5 KILLO
JAMMA'AH
One deen, one ummah

STATISTICS AND PROBABILITY FOR ENGINEERS(Stat 2171)

Worksheet Solutions

Addis Ababa University

Probability and Statistics for Engineers (Stat2171)

Work Sheet / Assignment II

3/ If X be the random variable with pmf given by:

X	-2	-1	0	1	2
P(x)	1/5	1/6	1/5	1/15	11/30

Then find the pmf of $Y = X^2$.

Soln

- ✓ $X = -2$, $Y = 4$
- ✓ $X = -1$, $Y = 1$
- ✓ $X = 0$, $Y = 0$
- ✓ $X = 1$, $Y = 1$
- ✓ $X = 2$, $Y = 4$

$P(Y=4) = P(X=-2) + P(X=2) = 1/5 + 11/30 = 17/30$

$P(Y=1) = P(X=-1) + P(X=1) = 1/6 + 1/15 = 7/30$

$P(Y=0) = P(X=0) = 1/5$

Then the probability mass function of Y is:

Y	0	1	4
P(Y)	1/5	7/30	17/30

4/ If **X** be the random variable with pdf given by

$$f(x) = \frac{3}{2}x^2 \; ; \; -1 \le x \le 1$$

Then find the pdf of $Y = 3 - X$ and $Y = X^2$

Soln

For $Y = 3 - X = H(x) \Rightarrow X = 3 - Y = H^{-1}(Y)$

let $g(y)$ be the pdf of Y, then

$$g(y) = f(H^{-1}(y)) \times \left| \frac{d}{dy} H^{-1}(y) \right|$$

$$= \frac{3}{2}(3-y)^2 \times \left| \frac{d}{dy}(3-y) \right|$$

$$= \frac{3}{2}(3-y)^2 \times |-1|$$

$$= \frac{3}{2}(3-y)^2$$

$$g(y) = \begin{cases} \frac{3}{2}(3-y)^2, & \text{if } 2 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$R_Y = \{y ; -1 \le 3-y \le 1\}$

$R_Y = \{y ; 2 \le y \le 4\}$

For $Y = X^2 = H(X) \Rightarrow X = \pm\sqrt{Y} = H^{-1}(Y)$

let $g(y)$ be the PDF of Y, then

$$g(y) = f(H^{-1}(y)) \times \left|\frac{d}{dy} H^{-1}(y)\right|$$

$$= \frac{3}{2}(\sqrt{y})^2 \times \left|\frac{d}{dy}(\sqrt{y})\right| + \frac{3}{2}(-\sqrt{y})^2 \times \left|\frac{d}{dy}(-\sqrt{y})\right|$$

$$= \frac{3}{2}\left(y \times \left|\frac{1}{2\sqrt{y}}\right| + y \times \left|\frac{-1}{2\sqrt{y}}\right|\right)$$

$$= \frac{3}{2}\left(\frac{y}{2\sqrt{y}} + \frac{y}{2\sqrt{y}}\right)$$

$$= \frac{3}{2}\left(\frac{y}{\sqrt{y}}\right)$$

$$= \frac{3}{2}\sqrt{y}$$

$R_Y = \{y : (-1)^2 \le \pm\sqrt{y} \le 1\}$

$= \{y : 0 \le y \le 1\}$

$$\therefore g(y) = \begin{cases} \frac{3}{2}\sqrt{y}, & 0 \le y \le 1 \\ 0, & \text{Otherwise} \end{cases}$$

5/ The joint probability mass function $P(x_i, y_j)$ is given by:

Y \ X	0	1	2
0	0.12	0.42	0.06
1	0.21	0.06	0.07
2	0.07	0.02	0.01

a) $P(X \le 1, Y = 2)$

$$P_Y(y_j) = \sum_i P_{XY}(x_i, y_j)$$

$$= P(X=1, Y=2) + P(X=0, Y=2)$$

$$= 0.02 + 0.07$$

$$= \underline{0.09}$$

b) $P(X = 1)$

$$P_X(x_i) = \sum_j P_{XY}(x_i, y_j)$$

$$= 0.42 + 0.06 + 0.02$$

$$= \underline{0.5}$$

c) $P(Y = 2)$

$$P_Y(y_j) = \sum_i P_{XY}(x_i, y_j)$$

$$= 0.07 + 0.02 + 0.01$$

$$= \underline{0.1}$$

d) $P(Y \ge 1)$

$$= P(Y=1) + P(Y=2)$$

$$= \sum_i P_{XY}(x_i, y_j) + \sum_i P_{XY}(x_i, y_j)$$

$$= 0.3 + 0.1$$

$$= \underline{0.4}$$

e) $P(X \geq 1, Y \geq 1)$

$\Rightarrow P(X=1, Y=1) + P(X=2, Y=1) + P(X=1, Y=2) + P(X=2, Y=2)$

$\Rightarrow 0.06 + 0.03 + 0.02 + 0.01$

$= \underline{0.12}$

7/ Suppose that the joint pdf of the two-dimensional random variable (X,Y) is given by

