ECE3103: Probability and Random Processes : WORKSHEET

Problem 1.

Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx+1 & x,y \geq 0, x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the constant C
2. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$
3. Find $P(Y < 2X^2)$

Answer:

1.

To find the constant c

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1 \int_0^{1-x} cx+1 \; dydx \\ &= \int_0^1 (cx+1)(1-x) \; dx \\ &= \frac{1}{2} + \frac{1}{6}c. \end{aligned}$$

Thus, we conclude $c = 3$

2.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{1-x} 3x+1 \; dy \\ &= (3x+1)(1-x), \quad \text{for } x \in [0,1]. \end{aligned}$$

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$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

$$= \int_{0}^{1-y} 3x+1 \ dx$$

$$= \frac{1}{2}(1-y)(5-3y), \quad \text{for } y \in [0,1].$$

$$f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(Y < 2X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f_{XY}(x,y)dydx$$

$$= \int_{0}^{1} \int_{0}^{\min(2x^2,1-x)} 3x+1 \ dydx$$

$$= \int_{0}^{1} (3x+1)\min(2x^2, 1-x) \ dx$$

$$= \int_{0}^{\frac{1}{2}} 2x^2(3x+1) \ dx + \int_{\frac{1}{2}}^{1} (3x+1)(1-x) \ dx$$

$$= \frac{53}{96}.$$

Problem 2.

Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x,y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

1. Are X and Y independent?
2. Find $E[Y|X>2]$

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3. Find $P(X > Y)$

1. We can write

$$f_X(x) = 2e^{-2x}$$

and

$$f_Y(y) = 3e^{-3y}$$

Thus, X and Y are independent

2. Since X and Y are independent, we have

$$E[Y|X > 2] = E[Y]$$
$$EY = \frac{1}{3}$$

3. We have

$$P(X > Y) = \int_0^\infty \int_y^\infty 6e^{-(2x+3y)}\, dxdy$$
$$= \int_0^\infty 3e^{-5y}\, dy$$
$$= \frac{3}{5}.$$

Problem 3

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le x \le 1, 0 \le y \le \sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

1.

Find $f_X(x)$ and $f_Y(y)$

2.

Are X and Y independent?

3.

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Find the conditional PDF of X given $Y = y$ $f_{X|Y}(x|y)$

4.

Find $E[X|Y = y]$

5.

Find $Var(X|Y = y)$

Q.N. 1

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy$$
$$= \int_{0}^{\sqrt{x}} 6xy\ dy$$
$$= 3x^2.$$

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx$$
$$= \int_{y^2}^{1} 6xy\ dx$$
$$= 3y(1-y^4).$$

$$f_Y(y) = \begin{cases} 3y(1-y^4) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2.

X and Y are not independent, since $f_{XY}(x,y) \neq f_x(x)f_Y(y)$

3.

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$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{1-y^4} & y^2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

4.

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_{y^2}^{1} x \frac{2x}{1-y^4}\, dx = \frac{2(1-y^6)}{3(1-y^4)}.$$

5.

$$E[X^2|Y = y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\, dx = \int_{y^2}^{1} x^2 \frac{2x}{1-y^4}\, dx = \frac{1-y^8}{2(1-y^4)}.$$

$$Var(X|Y = y) = E[X^2|Y = y] - (E[X|Y = y])^2 = \frac{1-y^8}{2(1-y^4)} - \left(\frac{2(1-y^6)}{3(1-y^4)}\right)^2.$$

Problem: 4

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} 2 & y + x \le 1, x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $Cov(X, Y)$

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- Solution
  - For $0 \le x \le 1$ $0 \le x \le 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$
$$= \int_{0}^{1-x} 2dy$$
$$= 2(1-x).$$

