

Unit 4 Complex Variable

$$\boxed{\sqrt{-3} \times \sqrt{-2} = -\sqrt{6}}$$

$z = x + iy$ (1) is a Complex number where $i = \sqrt{-1}$

$$i = \sqrt{-1} \quad i(\text{unit})$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$\text{Add. } i + i^2 + i^3 + i^4 = 0$$

note: $\sum_{j=0}^{\infty} i^j$ Converges

Summation of i with four consecutive power = 0

also, 4 consecutive even power = 0

4 consecutive odd power = 0

$i^k = \text{Real}$ when $k = \text{even}$
 $i^k = \text{Imaginary}$ when $k = \text{odd}$

$$(i^{59}) = (i^{58})i = (i^2)^{29} \cdot i = -i$$

$$\bar{z} = x - iy \quad \text{--- (2)}$$

$$z + \bar{z} = 2x$$

$$z - \bar{z} = 2iy$$

$$z\bar{z} = x^2 + y^2 \text{ (Real)}$$

$$\frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} + \frac{2ixy}{x^2+y^2}$$

Real Im = $\frac{2xy}{x^2+y^2}$

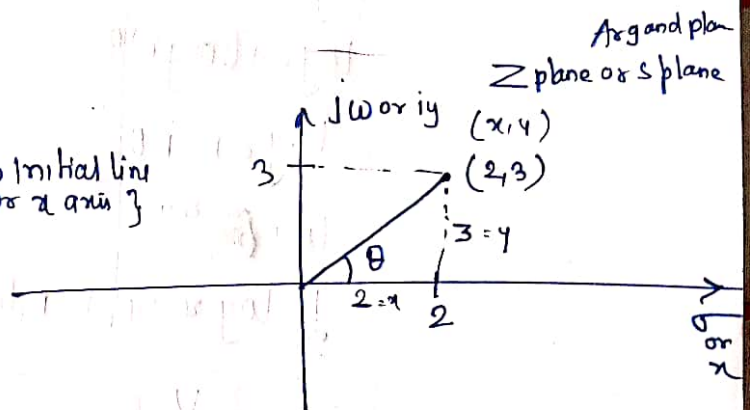
$$|z| = \sqrt{x^2+y^2}$$

$$|z|^2 = z \cdot \bar{z}$$

• argument / angle { w.r.t to initial line or x axis }

ARG(z) $z = x+iy$

$$\theta = \tan^{-1} \frac{y}{x}$$



$$(x, y) = (2, 3)$$

$$x+iy = 2+3i$$

Complex Variable

$$u+iv = f(z) = f(x+iy)$$

$$u+iv = \underbrace{\sin x \cosh y}_u + i \underbrace{\cos x \sinh y}_v$$

$$\left. \begin{aligned} \cos i\theta &= \cosh \theta \\ \sin i\theta &= i \sinh \theta \end{aligned} \right\}$$

$$u+iv = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$\Rightarrow e^x (\cos y + i \sin y)$$

$$= \underbrace{e^x \cos y}_u + i \underbrace{e^x \sin y}_v$$

Q $u+iv = \log(x+iy)$

$\Rightarrow e^{u+iv} = x+iy$

$u+iv = \log(r \cos \theta + i r \sin \theta)$

$\Rightarrow \log r (\cos \theta + i \sin \theta)$

$= \log r \cdot e^{i\theta}$

$= \log r + i \theta$

$= \log r + i \theta$

$= \log r$

$= \frac{1}{2} \log x^2 + y^2 + i \tan^{-1}(y/x)$

Q. find argument of $z = \frac{1+i}{1-i}$

$\frac{1-i+2i}{1+1}$

$= 0 + i$

$\tan^{-1} 1/0 = \pi/2$

Q2. $z = x+iy$ find y

$\frac{3+5i}{2-i}$

$= \frac{(3+5i)(2+i)}{4+1}$

$\frac{6+13i-5}{5}$

$= 13/5$

Q3. $z = i$

$\frac{1}{3} + i$

$\frac{1}{3} + i$

$\log z = i \log i$

$= i \log i$

$$z = i$$

$$\rightarrow \log z = i \log i$$

$$= i (\log 1)$$

$$\rightarrow a+ib = i$$

$$\log(a+ib)$$

$$z^2 = i^2$$

$$z = i$$

$$= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^i$$

$$\rightarrow (e^{i\pi/2})^i$$

$$= \frac{e^{-\pi/2}}{\text{or}}$$

$$e^i = e^{\log i}$$

$$= e^{i \log i}$$

$$= i \left[\frac{1}{2} \log 1 + i \tan^{-1} 1 \right]$$

$$= i \left[\frac{\pi}{2} \right] //$$

Q4. $a+ib = i^i i^i = \infty$

$$a+ib = i^{(a+ib)}$$

$$\rightarrow \log(a+ib) = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{a+ib}$$

