

SHORT ANSWER QUESTIONS(SAQ's) UNIT- V

Frequently Asked Questions(PART-A)

Marks: 5 x 3=15

Marks: 5 x 2=10

1. Evaluate the cross power spectral density given the cross correlation of two processes $X(t)$ and $Y(t)$ is $(AB/2)[\sin \omega_0 t + \cos \omega_0 (2t + \tau)]$ where A, B and ω_0 are constant.

[A109210401/Reg/Dec-2011/Set-4/Q7(b)]

Answer :

For answer refer Unit-5, Q.No.5.

2. Explain the concept of RMS bandwidth.

[A109210401/Sup./May/June-2012/Set-1/Q5(b)]

Answer :

For answer refer Unit-5, Q.No.6.

3. The output of a filter is given by $Y(t) = X(t + T) + X(t - T)$ where, $X(t)$ is a WSS process spectral density and T is a constant. Find the power spectrum of $Y(t)$.

[A109210401/Sup/May/June-2012/Set-3/Q6]

Answer :

For answer refer Unit-5, Q.No.7.

4. The power spectral density of $X(t)$ is given by $S_{xx}(\omega) = \begin{cases} 1 + \omega^2 & \text{for } |\omega| < 1 \\ 0 & \text{otherwise} \end{cases}$. Find the auto-correlation function.

[A109210401/Nov-2010/Set-2/Q7(b)]

Answer :

For answer refer Unit-5, Q.No.27.

5. Explain the power spectral density that can be obtained by using Wiener-Khinchine relation.

Answer :

For answer refer Unit-5, Q.No.33.

6. Let $X(t)$ be a zero mean, stationary, gaussian process with autocorrelation function $R_X(\tau)$. This process is applied to a square law device defined by the input output relation

$Y(t) = X^2(t)$ Where $Y(t)$ is the output.

(i) Show that the mean of $Y(t)$ is $R_X(0)$

(ii) Show that the auto covariance function of $Y(t)$ is $2R_X^2(t)$.

[A109210401/Reg/Dec-2011/Set-1/Q7(b)]

Answer :

For answer refer Unit-5, Q.No.42.

LONG ANSWER QUESTIONS(LAQ's) UNIT - V

Frequently Asked Questions(PART-B)

Marks: 5 x 10 = 50

1. Explain about power spectral density and derive the equations.

(or) Explain the concept of PSD of a Random process and derive the relation between its Power and PSD.

[A109210401/Sup/May/June-2012/Set-2/Q4(a)]

Answer :

For answer refer Unit-5, Q.No.1.

2. State and prove any three properties of power spectral density.

[A109210401/Reg/Dec-2011/Set-2/Q7(a)]

[A109210401/Sup/May/June-2012/Set-2/Q4(a)]

(or) State and prove the properties of power spectral density.

[Feb-08/Set-1,2,3/Q7(b)]

Answer :

For answer refer Unit-5, Q.No.2.

3. (a) For $X(t)$ and $Y(t)$ are random process and prove that $\xi_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T(\omega)Y_T(\omega)]}{2T}$

(b) $R_{XX}(\tau) = 4u(\tau)e^{-\alpha\tau}$. Find $\zeta_{XY}(\omega)$.

[Nov-8/Set-2/Q7]

Answer :

For answer refer Unit-5, Q.No.29.

4. If $R(\tau) = ae^{-b|\tau|}$. Find the spectral density function, where a and b are constants.

[Feb-08/Set-3,4/Q7(c)][Nov-07/Set-2,3/Q7(c)]

Answer :

For answer refer Unit-5, Q.No.38.

5. Find the cross correlation function corresponding to the cross the spectrum.

$$S_{XY}(\omega) = \frac{6}{(9 + \omega^2)(3 + j\omega)^2}$$

[Nov-09/Set-1/Q7(b)]

Answer :

For answer refer Unit-5, Q.No.39.

STOCHASTIC PROCESSES SPECTRAL CHARACTERISTICS

UNIT 5 Solutions

Structure

- 5.0 Objectives
- 5.1 Power Spectrum- Properties
 - 5.1.1 Relation between Power Spectrum and Autocorrelation Function
- 5.2 Cross Power Density Spectrum - Properties
 - 5.2.1 Relationship between Cross-power Spectrum and Cross-correlation Function
- 5.3 Spectral Characteristics of System Response- Power Density Spectrum of Response, Cross Power Spectral Density of Input and Output of a Linear System
- 5.4 Outcomes
- 5.5 Objective Type Questions
- 5.6 Student Activity

5.0 OBJECTIVES

After completion of this unit you should be able to understand :

- Power spectrum- properties
- Cross power density spectrum - properties
- Relationship between cross-power spectrum and cross correlation function
- Spectral characteristics of system response-Power density spectrum of response, Cross power spectral density of input and output of a linear system

5.1 POWER SPECTRUM PROPERTIES

Q1. Explain about power spectral density and derive the equations.

(or) Explain the concept of PSD of a random process and derive the relation between its power and PSD.

[A109210401/Sup/May/June-2012/Set-2/Q4(a)]

Sol.

A random variable varying with respect to times is a random process and if the variation is a known variation or

Probability Theory and Stochastic Processes

deterministic variation, the corresponding process is a deterministic process or a signal. For a periodic signal, Fourier series is used for the study of its spectral behaviour.

If it is non periodic, Fourier transform is used. It is given as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Where, $x(t)$ is the deterministic process.

If the variation of the random variable with respect to time is unknown or non-deterministic, the result of the variation is a stochastic process. Now, consider the spectral analysis of such a process.

A random process or a stochastic process $X(t)$ is a collection of number of sample functions. Let $x(t)$ be a sample function of the ensemble.

Let $x(t)$ be truncated, i.e., select a portion from $x(t)$ extending from $t = -T$ to $+T$, and remaining is neglected.

Let $x(t)$ be truncated, i.e., select a portion from $x(t)$ extending from $t = -T$ to $+T$, and remaining is neglected.

$$\text{Let } X_T(t) = x(t) \quad \text{for } -T < t < T; \\ = 0 \quad \text{elsewhere}$$

Now, $x_T(t)$ is of finite duration and we will apply Fourier Transformer to it.

$$\therefore F[X_T(t)] = X_T(\omega) = \int_{-T}^T x_T(t) \cdot e^{-j\omega t} dt$$

Since, over $(-T, +T)$, $x_T(t) = x(t)$

$$X_T(\omega) = \int_{-T}^T x(t) \cdot e^{-j\omega t} dt$$

Energy of a signal is defined as $E = \int_{-\infty}^{\infty} x^2(t) dt$

Consider the Parseval's theorem for energy signals.

$$E = \int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} x(t) \cdot x(t) dt$$

Using inverse Fourier transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\therefore E = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right] dt$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x(t) \cdot e^{j\omega t} \cdot dt \right] d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega) \cdot d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \cdot d\omega
 \end{aligned}$$

Using this relation for $x_T(t)$, its energy

$$= \int_{-T}^T x^2(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

Since, power is time average of energy, power available over the interval $(-T, T)$ is

$$= \frac{1}{2T} \int_{-T}^T x^2(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} \cdot d\omega$$

This $\frac{|X_T(\omega)|^2}{2T}$ is referred to as power density spectrum or power spectral density.

The above power computed is the power available in the interval $(-T, T)$ only, but not in the entire $x(t)$, which is of very large duration. So, to get the expression for the power in $x(t)$, consider the limit $T \rightarrow \infty$. But, this is also the power content in one sample function $x(t)$ of the process $X(t)$. To find the power of the process $X(t)$, its average i.e., expected value is considered. So, the average power of the random process $X(t)$ is,

$$\begin{aligned}
 P_{av} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x^2(t)] \cdot dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T} \cdot d\omega
 \end{aligned}$$

Thus, the average power of a random process is obtained by time averaging its mean square value.

If $\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$ is named as power spectral density, $S_{XX}(\omega)$, then the average power is given as,

$$P_{av} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot d\omega,$$

i.e., the area enclosed by power spectral density curve is the power.

Q2. State and prove any three properties of power spectral density.

[A109210401/Reg/Dec-2011/Set-2/Q7(a)]

[A109210401/Sup/May/June-2012/Set-2/Q4(a)]

(or) State and prove the properties of power spectral density. [Feb-08/Set-1,2,3/Q7(b)]

(or) State and prove any four properties of PSD.

[07A3EC10/Sup./May-2011/Set-1,4/Q7(b)]

[07A3EC10/Sup./May-2011/Set-2/Q2(b)]

[07A3EC10/Sup./May-2011/Set-3/Q6(b)]

(or) State and prove any four properties of power spectral density of random process.

[09A30401/Sup./May/June-2013/Q7(b)]

(or) State and prove Wiener Khinchin relation.

[Nov-06/Set-2,3,4/Q7(b)] [Nov-10/Set-4/Q7(a)]

[A109210401/Sup/May/June-2012/Set-1/Q5(a)]

(or) Let the auto correlation function of a certain random process $X(t)$ is given by,

$$R_n(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau). \text{ Obtain the expression for its power spectral density } S_n(\omega).$$

[A109210401/Nov-2010/Set-1/Q7(b)]

Sol.

Properties of PSD

1. Time average of auto correlation function and the power spectral density from a fourier transform pair.

$$\begin{aligned}
 \text{We have } PSD S_{XX}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} E[|X_T(\omega)|^2] \\
 &= \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) \cdot X_T(\omega)]}{2T}
 \end{aligned}$$

$$\text{We have } X_T(\omega) = \int_{-T}^T X(t) \cdot e^{-j\omega t} \cdot dt$$

$$\Rightarrow X_T^*(\omega) = \int_{-T}^T X(t) \cdot e^{j\omega t} \cdot dt$$

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T X(t_1) \cdot e^{-j\omega t_1} \cdot dt_1 \int_{-T}^T X(t_2) \cdot e^{j\omega t_2} \cdot dt_2 \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T \int_{-T}^T X(t_1) \cdot X(t_2) \cdot e^{j\omega(t_2 - t_1)} \cdot dt_1 \cdot dt_2 \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T E[X(t_1) \cdot X(t_2)] \cdot e^{j\omega(t_2 - t_1)} \cdot dt_1 \cdot dt_2$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) \cdot e^{j\omega(t_2 - t_1)} \cdot dt_1 \cdot dt_2$$

Consider the expression $\frac{1}{2T} \int_{-T}^T \int_{-T}^T S_{XX}(\omega) \cdot e^{j\omega \tau} \cdot d\omega$, which is the inverse fourier transform of $S_{XX}(\omega)$.

$$\frac{1}{2T} \int_{-\infty}^{\infty} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} \cdot dt_1 \cdot dt_2 \right] e^{j\omega \tau} \cdot d\omega$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega(t_2 - t_1) + j\omega \tau} \cdot d\omega \cdot dt_1 \cdot dt_2$$

$$\text{We have, } F[\delta(t)] = 1 \Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} \cdot dt = \delta(t)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_2 + t_1)} \cdot d\omega = \delta(\tau - t_2 + t_1)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) \cdot \delta(\tau - t_2 + t_1) \cdot dt_1 \cdot dt_2$$

$$\delta(\tau - t_2 + t_1) = 1 \text{ at } \tau - t_2 + t_1 = 0 \text{ i.e., } t_2 = \tau + t_1$$

$$\therefore F^{-1}[S_{XX}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-T}^{+T} R_{XX}(t_1, \tau + t_1) dt_1$$

Let $t_1 = t \Rightarrow dt_1 = dt$

$$\therefore F^{-1}[S_{XX}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt$$

The right hand side of the above equation is the time average of auto correlation function. Thus time average of the auto correlation function and the PSD form a Fourier transform pair. If the process is stationary, the time average of auto correlation $R_{XX}(t, t + \tau)$ will be $R_{XX}(\tau)$, since auto correlation function of stationary process is independent of time 't'.

$$\therefore F^{-1}[S_{XX}(\omega)] = R_{XX}(\tau)$$

Thus, for a wide sense stationary random process, auto correlation and power spectral density form a Fourier transform pair, i.e.,

$$R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega)$$

This is called Wiener - Khintchine relation.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot e^{j\omega\tau} \cdot d\omega$$

2. For a WSS process, PSD at zero frequency gives the area under the graph of auto correlation of the process.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{j\omega\tau} \cdot d\tau$$

$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot d\tau$$

Which is the area under auto correlation function.

3. The mean square value of WSS process equals the area under the graph of power spectral density.

$R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot d\omega$, which is the area under PSD curve. But, for a

WSS process, $R_{XX}(0) = E[X^2(t)]$

$$\therefore E[X^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot d\omega$$

4. A WSS process has a non negative power spectral density. Since $E[X^2(t)] \geq 0$, i.e., non negative, PSD is also nonnegative.

5. The PSD is an even function of frequency.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{j\omega\tau} \cdot d\tau = \int_{-\infty}^{\infty} R_{XX}(-\tau) \cdot e^{j\omega(-\tau)} \cdot d\tau$$

$$\text{Since } R_{XX}(\tau) = R_{XX}(-\tau)$$

$$\Rightarrow S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau = S_{XX}(\omega)$$

Hence, verified.

Q3. Determine which of the following functions are valid power density spectrum and why?

i) $\frac{\cos 8(\omega)}{2 + \omega^4}$ ii) $e^{-(\omega-1)^2}$

[A109210401/Nov-2010/Set-4/Q7(b)]

[A109210401/Reg./Dec-2011/Set-1/Q7(a)]

Sol.

The PSD is an even function of frequency

$$S_{XX}(\omega) = S_{XX}(-\omega)$$

$$\frac{\cos 8(-\omega)}{2 + (-\omega)^4} = \frac{\cos 8\omega}{2 + \omega^4}$$

$$S_{XX}(\omega) = S_{XX}(-\omega)$$

Thus the given function is a valid function.

ii) $e^{-(\omega-1)^2} = e^{(\omega+1)^2}$

Thus the given function is not a valid function.

Q4. A stationary random process $X(t)$ has special

density $S_{XX}(\omega) = 25/(\omega^2 + 25)$ and $S_{YY}(\omega) = \frac{\omega^2}{(\omega^2 + 25)}$. If $X(t)$ and $Y(t)$ are of the zero mean, find the:

a) PSD of $Z(t) = X(t) + Y(t)$

b) Cross spectral density of $X(t)$ and $Z(t)$. State and prove the result used.

[A109210401/Reg./Nov/Dec-2012/Q7]

Ans.

Given data,

$X(t)$ is a stationary random variable process, which has

$$S_{XX}(\omega) = \frac{25}{(\omega^2 + 25)} \text{ and } S_{YY}(\omega) = \frac{\omega^2}{(\omega^2 + 25)}$$

Also given that, $X(t)$ has zero mean, i.e., $\bar{X} = 0$

and $Y(t)$ has zero mean, i.e., $\bar{Y} = 0$

(a) PSD of $Z(t) = X(t) + Y(t)$

In the given problem, $Z(t) = X(t) + Y(t)$ is correct

To solve it, we need to consider $Z(t) = X(t) + Y(t)$

$$R_{ZZ}(t, t + \tau) = E[Z(t)Z(t + \tau)]$$

$$E\{[X(t) + Y(t)][X(t + \tau) + Y(t + \tau)]\}$$

$$R_{ZZ}(t, t + \tau) = E\{[X(t)X(t + \tau) + X(t)Y(t + \tau) + Y(t)X(t + \tau) + Y(t)Y(t + \tau)]\}$$

$$\therefore R_{ZZ}(t, t + \tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

Since, $X(t)$ and $Y(t)$ are of zero mean stationary random process then, we obtain,

$$R_{XX}(\tau) = 0 = R_{YX}(\tau) = 0$$

$$\text{Then, we have } R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) \quad \dots (1)$$

So by applying fourier transform on both sides, then we have,

$$S_{ZZ}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)$$

Hence, from the given data, we obtain $S_{ZZ}(\omega)$ we have,

$$S_{ZZ}(\omega) = \left[\frac{25}{\omega^2 + 25} \right] + \left[\frac{\omega^2}{\omega^2 + 25} \right] = \left[\frac{\omega^2 + 25}{\omega^2 + 25} \right]$$

$$\therefore S_{ZZ}(\omega) = 1$$

(b) Cross spectral density of $X(t)$ and $Z(t)$

The power spectral density $S_{ZZ}(\omega) = 1$

The relation between the auto correlation function and power density spectrum is given as,

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau) e^{-j\omega\tau} d\tau$$

So both forms a fourier transform pair, then

$$S_{ZZ}(\omega) \xleftrightarrow[\text{IFT}]{\text{FT}} R_{ZZ}(\tau)$$

The inverse fourier transform of '1' is an impulse function, $\delta(t)$ form the table of Fourier transforms.

$$\therefore R_{ZZ}(\tau) = \delta(\tau)$$

The power spectral density $S_{XX}(\omega) = \frac{25}{(\omega^2 + 25)}$

Then, by applying inverse fourier transform we get,

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{25}{(\omega^2 + 25)} \right) e^{j\omega\tau} d\omega \quad \dots\dots(1)$$

Then form the table of fourier transform functions we have

$$\text{F.T} [e^{-\alpha|t|}] = \frac{2\alpha}{\alpha^2 + \omega^2} \quad \dots\dots(2)$$

In the equation (2), consider $\alpha = 5$

$$\text{F.T} [e^{-\alpha|t|}] = \frac{2 \times 5}{5^2 + \omega^2} = \frac{10}{25 + \omega^2}$$

$$\text{Then, F.T} [2.5e^{-5|t|}] = 2.5 \times \frac{10}{25 + \omega^2} = \frac{25}{25 + \omega^2}$$

$$\text{The I.F.T} \left[\frac{25}{25 + \omega^2} \right] = 2.5e^{-5|\tau|}$$

$$\therefore R_{XX}(\tau) = 2.5e^{-5|\tau|}$$

By summarizing, we have,

$$R_{ZZ}(\tau) = \delta(\tau) \text{ and } R_{XX}(\tau) = 2.5e^{-5|\tau|}$$

But according to the properties of auto correlation, $X(t)$ is to zero mean from the given data, $E[X(t)] = 0$ and $E[X(t+\tau)] = 0$

$$\therefore \lim_{|\tau| \rightarrow \infty} R_{XX}(\tau) = 0$$

According to the properties of cross proper density spectrum. If $X(t)$ and $Z(t)$ are uncorrelated (i.e.,

independent (or) stationary) and have constant means i.e.,

$\bar{X}$ and $\bar{Z}$. Then, we have

$$S_{XZ}(\omega) = S_{ZX}(\omega) = 2\pi \bar{X} \bar{Y} \delta(\omega)$$

But from the given data $\bar{X} = 0$

$$\therefore S_{XZ}(\omega) = S_{ZX}(\omega) = 0$$

Q5. Evaluate the cross power spectral density given the cross correlation of two processes $X(t)$ and $Y(t)$ is $(AB/2)[\sin \omega_0 t + \cos \omega_0(2t + \tau)]$ where A ,

B and ω_0 are constant.

