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Unit-4

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Defn: A Metric space (X, d) is said to be Bolzano - Weierstrass property if every infinite subset of X has a limit point.

Defn: A Metric space is said to be sequentially compact if every sequence in it has convergent subsequence.

Thm: A Metric space is sequentially compact iff it has Bolzano-Weierstrass property.

Proof: Let X be a Metric space and suppose that X is sequentially compact.
Let A be an infinite subset of X .
Consider $x_1 \in A$
 $x_k \in A - \{x_1, x_2, \dots, x_{k-1}\} \quad \forall k \in \mathbb{N}, k \geq 2$

Note as A is infinite
 $A - \{x_1, x_2, \dots, x_{k-1}\} \neq \emptyset$
 $\forall k \in \mathbb{N} \quad k \geq 2$

$\Rightarrow \{x_n\}_{n \in \mathbb{N}}$ is an infinite sequence of distinct point of A .

As X is sequentially compact
 $\{x_n\}_{n \in \mathbb{N}}$ has convergent subsequence.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent

subsequence of $\{x_n\}_{n \in \mathbb{N}}$ converging to x

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of distinct points of A

$\{x_{n_k}\}_{k \in \mathbb{N}}$ is a sequence of

distinct point of A converging to x

$\Rightarrow x$ is a limit point of A

$\Rightarrow X$ has Bolzano Weierstrass property

Conversely suppose X has Bolzano Weierstrass property

i.e every infinite subset of X

has a limit point

Thm: X is sequentially compact

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X

If a point x of $\{x_n\}$ repeats infinitely many times then

$\{x_{n_k}\}_{k \in \mathbb{N}}$ where $x_{n_k} = x \forall k \in \mathbb{N}$

is a subsequence of $\{x_n\}_{n \in \mathbb{N}}$

converging to x

Suppose no point of $\{x_n\}_{n \in \mathbb{N}}$

repeats infinitely many times

Consider $A = \{x_n / n \in \mathbb{N}\}$

$\Rightarrow A$ is infinite subset of X

X has Bolzano Weierstrass property

A has a limit point in X

Let x be a limit point of A in X

then $\exists$ a sequence $\{y_n\}_{n \in \mathbb{N}}$ of point, of A converging in X

$\forall k \in A \forall k \in \mathbb{N} \rightarrow \exists k = x_{n_k}$ for some $n_k \in \mathbb{N}$

then $\{x_{n_k}\}_{k \in \mathbb{N}}$ is a subsequence

of $\{x_n\}_{n \in \mathbb{N}}$ converging to x

$\therefore X$ is sequentially compact

Thm: Let X is compact Metric Space then X is sequentially compact.

X is compact $\Rightarrow X$ is limit point

compactness $\Rightarrow$ i.e X has Bolzano

Weierstrass property

Def: A real number $\alpha > 0$ is called a Lebesgue number for a given cover $\{G_i / i \in I\}$ of a Metric

space X if each subset of X whose diameter is less than ' α ' is contained in atleast one G_i for $i \in I$

Def: Let (X, d) be a Metric Space
 $A \subseteq X$ diameter of A is
 defined as follows

$$\text{diam}(A) = \max \{d(x, y) / x, y \in A\}$$

$A = \emptyset$ then $\text{diam}(A) = 0$

$A = \{a\}$ $\text{diam}(A) = 0$

18/4/18 Thm] (Lebesgue covering lemma)

In a sequentially compact Metric Space every open cover has a Lebesgue number.

Proof: Let (X, d) be a sequentially compact Metric Space
 Let $\{G_i\}_{i \in I}$ be an open cover of X in (X, d)

We say that a subset of X is big if it is not contained in any G_i

If there is no big set in X

then any $a \in \mathbb{R}, a > 0$ is a Lebesgue number for the cover $\{G_i\}_{i \in I}$

for any $A \subseteq X$ (and $\text{diam}(A) < a$)