$$= (e^{i\pi/2})^{(a+ib)}$$

$$= e^{\pi/2 [ia - b]}$$

$$\rightarrow e^{i a \pi/2 - b \pi/2}$$

or

$$\rightarrow e^{ia\pi} \cdot e^{-b\pi}$$

$$= e^{(ia-b)\pi}$$

Sin Method

$$a+ib = e^{\log i (a+ib)}$$

$$= e^{(a+ib) \log i}$$

$$= e^{(a+ib) i \pi/2}$$

$$a+ib = e^{-\frac{\pi}{2} b} \cdot e^{i a \pi/2}$$

$$= e^{-\pi/2 b} \left[\cos \frac{a\pi}{2} + i \sin \frac{a\pi}{2} \right]$$

$$\rightarrow e^{-\pi/2 b}$$

Q4, Q20, Q27, Q28, Q34, Q51^{HW}, Q50, 59, 65, 66

Sol 4 $z = x^x$

$z = i$ ~~$e^{i\pi/2}$~~ $e^{-\pi/2}$

Sol 20 0

Sol 27 $2 - 5i + 2i + 5$
 $7 - 3i$

Q. 28. $\frac{(3+4i)(1+2i)}{5}$ $\frac{3+6i+4i-8}{5}$
 $-\frac{5}{5} + \frac{10i}{5}$
 $-1 + 2i$
 $\sqrt{1+4} = \sqrt{5}$

34.

Sol $\frac{(2-3i)(-5-i)}{25+1}$ $\frac{-10 - 2i + 15i - 3}{26}$
 $= \frac{-13}{26} + \frac{13i}{26}$
 $= -0.5 + 0.5i //$

50. $|e^{iz}|$

~~$|e^{iz}|$~~ $|\cos z + i \sin z|$

~~$\cos^2 z + \sin^2 z$~~

$|e^{i(x+iy)}|$

e^{ix-y}

$e^{ix} \cdot e^{-y}$

$e^{-y} //$

Q 66.
So

$$z = \frac{\sqrt{3}}{2} + i \sin \frac{1}{2} \quad z^4$$

$$(\omega)^4 \quad \omega^3 \cdot \omega$$

$$e^{i\pi/6}$$

$$= e^{2i\pi/3}$$

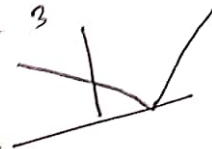
$$= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

$$\rightarrow -1/2 + i\sqrt{3}/2$$

Random Question test

$$Q1. f(x) = \begin{cases} |x-3| & x \geq 1 \\ \frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4} & x < 1 \end{cases}$$

$$x-3=0 \\ x=3$$



☒ a) Continuous at $x=1$

b) Diff. at $x=1$

☒ c) Both a & b

d) None of these

$$Q2 \quad f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx & x \geq 1 \end{cases} \quad x \in \mathbb{R}$$

which statement is true

☒ (1) $f(x)$ Not diff. at $x=1$ for any value and of a & b

☐ (2) $f(x)$ is differentiable for unique value of a and b

☐ (3) $f(x)$ is diff. at $x=1 \quad \forall a, b \quad ; a+b=e$

☒ (4) $f(x)$ is diff. at $x=1 \quad \forall a, b$

Q3. Max Value of fcn

(2/3)

$$f(x) = \frac{1}{3}x(x^2-3) \text{ in } -100 \leq x \leq 100$$

Q4. Find distance between the origin and the point at Sargare
ie nearest to its Sargare $z^2 = 1+xy$

a) 0 b) 1 c) 2 d) 3

Q5. for Complex no. z

$$\lim_{z \rightarrow i} \frac{z^2+1}{z^3+2z-i(z^2+2)}$$

a) $-2i$

b) -1

c) i

d) $2i$

$$\lim_{z \rightarrow i} \frac{2z}{3z^2+2-i(2z)}$$

$$\Rightarrow \frac{2i}{-3+2+2} = \frac{2i}{1} = 2i$$

Sol 3 $f(x) = \frac{1}{3}x(x^2-3)$

$$= \frac{1}{3}[3x^2-3] = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

$$1, -1$$

$$f''(x) = \frac{1}{3}[6x]$$

$$= 2x$$

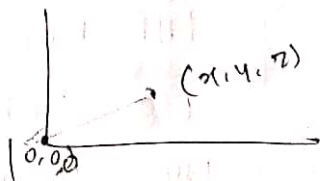
$$f(x) = -\frac{1}{3}(1-3) = \frac{2}{3}$$