Thus,

$$f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$EX = \int_{0}^{1} 2x(1-x)dx$$
$$= \frac{1}{3} = EY,$$

$$EX^2 = \int_{0}^{1} 2x^2(1-x)dx$$
$$= \frac{1}{6} = EY^2.$$

Thus,

$$Var(X) = Var(Y) = \frac{1}{18}.$$

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$$EXY = \int_0^1 \int_0^{1-x} 2xydydx$$

$$= \int_0^1 x(1-x)^2 dx$$

$$= \frac{1}{12}.$$

$$Cov(X,Y) = EXY - EXEY$$

$$= \frac{1}{12} - (\frac{1}{3})^2$$

$$= -\frac{1}{36},$$

Problem The random variable X has a range of $\{0,1,2\}$ and the random variable Y has a range of $\{1,2\}$. The joint distribution of X and Y is given by the following table:

x	y	$P(X=x, Y=y)$
0	1	0.2
0	2	0.1
1	1	0.0
1	2	0.2
2	1	0.3
2	2	0.2

1. Write down tables for the marginal distributions of X and of Y, i.e. give the values of $P(X=x)$ for all x, and of $P(Y=y)$ for all y.

x	$P(X=x)$
0	0.3
1	0.2
2	0.5

y	$P(Y=y)$
1	0.5
2	0.5

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2. Write down a table for the conditional distribution of X given that $Y = 2$, i.e. give the values of $P(X = x \,|\, Y = 2)$ for all x.

x	$P(X = x \,\|\, Y = 2)$
0	0.2
1	0.4
2	0.4

3. Compute $E(X)$ and $E(Y)$.

$$E(X) = 0 \times 0.3 + 1 \times 0.2 + 2 \times 0.5 = 1.2$$
$$E(Y) = 1 \times 0.5 + 2 \times 0.5 = 1.5$$

4. Compute $E(XY)$.

$$E(XY) = (0\times1)\times0.2 + (0\times2)\times0.1 + (1\times1)\times0.0 + (1\times2)\times0.2 + (2\times1)\times0.3 + (2\times2)\times0.2$$
$$= 1.8$$

5. Are X and Y independent? Explain why or why not.

No, they are not independent, as can be seen from the fact that $P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$. *(The left side is 0, but the right side is* $0.2 \times 0.5 = 0.1$.*)*

Note that the fact that $E(XY) = E(X)E(Y)$ *does NOT imply that* X *and* Y *are independent.* $E(XY) = E(X)E(Y)$ *is implied by* X *and* Y *being independent, but not the other way around.*

Problem

You roll one red die and one green die. Define the random variables X and Y as follows:

X = The number showing on the red die

Y = The number of dice that show the number six

For example, if the red and green dice show the numbers 6 and 4, then $X = 6$ and $Y = 1$.

Write down a table showing the joint probability mass function for X and Y, find the marginal distribution for Y, and compute $E(Y)$.

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Here is a table showing the joint probability mass function, with the marginal distribution for Y on the right:

		$X=$ 1	2	3	4	5	6	
	0	5/36	5/36	5/36	5/36	5/36	0	25/36
$Y=$	1	1/36	1/36	1/36	1/36	1/36	5/36	10/36
	2	0	0	0	0	0	1/36	1/36

The expectation of Y is

$$E(Y) = 0 \times 25/36 + 1 \times 10/36 + 2 \times 1/36 = 12/36 = 1/3$$

Question 3. Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

X = The number shown on the red die

$$Y = \begin{cases} 0 & \text{if the two dice show the same number} \\ 1 & \text{if the number on the green die is bigger than the number on the red die} \\ 2 & \text{if the number on the red die is bigger than the number on the green die} \end{cases}$$

a) Write down a table showing the joint probability mass function for X and Y.

		$X=$ 1	2	3	4	5	6
	0	1/36	1/36	1/36	1/36	1/36	1/36
$Y=$	1	5/36	4/36	3/36	2/36	1/36	0
	2	0	1/36	2/36	3/36	4/36	5/36

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b) Find the marginal probability mass function for Y, and compute its expected value.

y	$P(Y=y)$
0	6/36
1	15/36
2	15/36

$$E(Y) = 0\times(6/36)+1\times(15/36)+2\times(15/36) = 45/36$$

c) Find the conditional probability mass function for X given $Y=1$.