A is not big in X

$\Rightarrow A$ is contained in atleast one $G_i, i \in I$

Suppose X has big sets

$$\text{Let } a' = \text{glb} \{ \text{diam}(A) / A \subseteq X, A \text{ is big} \} = B$$

$$a' \in \mathbb{R}, 0 \leq a' \leq +\infty$$

Suppose $a' = +\infty$

Then for any $a \in \mathbb{R}, a > 0$ is a Lebesgue number for $\{G_i\}_{i \in I}$

$$a < \text{glb} \{ \text{diam}(A) / A \subseteq X, A \text{ is big} \} = B$$

$$\text{Let } A \subseteq X \text{ and } \text{diam}(A) < a$$

$$\Rightarrow \text{diam}(A) < a < \infty = \text{glb } B$$

$$A \notin B$$

$\Rightarrow A$ is not a big set

$\Rightarrow \exists$ atleast one $G_i, i \in I$ such that $A \subseteq G_i$

Suppose $a' \in \mathbb{R}$
 $0 \leq a' < \infty$

Thpt a' is a Lebesgue Me. number for $\{G_i\}_{i \in I}$

Let $A \subseteq X$ and $\text{diam}(A) < a'$

$\Rightarrow A \notin B$

$\Rightarrow A$ is not a big set in X

$\Rightarrow \exists$ atleast one $G_i, i \in I$ s.t. $A \subseteq G_i$

Suppose $a' = 0$

Note: single point set is not a big set in X as $\{G_i\}_{i \in I}$ is an

open cover for X .

$\Rightarrow$ Each big set in X must have at least two points.

$$a' < \frac{1}{n} + a' \quad \forall n \in \mathbb{N} \quad a' = \text{glb } B$$

$\Rightarrow \exists B_n \subseteq X, B_n$ is big and $0 < \text{diam}(B_n) < \frac{1}{n} + a'$

Let $x_n \in B_n \quad \forall n \in \mathbb{N}$

Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X .
As X is sequentially compact sequence $\{x_n\}_{n \in \mathbb{N}}$ in X has a convergent subsequence.

Let $\{x_{n_k}\}_{k \in \mathbb{N}}$ be a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ converging to x in (X, d)

$x \in X, \{G_i\}_{i \in I}$ is an open cover of X

$\Rightarrow x \in G_{i_0}$ for some $i_0 \in I$

Since G_{i_0} is open $\exists r \in \mathbb{R}, r > 0$ such that $B_d(x, r) \subseteq G_{i_0}$

As $x_{n_k} \rightarrow x$ as $k \rightarrow \infty$, $B_d(x, r/2)$ contains infinitely many points of $\{x_{n_k}\}_{k \in \mathbb{N}}$

As $x_{n_k} \rightarrow x$ as $k \rightarrow \infty \quad \exists n_{k_0} \in \mathbb{N}$ such that $x_{n_k} \in B_d(x, r/2) \quad \forall k \geq k_0$

Suppose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{r}{2}$

and $x_{n_0} \in B_d(x, r/2)$

$$\therefore \text{diam}(B_{n_0}) < \frac{1}{n_0} < \frac{r}{2}$$

and $x_{n_0} \in B_{n_0}$

$$B_{n_0} \subseteq B_d(x, r/2) \subseteq G_{i_0}$$

$$\forall y \in B_{n_0} \quad d(y, x) < \frac{1}{n_0}$$

$$d(x, y) \leq d(x, x_{n_0}) + d(x_{n_0}, y) < \frac{r}{2} + \frac{r}{2} = r$$

$$y \in B_d(x, r) \subseteq G_{i_0}$$

$$B_{n_0} \subseteq B_d(x, r) \subseteq G_{i_0}$$

a contradiction as B_{n_0} is a big set.
 $\therefore a' \neq 0$.

Defn: Let X be a Metric Space and $\varepsilon \in \mathbb{R}, \varepsilon > 0$. A subset 'A' of X is called ε -net if A is finite and $X \setminus A \subseteq \bigcup_{a \in A} B_d(a, \varepsilon)$.

Defn: A Metric Space is called totally Bounded if it has an ε -net for each $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

Thm Every sequentially compact metric space is totally bounded.

Proof: Let X be a sequentially compact metric space and $\epsilon \in \mathbb{R}, \epsilon > 0$
to prove: X has ϵ -net

Let $a \in X$ and consider $B(a, \epsilon)$
If $X = B(a, \epsilon)$ then $A = \{a\}$ is an ϵ -net in X

Suppose $X \neq B(a, \epsilon)$

$\Rightarrow X - B(a, \epsilon) \neq \emptyset$

Let $a_2 \in X - B(a, \epsilon)$

if $X = B(a, \epsilon) \cup B(a_2, \epsilon)$

then $A = \{a, a_2\}$ is an ϵ -net for X

Suppose this process continues indefinitely. Then we get a seqⁿ $\{a_n\}_{n \in \mathbb{N}}$ such that

$X - \bigcup_{i=1}^k B(a_i, \epsilon) \neq \emptyset \quad \forall k \in \mathbb{N}$

As X is sequentially compact the sequence $\{a_n\}_{n \in \mathbb{N}}$ has a convergent subsequence

