CHAPTER -3

Special probability distribution and densities

- A number of applied problems in experimental situation have been utilized to the formulation of various discrete probability distribution formulae.
- These distributions are probability distribution.

1. THE DISCRETE UNIFORM DISTRIBUTION

If a random variable can take on k different values with equal probability, we say that it has a discrete uniform distribution.

- **Definition:** A random variable X has a discrete uniform distribution and it is referred to as a discrete uniform random variable its probability distribution is given by $f(x) = {}^{1}/_{k}$,for $x = x_1 , x_2 \ldots\ldots x_k$, where $x_i \neq x_j$, when $i \neq j$.

- Th… …stribution are

$$\mu = \sum_{i=1}^{k} x_i \frac{1}{k} \text{ and } \sigma^2 = \sum_{i=1}^{k} (x_i - \mu)^2 \cdot \frac{1}{k}$$

2. THE BERNOULLI DISTRIBUTION

- In this type of distribution a success may be
  - getting heads with a balanced coin,
  - catching pneumonia,
  - passing /or failing an examination,
  - gaining/ or losing a profit in Business activities.
- This inconsistency is carried over from the days when probability theory was applied only to games of chance. For this reason we refer to an experiment to which the Bernoulli distribution applies as a “Bernoulli trial”. There fore if an experiment has two possible outcomes, “success” and “failure”, and their probabilities are, respectively P and (1 –P), then the number of successes, 0 or 1, has a Bernoulli distribution.

- **Definition:** A random variable X has a Bernoulli distribution and it is referred to as a Bernoulli random variable iff
- its probability distribution is given by:

$$f(x; p) = P^x (1 – p)^{1-x} , \text{ for } x = 0,1$$

- **Example :** In an industry's quality control system it was observed that there is a 95% success in producing quality product

i)What is the probability of obtaining defective products?

ii) List the probabilities for values x = 0,1.

- **Solution** : $f (0 ; 0.95) = 0.95^0 (1 – 0.95)^{1-0} = 0.05$

$f (0; 0.95) = 0.05$; $f(1;0.95) = (0.95)^1 (1 – 0.95)^{1-1} = 0.95$

- In general, if we have p – success and the random variable x = 0, 1 then $f(0;p)= p^0(1- P)^{1-0} = 1 – p,$

$f(1;p) = p^1(1-p)^{1-1} =p$

3. THE BINOMIAL DISTRIBUTION

- Repeated trials play a vital role in various disciplines in general and in probability and statistics in particular, especially when the number of trials is fixed,……

…….the parameter P (probability of success) is the same for each trial, and the trials are all independent.

This type of distribution has many applications, for instance, it applies if we want to know the probability of getting 6 heads in 16 flips of a coin, the probability that 35 of 60 persons will escape from a poverty line, the probability that 18 of 20 persons will recover from HIV. However, this is the case only if each of the 20 person has the same chance of recovering from the disease and their recoveries are independent (say, they are treated by different physicians in different hospitals). To derive the formula for the probability

- **Definition 3.3.** A random variable X has a binomial distribution and it is referred to as a binomial random variable iff…
- its probability distribution is given by:

$$b(x;n,p)=\binom{n}{x}p^{x}(1-p)^{n-x}, \quad \text{for } x=0,1,2,\ldots,n.$$

- Where: n-be the number of trials; x- be the number of observed success; p- be the probability of each trial;

$$\binom{n}{x}={}_nC_x=\frac{n!}{x!(n-x)!}$$

- **Characteristics of the binomial distribution**
  - An outcome of an experiment is classified into one of the two mutually exclusive categories i.e., success or failure.
  - The data collected are the results of counts.
  - The probability of success stays the same for each trial.
  - The trials are independent.

- ***Example1.*** In binomial situation n=4 and p=0.25, determine the probabilities of the following events. i) X=2; ii) x=3
- **So** $b(x;n,p)=\binom{n}{x}p^x(1-p)^{n-x}$, e given problems apply the fo
- for x=0, 1, 2
$$\binom{4}{2}(0.25)^2(1-0.25)^{4-2}=\frac{4!}{2!(4-2)!}(0.25)^2(0.75)^2=6\left(\frac{1}{16}\right)\left(\frac{9}{16}\right)=0.2109$$
- i) b(2;4,0.25)=
$$=\frac{4!}{3!(4-3)!}(0.25)^3(0.75)^{4-3}=0.0469$$
- ii) b(3;4,0.25)
- ***Example2.*** A tele center marketer makes six phone calls per hour and is able to make a sale on 30% of these contacts. During the next two hours, find:

  i) The probability of making exactly four sales.

  ii) The probability of making no sales.

- **Solution:** I. $b(4;12,0.3)=\frac{12!}{4!(12-4)!}(0.30)^4(0.70)^{12-4}=0.0576$

II. $b(0;12,0.3)=\frac{12!}{0!(12-0)!}(0.30)^0(0.70)^{12-0}=0.014$

- ***Example3.*** Suppose 50% of the people in the region prefer inferior goods, we select 10 people for further study. What is the probability 5 of those surveyed will prefer inferior goods?

Solution: $b(5;10,0.=\frac{10!}{5!(10-5)!}(0.5)^5(0.5)^{10-5}=\binom{10}{5}(0.5)^{10}=\frac{63}{256}$

- ***Example 4:*** **In binomial, a binomial distribution n=8 and p=0.30, find the probabilities' of the following** I. $x\le 2$ II. $x\ge 3$ **.**

Solution:

I. $b(x\le 2;8,0.3)=p(x=0)+p(x=1)+p(x=2)$

$\binom{8}{0}(0.3)^0(0.7)^8+\binom{8}{1}(0.3)^1(0.7)^7+\binom{8}{2}(0.3)^2(0.7)^5=$

$0.058+0.198+0.296=0.552$

II. $b(x\ge 3;8,0.3)=1-b(x\le 2;8,0.3)=1-0.552=0.448$

- **Theorem:** The mean and the variance of the binomial distribution are: = np and = npq respectively.
- ***Example 5:*** If the probability of defective goods in a particular factory is 0.1, find the mean and the variance, for the distribution of defective goods in a total (200.
- **Solution:-** Given: n = 200; p =0.1; (= 1-p = 1-0.1= 0.9, hence =np = 200 x1/10=20 & npq =200(0.1)(0.9)= 18
- ***Exampl*** ution, =4 and var = 2.4, fi .
- **Solutio** pq = 2.4,

Then,

$$q = (\sigma^2 / \mu) = (npq/np) = (2.4)/4 = 0.6$$

$$p = 1-q = 1-0.6 = 0.4$$

4. **The hyper geometric distribution**

- Here we are choosing, with out, replacement, n of the N elements contain in the set, there $\binom{M}{x}$ ways of choosing x of the M successes and $\binom{N-M}{n-x}$ ways of choosing (n–x) of the (N–M) failures and, hence $\binom{M}{x}\binom{N-M}{n-x}$ ways of choosing x successes and (n–x) failures.