$X =$

1	2	3	4	5	6
5/15	4/15	3/15	2/15	1/15	0

Correlation Coefficient:

$$\rho_{XY} = \mathrm{Cov}(U,V) = \mathrm{Cov}\left(\frac{X-EX}{\sigma_X}, \frac{Y-EY}{\sigma_Y}\right)$$
$$= \mathrm{Cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$
$$= \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

Problem: 5

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

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We know that given $X = x$ the random variable Y is uniformly distributed on $[-x, x]$

1. Find the joint PDF $f_{XY}(x, y)$
2. Find $f_Y(y)$
3. Find $P(|Y| < X^3)$

1. First note that, by the assumption

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2x} & -x \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$f_{XY}(x, y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} 1 & 0 \le x \le 1, -x \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$f_{XY}(x, y) = \begin{cases} 1 & |y| \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2. First, note that $R_Y = [-1, 1]$. To find $f_Y(y)$ we can write

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y)dx \\ &= \int_{|y|}^{1} 1dx \\ &= 1 - |y|. \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} 1 - |y| & |y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

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3. To find $P(|Y| < X^3)$, we can use the law of total probability

$$P(|Y| < X^3) = \int_0^1 P(|Y| < X^3 | X = x) f_X(x) dx$$

$$= \int_0^1 P(|Y| < x^3 | X = x) 2x dx$$

$$= \int_0^1 \left(\frac{2x^3}{2x}\right) 2x dx \qquad \text{since } Y|X = x \sim \text{Uniform}(-x, x)$$

$$= \frac{1}{2}.$$

Problem: 6

Let the probability density function of X be given by

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 \; otherwise \end{cases}$$

Now find the probability density of $Y = X^3$.

Let G(y) denote the value of the distribution function of Y at y and write

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(X^3 \leq y) \\ &= P\left(X \leq y^{1/3}\right) \\ &= \int_0^{y^{1/3}} 6x(1-x)\,dx \\ &= \int_0^{y^{1/3}} (6x - 6x^2)\,dx \\ &= (3x^2 - 2x^3)\,|_0^{y^{1/3}} \\ &= 3y^{2/3} - 2y \end{aligned}$$

Now differentiate G(y) to obtain the density function g(y)

$$\begin{aligned} g(y) &= \frac{dG(y)}{dy} \\ &= \frac{d}{dy}\left(3y^{2/3} - 2y\right) \\ &= 2y^{-1/3} - 2 \\ &= 2(y^{-1/3} - 1), \; 0 < y < 1 \end{aligned}$$

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Problem:7

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} 2 & y+x \le 1, x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Cov}(X,Y)$ and $\rho(X,Y)$

- Solution
  - For $0 \le x \le 1$ $0 \le x \le 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{1-x} 2dy \\ &= 2(1-x). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$\begin{aligned} EX &= \int_0^1 2x(1-x)dx \\ &= \frac{1}{3} = EY, \end{aligned}$$

$$\begin{aligned} EX^2 &= \int_0^1 2x^2(1-x)dx \\ &= \frac{1}{6} = EY^2. \end{aligned}$$

Thus,

$$Var(X) = Var(Y) = \frac{1}{18}.$$

$$\begin{aligned} EXY &= \int_0^1 \int_0^{1-x} 2xy\,dy\,dx \\ &= \int_0^1 x(1-x)^2\,dx \\ &= \frac{1}{12}. \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= EXY - EXEY \\ &= \frac{1}{12} - \left(\frac{1}{3}\right)^2 \\ &= -\frac{1}{36}, \end{aligned}$$

$$\begin{aligned} \rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= -\frac{1}{2}. \end{aligned}$$

Properties:

The covariance has the following properties:

1. $Cov(X,X) = Var(X)$
2. if X and Y are independent then $Cov(X,Y) = 0$
3. $Cov(X,Y) = Cov(Y,X)$
4. $Cov(aX,Y) = aCov(X,Y)$
5. $Cov(X+c,Y) = Cov(X,Y)$
6. $Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$

