

Estimation and Hypothesis Testing

Statistical Inference

- **Def**: The process of drawing conclusions about an entire population based on the data in a sample is known as **statistical inference.**
- Methods of inference usually fall into one of two broad categories: estimation or hypothesis testing

Sampling Distribution

- The sampling distribution of a statistic is defined as the distribution (a predictable pattern) of all possible values which can be taken by the statistic. It is formed by calculating the statistic for each large number of samples of the same size drown randomly.

- The frequency distribution of all these samples forms the sampling distribution of the sample statistic.

Cont...

- Three things about sampling distribution of a statistic
  - its mean
  - its variance
  - its shape
- Due to **random variation** different samples from the same population will have different sample means.
- If we repeatedly take sample of the same size n from a population the means of the samples form a sampling distribution of means of size n is equal to population mean.
- In practice we do not take repeated samples from a population, but it is necessary to know their properties in order to draw statistical inferences.

The Central Limit Theorem

- Regardless of the shape of the frequency distribution of a characteristic in the parent population, the means of a large number of samples (independent observations) from the population will follow a normal distribution (with the mean of means approaches the population mean μ, and standard deviation of σ/√n).

Sampling Distribution of the mean

- Suppose we choose a random sample of size n, the sampling distribution of the sample mean $\bar{x}$ posses the following properties.
  - The sample mean $\bar{x}$ will be an estimate of the population mean μ.
  - The standard deviation of $\bar{x}$ is $\sigma/\sqrt{n}$ (called the standard error of the mean).
  - Provided n is large enough the shape of the sampling distribution of $\bar{x}$ is normal.

Sampling Distribution of the proportion

- Suppose we choose a random sample of size n, the sampling distribution of the sample proportion p posses the following properties.
  - The sample proportion p will be an estimate of the population proportion P.
  - The standard deviation of p is = $\sqrt{p(1-p)/n}$ called the standard error of the proportion).
  - Provided n is large enough the shape of the sampling distribution of p is normal.

Standard deviation and Standard error

- **Standard deviation** is a measure of variability between individual observations (descriptive index relevant to mean)

- **Standard error** refers to the variability of summary statistics (e.g. the variability of the sample mean or the sample proportion)

- Standard error is a measure of uncertainty in a sample statistics i.e. precision of the estimate of the estimator

Estimation, Estimator & Estimate

- **Estimation is the computation of a statistic from sample data, yielding a value that is an approximation of its target, an unknown true population parameter value.**
- **The statistic itself is called an estimator and can be of two types - point or interval.**
- **estimates are the value or values that the estimator assumes are called.**

- Two methods of estimation are commonly used:
  - point estimation and interval estimation
- **Point estimation** involves the calculation of a **single number** to estimate the population parameter
- **Interval estimation** specifies **a range of reasonable values** for the parameter

Point versus Interval Estimators

- **An estimator that represents a "single best guess" is called a point estimator.**
- **When the estimate is of the form of a "range of plausible values", it is called an interval estimator.**
- **Thus,**
  - **A point estimate is of the form: [Value],**
  - **Whereas, an interval estimate is of the form:**

  [lower limit, upper limit]

Example -

The sample mean $\overline{X}_n$, calculated using data in a sample of size n, is a point estimator of the population mean μ. If $\overline{X}_n = 10$, the value 10 is called a point estimate of the population mean μ.

Point Estimates

Estimate Population Parameters ...		with Sample Statistics
Mean	μ	$\bar{X}$
Proportion	p	P_S
Variance	σ^2	S^2
Difference	$\mu_1 - \mu_2$	$\bar{X}_1 - \bar{X}_2$

Interval estimate

- The value of the sample statistic will vary from sample to sample
  - simply obtaining an estimate of the single value of the parameter is not generally acceptable.

  We need also a measure of how **precise** our estimate is likely to be
  - We need to take into account the sample to sample variation of the statistic

Estimating the Sampling Error

- Any estimates derived from samples are subject to the sampling error.
- This comes from the fact that only a part of the population was observed, instead of the whole.
- The **amount of variation** that exists among the estimates from the different possible samples is the sampling error.

- The set of sample means in repeated random samples of size *n* from a given populatiσ^2/nas variance .

- The standard deviation of this set of sample $\sigma/\sqrt{n}$ns is and is referred to as the *standard error* of the mean or the standard error.

