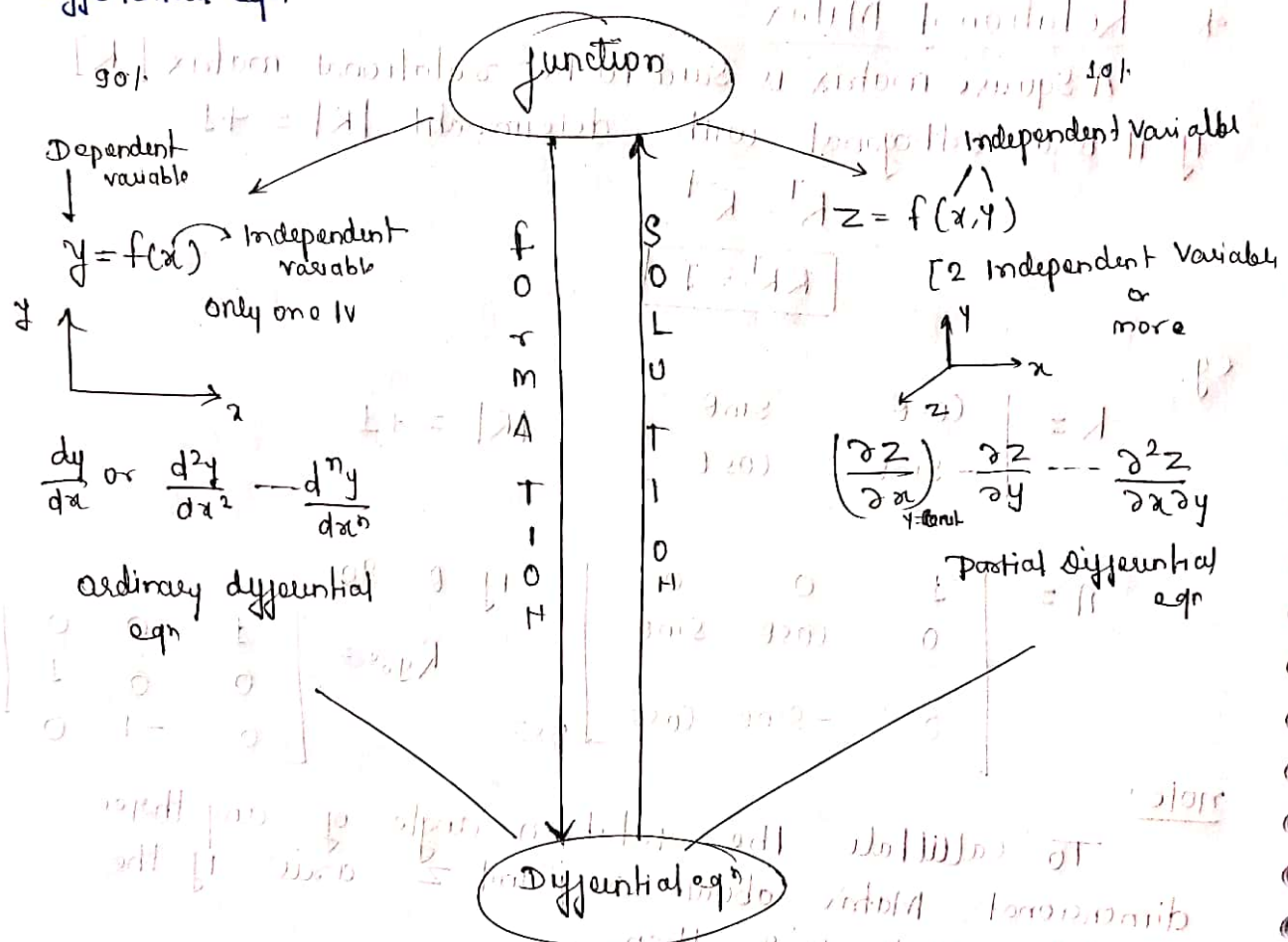


Chapter: 2 DIFFERENTIAL EQⁿ

An Equation in which differential terms are involved is known as differential eqⁿ.



* Representation of Differential eqⁿ

- Order** :- In D.E. order is the maximum no. of times of differentiation. Order is (or) always a positive Integer degree.
- Degree** :- The power of Higher order term in differential eqⁿ after removing the fraction value & radical sign.
 - There is no relation betⁿ order & degree.
 - A differential eqn can exist without degree but cannot exist without order.
 - By changing the representⁿ degree can be changed but order is fixed for every differential eqn.
 - Order of D.E can only change by differentiating or integrating the differential eqn.

Q1 find order & degree

$$(1) \quad \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = y \quad \Rightarrow \quad \begin{matrix} \text{order} = 2 \\ \text{degree} = 1 \end{matrix}$$

$$(2) \quad \left(1 + \frac{d^2 y}{dx^2}\right)^{3/2} = \left(\frac{d^2 y}{dx^2}\right)^2 \quad \Rightarrow \quad \begin{matrix} O = 2 \\ d = 2 \end{matrix}$$

$$(3) \quad \left(\frac{d^2 y}{dx^2} + \frac{dy}{dx}\right)^3 = \left(\frac{d^2 y}{dx^2}\right)^{2/3} \quad \Rightarrow \quad \left(\frac{d^2 y}{dx^2} + \frac{dy}{dx}\right)^9 = \left(\frac{d^2 y}{dx^2}\right)^2 \quad \begin{matrix} O = 2 \\ d = 9 \end{matrix}$$

$$(4) \quad y = e^{y'} \quad \Rightarrow \quad O = 1$$

$$(5) \quad \left(1 + \frac{1}{y'}\right)^{2/3} = (y'')^2$$

$$(6) \quad (y')^{2/3} + (y')^{3/2} = 1$$

$$(7) \quad y = \log(1 + y'')$$

Sol 2 $\left(1 + \frac{d^2 y}{dx^2}\right)^{3/2} = \left(\frac{d^2 y}{dx^2}\right)^2$

$$\Rightarrow \left(1 + \frac{d^2 y}{dx^2}\right)^3 = \left(\frac{d^2 y}{dx^2}\right)^4$$

$$\Rightarrow \left(\frac{d^2 y}{dx^2}\right)^9 + \dots = \left(\frac{d^2 y}{dx^2}\right)^4$$

Sol 3

$$y = e^{y'}$$

$$y = 1 + y' + \frac{(y')^2}{2!} + \dots \infty$$

$$\begin{matrix} O = 1 \\ d = \infty \end{matrix}$$

$\log y = y'$ $\times$
do not change
eqn

Sol 5

$$O = 2$$
$$d = 6$$

Sol 6

$$(y')^{2/3} + (y')^{3/2} = 1$$

LCM of deno.