Since there are $\binom{N}{n}$ ways of choosing n of the N elements in the set, assuming that they are all equally likely, there fore, the probability of "x successes in n trials" is $\binom{M}{x}\binom{N-M}{n-x}/\binom{N}{n}$

- **Definition:** A random variable X has a hyper geometric distribution and it is referred to as a hyper geometric random variable iff
- its probability distribution is given by:
- $$h(x;n,N,M) = \frac{{}_{M}C_{x} \cdot {}_{N-M}C_{n-x}}{{}_{N}C_{n}}$$, for x = 0, 1, 2, …, n.

- X ≤ M and n – x ≤ N – M
- N – The size of the population
- M – Number of successes in the population
- n – The size of sample (number of trials)
- x – Number of successes in the sample (n)
- C – The symbol for a combination

- **The hyper geometric distribution should be applied:**

- ✓ If the sample is selected from a finite population
- ✓ The size of the sample n is more than or equal to 5% of the population N
- ✓ It is especially appropriate when the size of the population is small

- ***Example 1:*** A factory employs 50 people in the assembly department 40 of the employees belong to a union. 5 employees are selected at random to form a committee to meet with management regarding shift starting times. What is the probability that 4 of the 5 selected for the committee belong to a union?
- **Solution:** N= 50 the number of employees
- M = 40 the number of union employees
- X = 4 the number of union employees selected
- n = 5 the number of employees selected

$$h(4;5,50,40) = \frac{{}_{40}C_4 \cdot {}_{50-40}C_{5-4}}{{}_{50}C_5} = \frac{\left(\frac{40!}{4!36!}\right)\left(\frac{10!}{1!9!}\right)}{\left(\frac{50!}{5!45!}\right)} = \frac{(91390(10))}{2118760} = 0.431$$

Or it is possible to solve the above problem using binomial distribution hence

let p=(M/N) =(40/50) = 0.8, n = 5, x = 4, then $b(4;5,0.8) = {}_5C_4 (0.8)^4 (1-0.8)^{5-1}$

$= 5!/4!/! (0.8)^4 (.2)^1$

$= 5 \text{ X } 0.4096 \text{ X } 0.2 = 0.410$

- **Theorem:** The mean and the variance of the hyper geometrical distribution are: $\mu = \frac{nM}{N}$, and $\sigma^2 = \frac{nM(N-M)(N-n)}{N^2(N-1)}$

1) Prove that $\mu = \frac{nM}{N}$?

Proof: $\mu = \sum x \cdot \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} = \frac{x \cdot \frac{m(m-1)!}{x(x-1)!(m-x)!}\binom{N-m}{n-x}}{\binom{N}{n}} = \frac{m}{\binom{N}{n}}\binom{m-1}{x-1}\binom{N-m}{n-x} =$

$= \frac{m}{\frac{N(N-1)!}{n(n-1)!(N-n)!}} \cdot \binom{m-1}{x-1}\binom{N-m}{n-x} = \frac{nm}{N} \cdot \frac{\binom{m-1+N-m}{x-1+n-x}}{\binom{N-1}{n-1}} = \frac{nm}{N} \cdot \frac{\binom{N-1}{n-1}}{\binom{N-1}{n-1}} = \frac{nm}{N}$

2) Variance= $\sigma^2 = \frac{nm(N-m)(N-n)}{N^2(N-1)}$

Proof: To prove it let us 1st evaluate $\mu_2^1 = E(x^2) = E[X(X-1)] + E(X)$ and then, $\mu^2 = [E(X)]^2$.

hence, $\sigma^2 = E[X(X-1)] + E(X) - [E(X)]^2 =$

$$= \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}\right] + \frac{nm}{N} - \left(\frac{nm}{N}\right)^2 = \frac{nm(n-1)(m-1)}{N(N-1)} + \frac{nm}{N} - \left(\frac{nm}{N}\right)^2 =$$

$$\frac{nm[N(n-1)(m-1) + N(N-1) - (N-1)nm]}{N^2(N-1)} = \frac{nm[Nnm - Nn - Nm + N + N^2 - N - Nnm + nm]}{N^2(N-1)} =$$

$$\frac{nm[N^2 - Nm - Nn + nm]}{N^2(N-1)} = \frac{nm[N(Nm) - n(N-m)]}{N^2(N-1)} = \frac{nm[N-m][N-n]}{N^2(N-1)}$$

- ***Example 2:*** Suppose the Economics Department of X- University wants to recruit four people for a teaching position and for this 16 applicants have applied. Among the 16 applicants for a teaching position, 10 have university degree. If 4 of the applicants are randomly chosen for interviews, what are the mean and the variance?
- **Solutio**

$$\sigma^2 = \frac{nm(N-m)(N-n)}{N^2(N-1)} = \frac{4X10(16-10)(16-4)}{16^2(16-1)} = \frac{40X6X12}{16X16X15} = \frac{3}{4} = 0.75$$

$$\mu = \frac{nm}{N} = \frac{4x10}{16} = \frac{10}{4} = 2.5.$$

Example 3: Based on the example above, if two of the four will be taken for the job, what will be the probability?

Solution:

$$b(2;4,16,10) = \frac{\binom{10}{2}\binom{16-10}{4-2}}{\binom{16}{4}} = \frac{\binom{10}{2}\binom{6}{2}}{\binom{16}{4}} = \frac{\left[\frac{10!}{2!8!}\cdot\frac{6!}{2!4!}\right]}{\left[\frac{16!}{4!12!}\right]} = \frac{135}{364} = 0.371$$

5. THE POISSON DISTRIBUTION

- We have seen the binomial distribution so far.
- As the number of trials (n) be come large the calculation of binomial probabilities with the formula of definition 3.3 will involve a prohibitive amount of work.
- For instance to calculate the probability that 2 of 10,000 persons watching (celebrating) a ceremony of" timket" on a very hot day will suffer from heat exhaustion,
- we first have to determine $\binom{10000}{2}$ and if the probability is 0.0005, that any one of the 10,000 persons cerebrating the ceremony will suffer from heat exhaustion we also have to calculate the valve of $[0.0005]^2(0.9995)^{9998}$.
- So in this case, since it gives a burden we are forced to look at the poison probability distribution.