- ***The standard error is dependent on sample size (n), the variability of individual sample points (σ), sampling and estimation methods.***
- As *n* increases, the sample mean ($\bar{x}$) and the sample variance s^2 **approach** the values of the true population parameters, μ and σ^2, respectively.

Confidence Intervals

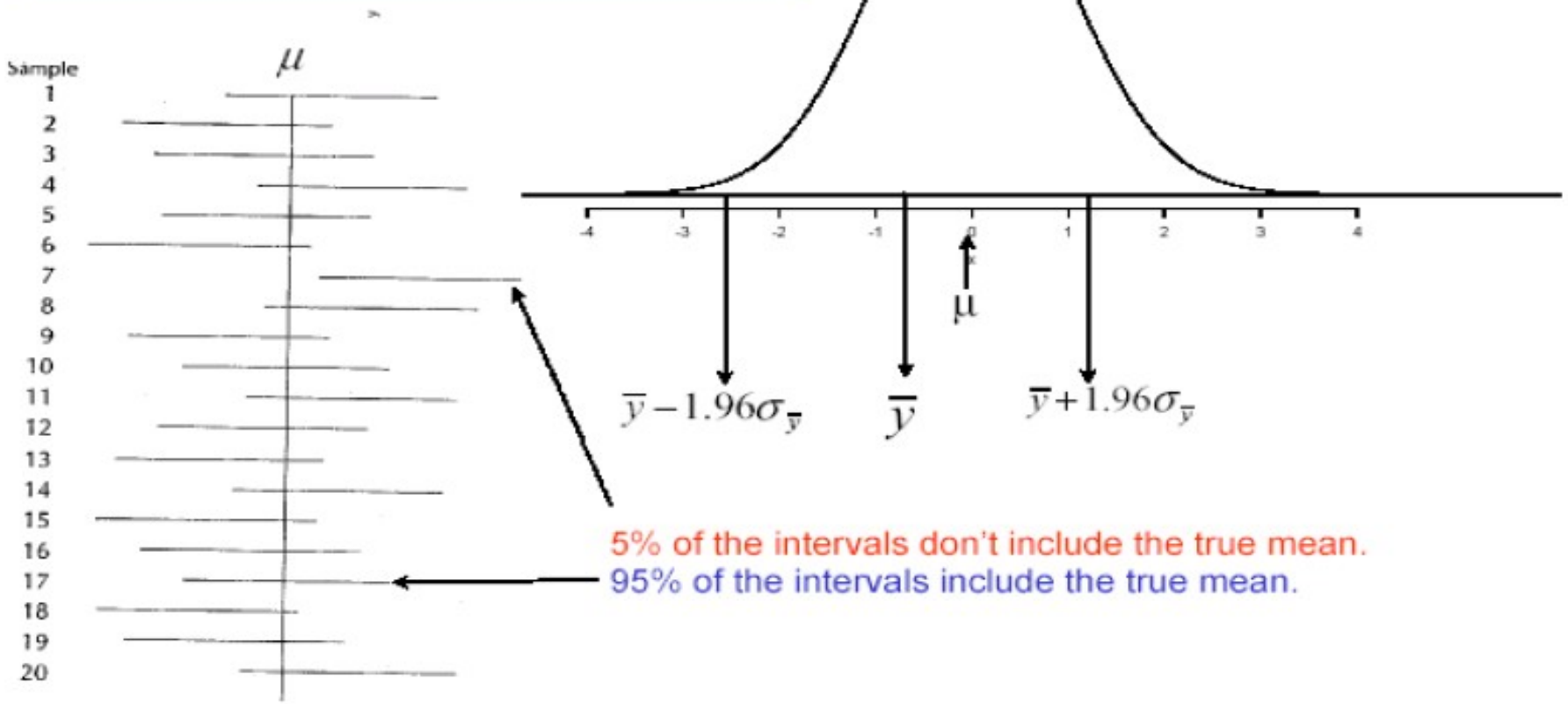
- is a particular type of interval estimator.
- Give a plausible range of values of the estimate likely to include the “true” (population) value with a given confidence level.
- An interval estimate provides more information about a population characteristic than does a point estimate

- CIs also give information about the precision of an estimate.
- When sampling variability is high, the CI will be wide to reflect the uncertainty of the observation.
- Wider CIs indicate less certainty.

- **A CI in general:**
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observation from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - Never 100% sure

Meaning of a 95% Confidence Interval

In repeated sampling, the interval would encompass the true parameter value 95% of the time.