$$\left(\frac{2}{3}, \frac{3}{2} \right)$$

$$= 6$$

$$(y')^{2/3 \times 6} + (y')^{3/2 \times 6} = \text{any value}$$

short trick
only when same order

$$(y')^4 + (y')^9$$

$$\text{order} = 1$$
$$\text{deg} = 9$$

Soln - $y = \log(1 + y'')$

$$\text{order} = 2$$
$$\text{deg} = \infty$$

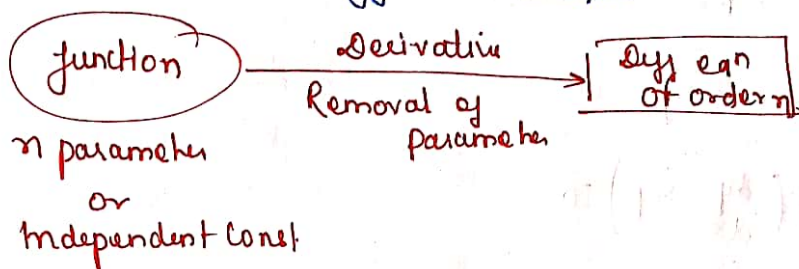
donot

$$e^y = 1 + y''$$

$$\text{order} = 2$$
$$\text{deg} = 1$$

X

* Formation of differential eqn



eg (1) $y = c_1 x$

$$\frac{dy}{dx} = c_1$$

$$\Rightarrow y = y' x$$

no. of parameters = order of DE

eg (2) $y = c_1 x + (c_1)^2$

$$\frac{dy}{dx} = c_1$$

$$\text{so } y = y' x + (y')^2$$

eg (3) $y = ax + bx^2$

sol $\frac{dy}{dx} = a + 2bx$

$$\frac{d^2y}{dx^2} = 2b$$

(3) so $y = \frac{dy}{dx} x - \frac{2}{2} \frac{d^2y}{dx^2} x^2 + \frac{d^2y}{dx^2} x^2$

$$y = \frac{dy}{dx} x - \frac{d^2y}{dx^2} x^2 + \frac{d^2y}{dx^2} x^2$$

$$y = x(y' - y'' x) + \frac{y''}{2} x^2$$

$$\Rightarrow y'' \left(\frac{x^2}{2} - x \right) + y' x - y = 0$$

eg ④

$$y = Ae^{-x} + Be^{3x}$$

$$\text{sol} \quad (D+1)(D-3)y = 0$$

$$\Rightarrow \left(\frac{dy}{dx} + y \right) \left(\frac{dy}{dx} - 3y \right) = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

Q. form diff. eqn for function
 $y = Ax^2 + Bx^3$

sol

$$\frac{dy}{dx} =$$

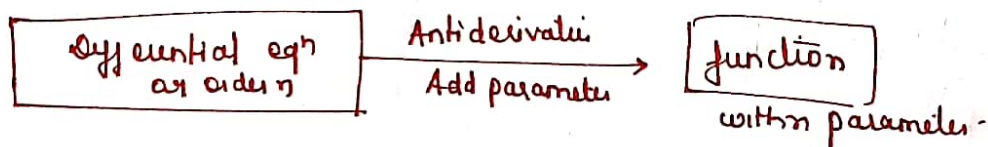
Standard form for $y = Ax^m + Bx^n$

$$\Rightarrow xy'' - xy'(m+n-1) + mny = 0$$

* Solution of Ordinary DEqn:

In general there are two type of soln

- i) General Solution
- ii) particular Soln



- General Soln of n order differential eqn is n parameter family.
i.e. n numbers of independent constants.
- General Soln is also known as Integral primitive or antiderivative.
- Assigning some particular values to the parameter gives the particular Soln.
- General Soln is a Unique Soln but particular Soln can be ∞ .

$$\text{eg ①} \quad \frac{dy}{dx} + \frac{x}{y} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y dy + x dx = 0$$

∫ both side

$$\int y dy + \int x dx = c$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c$$

Unique General soln, i.e. family is unique (Always get circle)

$$\boxed{y^2 + x^2 = 2c}$$

family of circle

$$\begin{cases} 2c = r \end{cases}$$

$$\text{eg ②} \quad \frac{dy}{dx} - \frac{y}{x} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$x \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{y^2} = \frac{1}{x^2} = -c$$

$$\Rightarrow \log y = \log x + c$$

$$\Rightarrow \boxed{y = x + e^c} \text{ straight line}$$

$$\text{eg ③} \quad \frac{dy}{dx} + \frac{y}{x} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

$$\log y = -\log x + c$$

$$\log y + \log x = c$$

$$\boxed{xy = c} \text{ rectangular hyperbola}$$

eg $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow y dy = x dx$

$\frac{y^2}{2} - \frac{x^2}{2} = c$

$y^2 - x^2 = 2c$ Hyperbola.

*** The condition of a function to be the soln of differential equation —

a) The soln of differential eqn is the linear combination of independent functions.

To check the dependency of function we use Wronskian Determinant $f_1(x)$ $f_2(x)$

$$|W| = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$|W|=0$ Dependent functions

$|W| \neq 0$ Independent function

eg- check the following functions whether the linear combination of the function form the soln of differential eqn or not

i) $x^2, 2x, 3$

ii) $x^2 + 2x + 6, x^2 + 2x, 3$

iii) $e^x, xe^x, (x-1)e^x$

Sol

$$\begin{vmatrix} x^2 & 2x & 3 \\ 2x & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$\Rightarrow 3(-4) = -12 \neq 0$ Independent functions

Short

check dependency of function by linear combination or scalar product — $x^2, 2x, 3$
no combination of any two gives the 3rd.

ii) x^2+2x+6 x^2+2x 3
check combⁿ

$$R_1 = R_2 + 2R_3 \quad \text{dependent}$$

$$\begin{vmatrix} x^2+2x+6 & x^2+2x & 3 \\ 2x+2 & 2x+2 & 0 \\ 2 & 2 & 0 \end{vmatrix}$$

$$3(4x+4-4x-4)$$

$$= 0 \quad \text{Hence dependent function}$$

iii) $e^x, xe^x, (x-1)e^x$

$$e^x, xe^x, (xe^x - e^x)$$

$$R_3 = R_2 - R_1 \quad \text{dependent}$$

$$\begin{vmatrix} e^x & xe^x & xe^x - e^x \\ e^x & xe^x + e^x & xe^x + e^x - e^x \\ e^x & xe^x + e^x + e^x & xe^x + e^x \end{vmatrix}$$

$$\begin{vmatrix} 0 & e^x & -e^x \\ 0 & e^x & -e^x \\ e^x & xe^x + e^x + e^x & xe^x + e^x \end{vmatrix} \quad \left. \vphantom{\begin{vmatrix} 0 & e^x & -e^x \\ 0 & e^x & -e^x \\ e^x & xe^x + e^x + e^x & xe^x + e^x \end{vmatrix}} \right\} \text{Same} = 0$$

Q. choose the correct set of function for not a solution for any ordinary differential eqⁿ

I a) $\sin x, \sin^2 x, \cos^2 x$

$$\sin x, 1 - \cos^2 x, 1 + \cos^2 x$$

II b) $\sin^2 x, \cos^2 x, \cos 2x$

$$\sin^2 x - \cos^2 x = -\cos 2x$$

III c) $\cos x, \sin x, \tan x$

IV d) $\cos 2x, \sin x, \cos x$

Solution of ODE:

$$y = f(x)$$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$$

General Differential eqn of order n with constant coeff.

$$a_n \frac{d^ny}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = \phi(x)$$

$$\frac{d}{dx} = D, \quad \frac{d}{dx^2} = D^2$$

$$\text{So } a_n D^n y + a_{n-1} D^{n-1} y + \dots + a_1 D y + a_0 y = \phi(x)$$

$$> [a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0] y = \phi(x)$$

$$> [D^n + D^{n-1} + \dots + D + D^0] y = \phi(x)$$

$$\left\{ \begin{array}{l} a's = 1 \end{array} \right.$$

$$> \boxed{f(D) y = \phi(x)}$$

$\phi(x) = 0$
Homogenous Diff. eqn

$\phi(x) \neq 0$
non Homogenous Diff. eqn

 Solution of Homogenous Differential eqn:
 $f(D) y = 0$

So, $y \neq 0$
 $\boxed{f(D) = 0}$ Auxiliary eqn

Value of D (roots)