$$-\frac{100}{3}(10000-3)$$

$$\frac{100}{3}(10000-3)$$

$$3 \frac{9997 \times 100}{3}$$

Sol 4



$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$\text{Distance}^2 = x^2 + y^2 + z^2$$

$$f(x, y) = x^2 + y^2 + 1 + xy$$

Minimizing

Sol 2

$$\lim_{x \rightarrow 1} e^x$$

$$= e$$

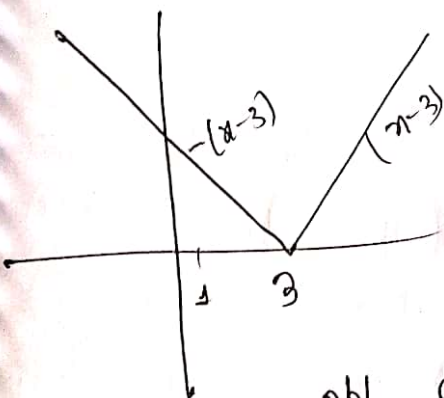
$$= e$$

$$\lim_{x \rightarrow 1} \ln x + ax^2 + bx$$

$$= \ln 0 + \underline{a+b}$$

Sol 1

$$f(x) = \begin{cases} -(x-3) & 1 < x < 3 \\ (x-3) & x > 3 \end{cases}$$



$$\frac{x^2}{4} = \frac{3}{2}x + \frac{13}{4} \quad x < 1$$

opt. C

$$L-f'(1) = -1$$

$$R-f'(1) = \frac{2(1)}{4} = \frac{3}{2} + \frac{13}{4} = \frac{2-6+13}{4} = \frac{9}{4} = 1\frac{1}{4}$$

Sol 2

$$LHO = Lf'(a)$$

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{e^x - (a+b)}{x-1} \quad \text{with } e^x \rightarrow e$$

L'Hospital

$$\lim_{x \rightarrow 1} e^x = e \text{ LHO}$$

RHO

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \rightarrow 1} \frac{(\ln x + ax^2 + bx) - (a+b)}{x-1}$$

L'Hospital

$$\lim_{x \rightarrow 1} \frac{\left(\frac{1}{x} + 2ax + b\right)}{1} = 1 + 2a + b$$

$$\text{for } LHO = RHO = f(a)$$

$$1 + 2a + b \text{ (must)} = e$$

$$\text{for } a = -\forall a, b \quad LHS \neq RHS$$

$$\text{for } c = a+b=e \quad LHS \neq RHS$$

$$\text{for } B = \left. \begin{array}{l} a, b \text{ are not unique} \\ a=0, b=1.27 \\ a=\frac{1}{2} \times 1.27, b=0 \end{array} \right\} = e \quad \underline{\text{optm}} \quad \neq$$

for continuity

$$LHL = RHL = f(a)$$

$$f(a) = a+b$$

$$LHL = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow 1} e^x = e$$

$$\text{Cont } a+b=e$$

$\frac{1}{0}$ form (Not apply L'Hospital)

L'Hospital only for $0/0$ & ∞/∞ form

H.W.
Q.

$$f(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x)$ is continuous but not differentiable at $x=0$
then

a) $0 < p \leq \infty$

b) $1 \leq p < \infty$

c) $-\infty < p < 0$

d) $p=0$

Complex Variable

Analytic function :- A Complex function $f(z) = u(x,y) + i v(x,y)$

then this function is a Single Valued function and differentiable at $z=z_0$ then it is said to be Analytic at $z=z_0$. (1)

If the function is analytic for all value of z in finite complex plane then it is referred as entire function.

for eg. $\sin z, \cos z, e^z, z^n$ etc are entire functions.

If the function is not differentiable at any point then the point is referred as singular point.

Necessary Condition for $f(z)$ to be Analytic :-

Cauchy Riemann Eqn

(1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

(2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Polar form of CR eqn

$$\frac{\partial u}{\partial x} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial y} = -r \frac{\partial v}{\partial \theta}$$

Q. check the functions Analytic or not

1. $\sin z$ 2) z^3 3) $|z| \cdot z$

Sol1 $\sin z = \sin(x+iy)$

$$\sin x \cosh y + i \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cosh y \cos x$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial v}{\partial y} = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore \sin z$ is Analytic function.