Let $\{a_{n_k}\}_{k \in \mathbb{N}}$ is a convergent subsequence of $\{a_n\}_{n \in \mathbb{N}}$ converging to a in X

Then $\exists k_0 \in \mathbb{N}$ such that $a_{n_k} \in B(a, \epsilon/2) \quad \forall k \geq k_0$

$\forall k_1, k_2, k_0 \in \mathbb{N}$ and

$k_2 > k_1 > k_0$

$a_{n_{k_1}}, a_{n_{k_2}} \in B(a, \epsilon/2)$

$d(a_{n_{k_1}}, a_{n_{k_2}}) \leq d(a_{n_{k_1}}, a) + d(a, a_{n_{k_2}}) < \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow a_{n_{k_2}} \in B(a_{n_{k_1}}, \epsilon)$

$k_2 > k_1$ then $n_{k_2} > n_{k_1}$
 $a_{n_{k_2}} \in X - B(a_{n_{k_1}}, \epsilon)$

$\Rightarrow a_{n_{k_2}} \notin B(a_{n_{k_1}}, \epsilon)$

$\therefore$ Contradiction

process must terminate

$\Rightarrow \exists A = \{a_1, a_2, \dots, a_n\} \subseteq X$ such that $X = \bigcup_{i=1}^n B(a_i, \epsilon)$

$\Rightarrow A$ is a ϵ -net for X
 $\Rightarrow X$ is totally bounded.

Thm Every sequentially compact metric space is compact.

Proof: Let X be a sequentially compact vector space and $\{G_i\}_{i \in \mathbb{I}}$ be an open cover of X . Since X is sequentially compact this open cover has a Lebesgue number. Let $\delta \in \mathbb{R}$ be a Lebesgue number $\delta > 0$

for the open cover $\{G_i\}_{i \in I}$
take $\epsilon = \frac{\delta}{3}$

As sequentially compact Metric space has an ϵ -net

Let $A = \{a_1, a_2, \dots, a_n\}$ be an ϵ -net for $X \Rightarrow X = \bigcup_{k=1}^n B(a_k, \epsilon)$

$$\text{diam}(B(a_k, \epsilon)) \leq 2\epsilon = \frac{2\delta}{3} < \delta$$

for $k=1, 2, \dots, n$

As δ is Lebesgue number for open cover $\{G_i\}_{i \in I}$

$$B(a_k, \epsilon) \subseteq G_{i_k} \text{ for some } i_k \in I$$

$$X = \bigcup_{k=1}^n B(a_k, \epsilon) \subseteq \bigcup_{k=1}^n G_{i_k} \quad k=1, 2, \dots, n$$

$\{G_{i_k}\}_{k=1}^n$ is a finite subcover

of $\{G_i\}_{i \in I}$ for X
 $\Rightarrow X$ is compact

Thm: Let (X, d) be a compact Metric space (Y, d_2) be a Metric space & $f: X \rightarrow Y$. If f is continuous then uniformly continuous

Proof: Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$.
As f is continuous for each $x \in X$

$f^{-1}(B_{d_2}(f(x), \epsilon/2))$ is open in (X, d)

So $\{f^{-1}(B_{d_2}(f(x), \epsilon/2))\}_{x \in X}$ is an open cover of X

As X is compact this open cover has Lebesgue no.

Let $\delta > 0$ be the Lebesgue no. of this open cover

Let $x, x' \in X$ such that $d_1(x, x') < \delta$
 $\Rightarrow$ the set $\{x, x'\}$ has diameter less than δ

As δ is a Lebesgue no. of an open cover $\{f^{-1}(B_{d_2}(f(y), \epsilon/2))\}_{y \in X}$ & $y \in X$ such that

$$\{x, x'\} \subseteq f^{-1}(B_{d_2}(f(y), \epsilon/2))$$

$$\Rightarrow \{f(x), f(x')\} \subseteq B_{d_2}(f(y), \epsilon/2)$$

$$\Rightarrow d_2(f(x), f(x')) \leq d_2(f(x), f(y)) + d_2(f(y), f(x'))$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$$\Rightarrow d_2(f(x), f(x')) < \epsilon$$

$\Rightarrow f$ is uniformly continuous.

Thm: A Metric space is compact iff it is totally bounded & complete.

Proof: Suppose that Metric space (X, d) is compact

Then (X, d) is totally bounded

Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in X

As (X, d) is compact, it is

Sequentially Compact

$\Rightarrow \{x_n\}_{n \in \mathbb{N}}$ has convergent subsequence
 Let $\{x_{n_k}\}_{k \in \mathbb{N}}$ is a convergent subsequence of $\{x_n\}_{n \in \mathbb{N}}$ converges to x .