Real

- a. all Value same $D = a, a, a$
- b. unequal or distinct $D = a, b, c$

Imaginary or complex

a. pair $a \pm ib$

Value of λ :

(1) Same or Equal
 $\lambda = a, a, a$

$$y = (C_1 + x C_2 + x^2 C_3) e^{ax}$$

(2) Unequal or distinct

$$y = C_1 e^{ax} + C_2 e^{bx} + C_3 e^{cx}$$

(3) Complex pair : $\lambda = a \pm ib$

$$y = e^{ax} [C_1 \cos bx + C_2 \sin bx]$$

Q. Solve the differential eqn.

(i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

(ii) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$

(iii) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

(iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

(v) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$

Sol 1: $(\lambda^2 + 2\lambda + 1)y = 0$

A-E $\lambda^2 + 2\lambda + 1 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$$\lambda = -1, -1$$

$$y = \cancel{C_1 e^{-x}} + \cancel{x C_2 e^{-x}} (C_1 + x C_2) e^{-x}$$

II

$$(D^2 + 7D + 10)y = 0$$

$$\Rightarrow D^2 + 5D + 2D + 10 = 0$$

$$D(D+5) + 2(D+5) = 0$$

$$\Rightarrow (D+5)(D+2) = 0$$

$$D = -5, -2$$

$$y = C_1 e^{-5x} + C_2 e^{-2x}$$

III

$$D^3 + 3D^2 + 3D + 1 = 0$$

$$(D+1)^3 = 0$$

$$D = -1, -1, -1$$

$$y = (C_1 + x C_2 + x^2 C_3) e^{-x}$$

IV

$$D^2 + D + 1 = 0$$

$$\Rightarrow \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

$$y = e^{-1/2x} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right]$$

V

$$D^2 + 4D = 0$$

$$D(D+4) = 0$$

$$D = 0, -4$$

$$y = C_1 + C_2 e^{-4x}$$

Q. $\frac{d^2y}{dx^2} + 9y = 0$

Sol A: $\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$y = C_1 \cos 3x + C_2 \sin 3x$ //

note: Soln of ^{Homogeneous} Ordinary differential eqn is known as Complementary function because it is derived by the auxiliary eqn which is complemented form of differential eqn

$f(\lambda)y = 0$ $f(0)y = 0$

. The solution is also known as natural soln or ZIR (Zero Input Response)

. The Initial Value or boundary Value is used to find the parameters for particular soln

Ex

Sol 2 $(\lambda + 2)^2 = 0$

$\lambda = -2, -2$

$y = (C_1 + x C_2) e^{-2x}$

put $y' = C_1 e^{-2x} (-2) + x C_2 e^{-2x} (-2) + C_2 e^{-2x}$

put $x=0$

$0 = C_1 + C_2 \times 0$

$1 = C_1$

put in y'

$1 = -2 e^0 + C_2$

$C_2 = 3$

$y = (1 + 3x) e^{-2x}$ //

$y = (1 + 3) e^{-2}$

$y = 4 e^{-2}$ //

Q9.

Sol. Given $y'' = y$

$$y'' - \frac{1}{L^2} y = 0$$

$$y'' = \pm \frac{1}{L}$$

$$y = u(x) = c_1 e^{-1/L x} + c_2 e^{1/L x}$$

$$u(0) = k$$

$$u(\infty) = 0$$

$$k = c_1 + c_2$$

— (2)

$$0 = c_1 e^{-\infty} + c_2 e^{\infty}$$

$$k = c_1$$

$$0 = c_2 \infty + 0$$

$$c_2 = 0$$

$$u(x) = k \exp\left(-\frac{x}{L}\right) //$$

Q14

Sol

$$y'' + k^2 y = 0$$

$$y'' = -k^2 y$$

$$y = \pm k i$$

$$y = c_1 \cos kx + c_2 \sin kx$$

— (1)

$$0 = c_1$$

Hence Ans (a)

$$0 = c_1 \cos ka + c_2 \sin ka$$

$$0 = c_2 \sin ka$$

$$\sin ka = 0 \quad | \quad c_2 = 0 \quad \text{not zero as non-zero mention}$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a}$$

$$y = \sum c_2 \sin\left(\frac{n\pi}{a} x\right)$$

$$\underline{c_2 = A_m}$$

Q23.

$$\theta^2 + 6\theta + 8 = 0$$

$$\Rightarrow \theta^2 + 4\theta + 2\theta + 8$$

$$\theta(\theta + 4) + 2(\theta + 4) = 0$$

$$y = C_1 e^{-4x} + C_2 e^{-2x}$$

$$y = C_1 + C_2$$

$$4 = 4C_1 + 4C_2$$

$$0 = 4C_1 - 2C_2$$

$$6C_2 = 4$$

$$C_2 = 4/6$$

$$y' = C_1 e^{-4x}(-4) + C_2 e^{-2x}(-2)$$

$$0 = -4C_1 + C_2(-2)$$

$$0 = -4C_1 - 2C_2$$

Ans. 2

Q37.

Sol

$$\theta^2 + 3 = 0$$

$$\theta = \sqrt{-3}$$

$$\theta = \pm \sqrt{3}i$$

$$y = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x \quad \text{--- (1)}$$

$$1 = C_1$$

$$y' = -\sin \sqrt{3}x \sqrt{3} C_1 + C_2 \sqrt{3} \cos \sqrt{3}x$$

$$0 = C_2 \sqrt{3}$$

$$C_2 = 0$$

$$y = \cos \sqrt{3}x$$

$$y = \cos \sqrt{3}$$

$$y = 0.99 \times \frac{\pi}{180} = -0.1611$$

Note All Calculation in Radian

Q8a.

Sol

$$\theta^2 + 2\theta + 17 = 0$$

$$\frac{-2 \pm \sqrt{4 - 68}}{2}$$

$$-1 \pm 4i$$

$$y = e^{-x} \left[\frac{C_1 \cos 4x + C_2 \sin 4x}{p} \right]$$

$$1 = C_1$$

$$2 \frac{dy}{dx} = e^{-x}(-1)[p] + e^{-x}[-4C_1 \sin 4x + 4C_2 \cos 4x]$$

$$0 = e$$

Q102

sol

$$y = \begin{cases} \frac{x^2}{2} & x > 0 \\ -\frac{x^2}{2} & x < 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$\frac{dy}{dx} = |x|$$

Q109

sol

$$D^2 + \lambda = 0$$

$$D = \pm \sqrt{\lambda} i$$

$$y = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x$$

$$\theta = \pi$$

$$y' = -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + C_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$$

$$\theta = 0$$

$$0 = C_2 \sqrt{\lambda}$$

$$C_2 = 0 \quad \left| \begin{array}{l} \sqrt{\lambda} \neq 0 \\ \lambda \neq 0 \end{array} \right.$$

$$0 = -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} x$$

$$\sin \sqrt{\lambda} x = 0$$

$$\sqrt{\lambda} x = \pm n\pi$$

$$\lambda = \pm n^2$$

Q120. $\lambda^2 + 2\lambda + 2 = 0$

$$\frac{-2 \pm \sqrt{4-8}}{2}$$

 $-1 \pm i$

Q122

sol $\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda = -1, -1$

~~$y = C_1 \cos x + C_2 \sin x$~~

~~$1 = C_1$~~

~~$y' = -C_1 \sin x + C_2 \cos x$~~

~~$-2 = C_2$~~

~~$y = \cos x - 2 \sin x$~~

$y = C_1 e^{-t} + x C_2 e^{-t}$

$y = (C_1 + x C_2) e^{-t}$

$1 = C_1$

$y' = x C_2 e^{-t} \cdot e^{-t} + C_2 e^{-t} + 1$

$-2 = C_2$

$C_2 = -2$

$y = (1 - 2t) e^{-t}$
 $= (1 - t) e^{-t}$

Q12. $D + 3 = 0$

$C_1 e^{-3x}$

$2 =$

Q16. $D^2 - 5D + 6 = 0$

$D^2 - 2D - 3D + 6 =$

$D(D-2) - 3(D-2)$

$(D-2)(D-3)$

* Solution of Non Homogeneous D.E eqn:-

$f(D)y = \phi(x) \neq 0$

$\phi(x)$ — Constant 10, 5, ...
 — exponential e^{ax}, e^{-bx}
 — $\sin ax / \cos bx$
 — x^m
 — Combination of above $e^{ax} g(x)$ or $x \cdot G(x)$