Sol2 $z^3 = (x+iy)^3$

$$= x^3 + y^3 i - 3x^2 y - 3xy^2 i$$

analytic

$$= \underbrace{x^3 + y^3 i - 3x^2 y}_{u} + i \underbrace{(-3xy^2 + y^3)}_{v}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial y} = -3x^2 + 3y^2$$

(11) $|z| \cdot z$

not analytic

Sol $[x+iy]z$

$\rightarrow (\sqrt{x^2+y^2})(x+iy)$

$(x+iy)^{3/2}$

$(\cosh(z) = \cosh x \cdot \cos y + i \sinh x \sin y)$

$\frac{\partial u}{\partial x} = \frac{x}{2\sqrt{x^2+y^2}} \cdot 2x + \sqrt{x^2+y^2}$

Consider
Q. $f(z) = \frac{1}{2} \log(x^2+y^2) + i \tan^{-1}\left(\frac{px}{y}\right)$

Sol find Value of p if $f(z)$ is analytic

a) 0 b) -1 c) 1 d) 2

$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{x}{x^2+y^2}$

$\frac{\partial v}{\partial y} = - \frac{1}{x^2+y^2} \cdot \frac{p}{y^2}$

$p = -1$

Method to form Analytic function

$\sin ix = i \sinh x$

Q- check $\sinh z$ analytic or not

Sol $-i \sin i(x+iy)$

$\rightarrow -i (\sin(ix-y)) = -i [\sin ix \cos y - \cos ix \sin y]$

$\rightarrow -i \sin ix \cos y + i \cos ix \sin y$

$\rightarrow \frac{\sinh x \cos y}{u} + i \frac{\cosh x \sin y}{v}$

$\frac{\partial u}{\partial x} = \cos y \cosh x$

$\frac{\partial v}{\partial y} = \cosh x \cos y$

Analytic

Milne Thomson Method :-

(approximate Method To form analytic function)

like Can form Analytic function with help of Real or Imaginary part.

$$f(z) = u(x, y) + i v(x, y) \quad \text{--- (i)}$$

diff wrt z

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

If only u is given

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

put $x = z$ & $y = 0$

and integrate with adding C .

$$f(z) = \int \text{Analytic function} + C$$

note:

$$\frac{\partial v}{\partial y} = f(x)$$

future use

$$\int dv = \int f(y) dy + \phi(x)$$

Q1. $u = x^2 + y^2$ then Analytic function $f(z) = u + iv$

- a) $z - c$ b) $z + c$ c) $z^2 + c$ d) $z^3 + c$

Q2. $f(z) = u + iv$ is analytic function
given $v = e^x [x \sin y + y \cos y]$

then $f(z)$ is

- a) $e^z + c$ b) $ze^z + c$ c) $e^{2z} + c$ d) $z + c$

Sol 1:

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

$$f'(z) = 2x - i 2y$$

$$= 2z$$

$$f(z) = \int 2z \, dz + c$$

$$= z^2 + c //$$

Sol 2: $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}$

$$= e^x [x \cos y + y(-\sin y) + \cos y] + i$$

$$e^x [x \sin y + y \cos y] + e^x [\sin y]$$

$$\Rightarrow e^z [z + 1] + i e^z$$

$$\Rightarrow z e^z + e^z + i e^z$$

$$\Rightarrow e^z [z + 1 + i]$$

$$\int f'(z) = e^z (z + 1 + i) - e^z \Rightarrow z e^z + c //$$

Q.28. find $f(z)$ if its Imaginary part is
 $v = x^2 \sin 2\theta + 4$

- a) $z^2 + c$ b) $-z^2 + c$ c) $z + z^2$ d) $-z + \frac{z^2}{2}$

Harmonic function : A function that satisfies Laplace equation is known as Harmonic function.

$$f(x, y)$$

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0$$

Conjugate Harmonic function : If $f(z) = u + iv$ is an analytic function and u and v are both Harmonic function then functions u & v are said to be conjugate harmonic.