[SAQ]

[A109210401/Reg/Dec-2011/Set-4/Q7(b)]

Sol.

The time average of cross correlation is,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XY}(t, t + \tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{AB}{2} [\sin \omega_0 \tau + \cos \omega_0(2t + \tau)] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{AB}{2} [\sin \omega_0 \tau + \cos \omega_0(2t + \tau)] dt$$

$$= \frac{AB}{2} \cdot \sin \omega_0 \tau \cdot \lim_{T \rightarrow \infty} \frac{1}{2T} [T] + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos \omega_0(2t + \tau) dt$$

The second term of above summation is zero.

$$\therefore \text{Time average of } R_{XY}(\tau) \text{ is } \frac{AB}{2} \cdot \sin \omega_0 \tau$$

$$\therefore \text{The cross power spectrum } S_{XY}(\omega) = F \left[\frac{AB}{2} \cdot \sin \omega_0 \tau \right]$$

$$= -\frac{j\pi AB}{2} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Q6. Explain the concept of RMS bandwidth. [SAQ]

[A109210401/Sup./May/June-2012/Set-1/Q5(b)]

Sol.

1. The variance of a random variable is a measure of the average amount by which a random variable varies with respect to its mean.
2. The positive square root of variance is called standard deviation. It is a measure of the spread in the density function of the random variable.
3. The density function of the random variable is always real and non negative and similarly, the PSD also.
4. The area enclosed by a valid density function is always unity. But, the PSD curve need not have unit area always and it can be normalized to have unit area, i.e., if $S_{xx}(\omega)$ is the PSD.

5. The area enclosed is $A = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$. By dividing this with its area gain, the result is 1. This is referred to as normalization, i.e., the curve $\frac{S_{XX}(\omega)}{A}$ always encloses unit area.
6. Similar to the measurement of the spread of the Pdf of a random variable using standard deviation, the spread of the power spectral density is measured by using RMS bandwidth.
7. The variance of a random variable X is given as,

$$\int_{-\infty}^{\infty} (x - m)^2 \cdot f(x) dx$$

Considering ' ω ' as a random variable and since PSD $S_{XX}(\omega)$ is even function of frequency and hence is of zero mean, the corresponding standard deviation is the square root of $\int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega) d\omega$ and is normalized to give the rms bandwidth i.e.,

$$\text{rms bandwidth} = \frac{1}{A} \int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega) d\omega = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega}$$

- Q7. The output of a filter is given by $Y(t) = X(t+T) + X(t-T)$ where, $X(t)$ is a WSS process spectral density and T is a constant. Find the power spectrum of $Y(t)$. [SAQ]

[A109210401/Sup/May/June-2012/Set-3/Q6]

Sol.

Given that,

The output of a filter is given by

$$Y(t) = X(t+T) + X(t-T)$$

$X(t)$ is a WSS process, power spectral density $S_{XX}(\omega)$ and T is a constant.

Let $R_{XX}(\tau)$ and $S_{XX}(\omega)$ be the auto correlation function and PSD of $Y(t)$ consider

$$\begin{aligned} R_{YY}(t, t+\tau) &= E[Y(t)Y(t+\tau)] \\ &= E\{[X(t+T) + X(t-T)][X(t+\tau+T) + X(t+\tau-T)]\} \\ &= E[X(t+T) \cdot X(t+\tau+T) + X(t+T) \cdot X(t+\tau-T) \\ &\quad + X(t-T) \cdot X(t+\tau+T) + X(t-T) \cdot X(t+\tau-T)] \\ &= E[X(t+T) \cdot X(t+\tau+T)] + E[X(t+T) \cdot X(t+\tau-T)] \\ &\quad + E[X(t-T) \cdot X(t+\tau+T)] + E[X(t-T) \cdot X(t+\tau-T)] \\ &= R_{XX}(\tau) + R_{XX}(\tau-T) + R_{XX}(\tau+T) + R_{XX}(\tau) \\ &= 2R_{XX}(\tau) + R_{XX}(\tau+T) + R_{XX}(\tau-T) \end{aligned}$$

PSD of $Y(t)$ is,

$$S_{YY}(\omega) = 2S_{XX}(\omega) - (e^{-j\omega T} + e^{j\omega T})S_{XX}(\omega)$$

Probability Theory and Stochastic Processes

$$= 2S_{XX}(\omega)[1 - \cos \omega T] = 4 \cdot S_{XX}(\omega) \cdot \sin^2\left(\frac{\omega T}{2}\right)$$

- Q8. Find the power spectrum of a random process whose Auto correlation function is $\exp[-\sigma|\tau|][1 + \sigma|\tau|]$.

Ans.

Given that,

Auto correlation function is $\exp[-\sigma|\tau|][1 + \sigma|\tau|]$

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} (e^{-\sigma|\tau|} \cdot [1 + \sigma|\tau|]) d\tau \\ &= \int_{-\infty}^{\infty} (e^{-\sigma|\tau|} + e^{-\sigma|\tau|} \cdot \sigma|\tau|) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} e^{-\sigma|\tau|} d\tau + \int_{-\infty}^{\infty} \sigma|\tau| e^{-\sigma|\tau|} e^{-j\omega\tau} d\tau \end{aligned}$$

By considering, $\int_{-\infty}^{\infty} e^{-\sigma|\tau|} e^{-j\omega\tau} d\tau$

$$\begin{aligned} &= \int_{-\infty}^0 e^{-\sigma\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} e^{-\sigma\tau} e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^0 e^{(\sigma-j\omega)\tau} d\tau + \int_0^{\infty} e^{-(\sigma+j\omega)\tau} d\tau \end{aligned}$$

$$\therefore R_{XX}(\tau) = e^{-\sigma(-\tau)}$$

$$\begin{aligned} &= \frac{1}{\sigma - j\omega} - \frac{1}{\sigma + j\omega} \\ &= \frac{(\sigma + j\omega) - (\sigma - j\omega)}{\sigma^2 + \omega^2} = \frac{j\omega + j\omega}{\sigma^2 + \omega^2} = \frac{2j\omega}{\sigma^2 + \omega^2} \dots (1) \end{aligned}$$

By considering 2nd term

$$\begin{aligned} \int_{-\infty}^{\infty} \sigma|\tau| e^{-\sigma|\tau|} e^{-j\omega\tau} d\tau &= \sigma \int_{-\infty}^{\infty} |\tau| e^{-\sigma|\tau|} e^{-j\omega\tau} d\tau \\ &= \sigma \left[\int_{-\infty}^0 |\tau| e^{\sigma\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} |\tau| e^{-\sigma\tau} e^{-j\omega\tau} d\tau \right] \end{aligned}$$

By considering $\int_{-\infty}^0 |\tau| e^{\sigma\tau} e^{-j\omega\tau} d\tau$

$$\begin{aligned} &\int_{-\infty}^0 -\tau e^{(\sigma-j\omega)\tau} d\tau \\ &= -\left[\frac{e^{(\sigma-j\omega)\tau}}{(\sigma-j\omega)} \right]_{-\infty}^0 - \left[\int \frac{e^{(\sigma-j\omega)\tau}}{(\sigma-j\omega)} (1) d\tau \right]_{-\infty}^0 = \frac{1}{(\sigma-j\omega)^2} \dots (2) \end{aligned}$$

By considering

$$\begin{aligned} \int_0^{\infty} \tau e^{-\sigma\tau} e^{-j\omega\tau} d\tau &= \int_0^{\infty} \tau e^{-(\sigma+j\omega)\tau} d\tau \\ &= \left[\tau \left(\frac{e^{-(\sigma+j\omega)\tau}}{-(\sigma+j\omega)} \right) \right]_0^{\infty} + \int_0^{\infty} \frac{e^{-(\sigma+j\omega)\tau}}{(\sigma+j\omega)} d\tau \end{aligned}$$

$$= \left[-\tau \left(\frac{e^{-(\sigma+j\omega)\tau}}{(\sigma+j\omega)} \right) \right]_0^\infty - \left[\frac{e^{-(\sigma+j\omega)\tau}}{(\sigma+j\omega)^2} \right]_0^\infty$$

$$= - \left[\frac{1}{(\sigma+j\omega)^2} (-1) \right] = \frac{1}{(\sigma+j\omega)^2} \quad \dots\dots(3)$$

From equation (1), (2) and (3), we get,

$$S_{XX}(\omega) = \frac{2j\omega}{\sigma^2 + \omega^2} + \frac{1}{(\sigma-j\omega)^2} + \frac{1}{(\sigma+j\omega)^2}$$

$$= \frac{2j\omega}{\sigma^2 + \omega^2} + \frac{(\sigma+j\omega)^2(\sigma-j\omega)^2}{(\sigma^2 + \omega^2 - 2j\omega\sigma)(\sigma^2 - \omega^2 + 2j\omega\sigma)}$$

$$= \frac{2j\omega}{\sigma^2 + \omega^2} + \frac{\sigma^2 - \omega^2 + 2\sigma j\omega + \sigma^2 + \omega^2 - 2j\sigma\omega}{(\sigma^2 + \omega^2 - 2j\omega\sigma)(\sigma^2 - \omega^2 + 2j\omega\sigma)}$$

$$= \frac{2j\omega}{\sigma^2 + \omega^2} + \frac{2\sigma^2}{(\sigma-j\omega)^2(\sigma+j\omega)^2}$$

Q9. For a random process, show that the "Time average of its auto correlation function and its power spectral density from a fourier transform pair."

[A109210401/Sup/May/June-2012/Set-4/Q4(a)]

Sol.

Let $x(t)$ be a random process, which is multiplied by sinusoidal signal $A \cos \omega_0 t$.

Let the product be $Y(t) = X(t) \cdot A \cos \omega_0 t$

The auto correlation function of $Y(t)$ given as,

$$R_{YY}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)]$$

$$= E[X(t) \cdot A \cos \omega_0 t \cdot X(t+\tau) \cdot A \cos \omega_0(t+\tau)]$$

$$= E[X(t) \cdot X(t+\tau)] \cdot A^2 \cos \omega_0 t \cdot \cos \omega_0(t+\tau)$$

$$= R_{XX}(t, t+\tau) \cdot \frac{A^2}{2} [\cos(2\omega_0 t + \omega_0 \tau) + \cos \omega_0 \tau]$$

If $X(t)$ is a WSS process, $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

$$R_{YY}(t, t+\tau) = R_{XX}(\tau) \cdot \frac{A^2}{2} [\cos(2\omega_0 t + \omega_0 \tau) + \cos \omega_0 \tau]$$

This indicates that, even though $X(t)$ is WSS process, $Y(t) \neq X(t)$.

$A \cos \omega_0 t$ is not stationary, since $R_{YY}(t, t+\tau)$ is not independent of 't'. Then, to find the PSD of $Y(t)$, we use relation that

Time average of auto correlation $\xrightarrow{\text{F.T.}}$ PSD.

$$R_{YY}(t, t+\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{YY}(t, t+\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XX}(\tau) \cdot \frac{A^2}{2} [\cos(2\omega_0 t + \omega_0 \tau) + \cos \omega_0 \tau] dt$$

$$= R_{XX}(\tau) \cdot \frac{A^2}{2} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos(2\omega_0 t + \omega_0 \tau) dt + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos \omega_0 \tau dt \right]$$

$$= \frac{A^2}{2} R_{XX}(\tau) \cdot \cos \omega_0 \tau$$

$\therefore$ The PSD of $Y(t)$ is,

$$S_{YY}(\omega) = F[\text{Time average of } R_{YY}(t, t+\tau)]$$

$$S_{YY}(\omega) = \frac{A^2}{2} \cdot F[R_{XX}(\tau) \cdot \cos \omega_0 \tau]$$

Now, we will use the modulation property of fourier transform stated as,

$$\text{If } f(t) \xrightarrow{\text{F.T.}} F(\omega)$$

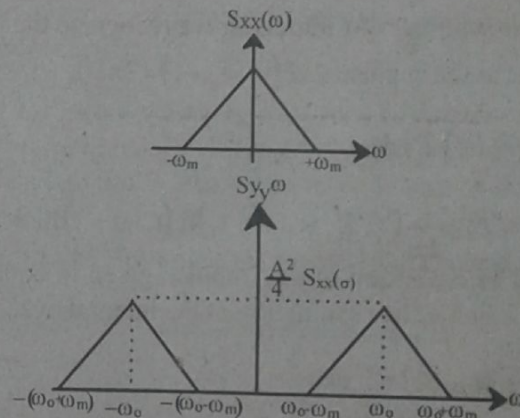
$$f(t) \cdot \cos \omega_0 t \xrightarrow{\text{F.T.}} \frac{1}{2} [F(\omega + \omega_0) + F(\omega - \omega_0)]$$

This results in frequency translation. So, the power spectral density of $Y(t)$ is,

$$S_{YY}(\omega) = \frac{A^2}{4} [S_{XX}(\omega + \omega_0) + S_{XX}(\omega - \omega_0)]$$

Where, $S_{XX}(\omega) = F[R_{XX}(\tau)]$

i.e., If the PSD of $X(t)$ is,



5.1.1 Relation Between Power Spectrum and Auto Correlation Function

Q10. What is the relationship between power spectrum and autocorrelation function.

[A109210401/Nov-2010/Set-1/Q7(a)]

[A109210401/Reg./Dec-2011/Set-3/Q7(a)]

(or) Derive the relationship between autocorrelation and power spectral density.

[09A30401/Sup./May/June-2013/Q7(a)]

Sol.

The inverse fourier transform of the power density spectrum is the time average of the autocorrelation function.

$$\text{i.e., } \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = A[R_{XX}(t, t+\tau)]$$

This expression will now be proved.

The defining equation for the power spectrum we have

$$\begin{aligned}\phi_{XX}(\omega) &= \lim_{T \rightarrow \infty} E \left[\frac{1}{2T} \int_{-T}^T X(t_1) e^{-j\omega t_1} dt_1 \int_{-T}^T X(t_2) e^{j\omega t_2} dt_2 \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T E[X(t_1)X(t_2)] e^{j\omega(t_2-t_1)} dt_2 dt_1\end{aligned}$$

The expectation in the integrand is identified as the autocorrelation function of $X(t)$.