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Let X and Y be two independent $N(0, 1)$ random variables and

$$Z = 1 + X + XY^2,$$
$$W = 1 + X.$$

Find $\mathrm{Cov}(Z, W)$

- Solution
  - $$\begin{aligned}\mathrm{Cov}(Z, W) &= \mathrm{Cov}(1 + X + XY^2, 1 + X)\\ &= \mathrm{Cov}(X + XY^2, X)\\ &= \mathrm{Cov}(X, X) + \mathrm{Cov}(XY^2, X)\\ &= \mathrm{Var}(X) + E[X^2Y^2] - E[XY^2]EX\\ &= 1 + E[X^2]E[Y^2] - E[X]^2E[Y^2]\\ &= 1 + 1 - 0 = 2.\end{aligned}$$

Problem: 8

A random Variable definded as $X(t) = A\cos(\omega t) + B\sin(\omega t)$, where A and B are random variables with density functions $f_A(a)$ and $f_B(b)$. We have to find the conditions under which $X(t)$ will be stationary:

1. *Mean*:

$$E[X(t)] = E[A\cos(\omega t) + B\sin(\omega t)] = \cos(\omega t)E[A] + \sin(\omega t)E[B]$$

If $E[X(t)]$ is to be independent of t, then $E[A] = E[B] = 0$, in which case $E[X(t)] = 0$.

2. *Autocorrelation*:

$$\begin{aligned}R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E\{[A\cos(\omega t_1) \\ &\quad + B\sin(\omega t_1)][A\cos(\omega t_2) + B\sin(\omega t_2)]\} \\ &= E\{A^2\cos(\omega t_1)\cos(\omega t_2) + B^2\sin(\omega t_1)\sin(\omega t_2) \\ &\quad + AB[\sin(\omega t_1)\cos(\omega t_2) + \cos(\omega t_1)\sin(\omega t_2)]\} \\ &= \frac{1}{2}E[A^2][\cos(\omega(t_2 - t_1)) + \cos(\omega(t_2 + t_1))] \\ &\quad + \frac{1}{2}E[B^2][\cos(\omega(t_2 - t_1)) - \cos(\omega(t_2 + t_1))] \\ &\quad + E[AB]\sin(\omega(t_2 + t_1))\end{aligned}$$

The conditions under which this equation will be dependent only on $(t_2 - t_1)$ are $E[A^2] = E[B^2]$ and $E[AB] = 0$. In this case

$$R_X(t_1,t_2) = E[A^2][\cos(\omega(t_2 - t_1))] = E[A^2][\cos(\omega\tau)]$$

We can now summarize the conditions for stationarity:

(a) $E[A] = E[B] = 0$

(b) $E[A^2] = E[B^2]$

(c) $E[AB] = 0$

Problem 9

Let a random variable X(t) defined as $X(t) = A\sin(\omega_0 t + \Phi)$, where A and Φ are independent and A is uniformly distributed over (0,1) and Φ is uniformly distributed over $(0,2\pi)$ and find whether it is irgodic or not

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We have already seen in Example 19.2.1 that

$$E[X(t)] = E[A]E[\sin(\omega_0 t + \Phi)] = 0$$

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[A^1]E[\sin(\omega_0 t + \Phi)\sin(\omega_0 t + \omega_0\tau + \Phi)]$$

$$= \frac{1}{2}E[A^2]\cos(\omega_0\tau)$$

and $X(t)$ is indeed stationary. In fact, it is also strict-sense stationary. We will check first whether it is mean-ergodic. Since the mean value is zero, $C_X(\tau) = \frac{1}{2}E[A^2]\cos(\omega_0\tau)$. The variance of the time average $\sigma^2_{\hat{\mu}_X}$ is given by

$$\sigma^2_{\hat{\mu}_X} = \frac{E[A^2]}{2T}\left\{\int_{-2T}^{2T}\left[1 - \frac{|\tau|}{2T}\right]\frac{\cos(\omega_0\tau)}{2}d\tau\right\}$$