- Therefore the limiting form of the binomial distribution where the probability of success p is very small and n is large is called the Poisson distribution [$[\text{ie}, p \to 0, n \to \infty]$

- **Definition:** A random variable X has a Poisson distribution and it is referred to as a poison random variable iff
- its probability distribution is given by:

$$P(x; m) = p(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \text{ for } x=0,1,2,3, \text{------}$$

- Where λ =np, e= 2.71828 is the base of the system of natural garithm.
- Poisson distribution will provide a good approximation to binomial probabilities when n≥30 and p ≤ 0.05 and np<5.
- Poison distribution may be obtained as a limiting case of Binomial probability distribution under the following conditions:
  - n, the number of trials is indefinitely large
  - P, the constant probability of success for each trial is indefinitely small $(\text{i.e., } p \to 0)$
  - np = λ, (say which is finite). $(\text{i.e., } n \to \infty)$

- ***Example 1:*** Calculate the probability that 2 of 10000 persons celebrating a ceremony of " Timket" on a very hot day will suffer from heat exhaustion, if the probability is 0.0005, that any one of the 10000 persons celebrating the ceremony will suffer from

Solution $\lambda = np = 10000.\frac{5}{10000} = 5; x = 2, p(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

$$p(2;5) = \frac{e^{-5}5^{2}}{2!} = \frac{0.0067x25}{2x1} = 0.08375$$

- ***Example 2:*** If the probability that an individual suffers from allergy from the by-product of tannery's factory is 0.001, d individuals.

a) x = 3 b) > 2

Solution: $\lambda = np = 2000x\frac{1}{1000} = 2 \Rightarrow p(3;2) = \frac{e^{-2}2^{3}}{3!} = \frac{e^{-2}x8}{6} = 0.1804$

b) $p(x>2; 2) = p(3;2) + p(4;2) + p(5;2) + \ldots + P(n; 2)$

$$= 1 - \left[\frac{e^{-2}2^{0}}{0!} + \frac{e^{-2}x2^{1}}{1!} + \frac{e^{-2}x2^{2}}{2!}\right]$$

$$= 1 - 0.1353 - 0.2706 - 0.2706 = 0.3233$$

Example 3: If 5% of the marble manufactured by a company are defective, what is the probability that in a sample of 100 marbles.

a) none is defective

Solution

$$\lambda = np = 100x\frac{5}{100} = 5$$

$$a)\ x = 0 \Rightarrow p(0;5) = \frac{e^{-5}5^{0}}{0!} = e^{-5} \approx 0.007$$

$$b)\ \ p(5;5) = \frac{e^{-5}5^{5}}{5!} = \frac{0.007x625x5}{5x24} = 0.1823$$

- **Therefore:** The mean and the variance of the poison distribution are $\mu = \lambda$ $\sigma^2 =$ =

Proof ??????

- **Examples 4:** In poison distribution, if p(x) for x = 0 is 10%, find the mean and the variance of the distribution.
- **Solution:** $P(x=0)=\frac{e^{-\lambda}\lambda^0}{0!}=10\%=e^{-\lambda};\Rightarrow \ln e^{-\lambda}=\ln 10\%$

$$\Rightarrow \lambda=\ln 10\approx 2.3026$$

Examples 5: If the standard deviation of a Poisson distributi ι and ?

Solution

$$p(x=0)=\frac{e^{-(\sqrt{2})^2}\cdot 2^0}{0!}=e^{-2}=\frac{1}{e^2}=0.135$$

$$\mu=\sigma^2=\left(\sqrt{2}\right)^2=2$$

DENSITIES

1. The uniform density (Rectangular distribution)

Definition. A random variable has a uniform distribution and it is referred to as a continuous uniform ran... probability density is g...

$$U(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta-\alpha}, & for\ \alpha<x<\beta,\\ 0, & elsewhere\end{cases}$$

the parameters α and β ...is probability density are real constants numbers, with $\alpha<\beta$.

Example 1. In a certain experiment, the error made in determining the density of a substance is a random variable having a uniform density with

α= - 0.015 and β= 0.015. Find the probability

Solution:

$$U[(-0.002, 0.003); -0.015, 0.015] = \left[\frac{1}{0.015-(-0.015)}\right]\int_{-0.002}^{0.003} dx = \frac{1}{0.03}\int_{-0.002}^{0.003} dx =$$

$$= \frac{1}{0.03}[x]_{-0.002}^{0.003} = \frac{0.005}{-0.03} = \frac{1}{6} = 0.167$$

Theorem 3 .4. The mean and the variance of the uniform distribution are

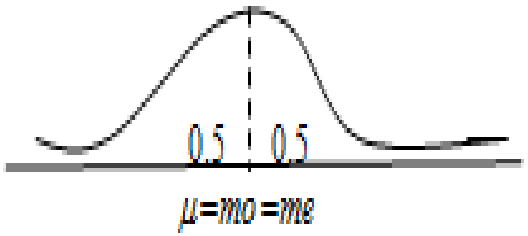
Given by: $\mu = \frac{\alpha+\beta}{2}; \sigma^2 = \frac{1}{12}(\beta-\alpha)^2.$

- **Definition:** A random variable X has a normal distribution and it is referred to as a normal random variable its probability density is given by:

$$n(x;\ \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, for, -\infty < x < \infty, where, \sigma > 0.$$

Characteristics of a normal probability distribution

i. This is bell shaped and has a simple peak at the exact center of the distrib ırtic



ii. Mean (μ), mode (M_o) and median (M_e) of the distribution are equal and located at the peak
iii. Half the area under the curve is above the peak ,and the other half is below it
iv. The normal probability distribution is symmetrical about its mean,
v. The normal probability distribution is asymptotic i.e., the curve gets closer and closer to the axis but never actually touches it.

The normal d $n(x;\ \mu,\sigma) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} = f(x), x \approx N(0,1)$ referred to as the standa ution).

Let us show it: we know that $Z = \frac{x-\mu}{\sigma}$, then we have

$$E(Z) = E\left[\frac{x-\mu}{\sigma}\right] = \frac{1}{\sigma}E(x-\mu) = \frac{1}{\sigma}\left[E(x) - E(\mu)\right] = \frac{1}{\sigma}\left[\mu - \mu\right] = 0, and$$

$$Var(Z) = Var\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(x-\mu) = \frac{1}{\sigma^2}Var(x), \therefore Var(Z) = \frac{1}{\sigma^2}\cdot\sigma^2 = 1$$

$Var(ax) = a^2 Var(x)$.