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2) \quad -T < (t_1 \text{ and } t_2) < T$$

Thus, becomes

$$\phi_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_1 dt_2$$

The inverse fourier transform will exist since the power spectrum is real, non-negative, and is absolutely integrable because we presume $P_{XX} < \infty$. The inverse transform is,

$$\begin{aligned}& \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t, t_1) e^{-j\omega(t_1-t)} dt dt_1 e^{j\omega\tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t, t_1) \frac{1}{2T} \int_{-\infty}^{\infty} e^{-j\omega(\tau+t_1-t)} d\omega dt_1 dt\end{aligned}$$

The event symmetry of impulses, we recognize the inner integral of as the impulse $2\pi\delta(\tau - t_1 + t) = 2\pi\delta(t_1 - t - \tau)$, so

$$\begin{aligned}& \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(\omega) e^{j\omega\tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \int_{-T}^T R_{XX}(t, t_1) \delta(t_1 - t - \tau) dt_1 dt\end{aligned}$$

The definition of the impulse given in equation allows the immediate evaluation of the integral over

$$\begin{aligned}t_1 &: \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(\omega) e^{j\omega\tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T R_{XX}(t, t+\tau) dt \quad -T < t+\tau < T\end{aligned}$$

In the limit as $T \rightarrow \infty$ the condition $-T < t+\tau < T$ has little meaning. On the right side is recognized as the time average of the process autocorrelation function

$$A[R_{XX}(t, t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XX}(t, t+\tau) dt$$

Thus, the inverse fourier transform of the power density spectrum is the line average of the process' autocorrelation function. The direct transform must also hold, giving

$$\phi_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

which shows that $\phi_{XX}(\omega)$ and $A[R_{XX}(t, t+\tau)]$ form a Fourier transform pair.

$$A[R_{XX}(t, t+\tau)] \leftrightarrow \phi_{XX}(\omega)$$

For the important case where $X(t)$ is at least wide-

sense stationary $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$ and we get,

$$\phi_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$(or) \quad R_{XX}(\tau) \leftrightarrow \phi_{XX}(\omega)$$

The expressions are usually called the Wiener-Khintchine relations. They form the basic link between the time domain description (correlation functions) of processes and their description in the frequency domain (power spectrum).

It is clear that knowledge of the power spectrum of a process allows complete recovery of the autocorrelation function when $X(t)$ is at least wide-sense stationary. For a non stationary process, only the time average of the autocorrelation function is recoverable.

Q11. Verify whether the following functions are a valid power density spectrum or not.

$$S_{XX}(\omega) = \exp[-(\omega-1)^2]. \quad [Nov-09/Set-2/Q7(b)]$$

Sol.

The given power density spectrum is,

$$S_{XX}(\omega) = \exp[-(\omega-1)^2]$$

For the above power density spectrum to be valid one, the following properties must be satisfied.

1. $S_{XX}(\omega) \geq 0$
2. $S_{XX}(-\omega) = S_{XX}(\omega)$
3. $S_{XX}(\omega)$ must be real
4. $S_{XX}(\omega) = \omega^2 S_{XX}(\omega)$

Consider the property $S_{XX}(\omega) \geq 0$, the given power density spectrum satisfies this condition for all values of ' ω ' i.e.,

$$\omega = 0 \Rightarrow S_{XX}(0) = \exp(-1) = 0.368$$

$$\omega = 5 \Rightarrow S_{XX}(5) = \exp(-16) = 1.125 \times 10^{-7}$$

$$\omega = 10 \Rightarrow S_{XX}(10) = \exp(-81) = 6.64 \times 10^{-36}$$

Now, consider $S_{XX}(-\omega) = S_{XX}(\omega)$

$$\begin{aligned}i.e., \quad S_{XX}(-\omega) &= \exp[-(-\omega-1)^2] \\ &= \exp[-(\omega+1)^2] \\ &= \exp[-(\omega+1)^2] \neq S_{XX}(\omega)\end{aligned}$$

The property (2) is not satisfied by the given power density spectrum. So the given function is not a valid power density spectrum.

Q12. A random process is given by $Y(t) = X(t) - X(t-a)$, where $X(t)$ is a WSS process and $a > 0$ is a constant. Find the auto correlation function and power density spectrum of $Y(t)$ in terms of the

corresponding quantities for $X(t)$.

[Nov-09/Set-3/Q7]

Sol.

The given random process is,

$$Y(t) = X(t) - X(t-a)$$

Where, $x(t)$ = Wide sense stationary random process

A = Positive constant

Then, the auto-correlation function of the random process $Y(t)$ in terms of the quantities of $X(t)$ can be obtained as,

$$\begin{aligned} R_{XX}(\tau) &= R_{YY}(t, t+\tau) \\ &= E[Y(t)Y(t+\tau)] \\ &= E[X(t) - X(t-a)][X(t+\tau) - X(t+\tau-a)] \\ &= E[X(t)X(t+\tau) - X(t)X(t+\tau-a) - X(t-a)X(t+\tau) \\ &\quad + X(t-a)X(t+\tau-a)] \\ &= E[X(t)X(t+\tau) - E[X(t)X(t+\tau-a)] \\ &\quad - E[X(t-a)X(t+\tau)] + E[X(t-a)X(t+\tau-a)]] \\ &= R_{XX}(\tau) - R_{XX}(\tau-a) - E[X(t-a)X(t-a+\tau+a)] \\ &\quad + E[X(t-a)X(t-a+\tau)] \\ &= R_{XX}(\tau) - R_{XX}(\tau-a) - R_{XX}(\tau+a) + R_{XX}(\tau) \\ &= 2R_{XX}(\tau) - R_{XX}(\tau-a) - R_{XX}(\tau+a) \\ \therefore R_{XX}(\tau) &= 2R_{XX}(\tau) - R_{XX}(\tau-a) - R_{XX}(\tau+a) \end{aligned}$$

And the power density spectrum of the random process $Y(t)$ in terms of the quantities of $X(t)$ can be obtained as,

$$\begin{aligned} S_{YY}(\omega) &= \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} [2R_{XX}(\tau) - R_{XX}(\tau-a) - R_{XX}(\tau+a)] e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} 2R_{XX}(\tau) e^{-j\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau-a) e^{-j\omega\tau} d\tau \\ &\quad - \int_{-\infty}^{\infty} R_{XX}(\tau+a) e^{-j\omega\tau} d\tau \\ &= 2 \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau-a) e^{-j\omega(\tau-a)} e^{-j\omega a} d\tau \\ &\quad - e^{-j\omega(\tau+a)} \int_{-\infty}^{\infty} R_{XX}(\tau+a) e^{-j\omega(\tau+a)} e^{j\omega a} d\tau \\ &= 2S_{XX}(\omega) - e^{-j\omega a} \int_{-\infty}^{\infty} R_{XX}(\tau-a) e^{-j\omega(\tau-a)} d\tau \\ &\quad - e^{j\omega a} + \int_{-\infty}^{\infty} R_{XX}(\tau+a) e^{-j\omega(\tau+a)} d\tau \end{aligned}$$

In the above expression, replace $(\tau-a)$ with ' k ' in the 2nd term and replace $(\tau+a)$ with ' m ' in the 3rd term. Then,

$$S_{YY}(\omega) = 2S_{XX}(\omega) - e^{j\omega a} \int_{-\infty}^{\infty} R_{XX}(k) e^{-j\omega k} dk$$

$$\begin{aligned} &- e^{-j\omega a} \int_{-\infty}^{\infty} R_{XX}(m) e^{-j\omega m} dm \\ &= 2S_{XX}(\omega) - e^{j\omega a} S_{XX}(\omega) - e^{-j\omega a} S_{XX}(\omega) \\ &= [2 - e^{j\omega a} - e^{-j\omega a}] S_{XX}(\omega) \\ &= [2 - (e^{j\omega a} + e^{-j\omega a})] S_{XX}(\omega) \\ &= [2 - 2\cos\omega a] S_{XX}(\omega) \\ &= 2(1 - \cos\omega a) S_{XX}(\omega) \\ \therefore S_{YY}(\omega) &= 2(1 - \cos\omega a) S_{XX}(\omega) \end{aligned}$$

Q13. A random process has the power density

$$\text{spectrum } S_{XY}(\omega) = \frac{6\omega^2}{(1+\omega^2)^3}, \text{ find the average power in the process. [Nov-09/Set-4/Q7(a)]}$$

Sol.

In the given question ' $S_{XY}(\omega)$ ' is misprinted as ' $S_{XX}(\omega)$ '. Given that, For a random process, $X(t)$

$$\text{Power density spectrum, } S_{XX}(\omega) = \frac{6\omega^2}{(1+\omega^2)^3}$$

Average power in the random process, $P_{XX} = ?$

Then, the expression for average power in the random process $X(t)$ in terms of power density spectrum is given by,

$$\begin{aligned} P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \\ \Rightarrow P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{6\omega^2}{(1+\omega^2)^3} d\omega = \frac{3}{\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^2)^3} d\omega \end{aligned}$$

By using the indefinite integral mathematics,

$$\int \frac{x^2 dx}{(x^2 + a^2)^2} = \frac{-x}{4(a^2 + x^2)^2} + \frac{x}{8a^2(a^2 + x^2)} + \frac{1}{8a^3} \tan^{-1}\left(\frac{x}{a}\right)$$

Then,

$$P_{XX} = \frac{3}{\pi} \left[\frac{-\omega}{4(1+\omega^2)^2} + \frac{\omega}{8(1+\omega^2)} + \frac{1}{8} \tan^{-1}(\omega) \right]_{-\infty}^{\infty}$$

For the terms $\frac{-\omega}{4(1+\omega^2)^2}$, $\frac{\omega}{8(1+\omega^2)}$,

The substitution of upper and lower limits results $\frac{\infty}{\infty}$ form. So, using L-hospital's rule. These two terms are simplified and reduced to zero.

$$\begin{aligned} \therefore P_{XX} &= \frac{3}{\pi} \left[0 + 0 + \frac{1}{8} (\tan^{-1}(\infty) - \tan^{-1}(-\infty)) \right] \\ &= \frac{3}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{3}{8} = 0.375 \text{ Watt} \end{aligned}$$

$$\therefore P_{XX} = 0.375 \text{ Watt.}$$

Q14. Calculate the PSD of a stationary random process for which the auto correlation is $R_{XX}(t) = 6^V \cdot e^{-\alpha|t|}$ auto correlation function of an a periodic power signal is $R_{XX}(\tau) = \exp(-\tau^2/26^2)$. Find the PSD and normalized average power content of the signal.
[May/June-09/Set-3/Q6(a)]

Sol.

The given auto correlation function of a stationary random process 'X' is,

$$R_{XX}(\tau) = 6^V e^{-\alpha|\tau|}$$

The relation between Power Spectral Density (PSD),

$$\xi_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$\Rightarrow \xi_{XX}(\omega) = \int_{-\infty}^{\infty} 6^V e^{-\alpha|\tau|} e^{-j\omega\tau} d\tau$$

$$= 6^V \left[\int_{-\infty}^0 e^{\alpha\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} e^{-\alpha\tau} e^{-j\omega\tau} d\tau \right]$$

$$= 6^V \left[\int_{-\infty}^0 e^{(-j\omega\tau + \alpha\tau)} d\tau + \int_0^{\infty} e^{-(\alpha + j\omega)\tau} d\tau \right]$$

$$\xi_{XX}(\omega) = 6^V \left[\frac{2\alpha}{\alpha^2 + \omega^2} \right] \quad (\because j^2 = -1)$$

$$\therefore \xi_{XX}(\omega) = \frac{(2\alpha)6^V}{\alpha^2 + \omega^2}$$

And the given autocorrelation function of an a periodic power signal is,

$$R_{XX}(\tau) = \exp\left(\frac{-\tau^2}{26^2}\right)$$

Then, the corresponding Power Spectral Density (PSD) of an a periodic power signal is obtained as,

$$\begin{aligned} \xi_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} \exp\left(\frac{-\tau^2}{26^2}\right) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \exp\left(\frac{-\tau^2}{26^2} - j\omega\tau\right) d\tau \end{aligned}$$

But, from the property of infinite integrals i.e.,

$$\int_{-\infty}^{\infty} e^{-a^2x^2 + bx} dx = \frac{\sqrt{\pi}}{a} e^{b^2/4a^2}, a > 0 \quad \dots\dots(1)$$

Here, $a = \frac{1}{26}$, $b = -j\omega$ and $x = \tau$

$$\begin{aligned} \text{Then, } \xi_{XX}(\omega) &= \sqrt{\pi} \times 26 \times e^{-\omega^2/4 \times (1/26)^2} \\ &= 26\sqrt{\pi} \exp\left(\frac{-676\omega^2}{4}\right) \end{aligned}$$

$$\therefore \xi_{XX}(\omega) = 26\sqrt{\pi} \exp(-169\omega^2)$$

And the normalized average power is given by the expression,

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_{XX}(\omega) d\omega$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 26\sqrt{\pi} \exp(-169\omega^2) d\omega$$

By substituting $b=0$ in equation (1), we have,

$$\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$$

Here, $A = 13$ and $x = \omega$

$$\text{Then, } P_{XX} = \frac{1}{2\pi} \times 26\sqrt{\pi} \times \frac{\sqrt{\pi}}{13}$$

$$P_{XX} = 1$$

Q15. (a) A WSS random process $X(t)$ has

$$R_{XX}(\tau) = A_0 \left[1 - \frac{|\tau|}{T} \right], \quad -T \leq \tau \leq T$$

$= 0$, elsewhere

Find power density spectrum.

$$R_{XX}(\tau) = \frac{A_0}{2} \sin \omega_0 \tau. \quad \text{(b) Find } \xi_{XX}(\omega).$$

[May/June-09/Set-1,2,3,4/Q7] [Nov-08/Set-1/Q7]

Sol.

(a) Given that,

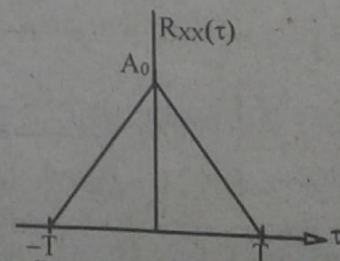
Auto correlation function,

$$R_{XX}(\tau) = A_0 \left[1 - \frac{|\tau|}{T} \right], \quad -T \leq \tau \leq T$$

$= 0$, elsewhere

By considering $A_0 > 0$ the sketch of given auto correlation function is,

The power spectrum, $\xi_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$



$$\xi_{XX}(\omega) = \int_{-\infty}^{\infty} A_0 \left[1 - \frac{|\tau|}{T} \right] e^{-j\omega\tau} d\tau$$

$$\xi_{XX}(\omega) = \int_{-\infty}^{\infty} A_0 \left[1 + \frac{\tau}{T} \right] e^{-j\omega\tau} d\tau + \int_0^T A_0 \left[1 - \frac{\tau}{T} \right] e^{-j\omega\tau} d\tau$$

Considering,

$$\begin{aligned}
 &= A_0 \int_{-T}^0 \left[1 + \frac{\tau}{T} \right] e^{-j\omega\tau} d\tau \\
 &= A_0 \left[\int_{-T}^0 e^{-j\omega\tau} d\tau + \int_{-T}^0 \frac{\tau}{T} e^{-j\omega\tau} d\tau \right] \left\{ \because \int e^{ax} dx = \frac{e^{ax}}{a} \right\} \\
 &= A_0 \left[\frac{1 - e^{-j\omega T}}{-j\omega} \right] + \frac{A_0}{T} \int_{-T}^0 \tau e^{-j\omega\tau} d\tau \\
 &\quad \left\{ \because \int x e^{ax} dx = e^{ax} \left[\frac{x}{a} - \frac{1}{a^2} \right] \right\} \\
 &= \frac{A_0 (e^{j\omega T} - 1)}{j\omega} + \frac{A_0}{T} \left[e^{-j\omega\tau} \left(\frac{\tau}{-j\omega} + \frac{1}{\omega^2} \right) \right]_{-T}^0 \\
 &= \frac{A_0}{\omega^2 T} - \frac{A_0}{j\omega} - \frac{A_0 e^{j\omega T}}{\omega^2 T}
 \end{aligned}$$

Considering,

$$\begin{aligned}
 &= A_0 \int_0^T \left[1 - \frac{\tau}{T} \right] e^{-j\omega\tau} d\tau \\
 &= A_0 \left[\int_0^T e^{-j\omega\tau} d\tau - \frac{1}{T} \int_0^T \tau e^{-j\omega\tau} d\tau \right] \\
 &= A_0 \left(\frac{e^{-j\omega\tau}}{-j\omega} \right)_0^T - \frac{A_0}{T} \left(e^{-j\omega\tau} \left(\frac{\tau}{-j\omega} + \frac{1}{\omega^2} \right) \right)_0^T \\
 &= A_0 \left(\frac{e^{-j\omega T} - 1}{-j\omega} \right) - \frac{A_0}{T} \left[\frac{T e^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega T}}{\omega^2} - \frac{1}{\omega^2} \right] \\
 &= \frac{A_0}{j\omega} - \frac{A_0 e^{-j\omega T}}{\omega^2 T} + \frac{A_0}{\omega^2 T}
 \end{aligned}$$

Substituting the above value in equation (1) we get,

$$\begin{aligned}
 \xi_{XX}(\omega) &= \frac{A_0}{\omega^2 T} - \frac{A_0}{j\omega} - \frac{A_0 e^{j\omega T}}{\omega^2 T} + \frac{A_0}{j\omega} + \frac{A_0}{\omega^2 T} - \frac{A_0 e^{-j\omega T}}{\omega^2 T} \\
 \xi_{XX}(\omega) &= \frac{2A_0}{\omega^2 T} - \frac{2A_0}{\omega^2 T} \left[\frac{e^{j\omega T} + e^{-j\omega T}}{2} \right] \\
 &\quad \left[\because \frac{e^{j\omega T} + e^{-j\omega T}}{2} = \cos \omega T \right] \\
 \xi_{XX}(\omega) &= \frac{2A_0}{\omega^2 T} [1 - \cos \omega T] \left\{ \because \cos \omega T = 1 - 2 \sin^2 \frac{\omega T}{2} \right\}
 \end{aligned}$$

$$\xi_{XX}(\omega) = \frac{2A_0}{\omega^2 T} \left[1 - 1 + 2 \sin^2 \frac{\omega T}{2} \right]$$