$x_{n_k} \rightarrow x$ as $k \rightarrow \infty$
 Let $\epsilon \in \mathbb{R} \quad \epsilon > 0 \quad \exists k_0 \in \mathbb{N}$ st
 $d(x_{n_k}, x) < \epsilon/2 \quad \forall k \in \mathbb{N} \quad k \geq k_0 \quad \text{--- (1)}$

Let $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy $\exists n_0 \in \mathbb{N}$ st

$d(x_m, x_n) < \epsilon/2 \quad \forall m, n \geq n_0 \quad \text{--- (2)}$

Let $n_0 = \max\{n_{k_0}, n_1\}$

for $n \in \mathbb{N} \quad n \geq n_0$

$$\begin{aligned} \therefore d(x_n, x) &\leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

$\therefore x_n \rightarrow x$ as $n \rightarrow \infty$

$\Rightarrow (X, d)$ is complete

Conversely,

Suppose (X, d) is complete & totally bounded.

t.p.t: (X, d) is compact

i.e t.p.t: (X, d) is sequentially compact

Let $\{x_n\}_{n \in \mathbb{N}}$ be a seq. in X

If $\{x_n\}_{n \in \mathbb{N}}$ has a term which repeats infinitely many times in $\{x_n\}_{n \in \mathbb{N}}$. then $\{x_{n_k}\} = \{y\}$ which is a constant subseq. of $\{x_n\}_{n \in \mathbb{N}}$ converges to y in (X, d)

Suppose no term of $\{x_n\}_{n \in \mathbb{N}}$ repeats infinitely many times
 Let X is totally bounded
 $\exists A \subset X, A = \{a_1, a_2, \dots, a_m\}$ such that
 $X = \bigcup_{i=1}^m B_d(a_i, 1)$

There is atleast one open ball $B_d(a_{i_1})$ which will contain infinitely many x_n 's
 wlog. Suppose $B_d(a_{i_1})$ contains infinitely many x_n 's
 Let $J_1 = \{n \in \mathbb{N} \mid x_n \in B_1\}$ $B_1 = B(a_{i_1}, 1)$
 J_1 is infinite

Now cover X by finitely many open balls of radius $1/2$
 Atleast one open ball of radius $1/2$ contains infinitely many x_n 's from B_1 call it B_2

$J_2 = \{n \in J_1 \mid x_n \in B_2\}$

We continue this process to get.

$J_n = \{n \in J_{n-1} \mid x_n \in B_n\} \quad n \in \mathbb{N}$

$J_0 = \mathbb{N}$ & $J_k \supset J_{k+1} \quad \forall k \in \mathbb{N}$

$B_k \supset B_{k+1}$

Choose $n_k \in J_k$ such that $n_k < n_{k+1}$
 This is possible as each of J_k

is infinite

$\Rightarrow \{x_{n_k}\}_{k \in \mathbb{N}}$ is a subseq of $\{x_n\}_{n \in \mathbb{N}}$

Thm: $\{x_{n_k}\}$ is Cauchy

Let $\varepsilon \in \mathbb{R}, \varepsilon > 0$ be given

(choose $k' \in \mathbb{N}$ such that $\frac{1}{k'} < \varepsilon/2$)

$\forall i, j \in \mathbb{N}, i, j \geq k'$

$n_i \in J_i \subseteq J_{k'}$

$n_j \in J_j \subseteq J_{k'}$

$\Rightarrow x_{n_i}, x_{n_j} \in B_{\varepsilon/2}$

$\text{diam}(B_{\varepsilon/2}) \leq 2 \left(\frac{1}{k'}\right) < \varepsilon$

$\Rightarrow d(x_{n_i}, x_{n_j}) < \varepsilon, \forall i, j \geq k'$

$\Rightarrow \{x_{n_k}\}_{k \in \mathbb{N}}$ is a Cauchy seqⁿ

As X is complete $\{x_{n_k}\}_{k \in \mathbb{N}}$ is convergent

$\therefore X$ is sequentially compact

$\Rightarrow X$ is compact

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Defn: Let (X, d_1) and (Y, d_2) be Metric spaces. A Map $f: X \rightarrow Y$ is called isometry if,

$$d_1(x, x') = d_2(f(x), f(x')) \quad \forall x, x' \in X$$

Rem: Isometry is one-one

$\forall x, x' \in X$ and $x \neq x', d(x, x') \neq 0$

$$d_1(x, x') = d_2(f(x), f(x')) \neq 0$$

$$\Rightarrow f(x) \neq f(x')$$

$\Rightarrow f$ is one-one

If isometry f is onto then $Y = f(X)$ and we say that X and Y are isometric (or X is isometric to Y)

If f is isometry (but $Y \neq f(X)$) i.e. f is not onto then $f(X) \subsetneq Y$. Then subspace $f(X)$ of Y is called isometric embedding of X into Y .