- Hence the probability density function of standard normal distribution

$$Z \text{ is given by: } \phi(Z) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{Z^2}{2}}, -\infty < Z < \infty$$

NORMAL PROBABILITY DISTRIBUTION

- This method calls for standardizing the distribution or changing from the X scale to Z- scale.
- To find the area between a value of interest (x) and the mean (μ),
- we first compute the difference between the value (x) and the mean (μ);
- then we express that difference in units of standard deviation.
- In other words we compute the valu $Z = \frac{x - \mu}{\sigma}$
- Finally, we find the desired area under the curve, or the probability, by referring to a table whose entry corresponds to the calculated value of Z.

- Therefore the valve of Z follows a normal probability distribution with a mean of zero and standard deviation of one unit.
- This probability distribution is called standard normal probability distribution.
- Thus we can converse any normal distribution to the standard normal distribution.

Using normal distribution steps

- Draw picture
- Transform the given value to Z-value
- Shade the area desired
- Read the area from standard normal distribution table

PROPERTIES OF THE STANDARD NORMAL DISTRIBUTION

- Same as a normal distribution, but also Mean is zero

Variance is one and Standard Deviation is one

- Areas under the standard normal distribution curve have been tabulated in various ways.
- The most common ones are the areas between Z=0 and a positive of Z
- Given a normal distributed random variable X with Mean (μ) and standard deviation (□)

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$\Rightarrow P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

Note:

$$P(a < X < b) = P(a \le X < b)$$

$$= P(a < X \le b)$$

$$= P(a \le X \le b)$$

- Examples 1: The monthly incomes of employees in a textile factory are normally distributed with a means of birr 2000 and a standard deviation of birr 200.
  a) What is the Z value for an in come of birr 2200?
  b) What is the Z value for an income of 1700?

Solution:

a) Given $x = 2200$; $\mu = 200$ and $\sigma = 200$

then for $x = 2200$; $Z = \frac{x-\mu}{\sigma} = \frac{2200-2000}{200} = 1$

b) For $x = 1700$; $Z = \frac{x-\mu}{\sigma} = \frac{1700-2000}{200} = -1.5$

- Hence a Z- $_{value}$ of 1 indicates that the value of 2200 is one standard deviation above the mean of 2000, while a Z- $_{value}$ of 1700 is 1.5 standard deviation below the mean of 2000.

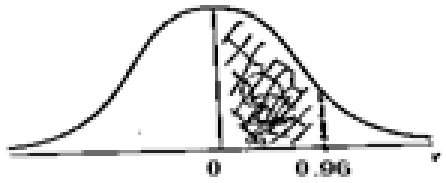
Examples 2: Find the area under the standard normal distribution which lies

a) Between Z=0 and Z=0.96
b) Between Z=-1.45 and Z=0
c) To the right of Z=-0.35
d) To the left of Z=-0.35
e) Between Z= -0.67 and Z=0.75
f) Between Z=0.25 and Z=1.25
g) P(Z>1.91) ?

a) Between $Z = 0$ *and* $Z = 0.96$

Solution:

$Area = P(0 < Z < 0.96) = 0.3315$



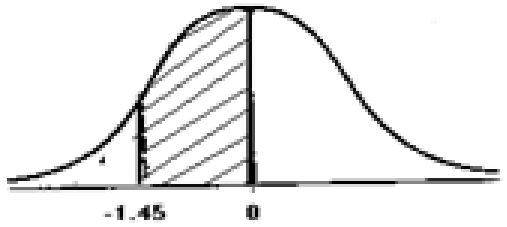
b) Between $Z = -1.45$ *and* $Z = 0$

Solution:

$Area = P(-1.45 < Z < 0)$

$= P(0 < Z < 1.45)$

$= 0.4265$



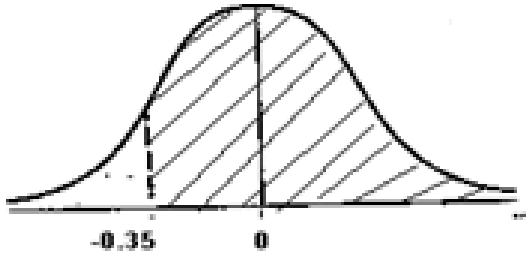
c) To the right of $Z = -0.35$

$Area = P(Z > -0.35)$

$= P(-0.35 < Z < 0) + P(Z > 0)$

$= P(0 < Z < 0.35) + P(Z > 0)$

$= 0.1368 + 0.50 = 0.6368$



d) To the left of $Z = -0.35$

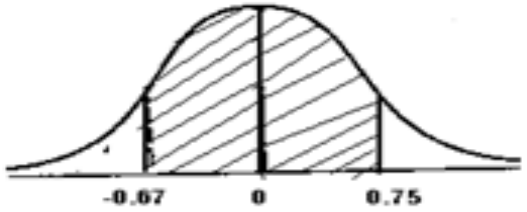
Solution:

$$\begin{aligned} Area &= P(Z < -0.35) \\ &= 1 - P(Z > -0.35) \\ &= 1 - 0.6368 = 0.3632 \end{aligned}$$

e) E) Between $Z = -0.67$ *and* $Z = 0.75$

Solution:

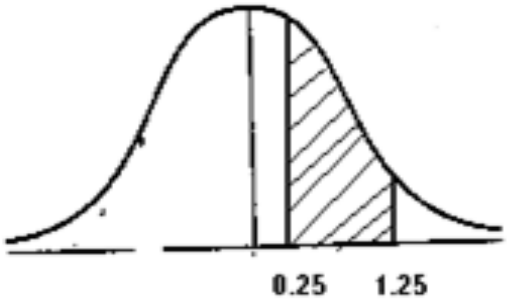
$$\begin{aligned} Area &= P(-0.67 < Z < 0.75) \\ &= P(-0.67 < Z < 0) + P(0 < Z < 0.75) \\ &= P(0 < Z < 0.67) + P(0 < Z < 0.75) \\ &= 0.2486 + 0.2734 = 0.5220 \end{aligned}$$



f) Between $Z = 0.25$ *and* $Z = 1.25$

Solution:

$$\begin{aligned} Area &= P(0.25 < Z < 1.25) \\ &= P(0 < Z < 1.25) - P(0 < Z < 0.25) \\ &= 0.3934 - 0.0987 = 0.2957 \end{aligned}$$



Examples 3: Find the value of Z if

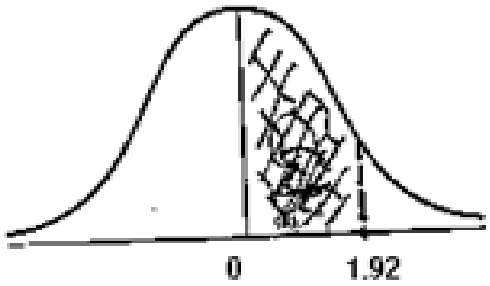
a) The normal curve area between 0 and z(positive) is 0.4726