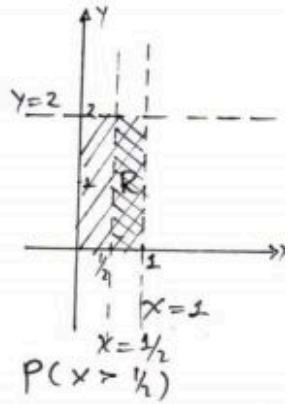
$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, \ 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

a) $P(X > 1/2)$

$$= \int_0^2 \int_{1/2}^1 \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)\Big|_{1/2}^1 dy = \int_0^2 \left(\frac{7}{24} + \frac{3y}{24}\right) dy$$

$$= \left(\frac{7}{24}y + \frac{3y^2}{48}\right)\Big|_0^2 = \left(\frac{7}{12} + \frac{3}{12}\right) = \underline{5/6}$$



$P(X > 1/2)$

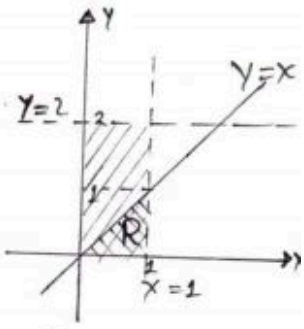
b) $P(Y < X) \Rightarrow 0 < Y < X < 1$

$$P(Y < X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{6}\right)\Big|_0^x dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx = \left(\frac{x^4}{4} + \frac{x^4}{24}\right)\Big|_0^1$$

$$= \left(\frac{1}{4} + \frac{1}{24}\right) = \underline{7/24}$$



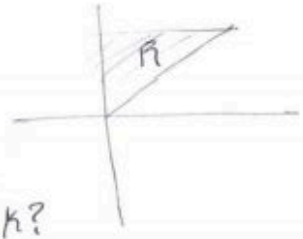
$P(Y < X)$

c) $P(Y < 1/2 \mid X < 1/2) = \dfrac{P(Y < 1/2, X < 1/2)}{P(X < 1/2)}$

$$\Rightarrow \frac{\int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dy\, dx}{\int_0^{1/2}\int_0^2 \left(x^2 + \frac{xy}{3}\right) dy\, dx} \Rightarrow \frac{\int_0^{1/2} \left(x^2 y + \frac{xy^2}{6}\right)\Big|_0^{1/2} dx}{\int_0^{1/2} \left(x^2 y + \frac{xy^2}{6}\right)\Big|_0^2 dx}$$

$$\Rightarrow \frac{\int_0^{1/2} \left(\frac{x^2}{2} + \frac{x}{24}\right) dx}{\int_0^{1/2} \left(2x^2 + \frac{4x}{6}\right) dx} \Rightarrow \frac{\left(\frac{x^3}{6} + \frac{x^2}{48}\right)\Big|_0^{1/2}}{\left(\frac{2x^3}{3} + \frac{2x^2}{6}\right)\Big|_0^{1/2}} = \frac{\frac{1}{48} + \frac{1}{192}}{\frac{1}{12} + \frac{1}{12}} = \underline{0.156}$$

11. Pdf $\Rightarrow f(x,y) = \begin{cases} Kxy^2 & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$



a) find K?

$$\int_0^1 \int_x^1 Kxy^2 \, dy\, dx = 1 \Rightarrow \int_0^1 \frac{Ky^3 x}{3} \, dx \Big|_x^1 = \int_0^1 \frac{Kx}{3} - \frac{Kx^4}{3}$$

$$\Rightarrow \frac{Kx^2}{6} - \frac{Kx^5}{15} \Big|_0^1 = \frac{K}{6} - \frac{K}{15} = \frac{3K}{30}$$

$$\frac{3K}{30} = 1 \Rightarrow K = \frac{30}{3} = 10$$

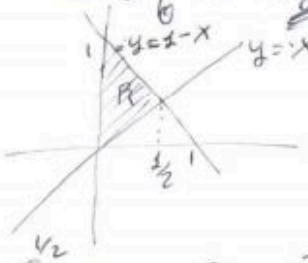
$\therefore f(x,y) = \begin{cases} 10xy^2 & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

b) Find marginal of x and y.