Solution of Non Homogeneous ODE

$$y = c.f + P.I \longrightarrow \text{Particular Integral}$$

complementary function

$$y = ZIR + ZSR$$

$$P.I = \frac{1}{f(\omega)} (\phi(x))$$

Case 1 :- $y \quad \phi(x) = e^{ax}$

$$P.I = \frac{1}{f(\omega)} \cdot e^{ax}$$

put $\omega = a$

$$P.I = \frac{1}{f(a)} e^{ax}$$

$$\{ f(a) \neq 0$$

$y \quad f(a) = 0$ Case fail

$$P.I = x \frac{1}{f'(a)} e^{ax}$$

$$P.I = x \frac{1}{f'(a)} e^{ax}$$

$y \quad f'(a) = 0$ — Again Case fail

$$P.I = x^2 \frac{1}{f''(a)} e^{ax}$$

$$\frac{1}{(\omega - a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

eg ① Solve $(\omega^2 - 2\omega + 1) y = e^x$

sol $c.f = (c_1 + x c_2) e^x$

$$P.I = \frac{1}{(\omega - 1)^2} e^x$$

Case fail

$$= \frac{x^2}{2} e^x$$

eg ② $(\omega^2 - 3\omega + 2) y = e^{2x}$

$$c.f = c_1 e^x + c_2 e^{2x}$$

$$P.I = \frac{1}{(\omega - 1)(\omega - 2)} e^{2x}$$

$$= \frac{1}{1} x e^{2x} \quad //$$

Case II $\phi(x) = \sin ax / \cos ax$

P.I = $\frac{1}{f(\theta^2)} \sin ax / \cos ax$
 $\swarrow$
 $-a^2$

P.I = $\frac{1}{f(-a^2)} \sin ax / \cos ax$ $[f(-a^2) \neq 0]$

$f(-a^2) = 0$ Case fail

P.I = $x \frac{1}{f'(\theta^2)} \sin ax / \cos ax$
 $\swarrow$
 $-a^2$

eg (1) $(\theta^2 + 4)y = \sin 2x$

$\Rightarrow$ P.I = $\frac{1}{\theta^2 + 4} \sin 2x$

put $\theta^2 = -a^2 = -4$

Case fail

$\Rightarrow x \frac{1}{2\theta} \sin 2x$

$\Rightarrow -\frac{x}{2} \frac{\cos 2x}{2}$

$= -\frac{x \cos 2x}{4} //$

Case III $\phi(x) = x^m$

P.I = $\frac{x^m}{f(\theta)}$
 $= [f(\theta)]^{-1} x^m$

Binomial

$(1-x)^{-1} = 1 + x + x^2 + x^3 - \dots \infty$

$(1+x)^{-1} = 1 - x + x^2 - x^3 - \dots \infty$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 - \dots \infty$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 - \dots \infty$

eg. $(D^2 + 2D + 2)y = x^2$

$$P.I. = \frac{x^2}{D^2 + 2D + 2}$$

$$= \frac{x^2}{2 \left(1 + \frac{D^2 + 2D}{2} \right)}$$

$$\Rightarrow \frac{1}{2} \left(1 + \frac{D^2 + 2D}{2} \right)^{-1} x^2$$

$$\Rightarrow \frac{1}{2} \left[1 - \frac{D^2 + 2D}{2} + \left(\frac{D^2 + 2D}{2} \right)^2 - \dots \right] x^2$$

$$= \frac{1}{2} \left[x^2 - 1 - 2x + \frac{D^4 + 4D^3 + 4D^2 + 4D}{4} x^2 + 0 \right]$$

$$\Rightarrow \frac{1}{2} [x^2 - 1 - 2x + 2]$$

$$\Rightarrow \frac{1}{2} [x^2 - 2x + 1] //$$

Case IV Combinations:-

a) $\phi(x) = e^{ax} G(x)$

$$P.I. = \frac{1}{f(D)} \cdot e^{ax} \cdot G(x)$$

② $\rightarrow D+a$

$$P.I. = e^{ax} \frac{1}{f(D+a)} G(x)$$

b) $\phi(x) = x \cdot G(x)$

$$P.I. = \frac{1}{f(D)} \cdot x \sin x$$

$$e^{ix} = \cos x + i \sin x$$

$$\sin x = \text{Im } e^{ix}$$

$$\cos x = \text{Re } e^{ix}$$

Q) Solve

$$(\theta^2 + 7\theta + 12)y = e^x \cos 2x$$

$$\Rightarrow \theta^2 + 7\theta + 12 = 0$$

$$\theta^2 + \cancel{7\theta} + \cancel{12}$$

$$\theta^2 + 4\theta + 3\theta + 12$$

$$\theta(\theta + 4) + 3(\theta + 4)$$

$$\Rightarrow (\theta + 4)(\theta + 3) = C_1 e^{-4x} + C_2 e^{-3x}$$

$$P.F. = \frac{1}{\theta^2 + 7\theta + 12} e^x \cos 2x$$

$$\Rightarrow e^x \frac{1}{(\theta + 1)^2 + 7(\theta + 1) + 12} \cos 2x$$

$$\Rightarrow e^x \frac{1}{\theta^2 + 2\theta + 1 + 7\theta + 7 + 12} \cos 2x$$

$$\Rightarrow e^x \frac{1}{-4 + 9\theta + 20} \cos 2x$$

$$e^x \frac{1}{9\theta + 16} \cos 2x$$

$\Rightarrow$ Rationalize

$$e^x \frac{9\theta - 16}{(9\theta)^2 - 16^2} \cos 2x$$

$$\Rightarrow e^x \left[\frac{9\theta - 16}{-81 \times 4 - 16^2} \right] \cos 2x$$

$$\Rightarrow e^x \left[\frac{9\theta - 16}{-324 - 256} \right] \cos 2x$$

$$\Rightarrow -e^x \left[\frac{9\theta - 16}{580} \right] \cos 2x$$

$$\frac{-e^x}{580} \left[-9 \times 2 \sin 2x - 16 \cos 2x \right]$$

$$\Rightarrow \frac{e^x}{580} \left[18 \sin 2x + 16 \cos 2x \right]$$

Q57

$$Q2 \quad \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 9y = e^{3x} + x + e^{-3x} \sin x$$