Q.29
 59

$$u = xy$$

$$\frac{\partial u}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = x$$

$$\therefore \frac{\partial v}{\partial x} = y$$

$$\frac{\partial v}{\partial y} = x$$

$$v = \frac{y^2}{2} + f(x)$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = x$$

$$\frac{\partial u}{\partial y} = x = \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = x$$

$$v = \frac{x^2}{2} + f(y)$$

So

$$v = \frac{y^2}{2} - \frac{x^2}{2} + c$$

$$v = \frac{y^2 - x^2}{2}$$

Q 2, 24, 25, 35, 88, 50

Complex Integration

$$I = \int_{z_1}^{z_2} f(z) dz$$

$\downarrow$ (x, y) $\downarrow$ $(x+iy)$ $\downarrow$ $(dx+idy)$

Replace $x \rightarrow y$ or $y \rightarrow x$ (one variable) (According to path)

Q $I = \int_0^{1+i} (x^2 - iy) dz$

(i) $y = x$ (ii) $y = x^2$

(A) 0 (B) $1-5-i$ (C) $\frac{5+i}{6}$ d $\frac{5-i}{6}$

Sol
(1) $y=x$

$$I = \int_{(0,0)}^{(1,1)} (x^2 - ix)(dx + i dy)$$

$$= \int_0^1 x^2 dx + i x^2 dx - i x dx + x dx$$

$$> \left(\frac{x^3}{3} \right)_0^1 + i \left(\frac{x^3}{3} \right)_0^1 - i \left(\frac{x^2}{2} \right)_0^1 + \left(\frac{x^2}{2} \right)_0^1$$

$$= \frac{1}{3} + \frac{i}{3} - \frac{i}{2} + \frac{1}{2}$$

$$\Rightarrow \frac{2+2i-3i+3}{6} = \frac{5-i}{6} //$$

11) $y = x^2$

$$\int_0^1 (x^2 - i \frac{x^2}{\sqrt{x}}) (dx + i dy)$$

Ans c.

$$\rightarrow \int_0^1 x^2 dx - i \int_0^1 x^{1/2} dx + i \int_0^1 x^{3/2} dx + \int_0^1 (x)^{1/2} dx$$

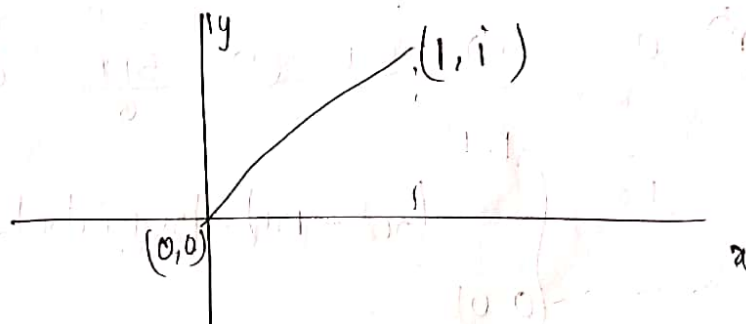
$$\rightarrow \frac{1}{3} - \frac{2i}{3} + \frac{i}{3} + \frac{2}{3}$$

$$\frac{1 - 2i + i + 2}{3} = \frac{3 - i}{3}$$

note: If $f(z)$ is entire function then between two points for line integration the integral is path independent.

Q. $\int (x^2 + iy^2) dz$, $z = x + iy$

for fig



Sol $\int x^2 + iy^2 (dx + i dy)$ Ans $3/4 i$

$$\int x^2 (1+i) dx (1+i)$$

$$\rightarrow (1+i)^2 \left[\frac{x^3}{3} \right]_0^1 \rightarrow \frac{1}{3} [1 - i + 2i]$$

Cauchy Integral formula

- Used to simplify complex integral
- only valid for closed complex integral

If $f(z)$ is analytic function at point $z=a$ inside or on closed curve C . Then

$$(1) \quad \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

(2) for multiple poles
(Repeated)

$$\oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} f(z) \right]_{z=a}$$

Residue Theorem

- Used to simplify
- Only for closed

If $f(z)$ is analytic inside or on closed curve C at point $z=a$ where a is a pole then

$$\oint_C f(z) dz = 2\pi i \times \left\{ \text{Sum of all residue at poles inside or on } C \right\}$$

Calculation of Residue

(i) To find pole $\leftarrow$ Denominator = 0

a) Simple pole $[z=a]$

$$\text{Residue of } [f(z)]_{z=a} = \lim_{z \rightarrow a} [(z-a) f(z)]$$

b) Multiple pole $z = a, a, a \dots n$

$$\text{Residue } [f(z)]_{z=a} = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right]_{z=a}$$

Q. Determine the pole and Residue at each pole for
function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Sol poles
 $(z-1)^2 = 0$
 $z = 1, 1$ Multiple pole
 $z+2 = 0$
 $z = -2$ Simple pole

$$\text{Residue } f(-2) = \lim_{z \rightarrow -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$$
$$\Rightarrow \frac{4}{9} //$$

$$\text{Residue } f(1) = \frac{1}{(1-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \frac{z^2}{(z-1)^2(z+2)}$$
$$\Rightarrow \frac{1}{1!} \lim_{z \rightarrow 1} \frac{(z+2)2z - z^2}{(z+2)^2}$$
$$= \frac{3 \times 2 - 1}{3^2} = \frac{5}{9} //$$