$$f_x = \int_x^1 10xy^2 \, dy = \frac{10xy^3}{3} \Big|_x^1 = \frac{10}{3}x - \frac{10x^4}{3}$$

$$f_y = \int_0^y 10xy^2 \, dx = 5x^2y^2 \Big|_0^y = 5y^4$$

c) $P(x+y<1)$



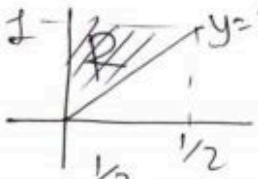
$$\int_0^{1/2} \int_x^{1-x} 10xy^2 \, dy\, dx$$

$$= \int_0^{1/2} \frac{10xy^3}{3} \Big|_x^{1-x} dx = \int_0^{1/2} \frac{10}{3}\left[x(1-x)^3 - x^4\right] = \frac{10}{3}\int_0^{1/2} (-2x^4 + 3x^3 - 2x^2 + x)\, dx \Rightarrow$$

$$\Rightarrow \frac{10}{3}\left[-\frac{2x^5}{5} + \frac{3}{4}x^4 - \frac{2x^3}{3} + \frac{x^2}{2}\right]\Big|_0^1 = \frac{10}{3}\left[-\frac{2}{5} + \frac{3}{4} - \frac{2}{3} + \frac{1}{2}\right]$$

$$= \frac{10}{3}\left[\frac{11}{60}\right] = \frac{11}{18}$$

(ii) $P(2x<y)$



$2x \le y \le 1$

$0 \le x \le 1/2$

$$\int_0^{1/2}\int_{2x}^{1} 10xy^2\,dy\,dx = \int_0^{1/2} \frac{10xy^3}{3}\Big|_{2x}^{1} dx = \frac{10}{3}\int_0^{1/2} x - 8x^4\,dx$$

$$= \frac{10}{3}\left(\frac{x^2}{2} - \frac{8}{5}x^5\right)\Big|_0^{1/2} = \underline{\underline{1/4}}$$

(iii) $P(X=Y)$

$$\int_0^1\int_x^x f(x,y)\,dy\,dx = \int_0^1\int_x^x 10xy^2\,dy\,dx = \int_0^1 \frac{10xy^3}{3}\Big|_x^x = \int_0^1 0\,dx$$

$$= \int_0^1 0\,dx = 1\Big|_0^1 dx = 1-1 = 0$$

$\therefore P(X=Y) = 0$

12

x \ y	1	2	3	P(x)
1	3/12	4/12	2/12	9/12
2	1/12	2/12	0	3/12
P(y)	4/12	6/12	2/12	

i. conditional probability mass function of X given the value of Y=1

$$P(x|y) = \frac{P(x,y)}{P(y)} \neq 1$$

$$P(X=1|Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{3/12}{4/12} = \frac{3}{4}$$

$$P(X=2|Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/12}{4/12} = \underline{\underline{1/4}}$$

$\therefore f(x|y=1) = \begin{cases} 3/4, & \text{if } x=1 \\ 1/4 & \text{if } x=2 \end{cases}$

ii) Y given the value of $x=2$

$P(y=1|x=2) = \frac{P(y=1, x=2)}{P(x=2)} = \frac{1/12}{3/12} = 1/3$ //

$P(y=2|x=2) = \frac{P(y=2, x=2)}{P(x=2)} = \frac{2/12}{3/12} = 2/3$ //

$P(y=3|x=2) = \frac{P(y=3, x=2)}{P(x=2)} = \frac{0}{3/12} = 0$ //

$f(y|x=2) = \begin{cases} 1/3 & \text{if } y=1 \\ 2/3 & \text{if } y=2 \\ 0 & \text{if } y=3 \end{cases}$ //

13. $f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y) & \text{if } 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$ //

a) Conditional pdf of X given Y=y

$f_{x,y}(x|y) = \frac{f(x,y)}{fy}$

but $fy = \int_0^1 \frac{12}{5} x(2-x-y)\,dx = \frac{12}{5}\left(\frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2}\right)\Big|_0^1$

