$.

c. Let $Y = \frac{1}{X}$. Find $f_Y(y)$.

d. The random variable X is uniform in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

If $Y = \tan X$, determine the pdf of Y .

Examples on Functions of One Random Variable.....

Solutions:

a. $Y = aX + b$

i. Using Method - I

Suppose that $a > 0$

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y - b}{a}\right)$$

$$F_Y(y) = F_X\left(\frac{y - b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y - b}{a}\right) \quad (i)$$

Examples on Functions of One Random Variable.....

Solutions:

a. $Y = aX + b$

i. Using Method - I

On the other if $a < 0$, then

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P\left(X > \frac{y - b}{a}\right)$$

$$F_Y(y) = 1 - F_X\left(\frac{y - b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{1}{a} f_X\left(\frac{y - b}{a}\right) \quad (ii)$$

Examples on Functions of One Random Variable.....

Solutions:

a. $Y = aX + b$

i. Using Method - I

From equations (i) and (ii), we obtain :

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right), \quad \text{for all } a$$

Examples on Functions of One Random Variable.....

Solutions:

a. $Y = aX + b$

ii. Using Method - II

The function $Y = aX + b$ is one - to - one and the range space of Y is IR

For any y , $x = \frac{y - b}{a} = h(y)$ is the principal solution

$$\frac{dx}{dy} = \frac{dh(y)}{dy} = \frac{1}{a} \Rightarrow \left| \frac{dx}{dy} \right| = \frac{1}{|a|}$$

$$f_Y(y) = \left| \frac{dx}{dy} \right| f_X(x) = \left| \frac{dh(y)}{dy} \right| f_X(h(y)) \Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$$

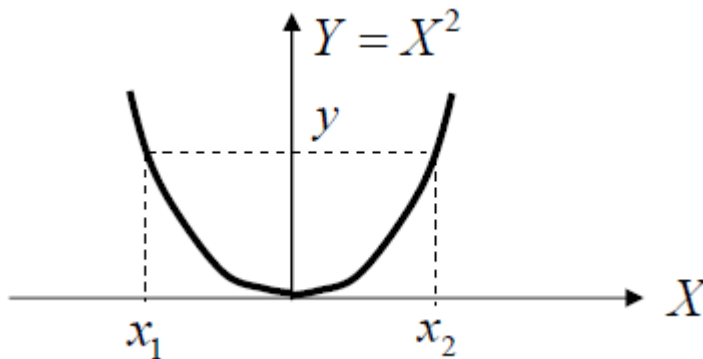
Examples on Functions of One Random Variable.....

Solutions:

b. The function $Y = X^2$ is not one - to - one and the range space of Y is $y > 0$

For each $y > 0$, there are two solutions given by

$$x_1 = -\sqrt{y} \text{ and } x_2 = \sqrt{y}$$



Examples on Functions of One Random Variable.....

Solutions:

$$b. \quad \frac{dx_1}{dy} = -\frac{1}{2\sqrt{y}} \Rightarrow \left| \frac{dx_1}{dy} \right| = \frac{1}{2\sqrt{y}} \quad \text{and}$$

$$\frac{dx_2}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow \left| \frac{dx_2}{dy} \right| = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \sum_i \left| \frac{dx_i}{dy} \right| f_X(x_i) \Rightarrow f_Y(y) = \left| \frac{dx_1}{dy} \right| f_X(x_1) + \left| \frac{dx_2}{dy} \right| f_X(x_2)$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})), & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Examples on Functions of One Random Variable.....

Solutions:

c. The function $Y = \frac{1}{X}$ is one - to - one and the range space of Y is $IR / \{0\}$

For any y , $x = \frac{1}{y} = h(y)$ is the principal solution

$$\frac{dx}{dy} = \frac{dh(y)}{dy} = - \frac{1}{y^2}$$

$$f_Y(y) = \left| \frac{dx}{dy} \right| f_X(x) = \left| \frac{dh(y)}{dy} \right| f_X(h(y)) \Rightarrow f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right)$$

$$\therefore f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right), \quad IR / \{0\}$$

Examples on Functions of One Random Variable.....