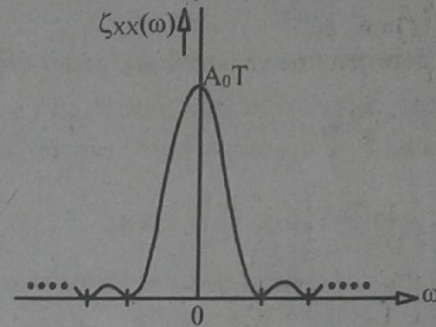
$$\xi_{XX}(\omega) = \frac{4A_0}{\omega^2 T} \sin^2 \frac{\omega T}{2}$$

Multiplying and dividing by T, we get,

$$\xi_{XX}(\omega) = A_0 T \frac{\sin^2 \left(\frac{\omega T}{2} \right)}{\left(\frac{\omega T}{2} \right)^2}$$

But, $\frac{\sin \theta}{\theta} = S_a \theta$

$$\xi_{XX}(\omega) = A_0 T S_a^2 \left(\frac{\omega T}{2} \right)$$



(b) Given function,

$$R_{XX}(\tau) = \left(\frac{A_0^2}{2} \right) \sin \omega_0 \tau \quad \text{.....(1)}$$

But, the function $\sin \omega_0 \tau$ is given as,

$$\sin \omega_0 \tau = \frac{e^{j\omega_0 \tau} - e^{-j\omega_0 \tau}}{2j}$$

$$\therefore R_{XX}(\tau) = \frac{A_0^2}{2} \times \left[\frac{e^{j\omega_0 \tau} - e^{-j\omega_0 \tau}}{2j} \right]$$

$$R_{XX}(\tau) = \frac{A_0^2}{4j} (e^{j\omega_0 \tau} - e^{-j\omega_0 \tau})$$

The power spectrum function $\xi_{XX}(\omega)$ is given as,

$$\xi_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \quad \text{.....(2)}$$

$$= \frac{A_0^2}{4j} \int_{-\infty}^{\infty} (e^{j\omega_0 \tau} - e^{-j\omega_0 \tau}) e^{-j\omega\tau} d\tau$$

$$= \frac{A_0^2}{4j} \int_{-\infty}^{\infty} 1 \cdot e^{-j(\omega - \omega_0)\tau} d\tau - \frac{A_0^2}{4j} \int_{-\infty}^{\infty} 1 \cdot e^{-j(\omega + \omega_0)\tau} d\tau$$

.....(3)

The inverse fourier transform $F(\omega)$ is given as,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

In equation (3) $f(t)$ is '1'

The inverse fourier transform of '1' is

$$1 \xrightarrow{F_s^{-1}} 2\pi \delta(\omega) \quad \text{.....(4)}$$

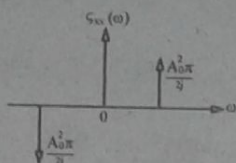
Inverse fourier transform of,

$$1 \cdot e^{j\omega_0 \tau} \xrightarrow{F_s^{-1}} 2\pi \delta(\omega - \omega_0) \quad \text{.....(5)}$$

Using equations (4) and (5) is equation (3), we get,

$$\begin{aligned} &= \frac{A_0^2}{4j} [2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)] \\ \xi_{XX}(\omega) &= \frac{A_0^2}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \end{aligned} \quad \dots\dots(6)$$

Equation (6) gives the required power spectrum and is plotted as follows.



Q16. A random process $X(t)$ whose mean value is 2 and auto correlation function $R_{XX}(Y) = 4e^{-2(\tau)}$ is applied to a system whose transfer function is $\frac{1}{2+j\omega}$. Find the mean value, auto correlation, power density and average power of output signal $Y(t)$.

[A109210401/Nov-10/Set-3/Q7(b)]

Sol.

Given that, Mean of random process $X(t) = 2$

Auto correlation function $R_{XX}(Y) = 4e^{-2(\tau)}$

Transfer function of the system $= \frac{1}{2 + j\omega}$

Mean of Input = Mean of output $\times \frac{1}{H(0)}$

Where, $H(0) = H(\omega)|_{\omega=0}$

$$H(\omega) = F[x(t)] = F[4e^{-2|\tau|}]$$

$$H(\omega) = 4 \times F(j\omega - jt) = \frac{4}{4 + \omega^2}$$

$$2 = \text{Mean of output} \times \frac{1}{H(\omega)}$$

$$\text{Mean of output} = 2 \times \frac{4}{4 + \omega^2} \times 4 = \frac{8 \times 2}{4 + \omega^2} = \frac{16}{4 + \omega^2}$$

$$S_{XX}(\omega) = \text{input PSD} = F[R_{XX}(\tau)] = F[4e^{-2|\tau|}] = \frac{16}{4 + \omega^2}$$

$$S_{YY}(\omega) = \text{output of PSD} = |H(\omega)|^2$$

$$\text{Input PSD} = \frac{16}{(4 + \omega^2)^2}$$

$$\text{Output of auto correlation} = F^{-1}[S_{YY}(f)]$$

$$= F^{-1}\left[\frac{16}{(4 + \omega^2)^2}\right] = F^{-1}\left[\frac{4 \times 4}{(2^2 + \omega^2)^2}\right]$$

$$= F^{-1}\left[\frac{4}{(4 + \omega^2)}\right]^2 = (e^{-2|\tau|})^2$$

$$\begin{aligned} \text{Average o/p power} &= S_{XX}(\omega) |H(\omega)|^2 \\ &= \frac{16}{4 + \omega^2} \times \frac{16}{4 + \omega^2} = \frac{32}{(4 + \omega^2)^2} \text{ s.} \end{aligned}$$

5.2 CROSS POWER DENSITY SPECTRUM PROPERTIES

Q17. Prove that, cross correlation function and cross spectral density are Fourier transform pair.

Sol.

Let $X(t)$ and $Y(t)$ be two random processes and let $x(t)$ and $y(t)$ be their sample functions respectively.

$$\text{Let } x_T(t) = x(t) \quad \text{for } -T < t < T$$

$$= 0 \quad \text{elsewhere}$$

$$\text{and } y_T(t) = y(t) \quad \text{for } -T < t < T$$

$$= 0 \quad \text{elsewhere}$$

The individual fourier transforms of $x_T(t)$ and $y_T(t)$ are,

$$X_T(\omega) = \int_{-T}^T x_T(t) \cdot e^{-j\omega t} \cdot dt = \int_{-T}^T x(t) \cdot e^{-j\omega t} \cdot dt$$

$$Y_T(\omega) = \int_{-T}^T y_T(t) \cdot e^{-j\omega t} \cdot dt = \int_{-T}^T y(t) \cdot e^{-j\omega t} \cdot dt$$

Consider the generalized Parseval's relation.

$$\text{Consider } \int_{-\infty}^{\infty} x(t)y(t) \cdot dt$$

$$\text{We have } y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} \cdot d\omega$$

$$\therefore \int_{-\infty}^{\infty} x(t)y(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y(\omega) \cdot e^{j\omega t} \cdot d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) \left[\int_{-\infty}^{\infty} x(t) \cdot e^{j\omega t} \cdot dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-\omega) \cdot Y(\omega) \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) \cdot d\omega$$

$$\therefore \int_{-\infty}^{\infty} x(t)y(t) \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot Y^*(\omega) \cdot d\omega$$

$$\text{The average power, } P_{XY} = \frac{1}{2\pi} \int_{-T}^T x_T(t) \cdot y_T(t) \cdot dt$$

$$= \frac{1}{2\pi} \int_{-T}^T x(t) \cdot y(t) \cdot dt$$

By using the above parseval's relation

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X_T^*(\omega) \cdot Y_T(\omega)}{2T} \cdot d\omega$$

Using the similar argument as made in the case of power spectral density

$$P_{XY(av)} = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T E[x(t) \cdot y(t)] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T} \cdot d\omega$$

If we define $\lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T} = S_{XY}(\omega)$,
i.e., the cross spectral density, the average power

$$P_{XY(av)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \cdot d\omega$$

$$\text{Similarly, } P_{YX(av)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) \cdot d\omega$$

Q18. Consider the Random process $x(t) = A \cos(\omega_0 t + \theta)$ where A and ω_0 are real constants and θ is a random variable uniformly distributed on the interval $(0, \frac{\pi}{2})$ find the average power P_{xx} in $x(t)$.

Sol.

Average Power P_{XX} = Average of

$$E[X^2(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt$$

$$E[X^2(t)] = E[A^2 \cos^2(\omega_0 t + \theta)] = \frac{A^2}{2} \cdot E[1 + \cos 2(\omega_0 t + \theta)]$$

$$= \frac{A^2}{2} + \frac{A^2}{2} \cdot E[\cos 2(\omega_0 t + \theta)]$$

$$E[\cos 2(\omega_0 t + \theta)] = \int_0^{\frac{\pi}{2}} \cos 2(\omega_0 t + \theta) \cdot f(\theta) \cdot d\theta$$

$$f(\theta) = \frac{2}{\pi}$$

$$\therefore E[\cos 2(\omega_0 t + \theta)] = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos 2(\omega_0 t + \theta) d\theta$$

$$= \frac{2}{\pi} \left[\frac{\sin 2(\omega_0 t + \theta)}{2} \right]_0^{\frac{\pi}{2}} = \frac{2}{\pi} \left[\sin 2 \left(\omega_0 t + \frac{\pi}{2} \right) - \sin 2\omega_0 t \right]$$

$$= \frac{-2 \sin 2\omega_0 t}{\pi}$$

$$\therefore E[X^2(t)] = \frac{A^2}{2} - \frac{A^2}{2} \cdot \sin 2\omega_0 t$$

$$P_{XX} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[\frac{A^2}{2} - \frac{A^2}{2} \cdot \sin 2\omega_0 t \right] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\frac{A^2}{2} \cdot t + \frac{A^2}{2} \frac{\cos 2\omega_0 t}{2\omega_0} \right]_{-T}^T = \frac{A^2}{2}$$

Q19. Consider a random process $x(t) = \cos(\omega t + \theta)$ where ω is a real constant and θ is a uniform

random variable in $(0, \frac{\pi}{2})$. Find the average power in the process.

Sol.

The average power P_{xx} in the process $x(t)$ is,

$$P_{xx} = \text{Average of } E[X^2(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt$$

$$E[X^2(t)] = E[\cos^2(\omega t + \theta)] = E\left[\frac{1 + \cos 2(\omega t + \theta)}{2}\right]$$

$$= E\left(\frac{1}{2}\right) + E\left[\frac{\cos 2(\omega t + \theta)}{2}\right]$$

$$\text{Consider } E\left[\frac{\cos 2(\omega t + \theta)}{2}\right] = \int_0^{\pi/2} f(\theta) \cdot \frac{\cos 2(\omega t + \theta)}{2} \cdot d\theta$$

$$\text{We have, } f(\theta) = \frac{2}{\pi}$$

$$\therefore E\left[\frac{\cos 2(\omega t + \theta)}{2}\right] = \frac{1}{\pi} \int_0^{\pi/2} \cos 2(\omega t + \theta) \cdot d\theta$$

$$= \frac{1}{2\pi} [\sin 2(\omega t + \theta)]_0^{\pi/2}$$

$$\therefore E[X^2(t)] = \frac{1}{2} - \frac{\sin 2\omega t}{\pi}$$

$$P_{xx} = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \left[\frac{1}{2} - \frac{\sin 2\omega t}{\pi} \right] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \left[\frac{1}{2} \int_{-T}^T dt - \frac{1}{\pi} \int_{-T}^T \sin 2\omega t dt \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \left[T + \frac{1}{\pi} \left(\cos \frac{2\omega t}{2\omega} \right)_{-T}^T \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2\pi} (T) = \frac{1}{2}$$

Q20. Mention the properties of cross spectral density.

[A109210401/Sup/May/June-2012/Set-2/Q4(b)]

Sol.

Time average of cross correlation function and the cross spectral density function form a fourier transform pair.

We have cross spectral density

$$S_{XY}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T}$$

Consider $X(t_1)$ and $Y(t_2)$

$$X_T^*(\omega) = \int_{-T}^T x(t_1) \cdot e^{-j\omega t_1} \cdot dt_1$$

$$Y_T(\omega) = \int_{-T}^T y(t_2) \cdot e^{-j\omega t_2} \cdot dt_2$$

$$\begin{aligned} \therefore S_{XY}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T x(t_1) \cdot e^{j\omega t_1} \cdot dt_1 \cdot \int_{-T}^T y(t_2) \cdot e^{j\omega t_2} \cdot dt_2 \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T E [x(t_1) y(t_2)] e^{-j\omega(t_2-t_1)} dt_1 \cdot dt_2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_1 \cdot dt_2 \end{aligned}$$

Consider the expression $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \cdot e^{j\omega\tau} \cdot d\omega$,

which is the inverse fourier transform of $S_{XY}(\omega)$.

$$\begin{aligned} F^{-1}[S_{XY}(\omega)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t_1, t_2) \cdot e^{-j\omega(t_2-t_1)} \cdot dt_1 \cdot dt_2 \cdot e^{j\omega\tau} \cdot d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t_1, t_2) \cdot \frac{1}{2T} \int_{-\infty}^{\infty} e^{-j\omega(t_2-t_1)} \cdot d\omega \end{aligned}$$

$$\text{Since } \frac{1}{2T} \int_{-\infty}^{\infty} e^{-j\omega(t_2-t_1)} d\omega = \delta(\tau - t_2 + t_1)$$

$$F^{-1}[S_{XY}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XY}(t_1, t_1 + \tau) dt_1$$

$$\text{Let } t_1 = t \Rightarrow dt_1 = dt$$

$$\therefore F^{-1}[S_{XY}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XY}(t, t + \tau) dt$$

The right hand side of the above equation is the time average of cross correlation function. It implies that time average of cross correlation function and cross spectral density form a fourier transform pair.

If the processes $X(t)$ and $Y(t)$ are jointly WSS processes, the time average of cross correlation function $R_{XY}(t, t + \tau)$ will be $R_{XY}(\tau)$.

$$\therefore F^{-1}[S_{XY}(\omega)] = R_{XY}(\tau) \Rightarrow R_{XY}(\tau) \stackrel{FT}{\leftrightarrow} S_{XY}(\omega)$$

$$2. S_{YX}(\omega) = S_{XY}(-\omega)$$

$$\text{We have } S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{j\omega\tau} \cdot d\tau$$

$$\text{Consider } S_{XY}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{j(-\omega)\tau} \cdot d\tau$$

$$\text{Since } R_{XY}(\tau) = R_{YX}(-\tau)$$

$$\begin{aligned} S_{XY}(-\omega) &= \int_{-\infty}^{\infty} R_{YX}(-\tau) \cdot e^{j\omega\tau} \cdot d\tau \\ &= \int_{-\infty}^{\infty} R_{YX}(-\tau) \cdot e^{-j\omega(-\tau)} \cdot d\tau = S_{YX}(\omega) \end{aligned}$$

3. Real part of $S_{XY}(\omega)$ is an even function of ω and imaginary part is an odd function of ω .

$$\text{We have } S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{j\omega\tau} \cdot d\tau$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot (\cos \omega\tau - j \sin \omega\tau) d\tau \\ &= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos \omega\tau \cdot d\tau - j \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin \omega\tau \cdot d\tau \end{aligned}$$

$$\therefore [S_{XY}(\omega)]_{\text{real}} = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos \omega\tau \cdot d\tau$$

$$\text{Consider } [S_{XY}(\omega)]_{\text{real}} = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos(-\omega\tau) d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos \omega\tau \cdot d\tau$$

Thus, $[S_{XY}(\omega)]_{\text{real}}$ is an even function of frequency.

$$[S_{XY}(\omega)]_{\text{imaginary}} = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin \omega\tau \cdot d\tau$$

$$\text{Consider } [S_{XY}(-\omega)]_{\text{imaginary}} = - \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin(-\omega\tau) \cdot d\tau$$

$$= - \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin \omega\tau \cdot d\tau = -[S_{XY}(-\omega)]_{\text{imaginary}}$$

Thus, $[S_{XY}(-\omega)]_{\text{imaginary}}$ is an odd function of frequency.