Defn: Let (X, d) be a Metric Space and $\beta(X, \mathbb{R}) = \{f \mid f: X \rightarrow \mathbb{R} \text{ f is bounded}\}$

$\rho(f, g) = \sup \{|f(x) - g(x)| \mid x \in X\}$
 $f, g \in \beta(X, \mathbb{R})$. Then ρ is a metric on $\beta(X, \mathbb{R})$.

Propⁿ: Metric space $(\mathcal{B}(X, \mathbb{R}), \rho)$ is a complete Metric Space.

Proof: Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy Seq in $\mathcal{B}(X, \mathbb{R})$ and $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

$\Rightarrow \exists n_0 \in \mathbb{N}$ such that $\rho(f_m, f_n) < \varepsilon \quad \forall m, n \geq n_0, m, n \in \mathbb{N}$.

$\Rightarrow \sup \{ |f_m(x) - f_n(x)| / x \in X \} < \varepsilon \quad \forall m, n \geq n_0$.

$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon \quad \forall m, n \geq n_0 \quad \forall x \in X$.

$\Rightarrow \text{seq}^n \{f_n(x)\}_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$.

As $\mathbb{R}$ is complete this sequence converges in $\mathbb{R}$.

Suppose $f_n(x) \rightarrow f(x)$ for $f(x) \in \mathbb{R}$.

$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in X$.

Mod $\hookrightarrow 1: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

$\Rightarrow \exists n_1 \in \mathbb{N}$ such that

$\therefore |f_m(x) - f(x)| < \varepsilon \quad \forall x \in X \quad \forall m \geq n_1$.

$\Rightarrow f_{n_1}(x) - \varepsilon < f(x) < f_{n_1}(x) + \varepsilon \quad \forall x \in X$.

$f_{n_1} \in \mathcal{B}(X, \mathbb{R})$ f_{n_1} is bounded.

$\Rightarrow f$ is bounded.

$\Rightarrow f \in \mathcal{B}(X, \mathbb{R})$.

Let $m \geq n_1$.

$\rho(f_m, f) = \sup \{ |f_m(x) - f(x)| / x \in X \}$.

$\Rightarrow \rho(f_m, f) < \varepsilon \quad \forall m \geq n_1$. By (1) $\{f_n\}_{n \in \mathbb{N}}$ converges in a Metric Space $(\mathcal{B}(X, \mathbb{R}), \rho)$.

$\therefore (\mathcal{B}(X, \mathbb{R}), \rho)$ is complete.

Def: Let X and Y be Metric Spaces if $h: X \rightarrow Y$ is isometric embedding of X into Y . If Y is complete Metric space then the subspace $h(X)$ of Y is a complete Metric space. $h(X)$ is called completion of X .

Thm: [Existence]

Let (X, d) be a Metric Space. There is an isometric embedding of X into a complete Metric space.

Proof: Consider $(\mathcal{B}(X, \mathbb{R}), \rho)$.

Let x_0 be a fixed point in X . Given $a \in X$ define $\phi_a: \mathbb{R} \rightarrow \mathbb{R}$.

$\phi_a: X \rightarrow \mathbb{R}$ by $\phi_a(x) = d(x, a) - d(x, x_0)$.

Jpt: $\phi_a \in \mathcal{B}(X, \mathbb{R}) \quad \forall a \in X$.

$|\phi_a(x)| = |d(x, a) - d(x, x_0)| \leq |d(a, x_0)| = d(a, x_0)$.

$$\begin{aligned}
 d(x,a) &\leq d(x,x_0) + d(x_0,a) \\
 d(x,a) - d(x,x_0) &\leq d(x_0,a) \\
 d(x,x_0) &\leq d(x,a) + d(a,x_0) \\
 d(x,x_0) - d(x,a) &\leq d(a,x_0)
 \end{aligned}$$

ϕa is bounded.
 $\Rightarrow \phi a \in \beta(x, \mathbb{R}) \quad \forall a \in X$
 Define $\phi: X \rightarrow \beta(x, \mathbb{R})$
 as $\phi(a) = \phi a \quad \forall a \in X$.