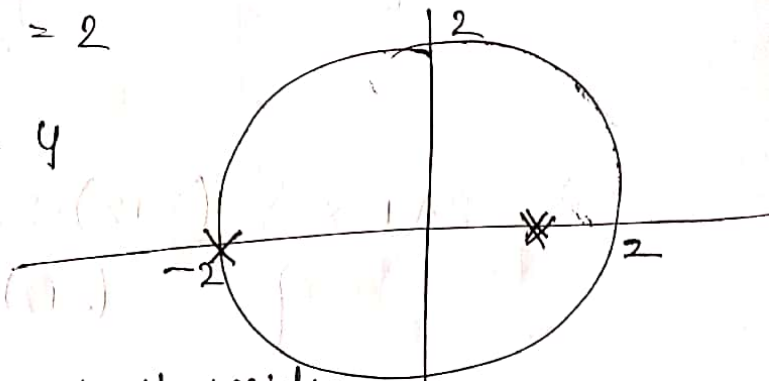
Q. find $\oint \frac{z^2}{(z-1)^2(z+2)} dz$ when
i) $C = |z| = 2$
ii) $C = |z-1| = 1$

Sol

$$\textcircled{1} \quad |z| = 2 \quad |x+iy| = 2$$

$$\sqrt{x^2+y^2} = 2$$

$$x^2+y^2 = 4$$



Residue

$$\oint_C f(z) dz = 2\pi i \left\{ \begin{array}{l} \text{Sum of all residue} \\ \text{at pole inside or on } C \end{array} \right\}$$

$$= 2\pi i \times [\text{Res } f(1) + \text{Res}(-2)]$$

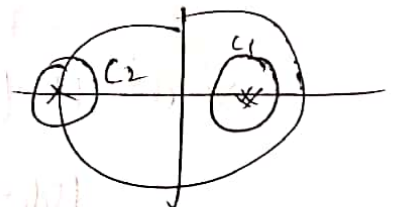
$$= 2\pi i \left[\frac{5}{9} + \frac{4}{9} \right]$$

{ from previous

$$= 2\pi i //$$

Solving with Cauchy's theorem

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$



$$\oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} f(z) \right]$$

$$I = \oint_C \frac{z^2}{(z-1)^2(z+2)} dz = \oint_{C_1} \frac{z^2}{(z-1)^2} dz + \oint_{C_2} \frac{z^2}{(z-1)^2(z+2)} dz$$

$$I = 2\pi i \left[\frac{d}{dz} \left(\frac{z^2}{z+1} \right) \right]_{z=1} + \frac{(-2)}{(-3)^2}$$

$$\Rightarrow 2\pi i \times \left\{ \frac{(z+2) \cdot 2z - z^2}{(z+2)^2} + \frac{4}{9} \right\}$$