4. $S_{XY}(\omega) = 0$ if $X(t)$ and $Y(t)$ are orthogonal.

$$S_{XY}(\omega) = F[R_{XY}(\tau)]$$

For two orthogonal processes $X(t)$ and $Y(t)$, $R_{XY}(\tau) = 0$

$$\text{Hence, } S_{XY}(\omega) = 0$$

5. If $X(t)$ and $Y(t)$ are uncorrelated and of constant means $E(X)$ and $E(Y)$, respectively,

$$S_{XY}(\omega) = 2\pi E(X) \cdot E(Y) \cdot \delta(\omega)$$

$$\text{We have } S_{XY}(\omega) = F[R_{XY}(\tau)] = F[E\{X(t) \cdot Y(t + \tau)\}]$$

Since $X(t)$ and $Y(t)$ are uncorrelated,

$$S_{XY}(\omega) = F[E\{X(t)\} \cdot E\{Y(t + \tau)\}]$$

Since mean of $Y(t)$ is constant,

$$E[Y(t)] = E[Y(t + \tau)] = E[Y]$$

$$\therefore S_{XY}(\omega) = F[E(X) \cdot E(Y)] = \int_{-\infty}^{\infty} E(X) \cdot E(Y) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$= E(X) \cdot E(Y) \int_{-\infty}^{\infty} e^{-j\omega\tau} \cdot d\tau$$

$$\text{Since } K \stackrel{FT}{\leftrightarrow} 2\pi k \cdot \delta(\omega) \Rightarrow 1 \stackrel{FT}{\leftrightarrow} 2\pi \cdot \delta(\omega)$$

$$\therefore S_{XY}(\omega) = 2\pi \cdot E(X) \cdot E(Y) \cdot \delta(\omega)$$

Q21. "The power spectral density of any random waveform and its auto correlation function are related by means of a Fourier transform". Prove and illustrate the above statement.

[A109210401/Nov-2010/Set-2/Q7(a)]

Sol.

Let $X(t)$ be a random process and Let $X_T(t)$ be the trun-

cated $X(t)$ between $t = -T$ and $t = +T$.

$$\text{Then } F[X_T(t)] = \int_{-T}^T X_T(t) \cdot e^{-j\omega t} \cdot dt$$

The energy content of $X_T(t)$ is,

$$E = \int_{-T}^T X_T^2(t) dt = \int_{-T}^T X^2(t) dt$$

According to Parseval's Theorem

$$\int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

Average power in the random process $X(t)$ is,

$$P = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E|X_T(\omega)|^2}{2T} d\omega = \frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{E|X_T(\omega)|^2}{2T} d\omega$$

$$\text{Let } S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E|X_T(\omega)|^2}{2T}$$

$$\therefore P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot d\omega$$

Where, $S_{XX}(\omega)$ = PSD of the random process $X(t)$

$$\text{Consider } X_T(\omega) = \int_{-T}^T X_T(t) \cdot e^{-j\omega t} \cdot dt$$

$$\text{Since } S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E|X_T(\omega)|^2$$

$$\begin{aligned} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot E \left[\int_{-T}^T X(t_1) \cdot e^{j\omega t_1} dt_1 \cdot \int_{-T}^T X(t_2) \cdot e^{-j\omega t_2} dt_2 \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \int_{-T}^T \int_{-T}^T E[X(t_1) \cdot X(t_2)] e^{-j\omega(t_2 - t_1)} dt_1 \cdot dt_2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \int_{-T}^T \int_{-T}^T R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_1 \cdot dt_2 \end{aligned}$$

$$\text{Let } t_1 = t, t_2 = t_1 + \tau \Rightarrow dt_1 = dt; dt_2 = d\tau$$

$$\begin{aligned} \text{When } t_1 = T &\Rightarrow t = T \text{ and } t_1 = \tau \Rightarrow t = -T \\ t_2 = T &\text{ and } t_2 - t_1 = \tau \Rightarrow t_2 - t = \tau \\ &\text{and } T - t = \tau \end{aligned}$$

$$\text{When } t_2 = -T \Rightarrow -T - t = \tau$$

$$\therefore S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t, t+\tau) e^{-j\omega\tau} \cdot dt \cdot d\tau$$

$$= \int_{-\infty}^{\infty} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XX}(t, t+\tau) dt \right] e^{-j\omega\tau} \cdot d\tau$$

$$\text{Since } \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XX}(t, t+\tau) dt \text{ is the time average value}$$

$$R_{XX}(t, t+\tau).$$

When the process is stationary,

$$R_{XX}(t, t+\tau) = R(\tau)$$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$R(\tau) \xleftrightarrow{F.T} S_{XX}(\omega).$$

Q22. The power spectral density of a stationary random process is given by

$$S_{XX}(\omega) = A \quad -k < \omega < k \\ = 0 \quad \text{otherwise}$$

Find the auto correlation function. [SAQ]

[A109210401/Reg./Dec-2011/Set-2/Q7(b)]

Sol.

Given that,

$$S_{XX}(\omega) = A \quad -k < \omega < k \\ = 0 \quad \text{otherwise}$$

$$\text{Autocorrelation function} = F^{-1}[S_{XX}(\omega)]$$

$$\therefore R_{XX}(\tau) = \int_{-K}^K e^{j\omega\tau} d\omega = A \left[\frac{e^{j\omega\tau}}{j\tau} \right]_{-K}^K$$

$$= A \left[\frac{e^{jK\tau} - e^{-jK\tau}}{j\tau} \right] = \frac{2A}{\tau} \sin(K\tau)$$

$$\therefore R_{XX}(\tau) = 2AK \cdot \text{Sa}(K\tau).$$

Q23. Find the autocorrelation function for white noise. [SAQ]

Sol.

$$\text{Let } f(t) = A + \left(\frac{2A}{\tau} \right) t \quad \text{for } -\frac{\tau_0}{2} \leq t \leq 0$$

$$= A - \left(\frac{2A}{\tau} \right) t \quad \text{for } 0 \leq t \leq \frac{\tau_0}{2}$$

We know that,

$$R(\tau) = \int_{-\infty}^{\infty} f(t) \cdot f(t+\tau) dt$$

$$= \int_{-\tau/2}^0 \left[A + \left(\frac{2A}{\tau} \right) t \right] \left[A + \left(\frac{2A}{\tau} \right) (t+\tau) \right] dt$$

$$+ \int_0^{\tau/2} \left[A - \left(\frac{2A}{\tau_0} \right) t \right] \left[A - \left(\frac{2A}{\tau_0} \right) (t+\tau_0) \right] dt$$

$$= \int_{-\tau/2}^0 \left[A^2 + \frac{2A}{\tau} (t+\tau) + \frac{2A^2}{\tau} t + \left(\frac{2A}{\tau} \right)^2 t(t+\tau) \right] d\tau$$

$$+ \int_0^{\tau/2} \left[A^2 - \left(\frac{2A^2}{\tau_0} \right) (t+\tau) - \frac{2A^2}{\tau} t + \left(\frac{2A}{\tau_0} \right)^2 t(t+\tau) \right] d\tau$$

$$= \left[A^2 + \frac{A^2 t^2}{\tau} - 2A^2 t + \frac{2A^2}{\tau} t + \left(\frac{2A}{\tau} \right)^2 t(t+\tau) \right] d\tau$$

$$+ \left[A^2 t - \frac{A^2 t^2}{\tau} - 2A^2 t - \frac{A^2 t^2}{\tau_0} + \frac{4A^2}{\tau_0^2} \cdot \frac{t^3}{3} + \frac{2A^2}{\tau_0} \cdot t^2 \right]_0^{\pi/2}$$

$$= \frac{A^2 \tau_0}{4}$$

Q24. A random process has the power density spectrum $S_{XX}(\omega) = \frac{6\omega^2}{1+\omega^4}$. Find the average power in the process.

[SAQ][Nov-09/Set-1/Q7(a)]

Sol.

Given that,

$$S_{XX}(\omega) = \frac{6\omega^2}{1+\omega^4} = \frac{6\omega^2}{1+(\omega^2)^2}$$

$$\text{Let } \omega^2 = x$$

$$\therefore S_{XX}(\omega) = \frac{6x}{1+x^2} = 3 \times \frac{2x}{1+x^2} = \frac{3 \times 2x}{x^2+1^2}$$

$$R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)] = F^{-1}\left[\frac{3 \times 2x}{x^2+1^2}\right] = 3e^{\omega^2} t$$

$\therefore$ Average power $R_{XX}(0) = 3$

Q25. Determine which of the following functions are valid PSDs,

(i) $\frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$

(ii) $\exp[-(\omega-1)^2]$

(iii) $\frac{\omega^2}{\omega^4 + 1} - \delta(\omega)$

(iv) $\frac{\omega^4}{1 + \omega^2 + j\omega^6}$

Sol.

If $S_{XX}(\omega)$ denote the power spectral density then

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

This property is used in verifying whether a given function is a valid power spectral density function are not.

(i) $\frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$

Let $S_{XX}(\omega) = \frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$

$$S_{XX}(-\omega) = \frac{(-\omega)^2}{(-\omega)^6 + 3(-\omega)^2 + 3} = \frac{\omega^2}{\omega^6 + 3\omega^2 + 3} = S_{XX}(\omega)$$

This function is valid PSD function.

(ii) $S_{XX}(\omega) = e^{-(\omega-1)^2}$

$$S_{XX}(-\omega) = e^{-(\omega-1)^2} \neq S_{XX}(\omega)$$

This function is not a valid PSD.

(iii) $\frac{\omega^2}{\omega^4 + 1} - \delta(\omega)$

Let $S_{XX}(\omega) = \frac{\omega^2}{\omega^4 + 1} - \delta(\omega)$

$$S_{XX}(-\omega) = \frac{(-\omega)^2}{(-\omega)^4 + 1} - \delta(-\omega) = \frac{\omega^2}{\omega^4 + 1} - \delta(\omega) = S_{XX}(\omega)$$

$\therefore$ This function is valid PSD function.

(iv) $\frac{\omega^4}{1 + \omega^2 + j\omega^6}$

$$S_{XX}(\omega) = \frac{(-\omega)^4}{1 + (-\omega)^2 + j(-\omega)^6} = \frac{\omega^4}{1 + \omega^2 + j\omega^6} = S_{XX}(\omega)$$

$\therefore$ This function is valid PSD function.

Q26. Find the ACF of the following PSD's.

(i) $S_{XX}(\omega) = \frac{157 + 12\omega^2}{(16 + \omega^2)(9 + \omega^2)}$

(ii) $S_{XX}(\omega) = \frac{8}{(9 + \omega^2)^2}$ [Nov-09/Set-4/Q7(b)]

Sol.

If $X(t)$ is a random process with probability density function $S_{XX}(\omega)$ and auto correlation function $R_{XX}(\tau)$. Then relation between $S_{XX}(\omega)$ and $R_{XX}(\tau)$ is given by Wiener-Khinchin as,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

(i) $R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{157 + 12\omega^2}{(16 + \omega^2)(9 + \omega^2)} e^{j\omega\tau} d\omega$$

Applying fraction to,

$$S_{XX}(\omega) = \frac{157 + 12\omega^2}{(16 + \omega^2)(9 + \omega^2)}$$

We get, $S_{XX}(\omega) = \frac{5}{(16 + \omega^2)} + \frac{7}{(9 + \omega^2)}$

$$\therefore R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{5}{16 + \omega^2} + \frac{7}{9 + \omega^2} \right) e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{5}{16 + \omega^2} e^{j\omega\tau} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{7}{9 + \omega^2} e^{j\omega\tau} d\omega$$

We have a fourier transform pair as,

$$e^{-a|\tau|} \xleftrightarrow{F.T.} \frac{2a}{a^2 + \omega^2}$$

Using this equation we get,

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{5(2)(4)}{8(4^2 + \omega^2)} e^{j\omega\tau} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{7(3)(2)}{6(3^2 + \omega^2)} e^{j\omega\tau} d\omega$$

$$\Rightarrow R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|}$$

$$(ii) S_{XX}(\omega) = \frac{8}{(9 + \omega^2)^2}$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{8}{36} \left(\frac{(3)(2)}{(9 + \omega^2)} \right)^2 e^{j\omega\tau} d\omega$$

$$\text{We have } e^{-a|\tau|} \xleftrightarrow{FT} \frac{2a}{a^2 + \omega^2}$$

Inverse transform of $\left(\frac{2a}{a^2 + \omega^2} \right)^2$ is convolution

of $e^{-a|\tau|}$ with $e^{-a|\tau|}$

$$\text{i.e., } \left(\frac{2a}{a^2 + \omega^2} \right)^2 \xleftrightarrow{FT} \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-a(t-\tau)} d\tau$$

There are two cases in the above integration.

Case (i): If $t > 0$ then

$$= \int_{-\infty}^0 e^{-a(-\tau+t-\tau)} d\tau + \int_0^t e^{-a(-\tau+t-\tau)} d\tau + \int_t^{\infty} e^{-\tau(t-\tau-\tau)} d\tau$$

Since $|\tau| = \tau$, $\tau > 0$ and $|t - \tau| = t - \tau$

for $t - \tau > 0$ i.e., $\tau < t$

$$= -\tau \tau > 0 \quad |t - \tau| = -(t - \tau)$$

for $-(t - \tau) < 0$ i.e., $\tau > t$

$$= e^{-at} \left[\frac{e^{2a\tau}}{2a} \right]_0^t + e^{-at} (\tau)_0^t + e^{at} \left[\frac{e^{-2a\tau}}{-2a} \right]_t^{\infty}$$

$$= \frac{e^{-at}}{a} (1 + ta)$$

Case (ii): If $t < 0$, then the convolution is,

$$= e^{-a|\tau|} * e^{-a|\tau|} = \frac{e^{-at}}{a} (1 - ta)$$

$$\therefore R_{XX}(\tau) = \frac{e^{-a|\tau|}}{a} (1 + |\tau|a)$$

Q27. The power spectral density of $X(t)$ is given by

$$S_{XX}(\omega) = \begin{cases} 1 + \omega^2 & \text{for } |\omega| < 1 \\ 0 & \text{otherwise} \end{cases} \text{ Find the auto-correlation function. [SAQ]}$$

[A109210401/Nov-2010/Set-2/Q7(b)]

Sol.

Given that, power spectral density of $X(t)$

$$S_{XX}(\omega) = \begin{cases} 1 + \omega^2 & \text{for } |\omega| < 1 \\ 0 & \text{otherwise} \end{cases}$$

The wiener-kitchen relation is,

$$S_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{-j\omega\tau} d\omega$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi} \int_{-1}^1 (1 + \omega^2) e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^1 e^{j\omega\tau} d\omega + \frac{1}{2\pi} \int_{-1}^1 \omega^2 e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega\tau}}{j\tau} \right]_{-1}^1 + \frac{1}{2\pi} \left[\int_{-1}^1 \omega^2 e^{j\omega\tau} d\omega \right]$$

$$= \frac{1}{2\pi} \times \frac{1}{j\tau} [e^{j\tau} - e^{-j\tau}] + \frac{1}{2\pi} \left[\frac{\omega^3}{3} e^{j\omega\tau} + \omega^2 \frac{e^{j\omega\tau}}{j\tau} \right]$$

$$= \frac{1}{2\pi j\tau} [e^{j\tau} - e^{-j\tau}] + \frac{1}{2\pi} \left[\frac{1}{3} e^{j\tau} + \frac{e^{j\tau}}{j\tau} - \frac{1}{3} e^{-j\tau} - \frac{e^{-j\tau}}{j\tau} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\tau} - e^{-j\tau}}{j\tau} + \frac{1}{3} e^{j\tau} + \frac{e^{j\tau}}{j\tau} + e^{-j\tau} - \frac{e^{-j\tau}}{j\tau} \right]$$

$$= \frac{1}{2\pi j\tau} [2e^{j\tau} - 2e^{-j\tau} + j\tau e^{j\tau} - j\tau e^{-j\tau}]$$

$$= \frac{1}{2\pi j\tau} [e^{j\tau} (2 + j\tau) - e^{-j\tau} (2 + j\tau)]$$

$$= \frac{2 + j\tau}{2\pi j\tau} [e^{j\tau} - e^{-j\tau}]$$

Q28. The PSD of random process is given by

$$\delta_{XX}(\omega) = \begin{cases} \pi & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find its autocorrelation function. [SAQ]

[07A3EC10/Sup./May-2011/Set-1/Q7(a)]

[07A3EC10/Sup./May-2011/Set-2,4/Q2(a)]

[07A3EC10/Sup./May-2011/Set-3/Q6(a)]

Sol.

Given that,

$$\delta_{XX}(\omega) = \begin{cases} \pi & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

From the Wiener Khintchin relation

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-1}^1 (\pi) e^{j\omega\tau} d\omega \int_{-1}^1 e^{j\omega\tau} d\omega$$

$$= \frac{1}{2} \left[\frac{e^{j\omega\tau}}{j\tau} \right]_{-1}^1 = \frac{e^{j\tau} - e^{-j\tau}}{2j\tau} = \frac{2j \sin \tau}{2j\tau}$$

$$\therefore R_{XX}(\tau) = \frac{\sin \tau}{\tau}$$

Q29. (a) For $X(t)$ and $Y(t)$ are random process and

prove that $\xi_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T(\omega)Y_T(\omega)]}{2T}$

(b) $R_{XX}(\tau) = 4u(\tau)e^{-\alpha\tau}$. Find $\zeta_{XY}(\omega)$

[Nov-8/Set-2/Q7]

Sol.