Solution

$$P(0 < Z < z) = 0.4726 \text{ and from table}$$

$$P(0 < Z < 1.92) = 0.4726$$

$$\Leftrightarrow z = 1.92 \ldots\ldots \text{uniqueness of Areea.}$$



b) The area to the left of z is 0.9868

Solution

$$P(Z < z) = 0.9868$$

$$= P(Z < 0) + P(0 < Z < z)$$

$$= 0.50 + P(0 < Z < z)$$

$$\Rightarrow P(0 < Z < z) = 0.9868 - 0.50 = 0.4868$$

$$\text{and from table}$$

$$P(0 < Z < 2.2) = 0.4868$$

$$\Leftrightarrow z = 2.2$$

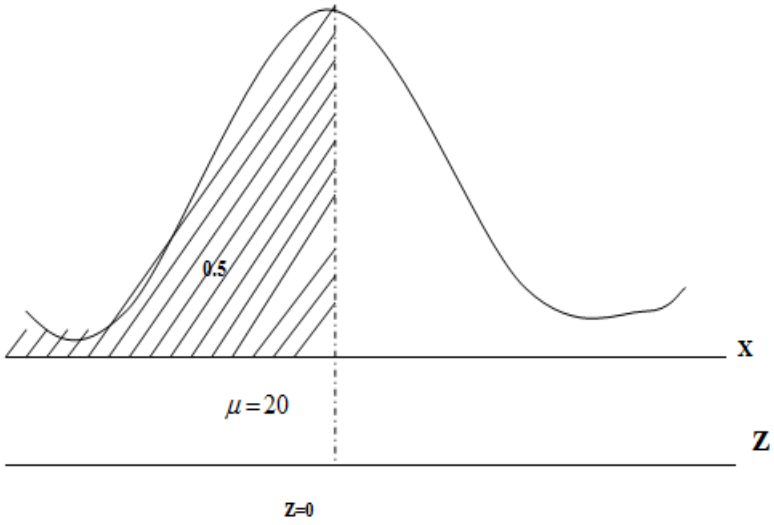
Example 4:The daily water usage per person in Ethiopia is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

i. *What is the probability that a person from Ethiopia selected at random will use less than 20 gallons per day?*

ii. *What percentage use between 20 and 24 gallons?*

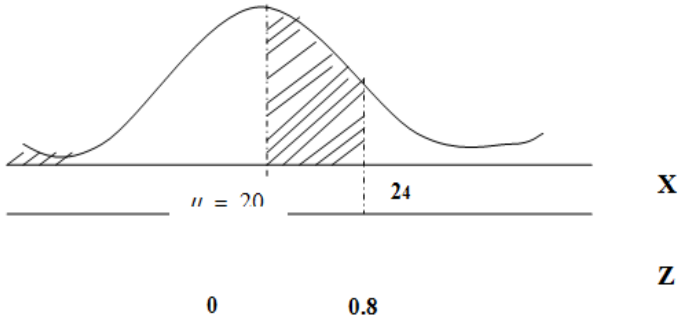
iii. *What percentage of the population uses between 18 and 26*

i) $Z\text{-value is} = Z = \frac{x-\mu}{\sigma} = 0, \text{thus } P(x<20) = p(Z<0) = 0.5$



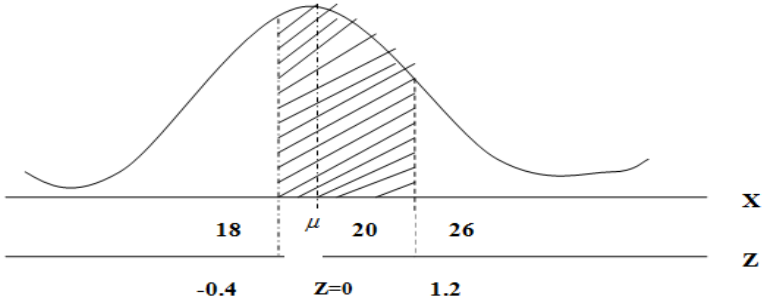
ii) The Z_{value} with $X = 20$ is $Z = \frac{x-\mu}{\sigma} = \frac{20-20}{5} = 0$, and with $X=24$ is , $Z = \frac{24-20}{5} = \frac{4}{5} = 0.8$,

then $P(20 < x < 24) = p(0 < Z < 0.8) = 0.2881$,using the Z table $\therefore$ it is 28.81%.



iii) The $Z -_{value}$ with $X = 18$ is $Z = \frac{18-20}{5} = -0.4$ and for $X = 26$, $Z = \frac{18-20}{5} = 1.2$, thus

$p(18 < x < 26) = p(-0.4 < x < 1.2)$, using the Z -table $\Rightarrow$ $0.1554 + 0.3849 = 0.5403$ it is 54.03%



Relation between Poisson and normal distribution

If $E(x) = \lambda =$ mean and var $(x) = \sigma^2 = \lambda$, for $\lambda \to \infty$,

thus standard poison variate becomes (Spv): $Z = \frac{x - E(x)}{\sigma} = \frac{x - \lambda}{\sqrt{\lambda}}$

Relation between binomial and normal distribution

Normal distribution is a limiting case of binomial probability distribution under the condition that n, the number of trials, n close to large, and either P or q is very small, then since ....

E(x) =np a... ...der binomial probabilit... $Z = \frac{x - E(x)}{\sigma} = \frac{x - np}{\sqrt{npq}}$ ...e standard Binomial ...