Sol 57 $D^2 + 3D + 2 = 5 \cos x$

$$\rightarrow P.I = \frac{5 \cos x}{D^2 + 3D + 2}$$

$$\rightarrow \frac{5 \cos x}{-1 + 3D + 2}$$

$$\frac{5 \cos x}{1 + 3D}$$

$$5 \left(\frac{1 - 3D}{1 - 9D^2} \right) \cos x$$

$$5 \left(\frac{1 - 3D}{1 + 9} \right) \cos x$$

$$\rightarrow \frac{5}{10} [\cos x + 3 \sin x]$$

$$= 0.5 \cos x + 1.5 \sin x //$$

Sol 2 $(D^2 + 6D + 9)y = e^{3x} + x + e^{-3x} \sin x$

$$\rightarrow \frac{e^{3x} + x + e^{-3x} \sin x}{D^2 + 6D + 9}$$

$$\rightarrow \frac{1}{D^2 + 6D + 9} e^{3x} + \frac{x}{9 \left(1 + \frac{D^2 + 6D}{9} \right)} + e^{-3x} \left[\frac{1}{(D-3)^2 + 6(D-3) + 9} \right] \sin x$$

$$\rightarrow \frac{1}{36} e^{3x} + \frac{1}{9} \left[1 - \frac{D^2 + 6D}{9} + \left(\frac{D^2 + 6D}{9} \right)^2 - \right] x +$$

$$\Rightarrow \frac{e^{3x}}{36} + \frac{1}{9} \left[x - \frac{6}{9} \right] + e^{-3x} \left[\frac{1}{D^2+9-6D+6D-18+9} \right] \sin x$$

$$\Rightarrow \frac{e^{3x}}{36} + \text{mistake } 9 \left[x - \frac{2}{3} \right] + e^{-3x} \left[\frac{1}{-1+9-9} \right] \sin x$$

$$= \frac{e^{3x}}{36} + 9x - 6 + e^{-3x} [\sin x] //$$

$$\Rightarrow \frac{e^{3x}}{36} + 9x - 6 - e^{-3x} \sin x //$$

after

$$(D+3)^2 y = e^{3x} + x + e^{-3x} \sin x$$

$$= \frac{1}{36} e^{3x} + \frac{1}{9} \left(x - \frac{2}{3} \right) - e^{-3x} \sin x //$$

Q. $(D^2+4)y = x \cos 3x$ — (i)

C.F. = $C_1 \cos 2x + C_2 \sin 2x$ — (ii)

P.I. = $\frac{1}{D^2+4} x \cos 3x$

$\cos 3x = \text{Real part of } e^{i3x}$

$e^{i3x} = \cos 3x + i \sin 3x$

P.I. = $\frac{1}{D^2+4} x (\text{R.P. of } e^{i3x})$

= $\text{Re of } \left\{ e^{i3x} \left[\frac{1}{(D+3i)^2+4} \right] x \right\}$

= $\text{Re of } \left\{ e^{i3x} \left(\frac{1}{D^2-9+6Di+4} \right) x \right\}$

$$\text{R.P of } \left\{ e^{i3x} \left(\frac{1}{\partial^2 + 6\partial - 5} \right) x \right\}$$

$$\Rightarrow \text{R.P of } \left\{ \frac{e^{i3x}}{-5} \left(\frac{1}{1 - \frac{\partial^2 + 6\partial}{5}} \right) x \right\}$$

$$\Rightarrow \text{R.P of } \left\{ \frac{e^{i3x}}{-5} \left(1 + \frac{\partial^2 + 6\partial}{5} \right) x \right\}$$

$$\Rightarrow \text{R.P of } \left\{ \frac{e^{i3x}}{-5} \left(x + \frac{6}{5}i \right) \right\}$$

$$\Rightarrow -\frac{1}{5} \left[\text{R.P of } \left((\cos 3x + i \sin 3x) \left(x + \frac{6i}{5} \right) \right) \right]$$

$$\Rightarrow -\frac{1}{5} \left[x \cos 3x - \frac{6}{5} \sin 3x \right]$$

$$\Rightarrow -\frac{x \cos 3x}{5} + \frac{6}{25} \sin 3x //$$

short trick

$$\text{If } \phi(x) = x G(x)$$

$$\boxed{P.I = x \left[\frac{1}{f(\omega)} - \frac{f'(\omega)}{[f(\omega)]^2} \right] G(x)}$$

check.

$$P.I = \left[\frac{x}{\partial^2 + 4} - \frac{2\partial}{(\partial^2 + 4)^2} \right] \cos 3x$$

$$= -\frac{x}{5} + \frac{6}{25} \sin 3x //$$

Q. Solve for c.f & P.I

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$$

$$\Rightarrow D^2 - 2D + 1 = x \sin x$$

$$c.f = (C_1 + x C_2) e^x$$

$$P.I = \left[\frac{x}{(D-1)^2} + \frac{2D-2}{(D-1)^4} \right] \sin x$$

$$= \left[\frac{x}{-1-2D+1} + \frac{2D-2}{(-1-2D+1)^2} \right] \sin x$$

$$\Rightarrow \left[+\frac{x}{2} \cos x + \left(-1 + \frac{1}{2D}\right) \sin x \right] \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2} \sin x + \frac{1}{2} \cos x - \frac{1}{4} \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2} \cos x - \frac{1}{4} \sin x$$

$$= \frac{2D-2}{4D^2} \sin x$$

$$= \left(-\frac{2D-2}{-4}\right) \sin x$$

$$= \frac{1}{2} D + \frac{1}{2} \sin x$$

$$= \frac{1}{2} \cos x + \frac{1}{2} \sin x$$

* Cauchy Linear Differential Equation:

n^{th} order General form

$$x^n \frac{d^n y}{dx^n} + x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \phi(x)$$

order = n
 x power = n

$$\left\{ \begin{array}{l} x \frac{dy}{dx} = \frac{dy}{dt} = \phi y \quad \frac{d}{dt} = \phi \\ \text{chain rule} \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \end{array} \right.$$

$$\therefore \frac{dx}{dt} \cdot \frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{dx}{dt} = x \Rightarrow \frac{dx}{x} = dt$$

$$x = e^t$$

Step ① $x = e^t \Rightarrow (t = \log x)$ at last

we get $x \frac{dy}{dx} = \phi y$

$$x^2 \frac{dy}{dx^2} = \phi(\phi-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = \phi(\phi-1)(\phi-2)y$$

$$Q1. \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4y = 0$$

$$\text{Sol} \quad x = e^t \quad \text{or } t = \log x$$

$$\text{d} \quad x \frac{dy}{dx} = 0 y$$

$$x^2 \frac{d^2 y}{dx^2} = 0(0-1)y$$

$$\text{So} \quad [0(0-1) + 0+4] y = 0$$

$$\cdot \quad (0^2 - 0 + 0 + 4) y = 0$$

$$\Rightarrow \quad (0^2 + 4) y = 0$$

$$0^2 + 4 = 0$$

$$0 = \pm 2i$$

$$C \cdot t = C_1 \cos 2t + C_2 \sin 2t$$

{ must be in form of t }

$$y = C_1 \cos(2 \log x) + C_2 \sin(2 \log x) \quad //$$

Condition if $y(1) = 0$ $y(1) = 0$

$$\log 1 = 0$$

$$0 = C_1 \cos 0 + C_2 \times 0$$

$$C_1 = 0$$

$$y = a \sin(2 \log x)$$

$$y = a \sin \log x^2 \quad //$$

note: Legendre's form:

$$(ax+b)^n \frac{d^n y}{dx^n} + \dots = \phi(x)$$

$$(ax+b) = e^t$$

we get

$$(ax+b) \frac{dy}{dx} = a \theta y$$

$$\theta = \frac{d}{dt}$$

$$(a+b)^2 \frac{d^2 y}{dx^2} = a^2 \theta(\theta-1)y$$

$$\vdots$$

Q.19, Q.25, Q.24, Q.27, Q.33, Q.46, Q.113

Sol 19.