$= \frac{12}{5}\left(\frac{2}{3} - \frac{y}{2}\right)$ //

$f_{x|y}(x|Y=y) = \frac{\frac{12}{5} x(2-x-y)}{\frac{12}{5}(2/3 - y/2)} = \frac{2x - x^2 - xy}{2/3 - y/2} \times \frac{6}{6}$ //

$= \frac{6x(2-x-y)}{4-3y}$ //

Q33. Given

$\mu = 62.4$, $X = 72.9$

Requested - $\sigma = ?$

$p\left(Z > \frac{X-\mu}{\sigma}\right)$

$p\left(Z > \frac{72.9 - 62.4}{\sigma}\right)$

Let A be the area from 0 to 72.9

then 20.05% means 0.2005 is the area above 72.9

then $0.5 - A = 0.2005$

$A = 0.5 - 0.2005$

$A = 0.2995$

From the table the area 0.2995 represents 0.84

Since $\frac{72.9 - 62.4}{\sigma} = 0.84 \Rightarrow \frac{10.5}{\sigma} = 0.84$

$\frac{0.84\sigma}{0.84} = \frac{10.5}{0.84} \Rightarrow \sigma = 12.5$

∴ the Standard deviation is 12.5

Q35 Suppose X is a discrete random variable with possible value 0, 1, 2, ... and such that its probability mass function is defined as

$$p(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the moment generating function of a random variable X.

$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$, when $x=0$ $\sum_{x=0}^{\infty} e^0 \cdot \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$

$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{(e^t\lambda)^x}{x!}$

$= e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t - 1)}$

(b) Find the average and the variance of a random variable X

Average = Mean = $E(X) = \sum x_i p(x_i)$ ..... Discrete Case

$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$ when $x=0$ $\frac{x e^{-\lambda}\lambda^x}{x!} = 0$

$x=1$ $\frac{x e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\lambda$

$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{(x+1)}}{(x-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda e^{-\lambda+\lambda} = \lambda$ = Average

OR by taking $M'_{(x)}(0)$, $E(X)$ = mean = average $= \lambda e^0 e^{\lambda(e^0 - 1)} = \lambda$

TO find Variance taking the Second order derivative of $M(X)^{(t)}$ at 0

$E(X^2) = M''_X(t) = (\lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}$

$E(X^2) = M''(X)(0) = \lambda e^0 e^{\lambda(e^0-1)} + \lambda^2 e^{2(0)} e^{\lambda(e^0-1)}$

$= \lambda + \lambda^2$ //

$Var(X) = E(X^2) - (E(X))^2 = \lambda + \lambda^2 - (\lambda)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$ //

(c) If $Y = 3X+5$ (i) Find moment generating fun^n of Y.

for the form of $Y = aX+b$ we use Moment generating fun^n as

$M_Y(t) = M_X(at) \cdot e^{bt}$ .. then $M_Y(t) = e^{5t} \cdot M_X(3t)$

$= e^{\lambda(e^{3t}-1)} \cdot e^{5t}$

$= e^{\lambda e^{3t}} \cdot e^{-\lambda} \cdot e^{5t}$

$= e^{5t + \lambda e^{3t} + (-\lambda)} = \underline{e^{5t + \lambda(e^{3t}-1)}}$

(ii) Find the mean and Variance of Y.

Mean $= E(Y) = E(3X+5) = 5 + 3E(X)$ ..... from properties of $E(X)$

since $E(X) = \lambda$, $5 + 3(\lambda)$

$= \underline{5 + 3\lambda} =$ Mean

$Var(Y) = Var(3X+5) = 3^2 Var(X)$ ... Since adding a Constant on Values has no effect on the Variance.

$= 3^2 \lambda$

$= \underline{9\lambda}$

(d) Let a random Variable X be the number of phone Calls received at a Real estate form per hour with hypothesized mean of 6. Suppose that the Secretary takes a 30 minute lunch break. What is the probability that no Calls will be received in this time?