We can use the result from Fourier transform that time multiplication is frequency convolution and evaluate the integral shown above. Substituting the Fourier transforms of the triangular and the cosine functions

$$\left[1 - \frac{|\tau|}{2T}\right] \Longleftrightarrow 2T\left[\frac{\sin(\omega T)}{\omega T}\right]^2 \quad \text{and} \quad \frac{\cos(\omega_0\tau)}{2} \Longleftrightarrow \frac{\pi}{2}[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

in the equation for the variance, we obtain

$$\sigma^2_{\hat{\mu}_X} = \frac{1}{2\pi}\frac{E[A^2]}{2T}\int_{-\infty}^{\infty} 2T\left[\frac{\sin(\omega T)}{\omega T}\right]^2\frac{\pi}{2}[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]d\omega$$

$$= \frac{E[A^2]}{2\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 T}\right]^2$$

and as $T \to \infty$, $\sigma^2_{\hat{\mu}_X} \to 0$, for $\omega_0 \neq 0$. Thus $X(t)$ is mean-ergodic.

Substituting $E[A^2] = \frac{1}{3}$ in the variance equation, we obtain

$$\sigma^2_{\hat{\mu}_X} = \frac{1}{6\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 T}\right]^2$$

To check for power ergodicity, it is simpler to use Eq. (19.3.16) rather than Eq. (19.3.17). Thus

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} X^2(t)\,dt\right] = \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} A^2\cos^2(\omega_0 t + \Phi)\,dt\right] = \frac{A^2}{2} \quad \text{for} \quad \omega_0 \neq 0$$

Since this expression does not converge to $R_X(0) = \frac{1}{6}$, the process $X(t)$ is neither power-ergodic nor correlation-ergodic.

Problem 10

consider a random process $X(t) = B + A\sin(\omega_0 t + \Phi)$, where B, A, and Φ are independent random variables. A and B are uniformly distributed over (0,1] and (0,2] respectively. We will check whether $X(t)$ is mean-ergodic.

We first determine whether $X(t)$ is stationary. The mean value μ_X is given by

$$E[B + A\sin(\omega_0 t + \Phi)] = \mu_B + E[A\sin(\omega_0 t + \Phi)] = \mu_B$$

since $E[A\sin(\omega_0 t + \Phi)] = 0$ from previous examples. The autocorrelation function is

$$\begin{aligned} R_X(\tau) &= E\{[B + A\sin(\omega_0 t + \Phi)][B + A\sin(\omega_0 t + \omega_0\tau + \Phi)]\} \\ &= E[B^2] + 0 + 0 + E[A^2]E[\sin(\omega_0 t + \Phi)A\sin(\omega_0 t + \omega_0\tau + \Phi] \\ &= E[B^2] + E[A^2]\frac{1}{2}E[\cos(\omega_0\tau) - \cos(2\omega_0 t + \omega_0\tau + 2\Phi] \\ &= E[B^2] + \frac{1}{2}E[A^2]\cos(\omega_0\tau) \end{aligned}$$

Hence the process $X(t)$ is stationary. The autocovariance function $C_X(\tau)$ is given by

$$\begin{aligned} C_X(\tau) &= R_X(\tau) - \mu_X^2 = E[B^2] + \frac{1}{2}E[A^2]\cos(\omega_0\tau) - \mu_B^2 \\ &= \sigma_B^2 + \frac{1}{2}E[A^2]\cos(\omega_0\tau) \end{aligned}$$

and substituting this equation in Eq. (19.3.7), we obtain the variance of the time average as

$$\begin{aligned} \sigma_{\hat{\mu}_X}^2 &= \frac{1}{2T}\left\{\int_{-2T}^{2T}\left[\sigma_B^2 + \frac{1}{2}E[A^2]\cos(\omega_0\tau)\right]\left[1 - \frac{|\tau|}{2T}\right]d\tau\right\} \\ &= \frac{1}{2T}\left[2T\sigma_B^2 + E[A^2]\frac{2T}{2\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 T}\right]^2\right] = \sigma_B^2 + \frac{E[A^2]}{2\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 t}\right]^2 \end{aligned}$$

and

$$\lim_{T\to\infty}\sigma_{\hat{\mu}_X}^2 = \lim_{T\to\infty}\left\{\sigma_B^2 + \frac{E[A^2]}{2\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 T}\right]^2\right\} = \sigma_B^2 \quad \text{if} \quad \omega_0 \neq 0$$