(a) Given that $X(t)$ and $Y(t)$ are random processes. Here we define two variables $x_1(t)$ and $y_1(t)$, which are shortened ensemble members. The two variables are magnitude integrable and have bounded variation under limits $(-T, T)$.

$$x_1(t) = \begin{cases} x_1(t) & -T \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

$$y_1(t) = \begin{cases} y_1(t) & -T \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

The fourier transforms of above two variables exist and are given as,

$$x_1(t) \xrightarrow{F} X_T(\omega)$$

$$y_1(t) \xrightarrow{F} Y_T(\omega)$$

The cross power $P_{XY}(T)$ under the limits $(-T, T)$, within the process is given as,

$$\begin{aligned} P_{XY}(T) &= \frac{1}{2T} \int_{-T}^T x_1(t)y_1(t)dt \\ &= \frac{1}{2T} \int_{-T}^T x(t)y(t)dt \end{aligned} \quad \dots\dots(1)$$

Since, the fourier transform of $x_1(t)$ and $y_1(t)$ exist, hence we can apply parseval's theorem. It is given as,

$$\int_{-T}^T x_1(t)y_1(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_T^*(\omega)Y_T(\omega)d\omega$$

$\therefore$ Equation (1) now becomes,

$$P_{XY}(T) = \frac{1}{2T} \int_{-T}^T x(t)y(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X_T^*(\omega)Y_T(\omega)}{2T} d\omega \quad \dots\dots(2)$$

The value of cross power depends upon the type of ensemble member, hence it is not a consistent factor. The expected value of cross power is,

$$\bar{P}_{XY}(T) = \frac{1}{2T} \int_{-T}^T R_{XY}(t, t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} d\omega \quad \dots\dots(3)$$

The cross power formula is also given as,

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{XY}(\omega) d\omega \quad \dots\dots(4)$$

By comparing equation (3) and (4), we get finally,

$$\xi_{XY}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T}$$

(b) Given that, $R_{XY}(\tau) = 4u(\tau)e^{-\alpha\tau}$

The cross power density spectrum is given as,

$$\zeta_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau} d\tau \quad \dots\dots(1)$$

$$\zeta_{XY}(\omega) = \int_{-\infty}^{\infty} 4u(\tau)e^{-\alpha\tau} \cdot e^{-j\omega\tau} d\tau$$

$$\left[\because u(\tau) = 1 \text{ for } \tau \geq 0 \right. \\ \left. = 0 \text{ elsewhere} \right]$$

$$\zeta_{XY}(\omega) = 4 \int_0^{\infty} e^{-\alpha\tau} \cdot e^{-j\omega\tau} d\tau \quad \zeta_{XY}(\omega) = 4 \int_0^{\infty} e^{-(\alpha+j\omega)\tau} d\tau$$

$$\zeta_{XY}(\omega) = 4 \times \frac{[e^{-(\alpha+j\omega)\tau}]_0^{\infty}}{-(\alpha+j\omega)} \quad \zeta_{XY}(\omega) = 4 \times \frac{-4}{\alpha+j\omega} [-1]$$

$$\therefore \zeta_{XY}(\omega) = 4 \times \frac{4}{\alpha+j\omega}$$

5.2.1 Relationship Between Cross-power Spectrum and Cross Correlation Function

Q30. Describe the relationship between cross power spectrum and cross correlation function.

[A109210401/Reg/Dec-2011/Set-4/Q7(a)]

Sol.

The relationship between cross power spectrum and cross correlation function is as follows;

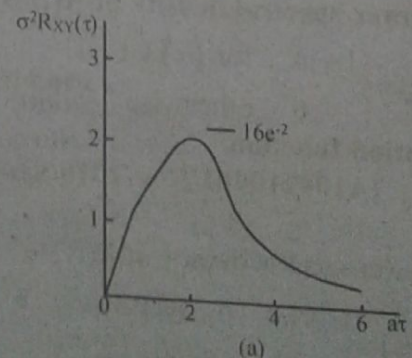
$$\phi_{XX}(\omega) = \lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-T}^T R_{XX}(t, t+\tau) d\tau \right\} e^{-j\omega\tau}$$

The development consists of using the transforms of the truncate processes given by,

$$X_T(\omega) = \int_{-T}^T X(t)e^{-j\omega t} dt \quad Y_T(\omega) = \int_{-T}^T Y(t_1)e^{-j\omega t_1} dt_1$$

This result is used to form the cross-power density spectrum.

$$\begin{aligned} \phi_{XX}(\omega) &= \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T} \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\int_{-T}^T X(t)e^{j\omega t} dt \int_{-T}^T Y(t_1)e^{-j\omega t_1} dt_1 \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t, t_1) e^{-j\omega(t_1-t)} dt dt_1 \end{aligned}$$



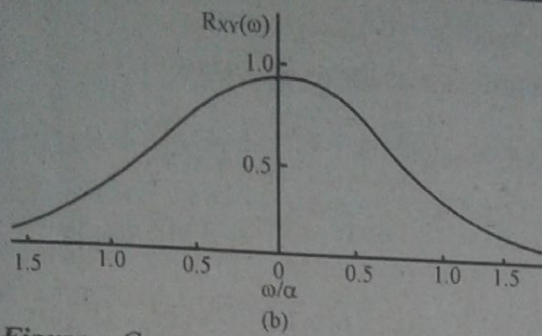


Figure : Cross-correlation function (a) and magnitude of cross-power spectrum (b) for the process of example

$$X^*_{-T}(\omega) Y_T(\omega) = \int_{-T}^T X(t) e^{j\omega t} dt \int_{-T}^T Y(t_1) e^{-j\omega t_1} dt_1$$

First, we inverse fourier transform both sides and identify one integral as an impulse function.

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XY}(\omega) e^{j\omega \tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t, t_1) e^{-j\omega(t_1 - t)} dt dt_1 e^{j\omega \tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t, t_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_1 + t)} d\omega dt_1 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XY}(t, t_1) \delta(t_1 - \tau - t) dt_1 dt \end{aligned}$$

The definition of the impulse allows the immediate solution for the integral over t_1 .

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XY}(\omega) e^{j\omega \tau} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{XY}(t, t + \tau) dt$$

which is valid for $-T < t + \tau < T$. This condition arises from the requirement that the impulse be within the range of integration. The condition can be ignored as $T \rightarrow \infty$ in the limit. The above equation indicates that the cross-power spectrum and the time-average cross-correlation function form a fourier transform pair. The result proves because it is the direct transform part of the pair, while it is the inverse part.

It should be noted that given cross-power spectrum, the cross correlation function cannot in general be recovered, only its time average can. For jointly wide-sense stationary processes, however, the cross-correlation function $R_{XY}(\tau)$ can be found from $\phi_{XY}(\omega)$ since its time average is just $R_{XY}(\tau)$.

Q31. Prove that, cross correlation function and cross spectral density are Fourier transform pair.

Sol.

Let $X(t)$ and $Y(t)$ be two random processes and let $x(t)$ and $y(t)$ be their sample functions respectively.

$$\text{Let } x_T(t) = x(t) \quad \text{for } -T < t < T \\ = 0 \quad \text{elsewhere}$$

$$\text{and } y_T(t) = y(t) \quad \text{for } -T < t < T \\ = 0 \quad \text{elsewhere}$$

The individual, fourier transforms of $x_T(t)$ and $y_T(t)$ are,

$$X_T(\omega) = \int_{-T}^T x_T(t) \cdot e^{-j\omega t} dt = \int_{-T}^T x(t) \cdot e^{-j\omega t} dt$$

$$Y_T(\omega) = \int_{-T}^T y_T(t) \cdot e^{-j\omega t} dt = \int_{-T}^T y(t) \cdot e^{-j\omega t} dt$$

Consider the generalized Parseval's relation.

$$\text{Consider } \int_{-\infty}^{\infty} x(t) y(t) dt = 2$$

$$\text{We have, } y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega$$

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} x(t) \cdot y(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y(\omega) \cdot e^{j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) \left[\int_{-\infty}^{\infty} x(t) \cdot e^{j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(-\omega) \cdot Y(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) d\omega \\ \therefore \int_{-\infty}^{\infty} x(t) y(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot Y(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot Y^*(\omega) d\omega \end{aligned}$$

$$\begin{aligned} \text{The average power } P_{XY} &= \frac{1}{2\pi} \int_{-T}^T x_T(t) \cdot y_T(t) dt \\ &= \frac{1}{2\pi} \int_{-T}^T x(t) \cdot y(t) dt \end{aligned}$$

By using above parseval's relation

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X^*_T(\omega) \cdot Y_T(\omega)}{2T} d\omega$$

Using the similar arguments as made in the case of power spectral density

$$\begin{aligned} P_{XY(av)} &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T E[x(t) \cdot y(t)] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \int_{-T}^T \frac{E[X^*_T(\omega) \cdot Y_T(\omega)]}{2T} d\omega \end{aligned}$$

If we define $\lim_{T \rightarrow \infty} \int_{-T}^T \frac{E[X^*_T(\omega) \cdot Y_T(\omega)]}{2T} = S_{XY}(\omega)$, i.e., the cross spectral density, the average power

$$P_{XY(av)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) d\omega$$

$$\text{Similarly, } P_{YX(av)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) d\omega$$

Q32. Consider the Random process $x(t) = A \cos$

$(\omega_0 t + \theta)$ where A and ω_0 are real constants and θ is a random variable uniformly distributed on the interval $(0, \frac{\pi}{2})$ find the average power P_{xx} in $x(t)$.

Sol.

Given that, $x(t) = A \cos(\omega_0 t + \theta)$

Average Power P_{xx} = Average of

$$E[X^2(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt$$

$$E[X^2(t)] = E[A^2 \cos^2(\omega_0 t + \theta)] = \frac{A^2}{2} E[1 + \cos 2(\omega_0 t + \theta)]$$

$$= \frac{A^2}{2} + \frac{A^2}{2} E[\cos 2(\omega_0 t + \theta)]$$

$$E[\cos 2(\omega_0 t + \theta)] = \int_0^{\pi/2} \cos 2(\omega_0 t + \theta) f(\theta) d\theta$$

We have, $f(\theta) = \frac{2}{\pi}$

$$\therefore E[\cos 2(\omega_0 t + \theta)] = \frac{2}{\pi} \int_0^{\pi/2} \cos 2(\omega_0 t + \theta) d\theta$$

$$= \frac{2}{\pi} \left[\frac{\sin 2(\omega_0 t + \theta)}{2} \right]_0^{\pi/2} = \frac{1}{\pi} \left[\sin 2\left(\omega_0 t + \frac{\pi}{2}\right) - \sin 2\omega_0 t \right]$$

$$= \frac{-2 \sin 2\omega_0 t}{\pi}$$

$$\therefore E[X^2(t)] = \frac{A^2}{2} - \frac{A^2}{\pi} \sin 2\omega_0 t$$

$$P_{xx} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[\frac{A^2}{2} - \frac{A^2}{\pi} \sin 2\omega_0 t \right] dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\frac{A^2}{2} t + \frac{A^2}{\pi} \frac{\cos 2\omega_0 t}{2\omega_0} \right]_{-T}^T = \frac{A^2}{2}$$

5.3 SPECIAL CHARACTERISTICS OF SYSTEM RESPONSE POWER DENSITY SPECTRUM OF RESPONSE, CROSS POWER SPECTRAL DENSITY AND INPUT AND OUTPUT OF A LINEAR SYSTEM

Q33. Explain the power spectral density that can be obtained by using Wiener-Khinchine relation.

[SAQ]

Sol.

Spectral Characteristics of the Output $Y(t)$ PSD of $Y(t)$

The power spectral density of $Y(t)$ can be obtained using Wiener-Khinchine relation and is given as,

$$S_{YY}(\omega) = F[R_{YY}(\tau)] = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau$$

Taking Fourier transform on both sides

$$F[R_{YY}(\tau)] = S_{YY}(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) \cdot h(\tau_2) \cdot R_{XX}(\tau - \tau_1 + \tau_2) d\tau_1 \cdot d\tau_2 \right] e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \cdot e^{+j\omega\tau_1} d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) \cdot e^{-j\omega\tau_2} d\tau_2 \cdot \int_{-\infty}^{\infty} R_{XX}(\tau - \tau_2 + \tau_1) \cdot e^{-j\omega(\tau - \tau_2 + \tau_1)} d\tau$$

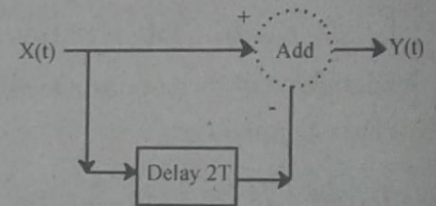
$$= H^*(\omega) \cdot H(\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega) \quad \dots (1)$$

Q34. Derive the relationship between power spectral density of input and output random process of an LTI system.

[A109210401/Nov-2010/Set-3/Q7(a)]

(or) Derive the expression for the cross spectral density $S_{XY}(\omega)$ between the input process $X(t)$ and the output process $Y(t)$ of an LTI system.



[A109210401/Sup/May/June-2012/Set-3/Q5(b)]

Sol.

Cross Spectral Density of Input $X(t)$ and Output $Y(t)$

We know that,

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

Take Fourier transform on both sides

$$F[R_{XY}(\tau)] = S_{XY}(\omega) = F[h(\tau) * R_{XX}(\tau)]$$

$$= F[h(\tau)] \cdot F[R_{XX}(\tau)] = H(\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{XY}(\omega) = H(\omega) \cdot S_{XX}(\omega)$$

We have then as $R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau)$

$$\therefore F[R_{YX}(\tau)] = S_{YX}(\omega) = F[h(-\tau) * R_{XX}(\tau)]$$

$$= F[h(-\tau)] \cdot F[R_{XX}(\tau)] = H(-\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{YX}(\omega) = H^*(\omega) \cdot S_{XX}(\omega)$$

We have $R_{YY}(\tau) = h(-\tau) * R_{XY}(\tau)$

$$\Rightarrow F[R_{YY}(\tau)] = F[h(-\tau) * R_{XY}(\tau)]$$

$$\Rightarrow S_{YY}(\omega) = H^*(\omega) \cdot S_{XY}(\omega)$$

But,

$$S_{YY}(\omega) = H(\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{YY}(\omega) = H^*(\omega) \cdot H(\omega) \cdot S_{XX}(\omega)$$

$$\Rightarrow S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega)$$

We have $R_{YY}(\tau) = h(\tau) * R_{YX}(\tau)$

$$\Rightarrow F[R_{YY}(\tau)] = F[h(\tau) * R_{YX}(\tau)]$$

$$\Rightarrow S_{YY}(\omega) = H(\omega) \cdot S_{YX}(\omega)$$

$$\text{But } S_{YX}(\omega) = H^*(\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{YX}(\omega) = H^*(\omega) \cdot S_{XX}(\omega)$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega)$$

Cross Spectral Density between the O/P Random Process of Two Different LTI Systems : Let $X(t)$ be the input process and $Y(t)$ be the output process of an LTI system with impulse response $h_1(t)$.

Let $Z(t)$ be the input process and $V(t)$ be the output process of another LTI system with impulse response $h_2(t)$.

The correlation between $Y(t)$ and $V(t)$ is

$$R_{YV}(t_1, t_2) = E[Y(t_1) \cdot V(t_2)]$$

$$= E\left[\int_{-\infty}^{\infty} h_1(\tau_1) \cdot X(t_1 - \tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot Z(t_2 - \tau_2) d\tau_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot h_2(\tau_2) \cdot E[X(t_1 - \tau_1) \cdot Z(t_2 - \tau_2)] d\tau_1 \cdot d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot h_2(\tau_2) \cdot R_{XZ}(t_1 - \tau_1, t_2 - \tau_2) d\tau_1 \cdot d\tau_2$$

Considering $X(t)$ and $Z(t)$ as Jointly WSS processes,

$$R_{YV}(t_1, t_2) = R_{YV}(t_2 - t_1) = R_{YV}(\tau)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot h_2(\tau_2) \cdot R_{XZ}(t_2 - \tau_2 - t_1 + \tau_1) d\tau_1 \cdot d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot h_2(\tau_2) \cdot R_{XZ}(\tau - \tau_2 + \tau_1) d\tau_1 \cdot d\tau_2$$

Take fourier transform on both sides

$$F[R_{YV}(\tau)] = S_{YV}(\omega)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot h_2(\tau_2) \cdot R_{XZ}(\tau - \tau_2 + \tau_1) d\tau_1 \cdot d\tau_2 \right] e^{-j\omega\tau} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1) \cdot e^{j\omega\tau_1} \cdot d\tau_1 \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot e^{-j\omega\tau_2} \cdot d\tau_2 \cdot \int_{-\infty}^{\infty} R_{XZ}(\tau - \tau_2 + \tau_1) \cdot e^{-j\omega(\tau - \tau_2 + \tau_1)} \cdot d\tau$$

$$\therefore S_{YV}(\omega) = H_1^*(\omega) \cdot H_2(\omega) \cdot S_{XZ}(\omega)$$

Q35. Derive the relation between the input PSD and the output PSD of an LTI system.