Given Mean $= \lambda = 6$ ... i/e means 6 Calls / 60 min

When it is 30 minutes $\mu = \lambda/2 = 6/2 = 3$,,

Required: $p(X=0) = ?$ from poisson distribution

$p(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$p(X=0) = \frac{e^{-3} \lambda^0}{0!}$ .... $\lambda = 3$ .... because the time has reduced by half

$= \frac{e^{-3}}{1} = e^{-3} = \frac{1}{e^3} = \underline{0.04978}$

Q37. Suppose X is a continuous random variable with probability density fun defined as $p(X=x) = \begin{cases} \lambda e^{-\lambda x} & \lambda > 0 \text{ & } 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$

(a) Find moment generating fun of the random variable X

$$M_{(x)}^{(t)} = E(e^{tx}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx = \int_0^\infty \lambda e^{tx-\lambda x}\, dx$$

$$= \int_0^\infty \lambda e^{x(t-\lambda)}\, dx = \frac{\lambda}{t-\lambda} e^{x(t-\lambda)} \Big|_0^\infty = 0 - \left(\frac{\lambda e^0}{t-\lambda}\right) = \frac{-\lambda}{t-\lambda}$$

$$= \frac{\lambda}{\lambda - t} = M_{(x)}^{(t)}$$

(B) If $P(X \le 1) = P(X > 1)$, find Variance of X

$$P(X \le 1) = \int_0^1 \lambda e^{-\lambda x}\, dx = \lambda \int_0^1 e^{-\lambda x} = \lambda \left(\frac{-e^{-\lambda x}}{\lambda}\Big|_0^1\right) = -e^{-\lambda} + 1$$

$$P(X > 1) = \int_1^\infty \lambda e^{-\lambda x}\, dx = \lambda \int_1^\infty e^{-\lambda x}\, dx = \lambda \left(\frac{-e^{-\lambda x}}{\lambda}\Big|_1^\infty\right) = 0 + e^{-\lambda}$$

$$P(X \le 1) = P(X > 1) \Rightarrow e^{-\lambda} + 1 = e^{-\lambda} = 2e^{-\lambda} = 1, \ln(2e^{-\lambda}) = \ln \tfrac{1}{2}$$

$$= -\lambda \ln e = \ln \tfrac{1}{2} \quad : \lambda = -\ln(\tfrac{1}{2}) = \ln(2) = \underline{0.693}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X) = \int_0^\infty X\, 0.693 e^{-0.693X}\, dx = 0.693 \int_0^\infty X e^{-0.693X}\, dx$$

let $u = x$, $dv = e^{-0.693x}$

$du = dx$, $v = \frac{-e^{-0.693x}}{0.693}$

$$\Rightarrow -\frac{Xe^{-0.693X}}{0.693} + \int \frac{e^{-0.693X}}{0.693} = \left(\frac{X 0.693^{-0.693X}}{0.693} - \frac{e^{-0.693X}}{(0.693)^2}\Big|_0^\infty\right) \times 0.693$$

$$= 0.693\left(0 - \left(\frac{-e^0}{(0.693)^2}\right)\right) = 0.693\left(\frac{1}{(0.693)^2}\right) = \frac{1}{0.693} = \frac{1}{\ln 2}$$

$$E(X^2) = \int_0^\infty x^2\, 0.693 e^{-0.693X}\, dx = 0.693 \int_0^\infty x^2 e^{-0.693X}\, dx$$

let $u = x^2$, $dv = e^{-0.693x}$

$du = 2x\,dx$, $v = \frac{-e^{-0.693x}}{0.693}$

$$\left(\frac{-x^2 e^{-0.693X}}{0.693}\right) + \frac{2}{0.693}\int \left(Xe^{-0.693X}\right) \times 0.693$$

$$= 0.693\left(\frac{-x^2 e^{-0.693X}}{0.693} - \frac{2Xe^{-0.693X}}{(0.693)^2} - \frac{2e^{-0.693X}}{(0.693)^3}\Big|_0^\infty\right)$$

$= 0.693\left(0 - \left(\frac{-2e^0}{(0.693)^3}\right)\right) = 0.693\left(\frac{2}{(0.693)^3}\right) = \ln 2\left(\frac{2}{(\ln 2)^3}\right) =$

$= \frac{2}{(\ln 2)^2} = \frac{2}{(0.693)^2} = \underline{\underline{\frac{2}{(0.693)^2}}}$

$V(X) = E(X^2) - ((E(X))^2)$

$= \frac{2}{(0.693)^2} - \left(\frac{1}{(0.693)^2}\right) = \frac{2-1}{(0.693)^2} = \frac{1}{(0.693)^2}$

$= \underline{\underline{2.08}}$

(c) Find the rth moment of the random variable X about the origin.