We find that even though $X(t)$ is stationary, it is not mean-ergodic. Hence the time average is not a good estimator of the ensemble average. We can now substitute $E[A^2] = \frac{1}{3}$ and $\sigma_B^2 = \frac{1}{3}$ in the variance equation and write

$$\sigma_{\hat{\mu}_X}^2 = \sigma_B^2 + \frac{E[A^2]}{2\pi}\left[\frac{\sin(\omega_0 T)}{\omega_0 T}\right]^2 = \frac{1}{3}\left[1 + \frac{\sin(\omega_0 T)}{2\pi\omega_0 T}\right]$$

Problem 11

A random process $Y(t)$ is given by $Y(t) = X(t)\cos(\omega t + \Phi)$, where $X(t)$, a zero mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = 2e^{-2\lambda|\tau|}$ is modulating the carrier $\cos(\omega t + \Phi)$. The random variable Φ is uniformly distributed in the interval $(0,2\pi)$, and is independent of $X(t)$.
to find the mean, variance, and autocorrelation of $Y(t)$:

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Mean. The independence of $X(t)$ and Φ allows us to write

$$E[Y(t)] = E[X(t)]E[\cos(\omega t + \Phi)]$$

and with $E[X(t)] = 0$ and $E[\cos(\omega t + \Phi)] = 0$ from Example 19.2.1, $E[Y(t)] = 0$.

Variance. Since $X(t)$ and Φ are independent, the variance can be given by

$$\sigma_Y^2 = E[Y^2(t)] = E[X^2(t)\cos^2(\omega t + \Phi)] = \sigma_X^2 E[\cos^2(\omega t + \Phi)]$$

However

$$E[\cos^2(\omega t + \Phi)] = \frac{1}{2}E[1 + \cos(2\omega t + 2\Phi)] = \frac{1}{2} \quad \text{and} \quad \sigma_X^2 = C_X(0) = R_X(0) = 2$$

and hence $\sigma_Y^2 = \sigma_X^2/2 = 1$.

Autocorrelation:

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[X(t)\cos(\omega t + \Phi)X(t+\tau)\cos(\omega t + \omega\tau + \Phi)] \\ &= R_X(\tau)\frac{1}{2}E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\Phi)] \\ &= \frac{R_X(\tau)}{2}\cos(\omega\tau) + \frac{R_X(\tau)}{2}E[\cos(2\omega t + \omega\tau + 2\Phi)] \end{aligned}$$

From Example 19.2.1 $E[\cos(2\omega t + \omega\tau + 2\Phi)] = 0$, and hence

$$R_Y(\tau) = \frac{R_X(\tau)}{2}\cos(\omega\tau) = e^{-2\lambda|\tau|}\cos(\omega\tau)$$

A graph of $R_Y(\tau)$ is shown in Fig. 19.2.7 with $\lambda = 0.5$ and $\omega = 2\pi$.

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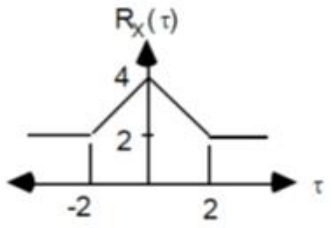
Problem 12

Consider a Gaussian random process X(t) with autocorrelation function::



a. Find the total average power, $E(X^2)$. $E(X^2) = R_X(0) = 4$

b. $var(X) = E(X^2) - (E(X))^2 = 4 - 2 = 2$

c. $E(X(3)\cdot X(4)) = R_X(t_1 = 3, t_2 = 4) = R_X(\tau = 1) = 3$ (from the graph, by inspection)

