Chapter 1 – Vectors (Fully Self-Contained Learning System)

1.1 Scalars and Vectors

A scalar is a quantity that has only magnitude.
Magnitude means "size" or "amount."
Scalars do not have direction.

Examples of scalars:
- Mass (5 kilograms)
- Temperature (30 °C)
- Time (10 seconds)
- Speed (60 km/h)

A vector is a quantity that has both magnitude and direction.
This means we must describe how large it is and also where it points.

Examples of vectors:
- Velocity (60 km/h north)
- Force (10 newtons upward)
- Displacement (5 meters east)
- Acceleration (2 m/s² downward)

1.1.2 Representation of Vectors

Vectors can be represented in two main ways:

Analytical (Algebraic) Representation

We write a vector using a letter with an arrow on top, like $\rightarrow A$, or in bold, like A.

Example:

- Force vector: F

- Momentum vector: P

This way is useful when we want to calculate using algebra.

Graphical (Geometrical) Representation

We draw a straight line with an arrow.

- The length of the line shows the magnitude.

- The arrowhead shows the direction.

- The tail is the starting point.

- The head is the ending point.

Example:

If a displacement is 5 meters east, we draw a line 5 units long pointing east.

1.1.3 Vector Addition and Subtraction

The sum of two or more vectors is called the resultant vector (R).

Rule 1: Same Direction

If two vectors point in the same direction, add their magnitudes.

Example:

A = 8 m east, B = 6 m east

$$R = A + B = 8 + 6 = 14 \text{ m east}$$

Rule 2: Opposite Direction

If two vectors point in opposite directions, subtract their magnitudes.

The resultant points in the direction of the larger vector.

Example:

A = 8 m east, B = 6 m west

$$R = A + (-B) = 8 - 6 = 2 \text{ m east}$$

Rule 3: Perpendicular Vectors

If two vectors act at right angles, use Pythagoras' theorem.

Example:

A = 8 m east, B = 6 m north

Step 1: Square both magnitudes.

$$A^2 = 8^2 = 64$$

$$B^2 = 6^2 = 36$$

Step 2: Add them.

$A^2 + B^2 = 64 + 36 = 100$

Step 3: Take square root.

$R = \sqrt{100} = 10$ m

Step 4: Find direction using tangent.

$\tan \theta = \text{opposite} / \text{adjacent} = B / A = 6 / 8 = 0.75$

$\theta = \tan^{-1}(0.75) = 36.87°$ north of east

So, R = 10 m at 36.87° north of East

Triangular Method

Place the tail of the second vector at the head of the first vector.

Draw the resultant from the tail of the first to the head of the last.

Parallelogram Method

Place both vectors starting at the same origin.

Complete a parallelogram.

The diagonal from the origin is the resultant vector.

Formula:

$R^2 = A^2 + B^2 + 2AB \cos \theta$

If $\theta = 90°$, $\cos \theta = 0$, ***so*** $R^2 = A^2 + B^2$.

This matches the Pythagoras theorem.

Example Problems (Step-by-Step)

1. A = 4 units east, B = 3 units east

$R = 4 + 3 = 7$ ***units east***

2. A = 4 units east, B = 3 units west

R = 4 – 3 = 1 unit east

3. A = 4 units east, B = 3 units north

R = $\sqrt{(4^2 + 3^2)} = \sqrt{(16 + 9)} = \sqrt{25} = 5$ units

Direction: $\tan \theta = 3 / 4 = 0.75 \rightarrow \theta = 36.87°$ north of east

4. A = 4 units east, B = 3 units at 60° north of east

Formula: $R^2 = A^2 + B^2 + 2AB \cos \theta$

$= 4^2 + 3^2 + 2(4)(3)\cos 60°$

$= 16 + 9 + 24(0.5)$

$= 16 + 9 + 12 = 37$

$R = \sqrt{37} \approx 6.08$ units

Direction:

$\tan \alpha = (B \sin \theta) / (A + B \cos \theta)$

$= (3 \sin 60°) / (4 + 3 \cos 60°)$

$= (3 \times 0.866) / (4 + 3 \times 0.5)$

$= 2.598 / (4 + 1.5)$

$= 2.598 / 5.5 \approx 0.472$

$\alpha = \tan^{-1}(0.472) \approx 25.3°$ north of east

So, R ≈ 6.08 units at 25.3° north of east.

1.1.4 Vector Components

A vector can be split into parts along the coordinate axes.

These parts are called components.

Suppose vector A makes angle θ with the x-axis.