$$= 2\pi i \times \left(\frac{5}{9} + \frac{4}{9} \right)$$

$$= 2\pi i //$$

$$(11) \quad |z-1| = 1$$

$$\Rightarrow |x+iy-1| = 1$$

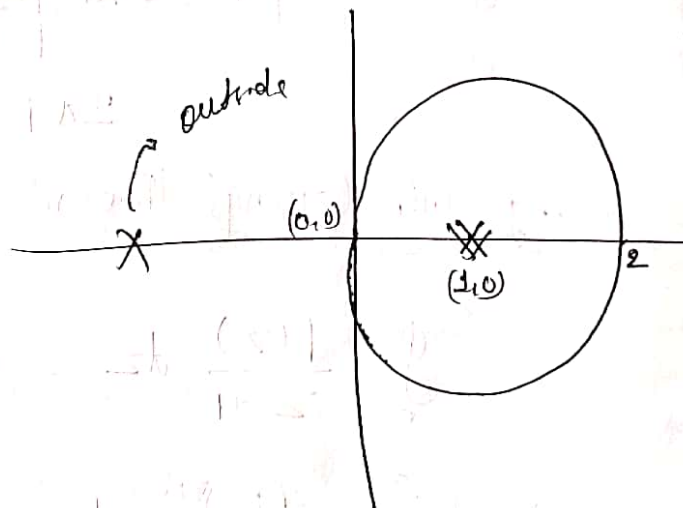
$$\sqrt{(x-1)^2 + y^2} = 1$$

$$(x-1)^2 + y^2 = 1$$

Centre (1,0) r=1

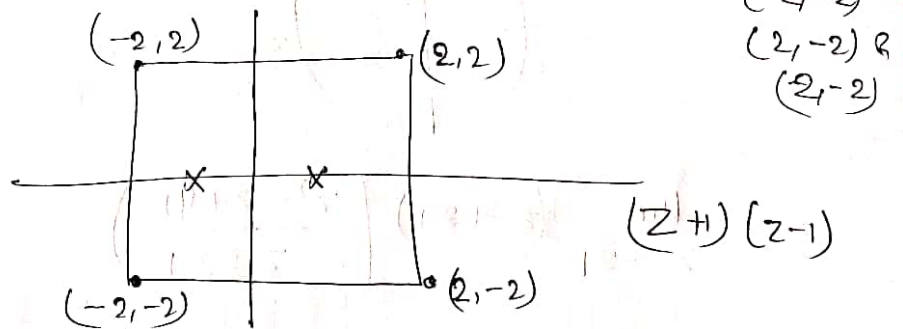
$$J = \int \frac{z^2}{(z+2)} \frac{dz}{(z-1)^2}$$

$$\Rightarrow 2\pi i \left[\frac{5}{9} \right] //$$



Q. Evaluate $\int_C \frac{e^z}{z^2-1} dz$ where C is a square bounded with vertices $(2,2)$

So



$$\int \frac{e^z}{(z+1)(z-1)} dz = \int \frac{e^z}{z-1} dz + \int \frac{e^z}{z+1} dz$$

$$\Rightarrow 2\pi i \frac{e^1}{1} + \frac{e^{-1}}{-1} 2\pi i$$

$$\frac{2\pi i}{2} [e - e^{-1}]$$

$$\pi i \left[e - \frac{1}{e} \right] \quad //.$$

2017 Q. The sum of all residues of a function $f(z) = \frac{1}{(z-4)(z+1)^3}$

Q.2. find $I = \int \frac{z^2-1}{z^2+1} e^z dz$ where C is defined as $|z| = 3$

Sol.1

a) $-\pi i \sin 1$ b) $-2\pi i \sin 1$ c) $-3\pi i \sin 1$ d) $-\pi i \sin 1$

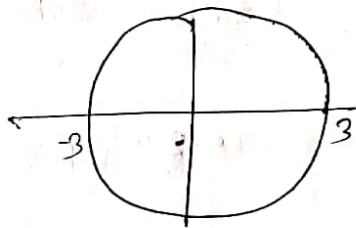
$$f(z) = \left(\frac{1}{(z+1)^3} \right)_{z=4} + \left(\frac{1}{z-4} \right)_{z=-1}$$

$$\Rightarrow 2\pi i \left[\frac{1}{125} + \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{z-4} \right]$$

$$\Rightarrow 2\pi i \left[\frac{1}{125} + \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{z-4} \right] = \frac{2\pi i}{2} \times \frac{1}{125} = \pi i / 125$$

Q4, Q5, 10, 12, 36, 47

Sol



$$\lim_{z \rightarrow -2j} (z+2j) \left(\frac{z^2 - z + 4j}{z+2j} \right)$$

$$\int z^2 - z + 4j \quad 2\pi j [(-2j)^2 + 2j + 4j]$$

$$\Rightarrow 2\pi j [-4 + 6j]$$

$$\lim_{z \rightarrow -2j} (z+2j) \frac{z^2 - z + 4j}{z+2j} \Rightarrow 4\pi [-6 - 4j]$$

$$\Rightarrow -4 + 2j + 4j$$

$$6j - 4$$

$$= 2(3j - 2)$$

$$= 2 \times 2\pi j [3j - 2]$$

$$= 4\pi [-2j - 3] = -4\pi [3 + 2j] //$$

Q5

$$z^2 + 4z + 5$$

$$\frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$\frac{-4 + 2j}{2} \quad \frac{-4 - 2j}{2}$$