$$(\theta(\theta-1) + \theta - 1) y = 0$$

$$\Rightarrow (\theta^2 - \theta + \theta - 1) y = 0$$

$$(\theta^2 - 1) y$$

$$\theta = \pm 1$$

$$C_1 e^{-t} + C_2 e^t$$

$$\text{for } C_1 e^{-\log x} + C_2 e^{\log x}$$

$$\Rightarrow \frac{1}{x} //$$

Q23

Sol

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{dt} + 3x = 0$$

$$d + 3 = 0$$

$$d = -3$$

$$C_1 e^{-3t} // C_1 = x_0$$

(A)

Q24

Sol

$$D^2 + 3D + 2 = 5$$

$$D^2 + 2D + D + 2 = 5$$

$$(D+2)(D+1) = 0$$

$$C_1 e^{-2x} + C_2 e^{-x}$$

$$\text{at } x \rightarrow \infty$$

$$C.F. \quad x(t) = 0$$

$$P.I. = \frac{5}{(D+2)(D+1)}$$

$$= \frac{5}{2}$$

$$\text{Hence } x(t) = 5/2 //$$

Q27

$$x(t) = x_1(t)$$

$$x(t) = x_2(t)$$

$$D^2 + 1 = 0$$

$$D = \pm i$$

$$C.F. = C_1 \cos x + C_2 \sin x$$

$$1 = C_1$$

$$y = \cos x$$

$$y' = -C_1 \sin x + C_2 \cos x$$

$$0 = C_2$$

$$0 = C_1$$

$$1 = C_2$$

$$y_2 = \sin x$$

Q93
Sol

$$\theta(\theta-1) + \theta - 4 = 0$$

$$\begin{aligned} \theta^2 - 4 &= 0 & \theta^2 &= 4 \\ \theta &= \pm 2 & \theta &= \pm 2 \end{aligned}$$

$$y = C_1 e^{2t} + C_2 e^{-2t}$$

$$y = C_1 e^{2 \log x} + C_2 e^{-2 \log x}$$

$$C_1 x^2 + C_2$$

$$0 = C_1 \infty$$

$$y = e^{-2 \log x}$$

=

$$(-2)^2 = 4$$

$$1 = C_1 e^{2 \log 1} + C_2 e^{-2 \log 1}$$

$$1 = C_1 + C_2$$

$$C_2 = 1$$

Q46. $\theta^2 + p\theta + q = 0$

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$\frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$y = C_1 e^{-x} + C_2 e^{-3x}$$

$$\theta = -1, -3$$

$$(\theta + 1)(\theta + 3) =$$

$$(\theta^2 + 4\theta + 3)y = 0$$

C//.

Q113

$$\theta(\theta-1) + \theta - 4 = 0$$

$$\theta^2 - 4 = 0$$

$$\theta = \pm 2$$

$$y = C_1 e^{2t} + C_2 e^{-2t}$$

$$y = C_1 e^{2 \log x} + C_2 e^{-2 \log x}$$

$$0 = C_1 e^{2 \log 0} + C_2 e^{-2 \log 0}$$

$$1 = C_1$$

* First order first degree differential eqn

→ Linear form (Leibnitz linear form)

→ Variable Separable form

→ Homogeneous form

→ Exact form.

{ Note: for solving
nonlinear eqn
try for substitution
 $\frac{dy}{dx} = 1+y^2$
 $y = \tan \theta$ }

I Leibnitz linear form:-

$$\frac{dy}{dx} + Py = Q$$

P, Q are function of x.

(1) I.F. = $e^{\int P dx}$
Integrating factor

$$\boxed{I.F. \times Y = \int I.F. \times Q dx + C}$$

eg. $\frac{dy}{dx} + \frac{y}{x} = 0$

$$I.F. = e^{\int \frac{1}{x}} = e^{\log x} = x$$

$$e^{\log x} = x$$

$$y \times x = \int x \times 0 dx + C$$

$$yx = C$$

$$\boxed{y = \frac{C}{x}}$$

Hyperbola

(2) Variable Separable

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$\int g(y) dy = \int f(x) dx + C$$

eg $\frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -dx$$

$$\int \frac{dy}{y} = -\int dx + c$$

$$y = c_1 e^{-x}$$

or

$$(D+1)y=0$$

$$y = c e^{-x}$$

or

$$If = e^{\int dx}$$

$$= e^x$$

$$y e^x = c$$

$$y = c e^{-x} //$$

③ If Variable Separable is not possible. then Homogenous:

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

total power of each term = same..

let If Homogenous

$$\text{let } y = vx$$

{ V is function of x

$$\boxed{\frac{dy}{dx} = v + x \frac{dv}{dx}}$$

Can be separated

in result put $v = y/x$

eg. $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

$$y = vx$$

$$\frac{dy}{dx} = \frac{vx \cdot v}{x^2 + v^2 x^2}$$

$$v + x \frac{dv}{dx} = \frac{v x^2}{x^2 + v^2 x^2}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v - v - v^3}{1+v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

$$\Rightarrow \frac{1+v^2}{v^3} dv = -\frac{1}{x} dx$$

$$\Rightarrow \int \frac{1}{v^3} dv + \int \frac{1}{v} dv = -\int \frac{1}{x} dx$$

$$= \frac{-1}{2v^2} + \log v = -\log x + C \quad v = y/x$$

$$\Rightarrow \frac{-x^2}{2y^2} + \log(y/x) = -\log x + C$$

④ Exact form.

$$M(x, y) dx + N(x, y) dy = 0$$

Exact necessary condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

If exact soln

$$\int_{y=\text{const.}} M dx + \int (\text{term of } N \text{ not containing } x) dy = C$$

Q1. $\frac{dy}{dx} + 3y = e^{2x}$

sol) $\lambda + 3 = 0$

$\lambda = -3$

$C_1 e^{-3x}$

$\frac{e^{2x}}{\lambda + 3} = \frac{e^{2x}}{5}$

$\frac{e^{2x}}{5} + C_1 e^{-3x} //$

Q2. $(x+1) \frac{dy}{dx} - y = (x+1)^2 e^x$

sol) $\frac{dy}{dx} - \frac{1}{(x+1)} y = (x+1) e^x$

if = $e^{\int \frac{-1}{x+1} dx}$
 $= e^{\log(x+1)^{-1}}$
 $= \frac{1}{x+1}$

$\frac{1}{x+1} \cdot y = \int \frac{1}{x+1} (x+1)^2 e^x + C$

$\frac{y}{x+1} = e^x + C //$

Q3. If $\frac{dy}{dx} + 2y \tan x = \sin x$ $y(\pi/3) = 0$ $y_{\max}$

a) $1/2$ b) $1/4$ c) $1/8$ d) $1/16$

$e^{\int 2 \tan x} = e^{2 \int \tan x} = e^{2 \log \sec x}$

$\pm f = \sec^2 x$

$y \sec^2 x = \int \sec^2 x \sin x + C$

$y \sec^2 x = \int \tan x + C$

$y \sec^2 x = \log(\sec x) + C$

$0 = \log\left(\frac{1}{\cos \pi/3}\right) + C$

$0 = \log(2) + C$

$C =$

$C = -2$

$y = \frac{1}{\cos x \cdot \sec^2 x} + \frac{C}{\sec^2 x}$

$y = \cos x - 2 \cos^2 x //$

Max Value y_m

$\frac{dy}{dx} = -\sin x + 4 \sin x$

$1 + 2 \cos 2x$
 $- 2 \sin 2x$

$y_{\max} = 1/8$

Q4. $x^2 \frac{dy}{dx} + 2xy = \frac{2 \log x}{x}$; $y(1) = 0$ then $y(e)$ is

So $x \frac{dy}{dx} + 2y = \frac{2 \log x}{x^2}$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{2 \log x}{x^3}$