- The x-component is $A_x = A \cos \theta$

- The y-component is $A_y = A \sin \theta$

So, vector $A = A_x \hat{i} + A_y \hat{j}$

Example:

A = 10 units at 30° above x-axis

$A_x = 10 \cos 30° = 10 \times 0.866 = 8.66$

$A_y = 10 \sin 30° = 10 \times 0.5 = 5$

So, $A = 8.66 \hat{i} + 5 \hat{j}$

1.1.5 Unit Vector

A unit vector has magnitude = 1.

It only shows direction.

If vector $A = A_x \hat{i} + A_y \hat{j} + A_z k$

Then unit vector in direction of A is:

$\hat{A} = (A / |A|) = (A_x \hat{i} + A_y \hat{j} + A_z k) / \sqrt{(A_x^2 + A_y^2 + A_z^2)}$

Example:

$A = 3 \hat{i} + 4 \hat{j}$

Magnitude $|A| = \sqrt{(3^2 + 4^2)} = \sqrt{25} = 5$

Unit vector $\hat{A} = (3/5) \hat{i} + (4/5) \hat{j}$

Class Work Example

Given vectors:

$A = 4\,\hat{\imath} + 3\,\hat{\jmath}$

$B = 2\,\hat{\imath} - 3\,\hat{\jmath}$

$C = 2\,\hat{\imath} + 3\,\hat{\jmath} - 2\,k$

$D = 1\,\hat{\imath} - 2\,\hat{\jmath} + 2\,k$

1. Find $|A|$

$|A| = \sqrt{(4^2 + 3^2)} = \sqrt{(16 + 9)} = \sqrt{25} = 5$

2. Find $2A + B - D$

$2A = 2(4\,\hat{\imath} + 3\,\hat{\jmath}) = 8\,\hat{\imath} + 6\,\hat{\jmath}$

$2A + B = (8\,\hat{\imath} + 6\,\hat{\jmath}) + (2\,\hat{\imath} - 3\,\hat{\jmath}) = (10\,\hat{\imath} + 3\,\hat{\jmath})$

Now subtract D: $(10\,\hat{\imath} + 3\,\hat{\jmath}) - (1\,\hat{\imath} - 2\,\hat{\jmath} + 2\,k)$

$= (10 - 1)\,\hat{\imath} + (3 - (-2))\,\hat{\jmath} + (0 - 2)\,k$

$= 9\,\hat{\imath} + 5\,\hat{\jmath} - 2\,k$

1.2 Multiplication of Vectors

Vector multiplication means performing operations between two vectors.

There are three main types:

1. Scalar–vector multiplication

2. Dot product (scalar product)

3. Cross product (vector product)

1.2.1 Scalar–Vector Multiplication

When a vector is multiplied by a scalar (a number), only its magnitude changes.

The direction stays the same.

Example:

Vector $A = 3\,\hat{\imath} + 4\,\hat{\jmath}$

Magnitude $|A| = \sqrt{(3^2 + 4^2)} = \sqrt{25} = 5$

If we multiply A by scalar 2:

$2A = 2(3\,\hat{\imath} + 4\,\hat{\jmath}) = 6\,\hat{\imath} + 8\,\hat{\jmath}$

Magnitude of $2A = \sqrt{(6^2 + 8^2)} = \sqrt{(36 + 64)} = \sqrt{100} = 10$

Notice: The magnitude doubled, but the direction is unchanged.

Rules of Scalar Multiplication

1. $(c + d)A = cA + dA$

2. $c(A + B) = cA + cB$

3. (cd)A = c(dA)

4. 1A = A

5. 0A = 0 (zero vector)

6. (–1)A = –A (same magnitude, opposite direction)

1.2.2 Dot Product (Scalar Product)

The dot product of two vectors gives a scalar (a number).

It measures how much one vector goes in the same direction as another.

Formula:

$A \cdot B = |A||B| \cos \theta$

Where θ is the angle between A and B.

Step-by-Step Example

Let $A = 3\,\hat{\imath} + 4\,\hat{\jmath}$

Let $B = 2\,\hat{\imath} + 5\,\hat{\jmath}$

Step 1: Multiply components.

$A \cdot B = (3)(2) + (4)(5)$

$= 6 + 20 = 26$

Step 2: Check with formula.