[A109210401/Reg./Nov/Dec-2012/Q8(b)]

Sol.

For answer refer Unit-5, Q.No.34.

Q36. A signal $x(t) = u(t)e^{-\alpha t}$ is applied to a network having an impulse response $h(t) = Wu(t)e^{-\omega t}$. Here α and ω are real positive constants. Find the network response?

Sol.

Let $y(t)$ be the response of the system which we need to find which is the convolution of $x(t)$ and $h(t)$ i.e.,

$$y(t) = x(t) * h(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

$$x(\tau) = u(\tau)e^{-\alpha\tau}$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} u(\tau) e^{-\alpha\tau} Wu(t - \tau) e^{-\omega(t - \tau)} d\tau$$

$$u(\tau) = 1 \Rightarrow \tau > 0 = 0 < \tau < 0$$

$$\text{and } u(t - \tau) = 1 \Rightarrow t - \tau > 0 \quad \text{or } \tau < t$$

$$= 0 \Rightarrow t - \tau < 0 \quad \text{or } \tau > t$$

The limits of $y(t)$ when $t > 0$ are,

$$y(t) = W \int_0^t e^{-\alpha\tau} e^{-\omega(t - \tau)} d\tau = We^{-\omega t} \int_0^t e^{(W - \alpha)\tau} d\tau$$

$$= We^{-\omega t} \left[\frac{e^{(W - \alpha)\tau}}{W - \alpha} \right]_0^t = \frac{W}{W - \alpha} (e^{-\alpha t} - e^{-\omega t})$$

$$\text{When } t < 0 \Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = 0$$

$$\therefore y(t) = \frac{W}{W - \alpha} (e^{-\alpha t} - e^{-\omega t})$$

Q37. The auto correlation function of a random process $X(t)$ is $R_{XX}(\tau) = 3 + 2\exp(-4\tau^2)$.

(a) Find the power spectrum of $X(t)$.

(b) What is the average spectrum of $X(t)$?

(c) What fractional power lies in the frequency

$$\text{band } \frac{-1}{\sqrt{2}} \leq \omega \leq \frac{1}{\sqrt{2}} ? \quad [\text{Nov-08/Set-3/Q7}]$$

Sol.

$$\text{Given that, } R_{XX}(\tau) = 3 + 2e^{-4\tau^2}$$

(a) The expression for power spectrum is given by,

$$\zeta_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} (3 + 2e^{-4\tau^2}) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 3 \cdot e^{-j\omega\tau} d\tau + \int_{-\infty}^{\infty} 2 \cdot e^{-4\tau^2} e^{-j\omega\tau} d\tau$$

We know that,

$$\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau} d\tau = 2\pi\delta(\omega)$$

$$\therefore \int_{-\infty}^{\infty} 3 \cdot e^{-j\omega\tau} d\tau = 6\pi\delta(\omega)$$

$$\therefore \zeta_{XX}(\omega) = 6\pi\delta(\omega) + \int_{-\infty}^{\infty} 2 \cdot e^{-4\tau^2} e^{-j\omega\tau} d\tau$$

By considering below standard integral expression,

$$\int_{-\infty}^{\infty} e^{-a^2 x^2 + bx} dx = \frac{\sqrt{\pi}}{a} e^{\frac{b^2}{4a^2}} \text{ if } a > 0 \quad \dots\dots(1)$$

Here, $a = 2 > 0$, $b = -j\omega$ and $x = \tau$
Hence,

$$\int_{-\infty}^{\infty} e^{-4\tau^2 - j\omega\tau} d\tau = \frac{\sqrt{\pi}}{2} e^{-\omega^2/16}$$

$$\text{Thus, } \zeta_{XX}(\omega) = 6\pi\delta(\omega) + 2 \times \frac{\sqrt{\pi}}{2} e^{-\omega^2/16}$$

$$\therefore \zeta_{XX}(\omega) = 6\pi\delta(\omega) + \sqrt{\pi} e^{-\omega^2/16}$$

(b) The expression for average power of $X(t)$ is given by,

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{XX}(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (6\pi\delta(\omega) + \sqrt{\pi} e^{-\omega^2/16}) d\omega$$

$$= \frac{1}{2\pi} \left[6\pi \times 1 + \sqrt{\pi} \int_{-\infty}^{\infty} e^{-\omega^2/16} d\omega \right]$$

Again using the standard integral equation (1), with

$b=0$ as,

$$\text{Here, } x = \omega, a = \frac{1}{4} > 0, b = 0$$

$$\therefore \int_{-\infty}^{\infty} e^{-\omega^2/16} d\omega = \frac{\sqrt{\pi}}{\left(\frac{1}{4}\right)} \times e^{0/4 \times 1/16} = 4\sqrt{\pi}$$

$$\therefore P_{XX} = \frac{1}{2} [6\pi + \sqrt{\pi}(4\sqrt{\pi})] = \frac{1}{2\pi} [6\pi + 4\pi] = 5$$

(c) The fractional power lies in the frequency band

$$\frac{-1}{\sqrt{2}} \leq \omega \leq \frac{1}{\sqrt{2}} \text{ is obtained as,}$$

$$P_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{XX}(\omega) d\omega$$

$$5 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{XX}(\omega) d\omega$$

$$\int_{-\infty}^{-1/\sqrt{2}} \zeta_{XX}(\omega) d\omega + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega + \int_{1/\sqrt{2}}^{\infty} \zeta_{XX}(\omega) d\omega = 31.416$$

However, the power contained in the region

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega \text{ is same as that in the region } \int_{1/\sqrt{2}}^{\infty} \zeta_{XX}(\omega) d\omega$$

Therefore,

$$2 \int_{1/\sqrt{2}}^{\infty} \zeta_{XX}(\omega) d\omega + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega = 31.416$$

Consider

$$\begin{aligned} \int_{1/\sqrt{2}}^{\infty} \zeta_{XX}(\omega) d\omega &= \int_{1/\sqrt{2}}^{\infty} [6\pi\delta(\omega) + \sqrt{\pi} e^{-\omega^2/16}] d\omega \\ &= \int_{1/\sqrt{2}}^{\infty} 6\pi\delta(\omega) d\omega + \int_{1/\sqrt{2}}^{\infty} \sqrt{\pi} e^{-\omega^2/16} d\omega \\ &= \sqrt{\pi} \int_{1/\sqrt{2}}^{\infty} e^{-\omega^2/16} d\omega \quad [\because \delta(0) = 1] \end{aligned}$$

$$\text{Substitute, } \omega = 2\sqrt{2}Z \Rightarrow d\omega = 2\sqrt{2} dZ$$

$$= \sqrt{\pi} \int_{1/4}^{\infty} e^{-Z^2/2} dZ \times 2\sqrt{2}$$

$$= 2\sqrt{2}\pi \int_{1/4}^{\infty} e^{-Z^2/2} dZ = 4\pi \int_{1/4}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} dZ$$

Using gaussian probability function i.e.,

$$Q(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} dZ$$

$$= 4\pi \times Q\left(\frac{1}{4}\right) = 4\pi \times 0.4013 = 5.043$$

$$[\because Q(0.25) = 0.4013]$$

Substituting above value in equation (1), we get,

$$\Rightarrow 2 \times 5.043 + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega = 31.416$$

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega = 21.33$$

$\therefore$ Fractional power

$$= \frac{1}{2\pi} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \zeta_{XX}(\omega) d\omega = \frac{21.33}{2\pi} = 3.395$$

Q38. If $R(\tau) = ae^{-b|\tau|}$. Find the spectral density function, where a and b are constants. [SAQ] [Feb-08/Set-3,4/Q7(c)] [Nov-07/Set-2,3/Q7(c)]

Sol.

In the given problem, $R(\tau) = ae^{-b|\tau|}$ is misprinted as,

$$R(\tau) = ae^{-b|\tau|}$$

The given auto correlation function is, $R(\tau) = ae^{-b|\tau|}$

Then, the power spectral density function, $S(\omega)$ obtained as,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} a e^{-b|\tau|} e^{-j\omega\tau} d\tau$$

$$S(\omega) = \int_{-\infty}^0 a e^{b\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} a e^{-b\tau} e^{-j\omega\tau} d\tau$$

$$= a \left[\int_{-\infty}^0 e^{(b-j\omega)\tau} d\tau + \int_0^{\infty} e^{-(b+j\omega)\tau} d\tau \right]$$

$$= a \left[\left[\frac{e^{(b-j\omega)\tau}}{(b-j\omega)} \right]_{-\infty}^0 + \left[\frac{e^{-(b+j\omega)\tau}}{-(b+j\omega)} \right]_0^{\infty} \right]$$

$$\therefore S(\omega) = \frac{2ab}{b^2 + \omega^2}$$

Q39. Find the cross correlation function corresponding to the cross the spectrum.

$$S_{XY}(\omega) = \frac{6}{(9 + \omega^2)(3 + j\omega)^2} \cdot [\text{Nov-09/Set-1/Q7(b)}]$$

Sol.

The given cross power density spectrum is,

$$S_{XY}(\omega) = \frac{6}{(9 + \omega^2)(3 + j\omega)^2}$$

The relation between the cross correlation function and cross power density spectrum is given by,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

$$\tilde{R}_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\tau) e^{j\omega\tau} d\omega$$

This implies $S_{XY}(\omega)$ and $R_{XY}(\tau)$ forms a fourier transform pair.

$$\text{i.e., } S_{XY}(\omega) \leftrightarrow R_{XY}(\tau)$$

$\therefore R_{XY}(\tau)$ is the inverse fourier transform of $S_{XY}(\omega)$.

Thus, the simplest way is by finding partial fractions.

$$\frac{6}{(9 + \omega^2)(3 + j\omega)^2} = \frac{A\omega + B}{9 + \omega^2} + \frac{C}{3 + j\omega} + \frac{D}{(3 + j\omega)^2}$$

$$(A\omega + B)(3 + j\omega)^2 + C(3 + j\omega)(9 + \omega^2) + D(9 + \omega^2) = 6$$

$$\Rightarrow (A\omega + B)(9 - \omega^2 + 6j\omega) + C(27 + 3\omega^2 + 9j\omega + j\omega^3) + D(9 + \omega^2) = 6$$

By comparing the constant terms, ω terms and ω^2 terms on both sides, we get,

$$9B + 27C + 9D = 6 \quad \dots\dots(1)$$

$$9A + 6jB + 9jC = 0 \quad \dots\dots(2)$$

$$-B + 6jA + 3C + D = 0 \quad \dots\dots(3)$$

From equation (2), we have,

$$A = 0, 6B + 9C = 0$$

$$3C = -2B \quad \dots\dots(4)$$

Similarly, from equation (3), we have,

$$-B + 0 + 3C + D = 0$$

$$3C + D = B \quad \dots\dots(5)$$

Substituting equation (5) in equation (1), we get,

$$9B + 9(3C + D) = 6 \Rightarrow 9B + 9B = 6$$

$$B = \frac{1}{3} \quad \dots\dots(6)$$

Substituting the value of B in equation (4), we have,

$$3C = -2 \times \frac{1}{3} \Rightarrow C = \frac{-2}{9}$$

Substituting B and C value in equation (5), we get,

$$3\left(\frac{-2}{9}\right) + D = \frac{1}{3} \Rightarrow D = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$\therefore \frac{6}{(9 + \omega^2)(3 + j\omega)^2} = \frac{1/3}{9 + \omega^2} + \frac{-2/9}{3 + j\omega} + \frac{1}{(3 + j\omega)^2}$$

We know that,

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a + j\omega} \text{ and}$$

$$\sin at \xleftrightarrow{F.T} \frac{a}{a^2 + \omega^2}$$

$\therefore$ I.F is

$$\left[\frac{6}{(9 + \omega^2)(3 + j\omega)^2} \right] = \text{I.F} \left[\frac{1/3}{9 + \omega^2} - \frac{2/9}{3 + j\omega} + \frac{1}{(3 + j\omega)^2} \right]$$

$$= \frac{1}{9} \sin 3t - \frac{2}{9} e^{-3t} + t e^{-3t}$$

$$\therefore R_{XY}(\tau) = \frac{1}{9} [\sin 3\tau - 2e^{-3\tau} + 9\tau e^{-3\tau}]$$

Q40. A random noise X(t) having power spectrum

$$S_{XX}(\omega) = \frac{3}{49 + \omega^2} \text{ is applied to a network for}$$

which $h(t) = u(t) t^2 \exp(-7t)$. The network response is denoted by Y(t).

(a) What is average power of X(t)?

(b) Find the power spectrum of Y(t).

(c) Find average power of Y(t).

[Feb-08/Set-1,2,3/Q8][Nov-07/Set-4/Q8]

[Feb-07Set-3/Q8]

Sol.

Given power spectral density of X(t) is,

$$S_{XX}(\omega) = \frac{3}{49 + \omega^2} \text{ and}$$

$$h_2(t) = u(t) t^2 \exp(-7t)$$

(a) Average Power of X(t)

Let P_{xx} denote the average of $X(t)$. Then

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{xx}(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3}{7^2 + \omega^2} d\omega = \frac{3}{2\pi} \left[\frac{\tan^{-1}\left(\frac{\omega}{7}\right)}{7} \right]_{-\infty}^{\infty} = \frac{3}{14}$$

$\therefore$ Average power of $X(t)$ is $\frac{3}{14}$

(b) Power of Spectrum of $Y(t)$

If $s_{yy}(\omega)$ denote power spectrum of $Y(t)$, then

$$s_{yy}(\omega) = s_{xx}(\omega) |H(\omega)|^2$$

Given that,

$$s_{xx}(\omega) = \frac{3}{49 + \omega^2} \text{ and } h_2(t) = u(t)t^2 \exp(-7t)$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} u(t)t^2 e^{-7t} e^{-j\omega t} dt = \int_0^{\infty} t^2 e^{-7t} e^{-(j\omega+7)t} dt$$

$$= t^2 \left(\frac{e^{-(j\omega+7)t}}{-(j\omega+7)} \right)_0^{\infty} - \left(\int_0^{\infty} \frac{2te^{-(j\omega+7)t}}{-(j\omega+7)} dt \right)$$

$$= t^2 \left(\frac{e^{-(j\omega+7)t}}{-(j\omega+7)} \right)_0^{\infty} - \left(\int_0^{\infty} \frac{2te^{-(j\omega+7)t}}{-(j\omega+7)} dt \right)$$

$$= 0 + \frac{2}{7+j\omega} \left[0 - \frac{(e^{-(7+j\omega)t})_0^{\infty}}{-(7+j\omega)^2} \right]$$

$$\therefore H\omega = \frac{2}{(7+j\omega)^3}$$

$$|H\omega| = \frac{2}{\sqrt{(7^2 + \omega^2)^3}} \Rightarrow |H\omega|^2 = \frac{4}{(7^2 + \omega^2)^3}$$

$$\therefore s_{yy}(\omega) = s_{xx}(\omega) |H(\omega)|^2 = \frac{3}{(7^2 + \omega^2)} \cdot \frac{4}{(7^2 + \omega^2)^3}$$

$$s_{yy}(\omega) = \frac{12}{(7^2 + \omega^2)^4}$$

$$\therefore \text{PSD of } Y(t) \text{ is, } s_{yy}(\omega) = \frac{12}{(7^2 + \omega^2)^4}$$

(c) Average Power of $Y(t)$

If $s_{yy}(\omega)$ is PSD of $Y(t)$, then average of $Y(t)$ is given by P_{YY} i.e.,

$$P_{YY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} s_{yy}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{12}{(7^2 + \omega^2)^4} d\omega$$

$$\text{Put } \omega = 7 \tan \theta \Rightarrow d\omega = 7 \sec^2 \theta d\theta.$$

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$$\Rightarrow P_{yy} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{12}{(7^2 + 7^2 \tan^2 \theta)^4} \cdot 7 \sec^2 \theta d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{12}{7^8 (1 + \tan^2 \theta)^4} \cdot 7 \sec^2 \theta d\theta$$

$$= \frac{12}{2 \cdot 7^7 \cdot \pi} \int_{-\pi/2}^{\pi/2} \cos^6 \theta d\theta = \frac{12}{2 \cdot 7^7 \cdot \pi} \int_{-\pi/2}^{\pi/2} (\cos^2 \theta)^3 d\theta$$

$$= \frac{12}{2 \cdot 7^7 \cdot \pi} \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^3 d\theta$$

$$= \frac{12}{16 \cdot 7^7 \cdot \pi} \int_{-\pi/2}^{\pi/2} \left(1 + \frac{3}{4} \cos 2\theta - \frac{1}{4} \cos 6\theta + 3 \cos 2\theta + \frac{3}{2} + \frac{3}{2} \cos 4\theta \right) d\theta$$

$$= \frac{12}{16 \cdot 7^7 \cdot \pi} \left[\int_{-\pi/2}^{\pi/2} 5 d\theta + 2 \int_{-\pi/2}^{\pi/2} \left(\frac{3}{4} \cos 2\theta - \frac{1}{4} \cos 6\theta + 3 \cos 2\theta + \frac{3}{2} + \frac{3}{2} \cos 4\theta \right) d\theta \right]$$

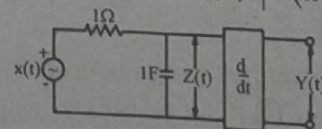
$$= \frac{12}{2 \cdot 7^7 \cdot \pi} \left[\frac{5}{2} \pi + 2 \left[\frac{3}{4} \frac{\sin 2\theta}{2} - \frac{1}{4} \frac{\sin 6\theta}{6} + \frac{3 \sin 2\theta}{2} + \frac{3 \sin 4\theta}{2 \cdot 4} \right]_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{12.5\pi}{16 \cdot 7^7 \cdot \pi \cdot 2} = \frac{3.5}{8 \cdot 7^7} = \frac{15}{8 \cdot 7^7}$$

$\therefore$ Average power of $Y(t) = \frac{15}{8 \cdot 7^7}$

Q41. A stationary random process $X(t)$ having an autocorrelation function $R_{xx}(\tau) = 2e^{-4|\tau|}$ is applied to the network shown in figure. Find

- (i) $S_{xx}(\omega)$ (ii) $|H(\omega)|^2$ (iii) $S(\omega)$.