5% of the intervals don't include the true mean.

95% of the intervals include the true mean.

General Formula:

The general formula for all CIs is:

The value of the statistic in my sample (eg., mean, odds ratio, etc.)

point estimate ± (measure of how confident we want to be) × (standard error)

Standard error of the statistic.

From a Z table or a T table, depending on the sampling distribution of the statistic.

Lower limit = Point Estimate - (Critical Value) x (Standard Error)

Upper limit = Point Estimate + (Critical Value) x (Standard Error)

- **A wide interval suggests imprecision of estimation.**
- **Narrow CI widths reflects large sample size or low variability or both.**

- ***Note: Measure of how confident we want to be = critical value = confidence coefficient***

Confidence interval ……

A (1-α)% confidence interval for unknown population mean and population proportion is given as follows;

$$\Rightarrow [\bar{x} - z_{\frac{\alpha}{2}} . \frac{s}{\sqrt{n}},\ \bar{x} + z_{\frac{\alpha}{2}} . \frac{s}{\sqrt{n}}] \qquad for estimating mean$$

$$\Rightarrow [p - z_{\frac{\alpha}{2}} . \sqrt{p(1-p)/n},\ \ p + z_{\frac{\alpha}{2}} . \sqrt{p(1-p)/n}] \quad for estimating proportic$$

Confidence intervals…

- For a given confidence level (i.e. 90%, 95%, 99%) the width of the confidence interval depends on
  - the standard error of the estimate which in turn depends on the

1. **Sample size:-**The larger the sample size, the narrower the confidence interval (this is to mean the sample statistic will approach the population parameter) and the more precise our estimate.

Confidence intervals…

- To increase precision, use a larger sample. You can make the precision as high as you want by taking a large enough sample. The margin of error decreases as √n increases.

2. **Standard deviation**:-The more the variation among the individual values, the wider the confidence interval and the less precise the estimate. As sample size increases SD decreases.

Confidence interval for single mean

- If the population standard deviation is known and the sample size is relatively large the confidence interval for sample mean is: $\bar{\mathbf{x}} \pm z(\sigma/\sqrt{n})$
  - $\bar{x}$ is the sample mean
  - σ is the population standard deviation
  - n is the sample size
  - Z is the value from SND
    - 90% CI, z=1.64
    - 95% CI, z=1.96
    - 99% CI, z=2.58

Confidence interval for single mean……

- If the population standard deviation is unknown and the sample size is small (<30), the formula for the confidence
- interval for sample mean is:

$x \pm t\,(s/\sqrt{n})$

  - x is the sample mean
  - s is the sample standard deviation
  - n is the sample size
  - t is the value from the t-distribution with (n-1) degrees of freedom

Confidence interval for single proportion

- For relatively large sample size the confidence interval for sample p is given by: $p \pm z \times(\sqrt{p(1-p)}/n)$
  - p is the sample proportion (r/n)
  - n is the sample size
  - Z is the value from SND

Proportion example

- In a survey of 300 automobile drivers in one city, 123 reported that they wear seat belts regularly. Estimate the seat belt rate of the city and 95% confidence interval for true population proportion.
- Answer :p= 123/300 =0.41=41%

n=300,

Estimate of the seat belt of the city at 95%

$$CI = p \pm z \times(\sqrt{p(1-p)/n}) = (0.35, 0.47)$$

Hypothesis testing

- **Hypothesis testing is:**
  - **One way of statistical inference**
- Is a claim (assumption) about a population parameter.

- The purpose of HT is to assist the clinician, researcher or administrator in reaching a decision (conclusion) concerning a population by examining a sample from that population.

Types of Hypothesis

1. The Null Hypothesis, H0

- Is a statement stating that there is no difference between the hypothesized value and the population value.
  - (The effect of interest is zero = no difference)
- States the assumption (hypothesis) to be tested

- **H_0 is a statement of agreement (or no difference)**
- **H_0 is always about a population parameter, not about a sample statistic**

- Begin with the assumption that the Ho is true
  - Always contains “=” , “ ≤” or “≥ ” sign
  - May or may not be rejected

2. **The Alternative Hypothesis, H_A**

- Is a statement of what we will believe is true if our sample data causes us to reject Ho.
- Is generally the hypothesis that is believed (or needs to be supported) by the researcher.