$|A| = \sqrt{(3^2 + 4^2)} = \sqrt{25} = 5$

$|B| = \sqrt{(2^2 + 5^2)} = \sqrt{(4 + 25)} = \sqrt{29} \approx 5.385$

$Cos\ \theta = (A \cdot B) / (|A||B|) = 26 / (5 \times 5.385) \approx 26 / 26.925 \approx 0.966$

$\Theta = cos^{-1}(0.966) \approx 15°$

So, the angle between A and B is about 15°.

Properties of Dot Product

1. Commutative: $A \cdot B = B \cdot A$

2. If $A \cdot B = 0$, then either one vector is zero or they are perpendicular.

3. $(pA) \cdot (qB) = pq(A \cdot B)$

4. $A \cdot A = |A|^2$

5. Distributive: $A \cdot (B + C) = A \cdot B + A \cdot C$

6. Not associative: $A \cdot (B \cdot C)$ is meaningless because $B \cdot C$ is a scalar, not a vector.

1.2.3 Cross Product (Vector Product)

The cross product of two vectors gives another vector.

This new vector is perpendicular to both original vectors.

Its magnitude equals the area of the parallelogram formed by the two vectors.

Formula:

$A \times B = |A||B| \sin \theta \, \hat{n}$

Where:

- θ = angle between A and B

- $\hat{n}$ = unit vector perpendicular to the plane of A and B (direction found using right-hand rule)

Step-by-Step Example

Let $A = \hat{\imath} + \hat{\jmath}$

Let $B = \hat{\imath} - \hat{\jmath}$

Step 1: Write determinant form.

$A \times B =$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 1 & 1 & 0 \end{vmatrix}$$

$$| 1 \ \ -1 \ \ 0 |$$

Step 2: Expand determinant.

$$= \hat{i}(1\times0 - 0\times-1) - \hat{j}(1\times0 - 0\times1) + k(1\times-1 - 1\times1)$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + k(-1 - 1)$$

$$= 0\,\hat{i} - 0\,\hat{j} + (-2)k$$

$$= -2k$$

So, $A \times B = -2k$

Step 3: Magnitude.

$|A \times B| = 2$

Step 4: Direction.

Vector points in negative z-direction.

Properties of Cross Product

1. Distributive: $A \times (B + C) = (A \times B) + (A \times C)$

2. Not commutative: $A \times B = -(B \times A)$

3. $A \times A = 0$

4. Unit vectors:

- $\hat{\imath} \times \hat{\jmath} = k$
- $\hat{\jmath} \times k = \hat{\imath}$
- $k \times \hat{\imath} = \hat{\jmath}$
- Reverse order gives negative result (e.g., $\hat{\jmath} \times \hat{\imath} = -k$)

Class Work Example

Given $M = 6\hat{\imath} + 2\hat{\jmath} - k$

$N = 2\hat{\imath} - 3\hat{\jmath} - 3k$

1. *Find* $M \cdot N$

$M \cdot N = (6)(2) + (2)(-3) + (-1)(-3)$

$= 12 - 6 + 3 = 9$

So, $M \cdot N = 9$

2. *Find* $M \times N$

Write determinant:

$M \times N =$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 6 & 2 & -1 \\ 2 & -3 & -3 \end{vmatrix}$$

Expand:

$$= \hat{\imath}((2)(-3) - (-1)(-3)) - \hat{\jmath}((6)(-3) - (-1)(2)) + k((6)(-3) - (2)(2))$$

$$= \hat{\imath}(-6 - (-3)) - \hat{\jmath}(-18 - (-2)) + k(-18 - 4)$$

$$= \hat{\imath}(-6 + 3) - \hat{\jmath}(-18 + 2) + k(-22)$$

$$= -3\,\hat{\imath} - (-16)\,\hat{\jmath} - 22k$$

$$= -3\,\hat{\imath} + 16\,\hat{\jmath} - 22k$$

So, $M \times N = -3\,\hat{\imath} + 16\,\hat{\jmath} - 22k$

3. ***Find angle between M and N***

Formula: $\cos\theta = (M \cdot N) / (|M||N|)$

$|M| = \sqrt{(6^2 + 2^2 + (-1)^2)} = \sqrt{(36 + 4 + 1)} = \sqrt{41} \approx 6.403$

$|N| = \sqrt{(2^2 + (-3)^2 + (-3)^2)} = \sqrt{(4 + 9 + 9)} = \sqrt{22} \approx 4.690$

$\cos\theta = 9 / (6.403 \times 4.690) = 9 / 30.04 \approx 0.299$

$\Theta = \cos^{-1}(0.299) \approx 72.6°$

So, angle between M and N ≈ 72.6°.