Solutions:

d. The function $Y = \tan X$ is one - to - one and the range space of Y is $(-\infty, \infty)$

For any y , $x = \tan^{-1} y = h(y)$ is the principal solution

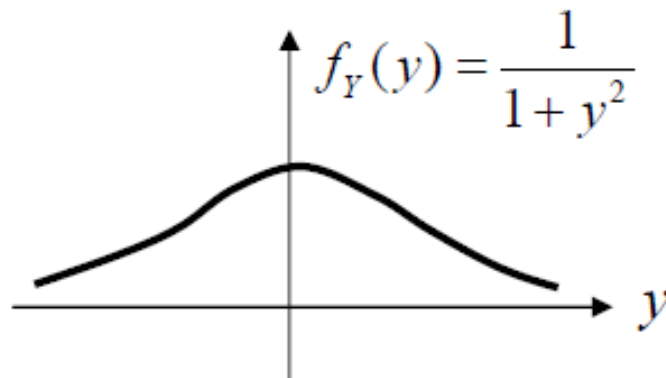
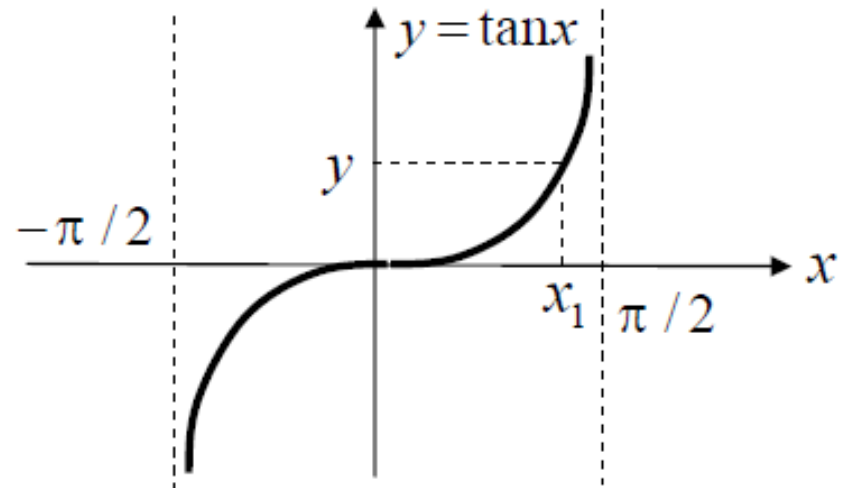
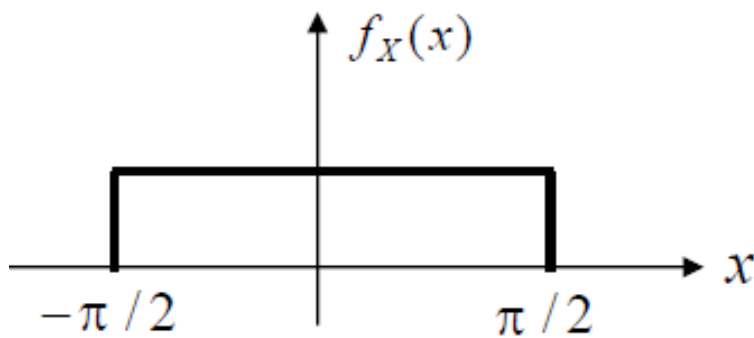
$$\frac{dx}{dy} = \frac{dh(y)}{dy} = \frac{1}{1+y^2}$$

$$f_Y(y) = \left| \frac{dx}{dy} \right| f_X(x) = \left| \frac{dh(y)}{dy} \right| f_X(h(y)) \Rightarrow f_Y(y) = \frac{1/\pi}{1+y^2}$$

$$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

Examples on Functions of One Random Variable.....

Solutions:



Assignment-II

1. The continuous random variable X has the pdf given by:

$$f_X(x) = \begin{cases} k(2x - x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

Find :

- a. the value of k .
- b. the cdf of X .
- c. $P(X > 1)$
- d. the mean and variance of X .

Assignment-II Cont'd.....

2. The cdf of continuous random variable X is given by:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ k\sqrt{x}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

where k is a constant.

Determine :

- a. the value of k .
- b. the pdf of X .
- c. the mean and variance of X .

Assignment-II Cont'd.....

3. The random variable X is uniform in the interval $[0, 1]$. Find the pdf of the random variable Y if $Y = -\ln X$.