Problem 13

A wide-sense stationary (WSS) random process X(t) has autocorrelation function

$$R_X(\tau) = 25\exp(-4|\tau|) + 16\cos(20\tau) + 36$$

a. $E[X^2(t)] = R_X(0) = 25 + 16 + 36 = 77$

b. $E[X(t)] = \pm 6$

c. $var(X(t)) = E[X^2(t)] - (E(X))^2 = 77 - 36 = 41$

Problem: 14

Random variables X and Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Note: The joint pdf factors to give

$f_X(x) = 2x \quad 0 \le x \le 1$, $f_Y(y) = 2y \quad 0 \le y \le 1$ $\Rightarrow$ X, Y indepdent, identically distributed (IID)

a. $$E(X) = E(Y) = 2\int_0^1 x^2 dx = \frac{2}{3}x^3\Big|_{x=0}^{1} = 2/3$$

b. $$E(X^2) = E(Y^2) = 2\int_{x=0}^{1} x^3 dx = \frac{2}{4}x^4\Big|_{x=0}^{1} = \frac{1}{2}$$

$$Var(X) = var(Y) = E(X^2) - (E(X))^2 = (1/2) - 4/9 = 1/18$$

c. $cov(X, Y) = 0$, since independent $\Rightarrow$ uncorrelated

$$E(X + Y) = E(X) + E(Y) = (2/3) + (2/3) = 4/3$$

d. $var(X + Y) = var(X) + var(Y) + 2\,cov(X, Y) = (1/18) + (1/18) + 0 = 1/9$

Final Exam Questions

1. (a)Explain the term central limit theory (CLT)
   (b) Define the term random process and random variable.
   (c) Define monotonic and non-monotonic function.
   (d) What are the conditions for any stochastic process will be Ergodic process.

2. A discrete random variable X and Y has joint PMF which is given **by**

$$p(x_i, y_j) = \begin{cases} k(4x_i + 2y_j), & x_i = 2,3 \text{ and } y_j = 2,3 \\ 0, & otherwise \end{cases}$$

Determine

a. The constant K values
b. The marginal PMF of X and Y
c. The conditional PMF of X and Y
d. Are X and Y independent?

3. Let X and Y are two jointly distributed continuous random variables representing voltages measured in Volts at two different nodes of a circuit. X has the marginal density function

$$f(x) = \begin{cases} \frac{2x+1}{2}, & 0\ volt \le x \le 1 volts \\ 0, & otherwise \end{cases}$$

You are also given the conditional density function

$$f(y|x) = \begin{cases} \frac{2(x+y)}{2x+1}, & 0\ volt \le x \le 1 volt \text{ and } 0\ volt \le y \le 1\ volt \\ 0, & otherwise \end{cases}$$

a. Compute $P(Y \le 0.5 \text{volts} \mid X = 0.5 \text{ volts})$
b. Compute $P(0.25 \text{ volts} \le X \le 0.5 \text{ volts}, Y \le 0.5 \text{ volts})$
c. Determine Mean of X and Y.

4. Let X has PMF given **by**

$$f_X(x_i) = \begin{cases} \frac{2}{3}, & x = 1 \\ \frac{1}{3}, & x = 2 \end{cases}$$

If $Y = X^2 + 1$, Find

a. PMF of Y
b. E[Y]

5. Two random variables X and Y have joint probability density function as

$$f_{X,Y}(x,y) = \begin{cases} 2-x-y\,, & 0<x<1, \quad 0<y<1 \\ 0\,, & 0therwise \end{cases}$$

Find Covariance of X and Y.

6. Let X be uniformly distributed random variable on [2, 4] and Y = e^{2X}

a. Compute the CDF of Y.
b. Compute pdf of Y.
c. Find the mean of Y.

7. Consider the random process $X(t) = B\cos(2\pi t)$, random variable B follows the exponential distribution as $f_B(b) = 2e^{-2b}$ where $b > 0$, the mean and variance of B are $\frac{1}{2}$ and $\frac{1}{4}$ respectively. Find

   a. The mean value of $X(t)$.
   b. The autocorrelation function of the random process X (t).
   c. Is the random process X (t) wide-sense stationary (WSS)? If yes or no, explain.