$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)\,dx$, but $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $0 < x < \infty$

$= \int_0^\infty x^r \cdot \lambda e^{-\lambda x}\,dx$

$= \lambda \int_0^\infty x^{r+1-1} \cdot e^{-\lambda x}\,dx$

$= \frac{\lambda \sqrt{r+1}}{\lambda^{r+1}}$

$= \frac{\lambda \sqrt{r+1}}{\lambda \cdot \lambda^r}$

$= \frac{\sqrt{r+1}}{\lambda^r}$

$\therefore \mu_r' = E(X^r)$

$= \frac{\sqrt{r+1}}{\lambda^r}$

b. pdf of $x|y=1/2$

$$f_{x|y}(x|y=1/2) = \frac{6x\left(2-x-\frac{1}{2}\right)}{4-3\left(\frac{1}{2}\right)} = \frac{18}{5}x - \frac{12}{5}x^2$$

$$f_{x|y=1/2} \begin{cases} \frac{18}{5}x - \frac{12}{5}x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

c. $P(x > 1/3 \mid y = 1/2)$

recall $f_{x|y}(x|y=1/2) = \frac{18}{5}x - \frac{12}{5}x^2$

$$P\left(x > \frac{1}{3} \mid y = \frac{1}{2}\right) = \int_{1/3}^{1} \frac{18}{5}x - \frac{12}{5}x^2 \, dx$$

$$= \frac{18}{10}x^2 - \frac{12}{15}x^3 \Big|_{1/3}^{1} = \frac{112}{135} = \underline{0.829}$$

31. (a) given that average of 3.2 customers every 4 minutes, Thus $\lambda = 3.2$

$$P(x=2) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3.2}(3.2)^2}{2!} = \underline{0.104}$$

(b) 4 min = 3.2 cus
8 min = ? $\Rightarrow$ 8 min = 6.4 cus
$\therefore \lambda = 6.4$

$$P(x=2) = \frac{e^{-6.4}(6.4)^2}{2!} = \underline{0.0169}$$

(c) $\lambda = \frac{12 \times 3.2}{4} = 9.6$

$P(x \ge 1) = 1 - P(x < 1)$ but $P(x<1) = P(x=0)$

$= 1 - 6.77 \times 10^{-5}$ $= \frac{e^{-9.6}(9.6)^0}{0!}$

$P(x \ge 1) = \underline{0.99993}$ $= 6.77 \times 10^{-5}$.

38. Suppose that the joint Pdf the two dimensional random Variable (x,y) is given by:

$$f_{(x,y)} = \begin{cases} x^2 + xy/3 & 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

a) Marginal Pdf of X

$$f_x = \int_0^2 x^2 + xy/3\, dy = x^2 y + xy^2/6 \Big|_0^2 = 2x^2 + 2/3 x$$

Pdf of $f_x = \begin{cases} 2x^2 + 2/3 x & \text{if } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$

b) Marginal Pdf of Y

$$f_y = \int_0^1 x^2 + xy/3\, dx = x^3/3 + x^2 y/6 \Big|_0^1 = 1/3 + y/6$$

Pdf of $f_y = \begin{cases} 1/3 + y/6 & \text{if } 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$

c) Find conditional Pdf of Y given X.

$$f_{(y/x)} = \frac{f(x,y)}{f_x} = \frac{x^2 + xy/3}{2x^2 + 2/3 x} = \frac{3x+y}{6x+2}$$

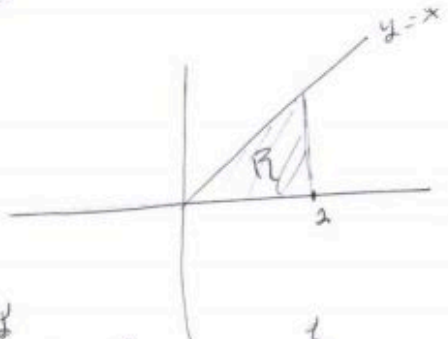
Pdf of $f_{(y/x)} = \begin{cases} \frac{3x+y}{6x+2}, & 0<x<1,\ 0<y<2 \\ 0, & \text{elsewhere.} \end{cases}$

(d) $P(X>1/2) = \int_{1/2}^{2} f_X dx = \int_{1/2}^{2} 2x^2 + \frac{2}{3}x\,dx = \int_{1/2}^{2} 2x^2 + \frac{2}{3}x\,dx$

$= \frac{2x^3}{3} + \frac{2x^2}{6}\Big|_{1/2}^{1} = \frac{2}{3} + \frac{1}{3} - \left(\frac{2}{24} + \frac{1}{12}\right)$

$P(X>1/2) = 1 - 1/6 = 5/6$

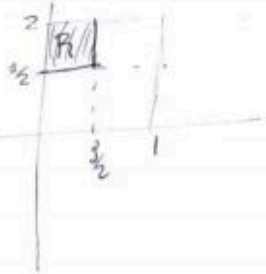
(e) $P(Y<X)$?