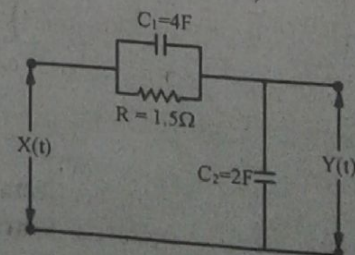


[Nov-07/Set-3/Q8(a)]

Sol.

Given that, $R_{xx}(\tau) = 2e^{-4|\tau|}$

Given network can be modified as,



(i) The power density spectrum is,

$$\begin{aligned}
 S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \\
 &= \int_{-\infty}^{\infty} 2e^{-4|\tau|} e^{-j\omega\tau} d\tau \\
 &= 2 \int_{-\infty}^{\infty} e^{4\tau} e^{-j\omega\tau} d\tau + 2 \int_{-\infty}^{\infty} e^{-4\tau} e^{-j\omega\tau} d\tau \\
 &= 2 \left[\int_{-\infty}^0 e^{(4-j\omega)\tau} d\tau + \int_0^{\infty} e^{-(4+j\omega)\tau} d\tau \right] \\
 &= 2 \left[\frac{1-0}{(4-j\omega)} + \frac{0-1}{-(4+j\omega)} \right] \\
 &= 2 \left[\frac{4+j\omega+4-j\omega}{16+\omega^2} \right] = \frac{16}{16+\omega^2}
 \end{aligned}$$

$$\therefore S_{XX}(\omega) = \frac{16}{16+\omega^2}$$

(ii) The transfer function of the given network (shown in figure) is obtained as,

$$\begin{aligned}
 H(\omega) = \frac{Y(\omega)}{X(\omega)} &= \frac{\left[\frac{1}{j2\omega} \right]}{\left[\frac{1.5 \times \frac{1}{j4\omega}}{1.5 + \frac{1}{j4\omega}} + \frac{1}{j2\omega} \right]} \\
 &= \frac{\left[\frac{1}{j2\omega} \right]}{\left[\frac{1.5}{1 + j6\omega} + \frac{1}{j2\omega} \right]} = \frac{1 + 54\omega^2 - j3\omega}{1 + 81\omega^2} \\
 |H(\omega)| &= \sqrt{\left[\frac{1 + 54\omega^2}{1 + 81\omega^2} \right]^2 + \left[\frac{-3\omega}{1 + 81\omega^2} \right]^2} \\
 &= \sqrt{\frac{1 + 2916\omega^4 + 108\omega^2 + 9\omega^2}{(1 + 81\omega^2)^2}} \\
 \therefore |H(\omega)|^2 &= \frac{1 + 36\omega^2}{1 + 81\omega^2}
 \end{aligned}$$

(iii) The power density spectrum of the output process $Y(t)$ can be easily obtained by using the following relation,

$$\begin{aligned}
 S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\
 &= \frac{1 + 36\omega^2}{1 + 81\omega^2} \times \frac{16}{16 + \omega^2} = \frac{16(1 + 36\omega^2)}{(1 + 81\omega^2)(16 + \omega^2)} \\
 \therefore S_{YY}(\omega) &= \frac{16(1 + 36\omega^2)}{(1 + 81\omega^2)(16 + \omega^2)}
 \end{aligned}$$

Q42. Let $X(t)$ be a zero mean, stationary, gaussian process with autocorrelation function $R_X(\tau)$.

This process is applied to a square law device defined by the input output relation

$$Y(t) = X^2(t) \text{ Where } Y(t) \text{ is the output.}$$

(i) Show that the mean of $Y(t)$ is $R_X(0)$

(ii) Show that the auto covariance function of $Y(t)$ is $2R_X^2(t)$. [SAQ]

[A109210401/Reg/Dec-2011/Set-1/Q7(b)]

Sol.

Let $x(t)$ be a zero mean, stationary, gaussian process with autocorrelation function $R_X(\tau)$. This process is applied to a square law device defined by the input output relation.

$$\therefore Y(t) = x^2(t)$$

In the above $Y(t)$ is the output.

(i) Mean value of $Y(t)$

$$E[Y(t)] = E[X^2(t)] - E[X(t)]^2$$

$$\text{Given } E[X(t)] = 0$$

$$\begin{aligned}
 R_X(t, t+\tau) &= E[X(t).Y(t+\tau)] \\
 &= E[X(t)]E[t+\tau] \\
 &= 0 \times E[t+\tau] = 0 = R_X(0)
 \end{aligned}$$

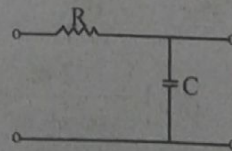
$\therefore$ The mean of $Y(t)$ is $R_X(0)$ is independent of 't'.

(ii) Auto covariance function of $Y(t)$ is given as,

$$\begin{aligned}
 R_Y(t, t+\tau) &= E[Y(t).Y(t+\tau)] \\
 &= E[X^2(t).X^2(t+\tau)] \\
 \text{From the above relation we can write} \\
 &= E[X^2(t)]E[X(t+\tau)]^2 = 2R_X(\tau)^2
 \end{aligned}$$

Which is independent of τ .

Q43. Determine the resultant output spectral density if a white noise of spectral density unity is passed through an RC low pass filter with transfer function $H(\omega)$.



[SAQ]

[A109210401/Reg./Dec-2011/Set-3/Q7(b)]

Sol.

The power spectral density of the random process is given as,

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$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-2\pi f\tau) d\tau$$

$$= \int_{-\infty}^0 \exp(2v\tau) \exp(-j2\pi f\tau) d\tau + \int_0^{\infty} \exp(-2v\tau) \exp(-j2\pi f\tau) d\tau$$

$$= \frac{1}{2(v-j\pi f)} + \frac{1}{2(v+j\pi f)} = \frac{v}{v^2 + \pi^2 f^2}$$

The transfer function of the filter is,

$$H(\omega) = \frac{1}{1+j2\omega R_c}$$

Therefore, PSD of the filter output is,

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega) = \frac{v}{(v^2 + \pi^2 f^2)(1+j2\pi f R_c)}$$

Substitute $2\pi f$ with ' ω ' in above then

$$= \frac{v}{\left(v^2 + \frac{4\pi^2 f^2}{4}\right)(1+j2\pi f R_c)}$$

$$S_y(\omega) = \frac{v}{\left(v^2 + \frac{\omega^2}{4}\right)(1+j\omega R_c)}$$

5.4 OUTCOMES

- Able to understand the Power spectrum- properties, Cross power density spectrum - properties
- Able to analyse the Relationship between cross-power spectrum and cross correlation function
- Able to understand the Spectral characteristics of system response-Power density spectrum of response
- Able to understand the Cross power spectral density of input and output of a linear system.

5.5 OBJECTIVE TYPE QUESTIONS

1. The PSD of the random process $X(t)$ with mean=1 and $R_{xx}(\tau) = 1 + e^{-\alpha|\tau|}$ is [c]

(a) $\frac{2\alpha}{\alpha^2 + \omega^2}$ (b) $1 + \frac{2\alpha}{\alpha^2 + \omega^2}$

(c) $\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$ (d) $\delta(\omega - \omega_0) + \frac{2\alpha}{\alpha^2 + \omega^2}$
2. The PSD of a random process whose auto correlation

function is $a \cdot e^{-b|\tau|}$ is [b]

- (a) $\frac{a}{a^2 + \omega^2}$ (b) $\frac{2ab}{a^2 + \omega^2}$
- (c) $\frac{2a}{b(a^2 + \omega^2)}$ (d) $\frac{2a}{a^2 + \omega^2}$
3. The autocorrelation function of a process with PSD of $\frac{4}{1 + 0.25\omega^2}$ [c]

(a) $4 \cdot \exp(-|\tau|)$ (b) $\exp(-|\tau|)$

(c) $4 \cdot \exp(-2|\tau|)$ (d) 1
 4. PSD of a WSS is always [b]

(a) negative (b) non-negative

(c) positive (d) None
 5. The mean square value of a random process whose auto correlation function is $\frac{\eta}{4RC} \cdot e^{-|\tau|/RC}$ [b]

(a) $\frac{\eta}{4}$ (b) $\frac{\eta}{4RC}$

(c) $\frac{\eta\tau}{4RC}$ (d) $\frac{\eta \cdot |\tau|}{4RC}$
 6. The random process $X(t)$ and $Y(t)$ are having their auto correlation functions as $e^{-2|\tau|}$ and $e^{-3|\tau|}$ respectively. If they are orthogonal processes, then the mean square value of $X(t) + Y(t)$ is [a]

(a) 2 (b) 3

(c) 5 (d) 6
 7. $X(t)$ is a random process with mean 'k' and auto correlation function as,
 $R_{xx}(\tau) = 12 + 4 \cdot e^{-0.2|\tau|}$. The variance of the random variable $X(5)$ is [c]

(a) $(4+k)^2$ (b) $(4-k)^2$

(c) $16 - k^2$ (d) $\frac{(4+k)}{(4-k)}$
 8. The cross spectral density of two random processes $X(t)$ and $Y(t)$ is $S_{xy}(\omega) = a + \frac{jb\omega}{k}$. The area enclosed by their auto correlation function is [a]

- (a) $\frac{a}{k}$ (b) a
 (c) ka (d) $a+1/k$
9. The auto correlation function of a WSS random process $X(t)$ is $R_{XX}(\tau) = 4 + 2e^{-2|\tau|}$. The area enclosed by the PSD curve of $X(t)$ is [a]
 (a) 6 (b) 2
 (c) 8 (d) 4
10. The means of two independent, WSS processes $X(t)$ and $Y(t)$ are 2 and 3 respectively. Their cross spectral density is [b]
 (a) $6\delta(\omega)$ (b) $12\pi \cdot \delta(\omega)$
 (c) $5\pi \cdot \delta(\omega)$ (d) $\delta(\omega)$
11. For a periodic signal _____ is used for the study of its spectral behaviour. [b]
 (a) Fourier spectrum (b) Fourier series
 (c) Fourier transform (d) none
12. For a non-periodic signal _____ is used for the study of its spectral behaviour. [c]
 (a) Fourier spectrum (b) Fourier series
 (c) Fourier transform (d) none
13. Parseval's, theorem for energy signals is given by $E =$ _____ [a]
 (a) $\int_{-\infty}^{\infty} x^2(t) dt$ (b) $\int_{-\infty}^{\infty} x(t) dt$
 (c) $-\int_{-\infty}^{\infty} x(t) dt$ (d) $-\int_{-\infty}^{\infty} x^2(t) dt$
14. Time average of auto correlation function and the power spectral density from a _____ [c]
 (a) Fourier spectrum pair (b) Fourier series pair
 (c) Fourier transform pair (d) none
15. $S_{XX}(\omega) =$ _____ [b]
 (a) $-\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$ (b) $\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$
 (c) $\int_{-\infty}^{\infty} R_{XX}(\tau) e^{j\omega\tau} d\tau$ (d) none
16. $R_{XX}(\tau) =$ _____ [c]
- (a) $-\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$ (b) $\int_{-\infty}^{\infty} S_{XX}(\omega) e^{-j\omega\tau} d\omega$
 (c) $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$ (d) none
17. The curve $\frac{S_{XX}(\omega)}{A}$ always encloses _____ area [a]
 (a) unit (b) square unit
 (c) zero unit (d) none
18. $S_{XY}(\omega) = 0$ if $X(t)$ and $Y(t)$ are _____ [b]
 (a) serial (b) parallel
 (c) diagonal (d) none
19. A random process $n(t)$ is called white noise if its power spectral density is _____ over the entire frequency range. [a]
 (a) uniform (b) non uniform
 (c) diagonal (d) none
20. $R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega)$ this is called Wiener - Khintchine relation is [a]
 (a) true (b) false
 (c) insufficient (d) none
21. In the power density spectrum $\zeta_{XX}(\omega)$ is [a]
 (a) ≥ 0 (b) ≥ 1 (c) ≥ 2 (d) none
22. One of the properties of the power spectrum density $\zeta_{XX}(-\omega) = \zeta_{XX}(\omega)$ is [a]
 (a) true (b) false
 (c) insufficient (d) none
23. In the cross power density spectrum $\zeta_{XY}(\omega) = \zeta_{XY}(-\omega) =$ [a]
 (a) $\zeta_{YX}(\omega)$ (b) $\zeta_{YX}(\omega)$
 (c) $-\zeta_{YX}(\omega)$ (d) none
24. In the cross power density spectrum $\zeta_{XY}(\omega) = 0$ and $\zeta_{YX}(\omega) = 0$ if $X(t)$ and $Y(t)$ are [b]
 (a) Parallel (b) orthogonal
 (c) serial (d) none
25. A sample function $n(t)$ of a wide-sense stationary noise random process $N(t)$ is called as a [b]
 (a) Gaussian noise (b) White noise

- (c) Spectral noise (d) none
26. In the White noise if the power density spectrum of $N(t)$ is a constant at all frequencies then $\zeta_{NM}(\omega) =$ [a]

(a) $\frac{N_0}{2}$ (b) $\frac{N_0}{4}$

(c) $\frac{N_0}{3}$ (d) none

27. Consider a random process

$X(t) = A_0 \cos(\omega_0 t + \Theta)$ then the average power P_{XX} is [a]

(a) $\frac{A_0^2}{2}$ (b) $-\frac{A_0^2}{2}$

(c) $-\frac{A_0^2}{4}$ (d) $\frac{A_0^2}{4}$

28. Given the power spectrum

$\zeta_{XX}(\omega) = \frac{10}{[1 + (\omega/10)^2]^2}$ find W_{rms} [a]

(a) 10rad/s (b) 20rad/s

(c) 30rad/s (d) 40rad/s

29. The ACF of the following PSD's.

$S_{XX}(\omega) = \frac{157 + 12\omega^2}{(16 + \omega^2)(9 + \omega^2)}$ is [a]

(a) $R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|}$

(b) $R_{XX}(\tau) = -\frac{5}{8}e^{-4|\tau|} - \frac{7}{6}e^{-3|\tau|}$

(c) $R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} - \frac{7}{6}e^{-3|\tau|}$ (d) None

30. $\frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{XX}(\omega) d\omega = A \{E[X^2(t)]\}$ which is the property of the [a]

(a) power density spectrum

(b) Cross-power density spectrum

(c) Discrete time process (d) None

5.6 STUDENT ACTIVITY

1. The cross correlation of jointly wide sense stationary process $X(t)$ and $Y(t)$ is assumed to be

$R_{XX}(\tau) = Bu(\tau) \exp(-W\tau)$

Where $B > 0$ & $W > 0$ are constants

(a) Find $R_{YX}(\tau)$

(b) Find $S_{XY}(\omega)$ & $S_{YX}(\omega)$

Ans. $[R_{YX}(\tau) = R_{XY}(-\tau) = Bu(-\tau) \exp(W\tau),$

$S_{YX}(\omega) = S_{XY}(-\omega) = \frac{B}{W - j\omega}]$

2. A WSS process $X(t)$ is applied to An ideal differentiator having the response $Y(t) = \frac{dx(t)}{dt}$. The cross-correlation of the input, output process is known

to be $R_{XY}(\tau) = \frac{\partial R_{XX}(\tau)}{\partial \tau}$. Find $S_{XY}(\omega)$ and $S_{YX}(\omega)$ in terms of power spectrum $S_{XX}(\omega)$ of $X(t)$

Ans. $[S_{YX}(\omega) = -j\omega S_{XX}(\omega)]$

3. Find the autocorrelation function corresponding to the power density spectrum

$S_{XX}(\omega) = \frac{157 + 12\omega^2}{(16 + \omega^2)(9 + \omega^2)}$

Ans. $[R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|}]$

4. Compare the difference between the power spectrum and autocorrelation function.

Ans. _____