Trpt: ϕ is isometry.
 Let $a, b \in X$ consider

$$\begin{aligned}
 f(\phi(a), \phi(b)) &= f(\phi a, \phi b) \\
 &= \sup \{ |\phi a(x) - \phi b(x)| : x \in X \}
 \end{aligned}$$

$$\begin{aligned}
 \phi a(x) - \phi b(x) &= d(x,a) - d(x,x_0) - d(x,b) + d(x,x_0) \\
 &= d(x,a) - d(x,b) \\
 &\leq d(a,b) \quad \forall x \in X \quad \text{--- (1)}
 \end{aligned}$$

$$\begin{aligned}
 &\leq d(a,b) \\
 f(\phi(a), \phi(b)) &\leq d(a,b) \quad \text{By (1) --- (2)} \\
 \text{Let } a, b \in X
 \end{aligned}$$

$$\begin{aligned}
 d(a,b) &= |d(a,x_0) - d(b,x_0)| \\
 &\leq \sup \{ |d(x,a) - d(x,b)| : x \in X \} \quad \text{replacing } a \text{ by } x \\
 &= \sup \{ |\phi a(x) - \phi b(x)| : x \in X \} \\
 &= f(\phi a, \phi b)
 \end{aligned}$$

$$= f(\phi(a), \phi(b)) \quad \text{--- (3)}$$

$\therefore$ By (2) & (3)

$$\begin{aligned}
 f(\phi(a), \phi(b)) &= d(a,b) \quad \forall a, b \in X \\
 \Rightarrow \phi \text{ is isometry}
 \end{aligned}$$

$\Rightarrow \phi(X)$ is an isometric embedding of X into a complete metric space $\beta(x, \mathbb{R})$.

$\therefore \phi(X)$ is a completion of X Metric space X

* Example of a complete Metric space where not all closed and bounded subsets are compact

$$\text{Ex] } \ell^2(\mathbb{R}) = \{ (a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{R}, \sum_{n=1}^{\infty} a_n^2 < \infty \}$$

$$x = (a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{R})$$

$$\text{Define } \|x\| = \sqrt{\sum_{n=1}^{\infty} a_n^2}$$

$$d(x,y) = \|x - y\|$$

$(\ell^2(\mathbb{R}), d)$ is a Metric space from

Note i) $\ell^2(\mathbb{R})$ is a vector space over the field $\mathbb{R}$

ii) $x \mapsto \|x\|$.

$$D = \{ x \in \ell^2(\mathbb{R}) \mid \|x\| < 1 \}$$

To prove that D is a compact subset of $\ell^2(\mathbb{R})$
 Non (Not a compact)

Consider n^{th} place
 $e_n = (0, 0, \dots, 1, 0, \dots)$
 $\|e_n\| = 1 \quad \forall n \in \mathbb{N}$
 $\Rightarrow e_n \in l^2(\mathbb{R}) \quad \forall n \in \mathbb{N}$
 $e_n \in D \quad \forall n \in \mathbb{N}$

$\|e_n - e_m\| = \sqrt{2} \quad \forall n, m \in \mathbb{N} \quad m \neq n$
 $\Rightarrow \{e_n\}_{n \in \mathbb{N}}$ is a sequence in D

which $\textcircled{a}$ has no Cauchy subsequence
 (take $0 < \varepsilon < \sqrt{2}$)
 $\therefore \{e_n\}$ does not converge in $l^2(\mathbb{R})$
 $\therefore D$ is not a compact

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[Lemma] If $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy seqⁿ
 in a Metric Space (X, d) and
 if $\{x_{n_k}\}$ is convergent subsequence
 of $\{x_n\}_{n \in \mathbb{N}}$ converges to l then

$\{x_n\}_{n \in \mathbb{N}}$ converges to l .

Proof

Let $\varepsilon > 0$ be given
 $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy so $\exists n_1 \in \mathbb{N}$
 such that

$$d(x_m, x_n) < \frac{\varepsilon}{2} \quad \forall m, n \geq n_1$$

$\{x_{n_k}\}_{k \in \mathbb{N}}$ converges to l .