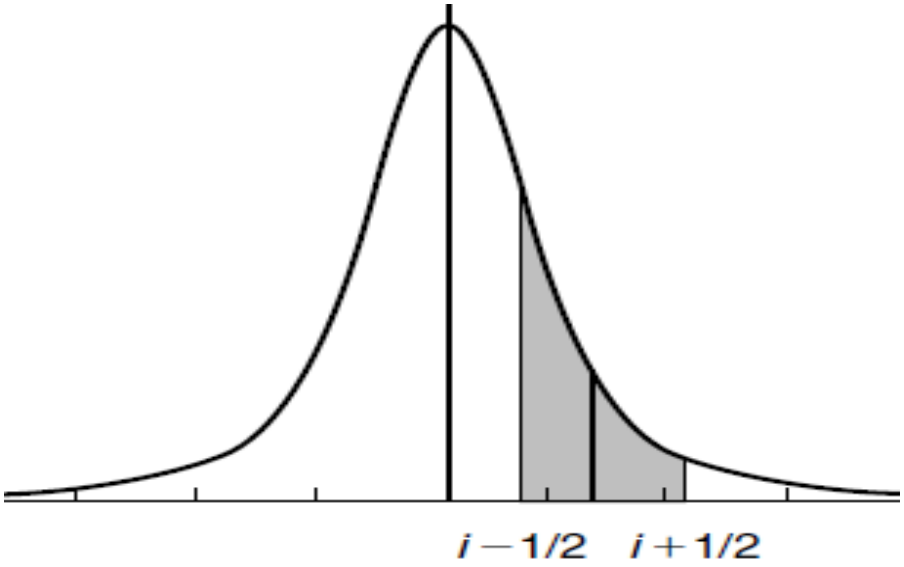
- The normal distribution is sometimes introduced as a continuous distribution, which provides a close approximation to the binomial, which is a discrete one
- **when n, number of trials is very large and P the probability of success on the individual trial is close to ½.**

WHEN TO USE THE NORMAL APPROXIMATION

- The normal probability distribution is a good approximate to the binomial probability distribution where np and n (1-p) are both at least 5. However,...
- before we apply the normal approximation, we must make sure that our distribution of interest is in fact a binomial distribution.
- So, when we approximate a discrete probability distribution by a continuous one we need to make an adjustment called the continuity correlation.
- In this adjustment, we add and/or subtract 0.5 to the value that the binomial distribution is approximated by the normal.

How to apply the correction factor

- For the probability at least X occur, use the area above (X – 0.5)
- For the probability that more than X occur, use the area above (X + 0.5)
- For the probability that X or fewer occur, use the area below (X + 0.5)
- For the probability that fewer than X occur, use the area below (X – 0.5)



Continuity correction for $P(X = i)$.

(a) To approximate $P(X \leq a)$ or $P(X > a)$, the correction for continuity is $(a + 0.5)$, that is,

$$P(X \leq a) = P\left(Z < \frac{(a + 0.5) - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(X > a) = P\left(Z > \frac{(a + 0.5) - np}{\sqrt{np(1-p)}}\right).$$

(b) To approximate $P(X \geq a)$ or $P(X < a)$, the correction for continuity is $(a - 0.5)$, that is,

$$P(X \geq a) = P\left(Z > \frac{(a - 0.5) - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(X < a) = P\left(Z < \frac{(a - 0.5) - np}{\sqrt{np(1-p)}}\right).$$

(c) To approximate $P(a \leq X \leq b)$, treat ends of the intervals separately, calculating two distinct z-values according to steps (a) and (b), that is,

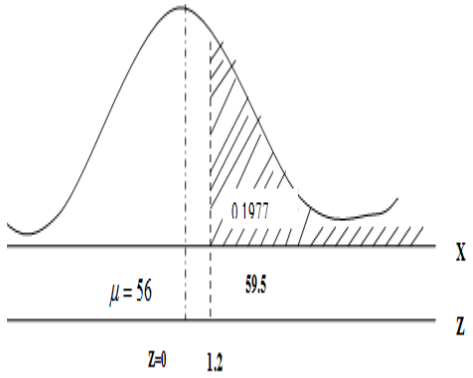
$$P(a \leq X \leq b) = P\left(\frac{(a - 0.5) - np}{\sqrt{np(1-p)}} < Z < \frac{(a + 0.5) - np}{\sqrt{np(1-p)}}\right).$$

Examples 5: A restaurant found that 70% of their new customers return for another meal. For a week in which 80 new customers were denied occurs at that restaurant's, what is the probability that 60 or more will return for another meal?

Solution: - given n=80; p=0.70; x ≥ 60,

$$\mu = np = 80 \times 0.70 = 56$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{80.\frac{70}{100}.\frac{30}{100}} = \sqrt{16.8} = 4.10$$



$$X = x_{given} = 0.5 = 60 - 0.5 = 59.5$$

$$Z = \frac{x-\mu}{\sigma} = \frac{59.5-56}{4.10} = 0.85, P(Z = 0.85) = 0.3023$$

$$P(60 \le x < 80) = P(Z \ge 0.85) = 0.5 - 0.3023 = 0.1977$$

*Examples 6:*Assume a binomial probability distribution with n=50 and p=0.25, compute the following:

A. the mean and the standard deviation of the random variable
B. the probably that x is 15 or more

Solution:

a. $\mu = np = 50 \times 0.25 = 12.5$

$$\sigma = \sqrt{np(1-p)} = \sqrt{50x0.25x0.75} = \sqrt{9.375} = 3.0619$$

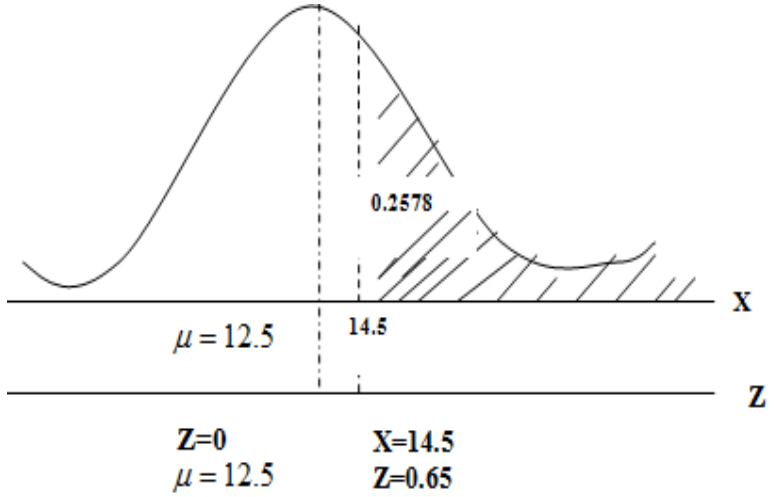
b. $X = 15 - 0.5 = 14.5$

$$Z = \frac{14.5 - 12.5}{3.0619} = 0.65$$

$P(Z = 0.65) = 0.2422$,

then $P(x \geq 15) = P(Z \geq 0.65) =$

$0.5 - 0.2422 = 0.2578$ is the area

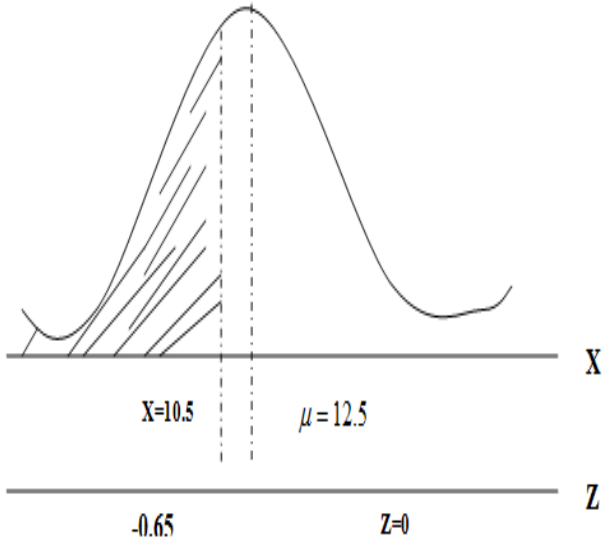


c. $x = 10 + 0.5 = 10.5$

$$z = \frac{10.5 - 12.5}{3.0619} = -0.65,$$ the area is $0.2422 = P(Z = 0.65)$

Then $P(x \leq 10) = P(Z \leq 0.65) = /-0.5 + 0.2422/$

$= P(0 < x \leq 10) = 0.5 - 0.2422 = 0.2578$



*Examples 7:*If *z* is a random variable having the standard normal distribution, find the probabilities that it will take on a value:

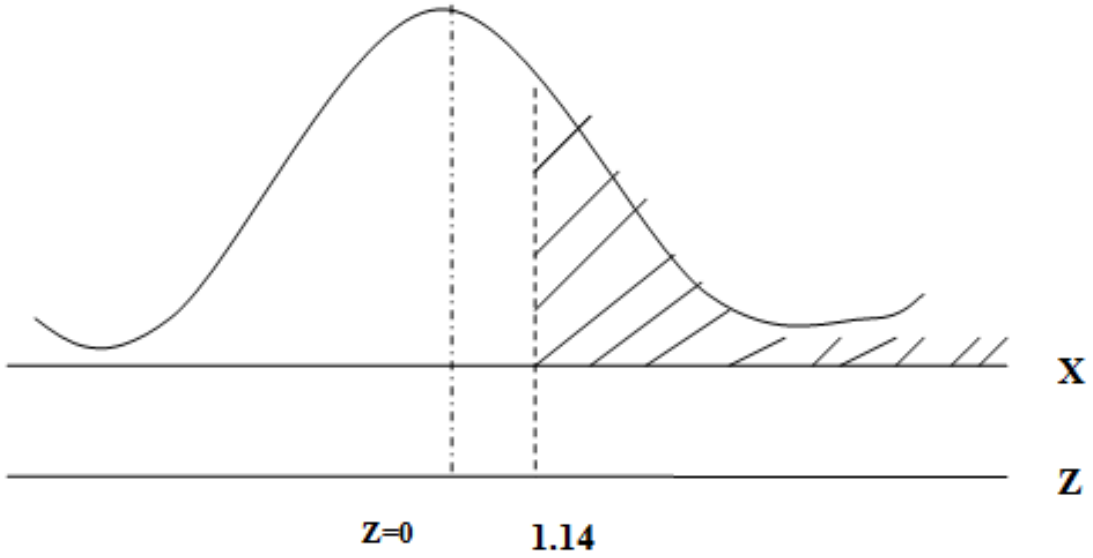
a) greater than 1.14

b) greater than – 0.36

- **Solution**:

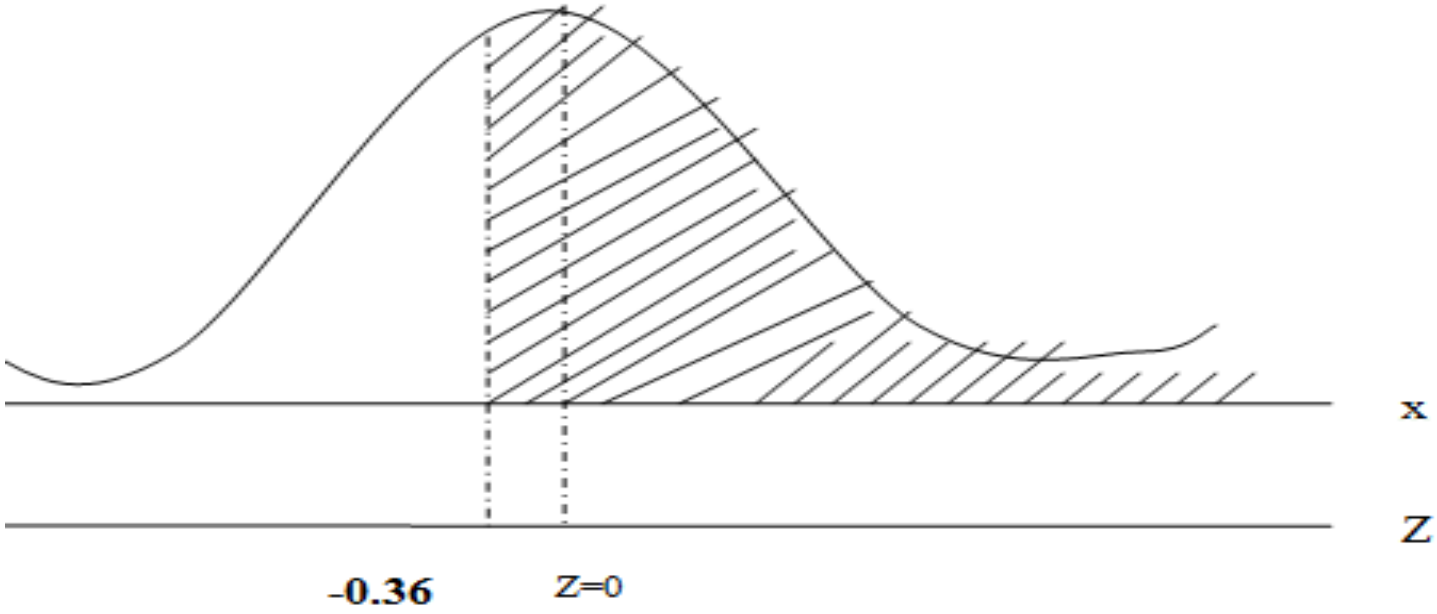
a. Z_{value}, $Z = 1.14$ from the normal distribution Z-table = 0.3729

Probability greater than $Z \Rightarrow P(Z > 1.14) = 0.5 - 0.3729 = 0.1271$



b. $/Z/ = / -0.36/ = 0.1406$

Therefore $P(Z > -0.36) = 0.5 + 0.1406 = 0.6406$



GAMMA PROBABILITY DISTRIBUTION

- The gamma probability distribution has found applications in various fields.
- We describe the gamma function before we introduce the gamma probability distribution.