$P(Y<X) = \int_0^1 \int_0^x x^2 + \frac{xy}{3}\,dy\,dx$

$= \int_0^1 \left(xy + \frac{xy^2}{6}\right)\Big|_0^x dx = \int_0^1 x^2 + \frac{x^3}{6}\,dx = \int_0^1 \frac{7x^3}{6}\,dx \Rightarrow$

$\Rightarrow \int_0^1 \frac{7x^4}{6}\,dx = \frac{7x^4}{24}\Big|_0^1 = 7/24$



(f) $P(Y>1/2 / X<1/2) = \frac{P(X<1/2,\ Y>1/2)}{P(X<1/2)}$

$P(X<1/2, Y>1/2) = \int_0^{1/2}\int_{1/2}^{2} x^2 + \frac{xy}{3}\,dy\,dx \Rightarrow$

$= \int_0^{1/2} x^2 y + \frac{xy^2}{6}\Big|_{1/2}^{2} dx = \int_0^{1/2} \frac{3x^2}{2} + \frac{15x}{24}\,dx$

$= \frac{x^3}{2} + \frac{15}{48}x^2\Big|_0^{1/2} = \frac{1}{16} + \frac{15}{48(4)} = \frac{9}{64}$

$$P(x<\tfrac{1}{2}) = \int_0^{1/2} f_x\,dx = \int_0^{1/2} 2x^2 + \tfrac{2}{3}x\,dx$$

$$= \tfrac{2}{3}x^3 + \frac{x^2}{3}\Big|_0^{1/2} = \tfrac{2}{3}\left(\tfrac{1}{2}\right)^3 + \frac{\left(\tfrac{1}{2}\right)^2}{3} = \underline{\tfrac{1}{6}}$$

$\therefore$ ~~$P(x<\tfrac{1}{2}, \ldots$~~

$$\therefore P(y>\tfrac{1}{2} / x<\tfrac{1}{2}) = \frac{9/64}{1/6} = \frac{27}{32} = \underline{0.843}$$

(2) Find the mean and Variance of X.

Sol/

$$\text{mean} = \mathcal{E}(x) = \int_{R_x} x f_x(x) = \int_0^1 x(2x^2 + \tfrac{2}{3}x)\,dx$$

$$= \int_0^1 2x^3 + \tfrac{2}{3}x^2\,dx = \left(\frac{x^4}{2} + \frac{2x^3}{9}\right)_0^1 = \tfrac{1}{2} + \tfrac{2}{9} = \tfrac{13}{18}$$

$\therefore$ The mean of $X = \underline{0.722}$

- Variance of X

$Var(x) = \mathcal{E}(x^2) - (\mathcal{E}(x))^2$, but $\mathcal{E}(x^2)$ is given by

$$\mathcal{E}(x^2) = \int_{R_x} x^2 f_x(x)\,dx = \int_0^1 x^2(2x^2 + \tfrac{2}{3}x)\,dx = \int_0^1 2x^4 + \tfrac{2}{3}x^3\,dx$$

$$= \left(\frac{2x^5}{5} + \frac{x^4}{6}\right)\Big|_0^1 = \tfrac{2}{5} + \tfrac{1}{6} = \tfrac{17}{30}$$

Now $Var(x) = \mathcal{E}(x^2) - (\mathcal{E}(x))^2 = \tfrac{17}{30} - \left(\tfrac{13}{18}\right)^2 = \underline{0.045}$