- Is a statement that disagrees (opposes) with Ho

  (The effect of interest is not zero)
- Never contains “=” , “ ≤” or “≥ ” sign
- May or may not be accepted

Steps in Hypothesis Testing

1. **Formulate the appropriate statistical hypotheses clearly**
   - **Specify Ho and HA**

     $H_0: \mu = \mu_0$ $H_0: \mu \leq \mu_0$ $H_0: \mu \geq \mu_0$

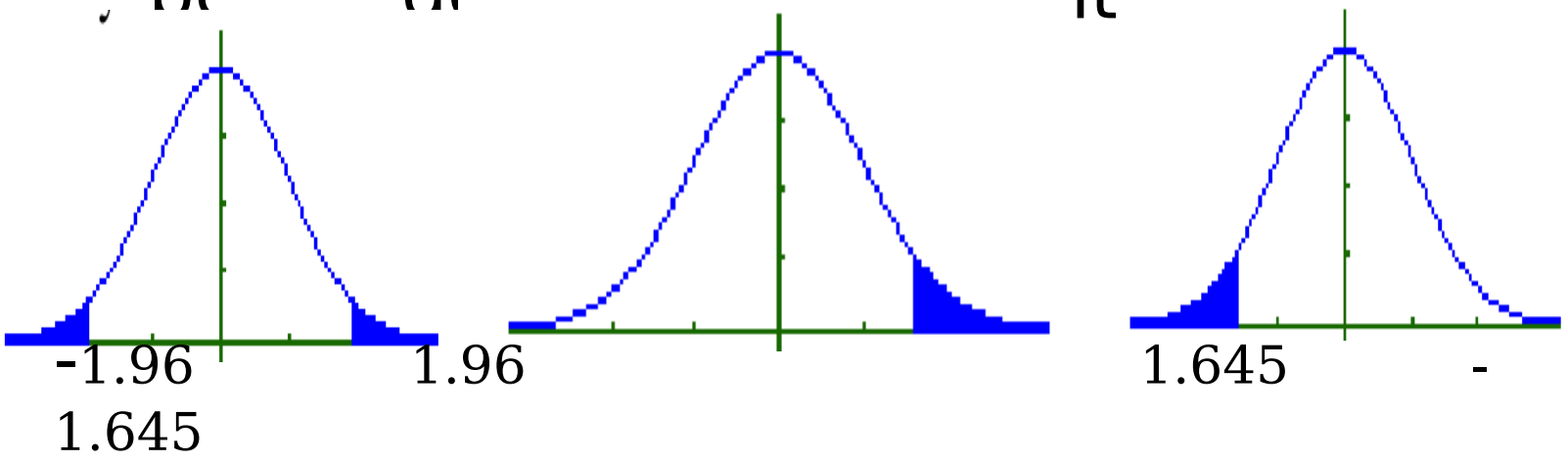
     $H_1: \mu \neq \mu_0$ $H_1: \mu > \mu_0$ $H_1: \mu < \mu_0$

     two-tailed **one-tailed** **one-tailed**
2. **State the assumptions necessary for computing probabilities**
   - **A distribution is approximately normal (Gaussian)**
   - **Variance is known or unknown**

3. Select a sample and collect data
   - Categorical, continuous
4. Decide on the appropriate test statistic for the hypothesis.
5. E.g., One population

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

5. Specify the desired level of significance for the statistical test (α=0.05, 0.01, etc.)
6. Determine the critical value.
   - A value the test statistic must attain to be declared significant



7. Obtain sample evidence and compute the test statistic
8. Decide on the result and draw the conclusion
   - ***If Ho is rejected, we conclude that H_A is true (or accepted).***
   - ***If Ho is not rejected, we conclude that Ho may be true.***

Example

- Researchers are interested in the mean age of a certain population.
- A random sample of 10 individuals drawn from the population of interest has a mean age of 27 yrs.
- Assuming that the population is approximately normally distributed with variance 20,
  - can we conclude that the mean age of the population is different from 30 years? ($\alpha=0.05$) .