$$(-2 + j)(-2 - j)$$

$$\int_c = 0$$

Sol 11b

$$\int \frac{1}{(s+1)^2}$$

$$\Rightarrow 2\pi i \left[\frac{1}{2} \right] = \pi i //$$

Sol 12

$$z^2 + 4$$

$$z = \pm 2i$$

$$(z+2i)(z-2i)$$

$$\frac{1}{z+2i}$$

$$\sum 2\pi i \left[\frac{1}{4i} \right] = \frac{1}{2}$$

$$\pi/2$$

Sol 36

$$z = 1/2$$

$$\frac{\cos 2\pi z}{z-3}$$

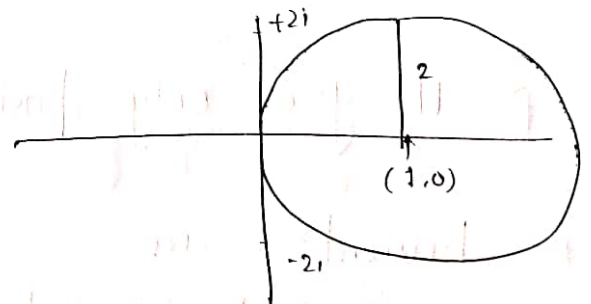
$$\frac{\cos \pi}{\frac{1}{2}-3}$$

$$\frac{-1}{-5/2}$$

$$\frac{6}{5} 2\pi$$

$$\left[\frac{-1}{\frac{1}{2}-3} \right] 2\pi i$$

$$\Rightarrow \frac{-1}{-5/2} \cdot \frac{2}{5} \times 2\pi i$$



Sol 41

$$Z = \frac{1.5j}{0.5} \quad 3j$$

$$\frac{2\pi j}{0.5} (-9 + 0)$$

$$\Rightarrow -4\pi j //$$

Series Expansion of Complex function

* Taylor Series:

→ Valid only for analytic function about $z=a$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

→ It gives only positive power of z .

* Laurent Series

- Applied for both analytic & non analytic $f(z)$
- It has +ve as well as -ve power of z
- Mainly followed by Binomial expansion.

$$f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \dots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \dots$$

Note

$$|z| < 1$$

ie $1-z$ expand.

$$\text{or } |z| < 2$$

$$\frac{|z|}{2} < 1$$

expand $1 - \frac{z}{2}$

Q. Common data given

$$f(z) = \frac{1}{(z-1)(z-2)} \quad \text{--- (1)}$$

Expansion is

Q1. $1 < |z| < 2$

a) $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots \infty$

b) $-z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{2}z - \frac{1}{4}z^2 - \frac{1}{8}z^3 - \dots$

c) $\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots$

d) Not

Q2. $|z| > 2$

a) $\frac{6}{z} + \frac{13}{z^2} + \frac{26}{z^3} + \dots$

b) $\frac{1}{z} + \frac{8}{z^2} + \frac{13}{z^3} + \dots$ d) NOT

c) $\frac{1}{z^2} + \frac{3}{z^3} + \frac{13}{z^4} + \dots$

Q3. $|z| < 1$

a) $1 + 3z + \frac{7}{2}z^2 + \dots$

b) $\frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \dots$

c) $\frac{1}{4} + \frac{9}{4}z + \frac{7}{8}z^2 + \dots \infty$

d) NOT

Q1.

Sol

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= -\frac{1}{(z-1)} + \frac{1}{(z-2)}$$

Sol 1

$$1 < |z|$$

$\Rightarrow$

$$\frac{1}{z} < 1$$

$$\left(1 - \frac{1}{z}\right)$$

$$|z| < 2$$

$\Rightarrow$

$$\frac{z}{2} < 1$$

$$\left(1 - \frac{z}{2}\right)$$

not in annulus $0 < |z-1| < 1$ given
take $z-1 = u$
so, $-1 < u < 0$
expand series in term of u

$$f(z) = \frac{-1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{-2\left(1 - \frac{z}{2}\right)}$$

$$\Rightarrow -\frac{1}{z} \left[\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{2} \left(1 - \frac{z}{2}\right)^{-1} \right]$$

$$\Rightarrow -\frac{1}{z} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] - \frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right)$$

$$\Rightarrow -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \dots$$

option B

Sol 2

$$|z| > 2$$

$$\frac{1}{(z-1)(z-2)}$$

$$|z| > 2$$

$$|z| > 1$$

$$\frac{2}{z} < 1$$

$$\frac{1}{z} < 1$$

$$\Rightarrow \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$\Rightarrow \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})}$$

$$\Rightarrow \frac{1}{z} \left[\left(1 - \frac{2}{z}\right)^{-1} \right] - \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-1}$$

$$\Rightarrow \frac{1}{z} \left[1 + \frac{2}{z} + \frac{4}{z^2} + \dots \right] - \frac{1}{z} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right]$$

$$\Rightarrow \left(\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \dots \right) - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3}$$

$$\Rightarrow \left(\frac{2}{z^2} - \frac{1}{z^2} \right) + \left(\frac{4}{z^3} - \frac{1}{z^3} \right) + \dots$$

$$\frac{1}{z^2} + \frac{3}{z^3} + \dots \quad // \text{ option c)$$

Sol 4

$$|z| < 1, |z| < 2$$