$\exists n_2 \in \mathbb{N}$ such that
 $d(x_{n_k}, l) < \frac{\varepsilon}{2} \quad \forall k \geq n_2$

$n_0 = \max\{n_1, n_2\}$
 $n \geq n_0 \geq n_1$ then $n \geq n_1$ and $n \geq n_2$
 $\Rightarrow n_k \geq k$
 $\Rightarrow n \geq n_{n_2} \geq n_{n_1}$

$$d(x_n, l) \leq d(x_n, x_{n_{n_0}}) + d(x_{n_{n_0}}, l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$n_0 = \max\{n_1, n_2\}$
 $n \geq n_0$ then $n \geq n_1$ and $n_{n_0} \geq n_0 \geq n_1$
 $n_0 \geq n_2$

$$d(x_n, l) \leq d(x_n, x_{n_{n_0}}) + d(x_{n_{n_0}}, l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by (I) by (II)

So $x_n \rightarrow l$ as $n \rightarrow \infty$

ex) $\mathbb{R}$ with Euclidean Metric is complete

sol: Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy seqⁿ in $\mathbb{R}$.

1st pt: $\{x_n\}$ is bounded.

for $\exists n_0 \in \mathbb{N}$ such that $m, n \geq n_0$
 $\Rightarrow |x_m - x_n| < 1 \quad \forall m, n \geq n_0$

$$|x_m - x_n| < 1 \quad \forall m \geq n_0$$

$$x_{n_0} - 1 \leq x_m \leq 1 + x_{n_0} \quad n \geq n_0$$

$$M = \max\{|x_1|, |x_2|, \dots, |x_{n_0}|, |1 + x_{n_0}|, |x_{n_0} - 1|\}$$

$|x_m| \leq M \quad \forall m \in \mathbb{N}$
 $\Rightarrow \{x_n\}$ is bounded.
 $S = \{x_n | n \in \mathbb{N}\}$
 S is bounded.

Case I] If $\{x_n | n \in \mathbb{N}\}$ is finite set
 then $\{x_n\}$ has a term which
 repeats infinitely many times
 so $\{x_n\}$ has constant subsequence
 which is convergent
 so $\{x_n\}$ converges by proven
 result.

Case II] Suppose $\{x_n\}$ does not have
 a term which repeats infinitely
 many times.
 $\{x_n | n \in \mathbb{N}\}$ is an infinite
 bounded set
 $\Rightarrow \{x_n | n \in \mathbb{N}\} \subset [a, b]$ for some
 $a, b \in \mathbb{R}$
 $[a, b]$ is compact so $\{x_n\}_{n \in \mathbb{N}}$
 has convergent subseq.
 $\Rightarrow \{x_n\}_{n \in \mathbb{N}}$ is convergent.

Ex $\mathbb{R}^n$ with Euclidean Metric is
 complete.
Don Let $\{x^m\}_{m \in \mathbb{N}}$ be a Cauchy Seq
 in $\mathbb{R}^n$
 Let $\epsilon > 0$ be given $\exists n_0 \in \mathbb{N}$
 such that

$$d(x^m, x^k) < \epsilon \quad \forall m, k \geq n_0$$

$$\leq \sqrt{\sum_{i=1}^n (x_i^m - x_i^k)^2} < \epsilon \quad \forall m, k \geq n_0$$

$\Rightarrow$ seq $\{x_i^m\}_{m \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$
 As $\mathbb{R}$ is complete $\{x_i^m\}_{m \in \mathbb{N}}$

Conversely, suppose it converges to
 x_i for $i = 1, 2, \dots, n$
 $x = (x_1, x_2, \dots, x_n)$

T.p.t $\{x^m\}_{m \in \mathbb{N}}$ converges to x
 Let $\epsilon > 0$ be given
 as $\{x_i^m\}_{m \in \mathbb{N}}$ converges to x_i
 $\exists N_i \in \mathbb{N}$ such that

$$|x_i^m - x_i| < \frac{\epsilon}{\sqrt{n}} \quad \forall m \geq N_i \quad (1)$$

$N = \max \{N_1, N_2, \dots, N_n\}$ and $m \geq N$
 $\Rightarrow m \geq N_i$ for $i = 1, 2, \dots, n$

$$d(x^m, x) = \sqrt{\sum_{i=1}^n (x_i^m - x_i)^2} \quad (\text{By 1})$$

$$\leq \sqrt{\sum_{i=1}^n \left(\frac{\epsilon}{\sqrt{n}}\right)^2} = \epsilon$$

$\Rightarrow \{x^m\}_{m \in \mathbb{N}}$ converges to x

$\Rightarrow \mathbb{R}^n$ is complete.

Ex) $\mathbb{Q}$ with Euclidean Metric is not complete.
 soln eg: $x_n = \left(1 + \frac{1}{n}\right)^n, n \in \mathbb{N}$.