Solution

1-Data: variable is age, n=10, $\bar{x}$ =27 ,σ^2=20,α=0.05

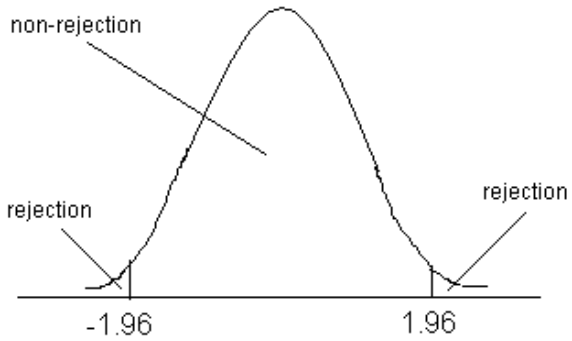
2-Assumptions: the population is approximately normally distributed with variance 20

3-Hypotheses:

- **H_0 : μ=30**
- **H_A: $\mu \neq 30$**

4-Test Statistic:

- **Z** = -2.12



5.Decision Rule

- if $Z > Z_{1-0.05/2}$ or $Z < -Z_{1-0.025/2}$=we reject H0
- $Z_{0.975}$=1.96 (from table D)

6.Decision:

- **We reject H_0, since -2.12 is in the rejection region .**

- We can conclude that μ is not equal to 30

- **Types of Errors in Hypothesis Tests**

- Whenever we reject or accept the Ho, we commit errors.
- Two types of errors are committed.
  - **Type I Error**
  - **Type II Error**

Type I Error

- The error committed when a true Ho is rejected
- Considered a serious type of error
- The probability of a type I error is the probability of rejecting the Ho when it is true
- The probability of type I error is α
- Called level of significance of the test
- Set by researcher in advance

Type II Error

- The error committed when a false Ho is not rejected
- The probability of Type II Error is β
- Usually unknown but larger than **α**

Power

- The probability of rejecting the Ho when it is false.

  Power = 1 – β = 1- probability of type II error

- **We would like to maintain low probability of a Type I error (α) and low probability of a Type II error (β) [high power = 1 - β].**

Type I and Type II Error in a box

Your Statistical Decision	True state of null hypothesis	
	H_0 True (example: the drug doesn't work)	H_0 False (example: the drug works)
Reject H_0 (ex: you conclude that the drug works)	*Type I error (α)*	*Correct*
Do not reject H_0 (ex: you conclude that there is insufficient evidence that the drug works)	*Correct*	*Type II Error (β)*

Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability (α) ⬆, then

Type II error probability (β) ⬇

An Alternative Decision Rule using the p - value Definition

- The p-value is defined as the smallest value of α for which the null hypothesis can be rejected.
- If the p-value is less than or equal to α, we reject the null hypothesis ($p \leq \alpha$)
- If the p-value is greater than α, we do not reject the null hypothesis ($p > \alpha$)

The P- Value…..

- By convention, a p-value of 0.05 or smaller is considered sufficient evidence for rejecting the null hypothesis.
- By using p-value of 0.05, we are allowing a 5% chance of wrongly rejecting the null hypothesis when it is in fact true.

- When the p-value is less than to 0.05, we often say that the result is *statistically significant.*