$\therefore$ The mean of $X = \underline{0.722}$

The Variance of $X = \underline{0.045}$

h) Find Conditional expectation of X given Y (i.e., $E(X/Y)$)

$$E(X/Y) = \int_{-\infty}^{\infty} x\, f(x/y)\, dx \quad \text{first let us find } f(x/y)$$

$$\Rightarrow f(x/y) = \frac{f(x,y)}{f_y} = \frac{3x^2+xy}{\frac{3}{6+3y}} = \frac{6x^2+2xy}{2+y}$$

$$\therefore E(x/y) = \int_0^1 x\left(\frac{6x^2+2xy}{2+y}\right)dx = \int_0^1 \frac{6x^3+2x^2y}{2+y}\, dx$$

$$\frac{1}{2+y}\left(\frac{6x^4}{4} + \frac{2x^3}{3}y\right)\Big|_0^1 = \frac{1}{2+y}\left(\frac{3}{2} + \frac{2}{3}y\right) = \frac{1}{2+y}\left(\frac{9+4y}{6}\right)$$

$$E(X/y) = \frac{9+4y}{12+6y}$$

i) Find the correlation coefficient of X and Y.

$$COV(x,y) = \frac{Cov(x,y)}{Sod(x).\, Sod(y)}$$

- First find Cov(x,y)

$$\Rightarrow Cov(x,y) = E(xy) - E(x)E(y)$$

$$E(xy) = \int_0^2 \int_0^1 xy\left(x^2 + \frac{xy}{3}\right)dx\,dy = \int_0^2 \int_0^1 x^3y + \frac{x^2y^2}{3}\, dx\,dy$$

$$= \int_0^2 \frac{x^4y}{4} + \frac{x^3y^2}{9}\Big|_0^1 dy = \int_0^2 \frac{y}{4} + \frac{y^2}{9}\, dy = \left(\frac{y^2}{8} + \frac{y^3}{27}\right)\Big|_0^2$$

$$E(xy) = \left(\frac{4}{8} + \frac{8}{27}\right) - 0 = \frac{43}{54}$$

From question 'g' we found $E(X) = \frac{13}{18}$

$$\Rightarrow E(y) = \int_0^2 y(f_y)\,dy = \int_0^2 y\left(\tfrac{1}{3} + \tfrac{y}{6}\right)dy = \int_0^2 \tfrac{y}{3} + \tfrac{y^2}{6}\,dy$$

$$= \left(\tfrac{y^2}{6} + \tfrac{y^3}{18}\right)\Big|_0^2 = \tfrac{4}{6} + \tfrac{8}{18} = \tfrac{10}{9}$$

So, $Cov(x,y) = E(xy) - E(x)E(y)$

$$= \tfrac{43}{54} - \tfrac{13}{18} \times \tfrac{10}{9} = -\tfrac{1}{162}$$

then Standard deviation of x = ?

$$S.d(x) = \sqrt{var(x)} = \sqrt{E(x^2) - (E(x))^2} = \sqrt{0.045} = 0.212$$

Standard deviation of y = ?

$$S.d(y) = \sqrt{Var(y)}, \quad Var(y) = E(y^2) - (E(y))^2$$

$$\Rightarrow E(y^2) = \int_0^2 y^2\left(\tfrac{1}{3} + \tfrac{y}{6}\right)dy = \int_0^2 \tfrac{y^2}{3} + \tfrac{y^3}{6}\,dy$$

$$= \tfrac{y^3}{9} + \tfrac{y^4}{24}\Big|_0^2 = \tfrac{8}{9} + \tfrac{16}{24} = \tfrac{14}{9}$$

$$Var(y) = E(y^2) - (E(y))^2 = \tfrac{14}{9} - \left(\tfrac{10}{9}\right)^2 = \tfrac{14}{9} - \tfrac{100}{81} = \tfrac{26}{81}$$

$Var(y) = 0.321$

$S.d(y) = \sqrt{0.321} = 0.567$

Finally $Cor(x,y) = \dfrac{Cov(x,y)}{S.d(x) \cdot S.d(y)} = -\dfrac{\frac{1}{162}}{0.212 \times 0.567} = -0.0514$

So, $Cor(x,y) = -0.0514$ It indicates that 'x' and 'y' are inversely Related.

j. Are X and Y independent?

NO, b/c their correlation is different from Zero.

Or. To be independent

$$f_{(x,y)}^{(x,y)} = \left(f_x(x)\right)\cdot\left(f_y(y)\right)$$

$$\text{But, } x^2 + \frac{xy}{3} \neq \left(2x^2 + \frac{2}{3}x\right)\left(\frac{1}{3} + \frac{y}{6}\right)$$

$$x^2 + \frac{xy}{3} \neq \frac{2}{3}x^2 + \frac{yx^2}{3} + \frac{2}{9}x + \frac{yx}{9}$$

5 KILLO JAMMA'AH

One deen, one ummah