$x_n \rightarrow e$ in $\mathbb{R}$.
 $\{x_n\}$ is Cauchy in $\mathbb{R}$.
 $x_n \in \mathbb{Q} \forall n \in \mathbb{N}$.
 $\Rightarrow \{x_n\}$ is Cauchy in $\mathbb{Q}$.
 $\{x_n\}$ does not have convergent subseqⁿ converging in $\mathbb{Q}$.

eg 2] $x_1 = 1$
 $x_2 = \frac{1}{1+x_1}$

$\{x_n\}$
 $x_{n+1} = \frac{1}{1+x_n}, n \in \mathbb{N}$

$\{x_n\}$ is a seq in $\mathbb{Q}$.
 $\{x_n\}$ is Monotonic decreasing seqⁿ
 bounded below so convergent.
 Suppose $x_n \rightarrow l$

$x_{n+1} (1+x_n) = 1$

$x_{n+1} x_n + x_{n+1} - 1 = 0$

Take $\lim_{n \rightarrow \infty}$.

$l \cdot l + l - 1 = 0$

$l^2 + l - 1 = 0$

$l = \frac{-1 \pm \sqrt{1+4}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$

$\frac{-1 - \sqrt{5}}{2}$ not possible

as l is a seq of the reals
 it cannot go to negative no.

so $l = \frac{-1 + \sqrt{5}}{2}$

Ex) $S = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$

for fixed $a, b, c \in \mathbb{R}, a > 0, b > 0, c > 0$
 S is compact in $\mathbb{R}^3$

Soln: Define $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
 $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

to show: f is continuous.

$\{a_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}^3$ converging seq
 then $f(a_n) \rightarrow f(a)$

$S = f^{-1}(\{1\})$

$\{1\}$ is closed and f is continuous
 $\Rightarrow S = f^{-1}(\{1\})$ is closed.

$S \subseteq [-a, a] \times [-b, b] \times [-c, c]$

S is closed and $[-a, a] \times [-b, b] \times [-c, c]$
 is compact

$\therefore S$ is compact
 $\left. \begin{array}{l} \text{closed subset} \\ \text{of a compact} \\ \text{set is compact} \end{array} \right\}$

Ex) $C = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}$
 not compact.

soln: $y=0$

$$z^2 = x^2$$

$$z = x \quad z = -x$$

$$A = \{(x, 0, x) \mid x \in \mathbb{R}\}$$

$$\subseteq \mathbb{C}$$

A is not bounded

$\Rightarrow \mathbb{C}$ is not bounded $\Rightarrow \mathbb{C}$ is not compact

ex 4/12 Examples on Quotient Spaces.

ex 10 $X = [0, 1]$

define equivalence relation $\sim$ on X

$$x, x' \in [0, 1]$$

$$x \sim x' \text{ iff } \{x, x'\} = \{0, 1\}$$

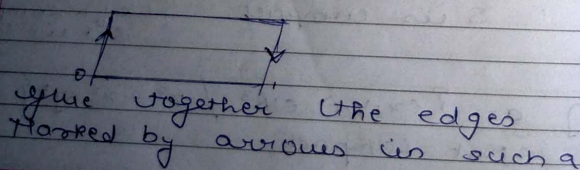
$$\text{or } x = x'$$

Then $X/\sim$ is a circle.

$$X/\sim = \{[x] = x\} \cup \{[0, 1]\}$$

Two sets are equivalent if they are homeomorphic to each other.

ex 11 Möbius Stripe (band)



way that the arrows point in same direction

$$X = [0, 1] \times [0, 1]$$

$\sim$ is defined by

$$(x, y) \sim (x', y') \text{ iff either}$$

$$\{x, x'\} = \{0, 1\} \quad \& \quad y + y' = 1$$

$$\text{or } (x, y) = (x', y')$$

$\therefore X$ is homeomorphic to $X/\sim$

ex 12

show that X is regular. every pair of points of X have nbds whose closures are disjoint.

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prob 10

ex 13 show that if X is normal every pair of disjoint closed sets have nbds whose closures are disjoint.

soln 1: suppose X is regular.

$$x, y \in X \quad x \neq y$$

$$\{x\}, \{y\} \text{ are closed in } X$$

$$x \notin \{y\}$$

X is regular $\Rightarrow \exists$ open sets U and V in X such that $x \in U$, $\{y\} \subseteq V$ and $U \cap V = \emptyset$

so X is regular

$\exists$ a nbd W of x such that $\overline{W} \subseteq U$

$y \in U' \Rightarrow U'$ is a nbd of y
 & nbd V' of y such that

$$\overline{V'} \subset U'$$

$$\overline{V} \cap \overline{V'} \subseteq U \cap U' = \emptyset$$