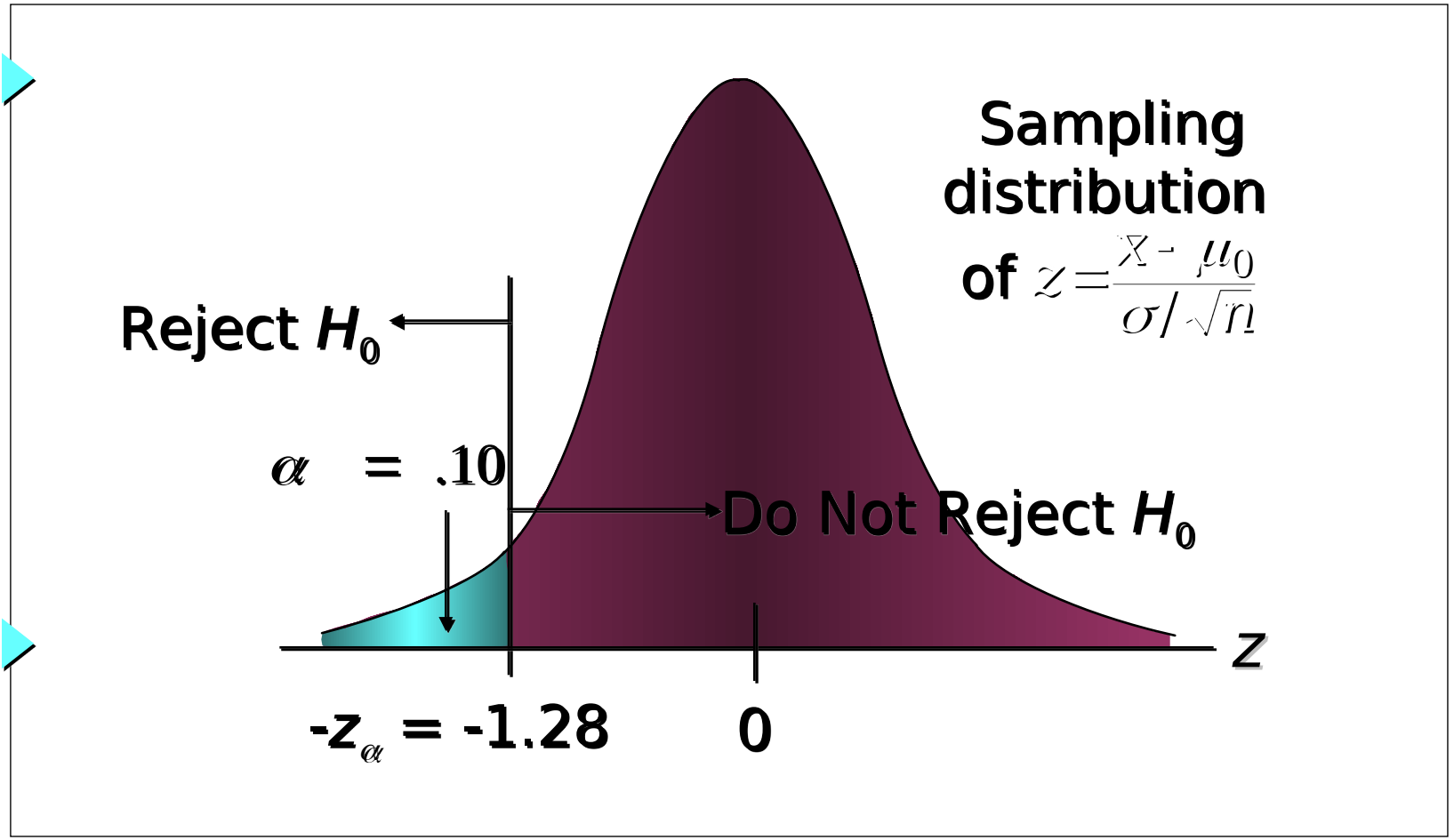
P-value and confidence interval

- Confidence intervals and p-values are based upon the same theory and will lead to the same conclusion about whether a population difference exists.
- Confidence intervals are preferable because they give information about the size of any difference in the population, and they also (very usefully) indicate the amount of uncertainty remaining about the size of the difference.

Lower-Tailed Test About a Population Mean:

σ Known

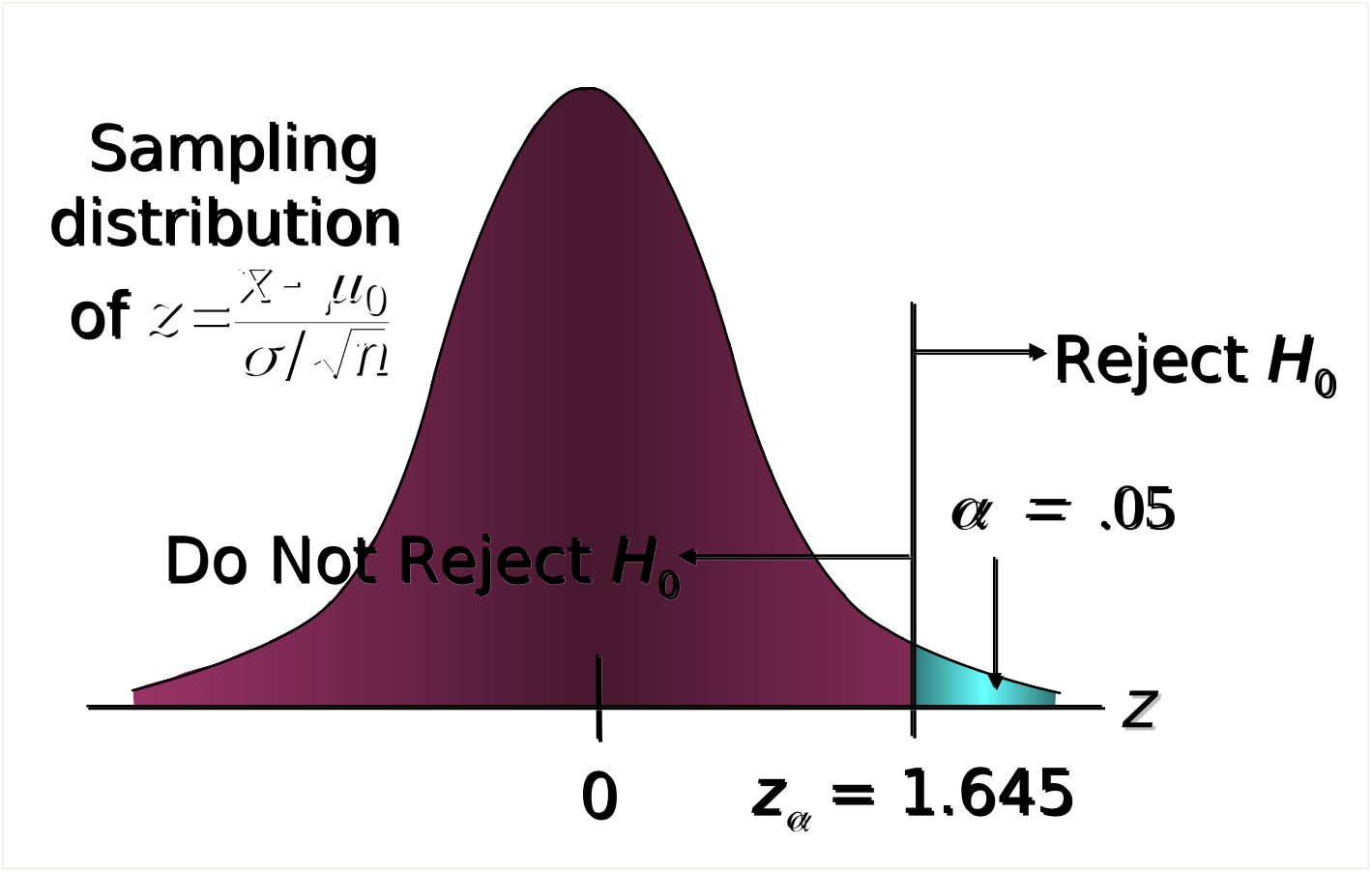
- Critical Value Approach



Upper-Tailed Test About a Population Mean:

σ Known

- Critical Value Approach



EXAMPLE

- A researcher claims that the mean of the IQ for 16 students is 110 and the expected value for all population is 100 with standard deviation of 10. Test the hypothesis if the average IQ for the student is different from 100 at $\alpha=0.05$.

Solution

1. Ho:$\mu=100$

   VS

- HA:$\mu\neq100$

2. Assume $\alpha=0.05$
3. $z=(110-100)4/10=4$

   z-critical at 0.025 is equal to 1.96.
4. $4\geq1.96$
5. Conclusion reject the null hypothesis

Hypothesis testing for proportions

Example

- In the study of childhood abuse in psychiatry patients, you found that 166 in a sample of 947 patients reported histories of physical or sexual abuse.

a. constructs 95% confidence interval

b. test the hypothesis that the true population proportion is 30%?

- Solution (a)
- The 95% CI for P is given by

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{p(1-p)}{n}}$$

$$\Rightarrow 0.175 \pm 1.96 \times \sqrt{\frac{0.175 \times 0.825}{947}}$$

$$\Rightarrow 0.175 \pm 1.96 \times 0.0124$$

$$\Rightarrow [0.151; 0.2]$$

Example…

- To test the hypothesis we need to follow the steps

Step 1: State the hypothesis

Ho: P=Po=0.3

Ha: P≠Po ≠0.3

Step 2: Fix the level of significant (α=0.05)

Step 3: Compute the calculated and tabulated value of the test statistic

$$z_{cal} = \frac{\hat{p} - Po}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.175 - 0.3}{\sqrt{\frac{0.3(0.7)}{947}}} = \frac{-0.125}{0.0149} = -8.39$$

$$z_{tab} = \pm 1.96$$

Example….

- Step 4: Comparison of the calculated and tabulated values of the test statistic
- Since the tabulated value is smaller than the calculated value of the test the we reject the null hypothesis.
- Step 6: Conclusion
- Hence we concluded that the proportion of childhood abuse in psychiatry patients is different from 0.3

Inference Process