$e^{\int \frac{2}{x} dx} = e^{2 \log x}$
 $= x^2$

$x^2 y = \int x^2 \cdot \frac{2 \log x}{x^3} dx + C$

$x^2 y = 2 \int \frac{\log x}{x} dx + C$

$x^2 y = 2 \int t dt + C$

$x^2 y = 2 \frac{t^2}{2} + C$

$x^2 y = (\log x)^2 + C$ //

$\log x = t$
 $\frac{1}{x} = \frac{dt}{dx}$

$0 = C$

$x^2 y = (\log x)^2$

$y = \frac{1}{e^2}$ //

Q5 Solve O.D.E

$\frac{dy}{dx} + 2y = 0$, $y(1) = 5$

A) ~~0 + 2~~

$0 + 2 = 6$

$0 = -2$

$C_1 e^{-2x}$

$y = C_1 e^{-2x}$

$5 = C_1 e^{-2}$

$5 e^2 e^{-2x}$

$C_1 = 5 e^2 = 36.95 e^{-2x}$ //

Q6. Given diff. eqn is exact -

$$(xy^2 + \lambda x^2 y) dx + (x+y) x^2 dy = 0$$

find λ

Sol

$$\frac{dy}{dx} = - \frac{xy^2 + \lambda x^2 y}{x^3 + yx^2}$$

$$y = vx$$

$$\frac{d(vx)}{dx} = - \frac{xv^2 x^2 + \lambda x^2 vx}{x^3 + vx^3}$$

$$v + x \frac{dv}{dx} = - \frac{v^2 + \lambda xv}{1+v}$$

$$\Rightarrow \frac{x dv}{dx} =$$

$$\lambda = 3$$

$$\frac{d}{dx} (xy^2 + \lambda x^2 y) = \frac{d}{dx} (x+y)x^2$$

$$\int (y^2 x + \lambda x^2 y) dx + \int x^3 dy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial y} (xy^2 + \lambda x^2 y) = \frac{\partial}{\partial x} (x^3 + x^2 y)$$

$$2xy + \lambda x^2 = 3x^2 + 2xy$$

Q7. Given diff. eqn is exact

$$(ay^2 + x + x^3) dx + (y^3 - y + bxy) dy = 0$$

a) $a=b$ b) $a=2b$ c) $a=b/2$ d) $a=3b$

Q8. $(x^2 + y^2) dx + 2xy dy = 0$

Sol 1) $2ay = + by$

$$\frac{a}{b} = + \frac{1}{2}$$

$$a = b/2 //$$

Sol 2) $\frac{dy}{dx} =$

$$y = \int x^2 dx + \int 0$$

$$\frac{x^3}{3}$$

Q9. $(1+y^2)dx = (\tan^{-1}y - x)dy$

$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$

$\Rightarrow \tan^{-1}y \frac{dy}{dx} - x \frac{dy}{dx} = 1+y^2$

$\Rightarrow (1+y^2) \frac{dx}{dy} = \tan^{-1}y - x$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

$\Rightarrow e^{\int \frac{dy}{1+y^2}} \cdot x = e^{\frac{\tan^{-1}y}{1+y^2}} \cdot \int \frac{\tan^{-1}y}{1+y^2} dy + C$

$e^{\tan^{-1}y} \cdot x = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2} dy + C$

$\tan^{-1}y = t$
 $\frac{1}{1+y^2} dt = dy$

$e^{\tan^{-1}y} \cdot x = \int e^t \cdot t \cdot dt + C$

$e^{\tan^{-1}y} \cdot x = t e^t - e^t + C$

$x e^{\tan^{-1}y} = \tan^{-1}y e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$

Q8
82)

$$(x^2 + y^2) dx + 2xy dy = 0$$

$$\Rightarrow \boxed{2y = 2y}$$

$$\int_{y=\text{const}} (x^2 + y^2) dx + \int 0$$

$$> \frac{x^3}{3} + y^2 x + C //$$

Q9
sol

$$z = xy \ln(xy)$$

$$\frac{dx}{dt} + \frac{x}{t} = 1$$

$$e^{\int \log t}$$

$$t, x = \int t$$

$$x^2 = \frac{t}{2} =$$

Q8.

$$\frac{dy}{dx} = ky$$

$$y(0) = c$$

sol

$$0 - k$$

$$c_1 e^{kx}$$

$$c_1 = c$$

$$y = y_2 e^{1/kx} + y_1 e^{-1/kx} + y_2$$

Q13

sg

$$k^2 \frac{d^2 y}{dx^2} - y + y_2 = 0$$

$$k^2 \frac{d^2 y}{dx^2} = 1$$

$$> k^2 \frac{d^2 y}{dx^2} = 1$$

$$-\frac{y_2}{k^2}$$

$$(0 - \frac{1}{k^2})(0 + \frac{1}{k^2}) = y_2$$

$$k^2 \frac{d^2 y}{dx^2} + y_2 = 0$$

$$y^2 - \frac{1}{k^2} = -\frac{y_2}{k^2} e^{ax}$$

$$\pm \frac{1}{k}$$

$$C_1 e^{1/kx} + C_2 e^{-1/kx} + y_2$$

$$y_1 = C_1 + C_2$$

$$y_2 = C_1 \infty + C_2 \cdot 0$$

$$C_1 = y_2$$

$$C_2 = y_1 - y_2$$

* Partial Differentiation

$$Z = f(x, y)$$

Independent Variable

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial x} = p$$

$$\frac{\partial^2 Z}{\partial x^2} = r$$

$$\frac{\partial Z}{\partial y} = q$$

$$\frac{\partial^2 Z}{\partial x \partial y} = s = \frac{\partial^2 Z}{\partial y \partial x}$$

$$\frac{\partial^2 Z}{\partial y^2} = t$$

eg ①

$$Z = 6xy^2 - 2xy + y^3$$

$$\frac{\partial Z}{\partial x} = 6y^2 - 2y$$

$$\frac{\partial Z}{\partial y} = 12xy - 2x + 3y^2$$

$$\frac{\partial^2 Z}{\partial x \partial y} = 12y - 2$$

$$\frac{\partial^2 Z}{\partial y \partial x} = 12y - 2$$

$$t = \frac{\partial^2 Z}{\partial y^2} = 6y$$

$$r = \frac{\partial^2 Z}{\partial x^2} = 0$$

② $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

The value of $x \frac{\partial y}{\partial x} + y \frac{\partial y}{\partial y} = u$

a) b)

$$\frac{1}{x^2 + y^2}$$

c)

$$\frac{1}{x^2 - y^2}$$

d)

$$\frac{1}{\sqrt{x^2 + y^2}}$$

So

$$\frac{\partial u}{\partial x} =$$

$$\frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} =$$

$$\frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right)$$

$$u = x \left[\frac{1}{\sqrt{1+x^2}} - \frac{1}{(x^2+y^2)} \right] + \left[\frac{-x}{\sqrt{y^2+y^2}} \right] + \frac{x}{x^2+y^2}$$

$$u = 0$$

Q. $u = e^{xyz}$
find $\frac{\partial^3 u}{\partial x \partial y \partial z}$ at $(1,1,1)$

sol) $\frac{\partial u}{\partial x} = e^x \cdot yz$

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot yz$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^x \cdot z$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{xyz} \cdot z + yz e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + e^{xyz} \cdot xy + ye^{xyz} + yze^{xyz} \cdot xy$$