The *gamma function*, denoted by $\Gamma(\alpha)$, is defined as $\Gamma(\alpha)=\int_{x=0}^{\infty} x^{\alpha-1}e^{-x}dx, \alpha>0$

It can be shown using the integration by parts that for $\alpha>1$, *if* $\alpha=1$, $\Gamma(1)=1$, but if $\alpha>1$

Let, u= $x^{\alpha-1}$, dv = $e^{-x}dx$

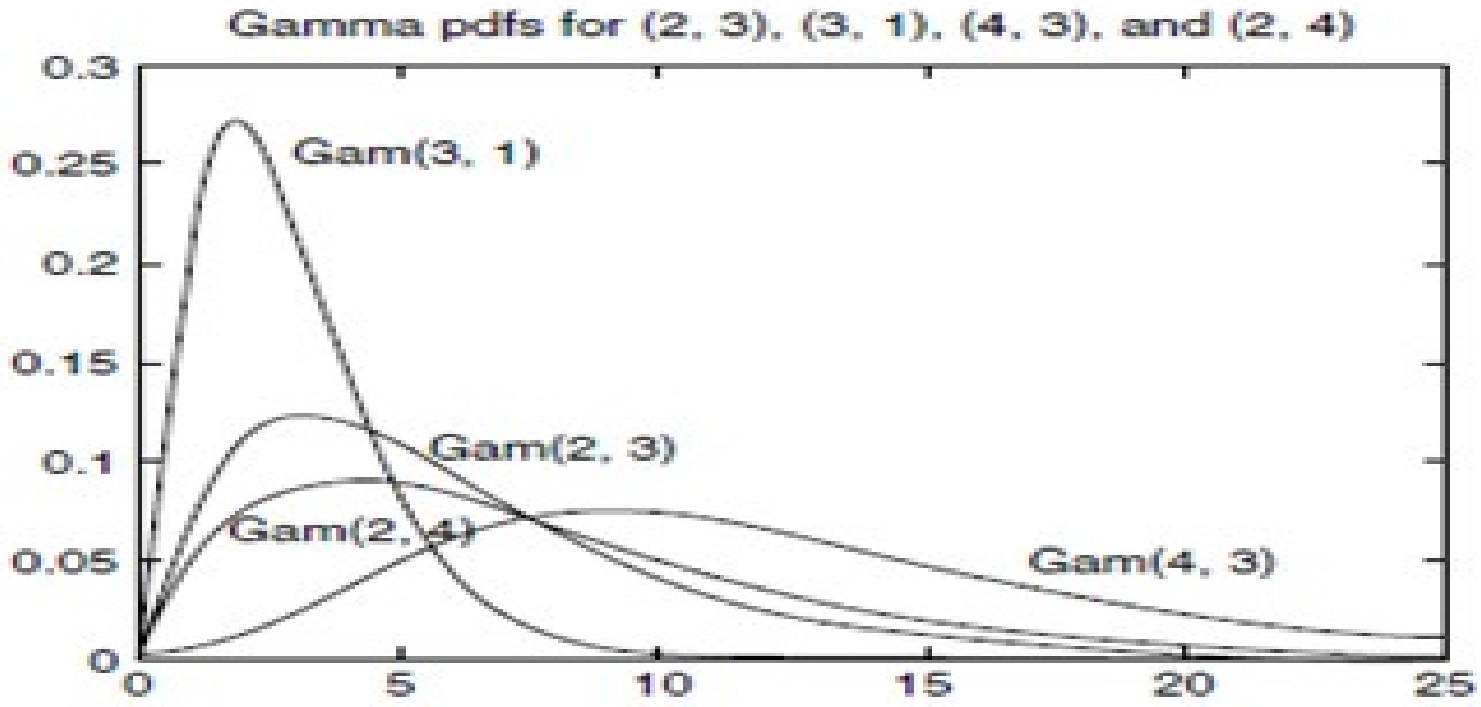
$$\Gamma(\alpha) = (\alpha-1)\int_0^{\infty} x^{\alpha-2}e^{-x}\,dx = (\alpha-1)\Gamma(\alpha-1).$$

If $\alpha=n$ is a positive integer, $\Gamma(\alpha)=(n-1)!$

A random variable X is said to possess a **gamma probability distribution** *with parameters* $\alpha>0$ *and* $\beta>0$ *if it has the pdf given by*

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}, & \text{if } x>0 \\ 0, & \text{otherwise.} \end{cases}$$

- The gamma density has two parameters, α and β. We denote this by *Gamma(α, β)*.
- The parameter α is called a *shape parameter*, and β is called a *scale parameter*.
- Changing α changes the shape of the density, whereas varying β corresponds to changing the units of measurement (such as changing from seconds to minutes).
- Varying these two parameters will generate different members of the gamma family.
- If we take α to be a positive integer, we get a special case of gamma probability distribution, known as the *Erlang distribution*.
- This is used extensively in queuing theory to model waiting times.



- *If X is a gamma random variable with parameters α > 0 and β > 0, then E(X) = αβ and Var(X) = αβ² and also, the moment-generating function is* $M_X(t) = \frac{1}{(1-\beta t)^{\alpha}}$ *,t<* $\frac{1}{\beta}$

EXPONENTIAL DISTRIBUTION

- In the pdf of the gamma, let α = 1, we get the pdf of an exponential random variable.
- A random variable X is said to have an **exponential probability distribution** with parameter β if the pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & \beta > 0; 0 \le x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Let } \frac{1}{\beta} = \lambda, \; f(x) = \begin{cases} \lambda e^{-\lambda x} \\ 0, \text{ otherwise} \end{cases}$$

- *If X is an exponential random variable with parameters β > 0, then E(X) = β and Var(X) = β². Also the moment-generating function is* $M_X(t) = \frac{1}{(1-\beta t)^{\alpha}}$, $t < \frac{1}{\beta}$

BETA DISTRIBUTION

$$f(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad 0<x<1$$

Here, the function in the denominator is the **Beta function** which is defined for α, $\beta>0$ and equals

$$B(\alpha,\beta)=\int_{x=0}^{1} x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The mean and variance of a beta random variable are $\mathbf{E}(X)=\frac{\alpha}{\alpha+\beta}$ &

$$\mathbf{Var}\,(X)=\frac{\alpha\beta}{((\alpha+\beta)^2(\alpha+\beta+1))}$$

The beta distribution has a wide range of shapes. If $\alpha=\beta$ it is symmetrical about the point $x=1/2$; otherwise it will be skewed. Contains the uniform distribution as a special case. Put $\alpha=\beta=1$ and we get a Uni (0, 1) distribution.