Ans 5e

$$= e^{xyz} (1 + xy + y + y^2 xz)$$

Q. $z = xy \ln(xy)$

sol) $\frac{\partial z}{\partial x} = y \left[\ln(xy) + \frac{1}{xy} \cdot y \cdot x \right]$

$$\frac{\partial z}{\partial x} = y [\ln(xy) + 1]$$

$$x \frac{\partial z}{\partial x} = y \frac{\partial z}{\partial y}$$

- * Jacobian: gives the functional Relationship between two independent Variables or functions, which depends on same independent Variables with the Variables.
- Jacobian is only valid for minimum two independent Variables.
 - Number of functions and independent Variables should be same.

for

$$u = f(x, y)$$

$$v = g(x, y)$$

Jacobian is defined by

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$(i) \quad \frac{\partial(u, v)}{\partial(x, y)} = J$$

$$(ii) \quad \frac{\partial(x, y)}{\partial(u, v)} = J^{-1}$$

$$(iii) \quad J \cdot J^{-1} = 1$$

$$(iv) \quad \frac{\partial(u, v)}{\partial(x, t)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(x, t)}$$

eg $x = r \cos \theta, \quad y = r \sin \theta$

find $\frac{\partial(x, y)}{\partial(r, \theta)}$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$\frac{\partial r}{\partial x} = \frac{1/2}{\sqrt{x^2 + y^2}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2 + y^2}}$$

length

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{vmatrix}$$

$$= \frac{1}{r}$$

Q.392

eg

sd

$$10x_2 \sin x_1 - 0.6 = 0$$

$$10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$$

$$\frac{\partial u_1}{\partial x_1} = 10x_2 \cos x_1$$

$$\frac{\partial u_1}{\partial x_2} = 10 \sin x_1$$

$$\frac{\partial u_2}{\partial x_1} = 10 \sin x_1$$

$$\frac{\partial u_2}{\partial x_2} = 20x_2 - 10 \cos x_1$$

$$J = \begin{bmatrix} 10 & 0 \\ 0 & 20 - 10 \end{bmatrix}$$

B//.

* Homogenous function and Euler's theorem

Homogenous function: A function in which all the terms are of same power or degree.

$$f(x, y) = a_0 x^n y^0 + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_{n-2} x^2 y^{n-2} + a_{n-1} x y^{n-1} + a_n y^n x^0.$$

all term with power (n)

$$f(x, y) = x^n \left[a_0 \left(\frac{y}{x} \right)^0 + a_1 \left(\frac{y}{x} \right)^1 + a_2 \left(\frac{y}{x} \right)^2 + \dots \right]$$

$$\boxed{f(x, y) = x^n \phi(y/x)}$$

(I) $u = \frac{x^3 + xy^2}{\sqrt{x} + \sqrt{y}}$

(II) (n can be any real value)

sol/ $u = \frac{x^3}{\sqrt{x}} \left[\frac{1 + y^2/x^2}{1 + \sqrt{y/x}} \right]$
 $= x^{5/2} \phi(y/x)$

Homogen

(2) $u = x/y^2$

$$= \frac{1}{x} \frac{x^2}{y^2}$$

$$= x^{-1} \left(\frac{y}{x} \right)^{-2} \quad \text{Homogen}$$

(3) $u = \frac{x^2 - y^2}{xy + x^2} = x^0 \phi(y/x)$

$$(iv) \quad u = \log \left(\frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} \right)$$

$$(v) \quad u = \sin^{-1} \left(\frac{x^3 - y^3}{x - y} \right)$$

m 4, 5 u is not Homogenous but in this case

$f(u)$ is Homogenous

$$iv) \quad e^u = \frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} = \frac{x^2}{x^{1/2}} \left[\frac{1 + (y/x)^2}{1 + (y/x)^{1/2}} \right] = x^{3/2} \left[\dots \right]$$

$$v) \quad \sin u = \frac{x^3 - y^3}{x - y}$$

$$= \frac{x^2 (1 - (y/x)^3)}{1 - y/x}$$

$f(u)$ is Homogenous

{ Numerator & denominator Same power }

* Euler's Theorem:— It is only Valid for Homogenous function.

$f(x, y) = x^n \phi(y/x)$ is a homogenous function then a/q to Euler's theorem.

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu}$$

• If u is not homogenous but $f(u)$ is Homogenous then 1st degree Euler's theorem

$$\text{1st degree} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{u f'(u)}{f(u)}$$

2nd order Euler theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u) [g'(u) - 1]$$

$$g(u) = \frac{x f(u)}{f'(u)}$$

Q1/ $u = \tan^{-1} \left(\frac{x^2 + y^2}{\sqrt{x} + \sqrt{y}} \right) \Rightarrow \tan u = x^{3/2}$

$$\frac{3}{2} \cdot \frac{\tan u}{\sec^2 u} //$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$$

Q2/ $u = \ln \left[\frac{x^4 + y^4}{x + y} \right]$

$$\frac{3u}{e^u} = 3$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$$

Q3/ $u = \tan^{-1} (y^2/x)$ then 2nd order Euler theorem

A) $-\sin 2u \cdot \sin 2u$

C) $\sin^2 2u \cdot \sin u$

B) $\sin 2u \cdot \sin 2u$

D) $-\sin^2 2u \cdot \sin u$

Sol $\tan u = x \cdot \frac{y^2}{x^2} \quad n=1$

$$\tan u = \frac{y^2}{x}$$

$$g(u) = \frac{\tan u}{\sec^2 u} = \frac{\sin}{\cos} \cos^2 u$$

$$\frac{1}{2} \sin 2u (\cos 2u - 1) > \frac{1}{2} \sin 4u$$

$$= \frac{1}{2} \sin u \cos u$$

$$= \frac{1}{2} \sin 2u$$

$$g(u) [g(u) - 1]$$

$$\Rightarrow \frac{1}{2} \sin 4u \left[\frac{1}{2} \cdot 2 \cos 2u - 1 \right] \Rightarrow \frac{1}{2} \sin 4u [1 - 2 \sin^2 2u - 1]$$

$$\Rightarrow \frac{1}{2} \sin 4u \cos 2u - \frac{1}{2} \sin 2u \Big| = -\sin^2 2u \sin 2u \quad // \quad A) \quad //$$

* LINEAR & Non linear Differential eqⁿ

1. A differential eqⁿ is said to be linear if the unknown function and its derivative appears to power 1.
2. The product of unknown function and its derivative are not allowed for linear differential eqⁿ
3. The coefficient of unknown function and its derivative ~~are~~ are allowed to be known or independent function

① $\frac{d^2 y}{dx^2} + 2y \frac{dy}{dx} = e^x$ nonlinear
nonHomo. order 2, deg 1

② $\frac{d^2 \eta}{d\theta^2} + \frac{\eta d\eta}{2d\theta} = 3\theta^2$ NL NH ord 2

③ $y'' + (y')^2 = 0$ NLH ord 2

④ $y^2 \frac{d^2 y}{dx^2} + x^2 = 0$ NLH ord 2

⑤ $\frac{dy}{dx} + xy = e^{-x}$ Linear NH ord 1

⑥ $\frac{dy}{dx} + xy = e^{-y}$ NLNH

⑦ $1 + y^2 = \frac{dy}{dx}$ NL

⑧ $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$ NL

⑨ $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = e^x$ Linear

⑩ $\frac{d^2 y}{dx^2} + xy = \sin y$ NL