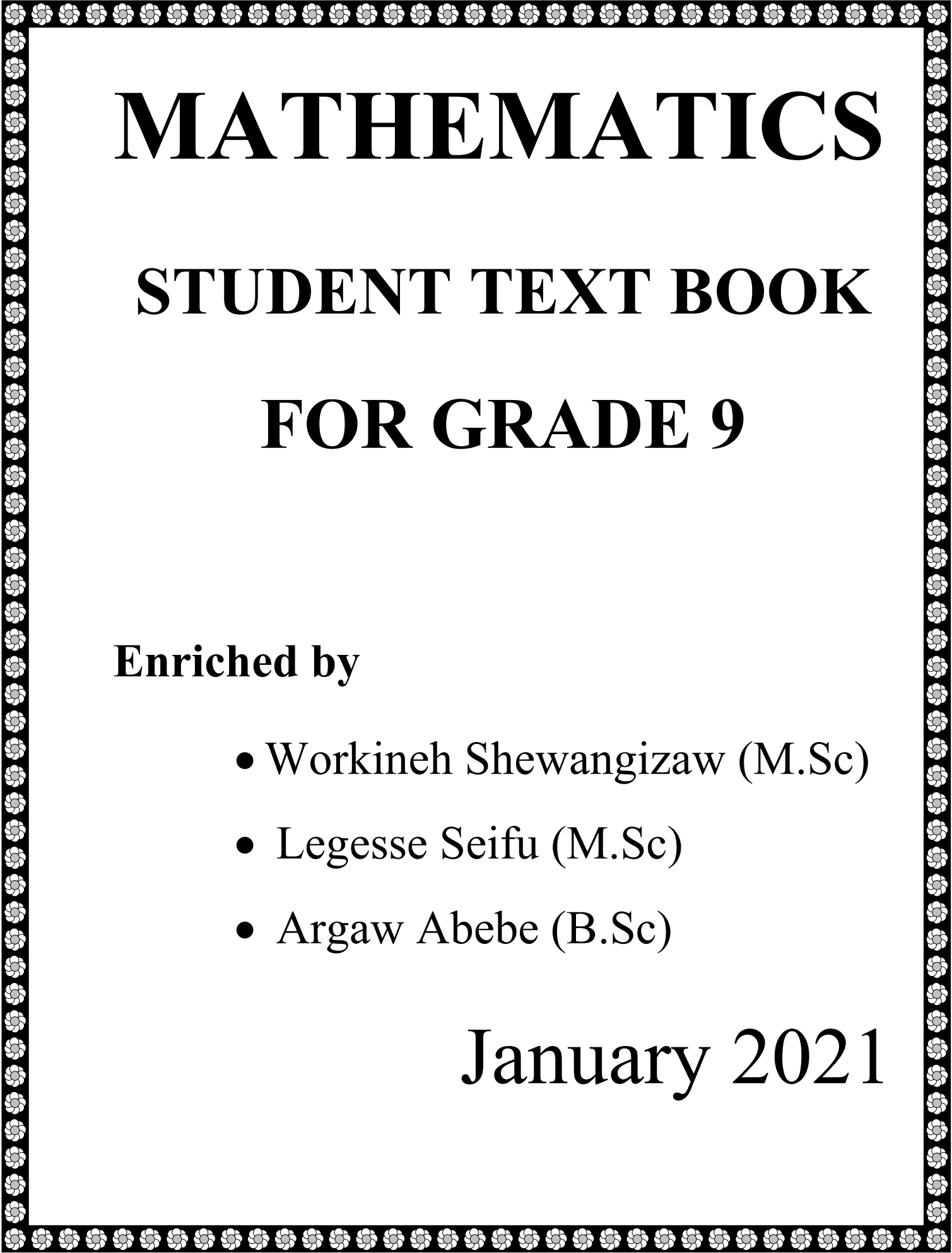




MATHEMATICS

STUDENT TEXT BOOK

FOR GRADE 9

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Unit 1

THE NUMBER SYSTEM

	Numerical Value							
Arabic Numeral	1	2	3	5	10	20	21	100
Babylonian	▼	▼▼	▼▼▼	▼▼▼▼▼	<	<<	<<▼	▼►
Egyptian Hieroglyphic	I	II	III	IIII	Λ	ΛΛ	IΛΛ	ϑ
Greek Herodianic	I	II	III	Γ	Δ	ΔΔ	ΔΔI	H
Roman	I	II	III	V	X	XX	XXI	C
Ethiopian Geez	፩	፪	፫	፬	፭	፮	፯	፰

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Unit Outcomes

After completing this unit, students will be able to;

- *know basic concepts and important facts about real numbers.*
- *justify methods and procedures in computation with real numbers.*
- *solve mathematical problems involving real numbers.*

Main contents covered in this unit:

- 1.1. Revision on the set of rational numbers
- 1.2. The real number system

1.1 REVISION ON THE SET OF RATIONAL NUMBERS

Competencies

At the end of this subunit, the students will be able to:

- *identify natural numbers, whole numbers and integers.*
- *define prime numbers and composite numbers.*
- *determine common factors and common multiples of pairs of numbers.*

ACTIVITY 1.1

1. Complete each of the following blank boxes with appropriate set of numbers listed below so that a set in the bottom is the subset of the set in the upper box.

- | | |
|---|-----------------------------|
| a. Set of rational numbers | g. Set of integers |
| b. Set of non-integer rational numbers | h. Set of whole numbers |
| c. Set of natural numbers | i. Set of prime numbers |
| d. Set of negative integers | j. Set of composite numbers |
| e. $\left\{\frac{1}{2}, -\frac{3}{5}, -\frac{11}{7}, 0.901, 4\frac{5}{9}, 12.\overline{05}\right\}$ | k. $\{0\}$ |
| f. $\{1\}$ | |

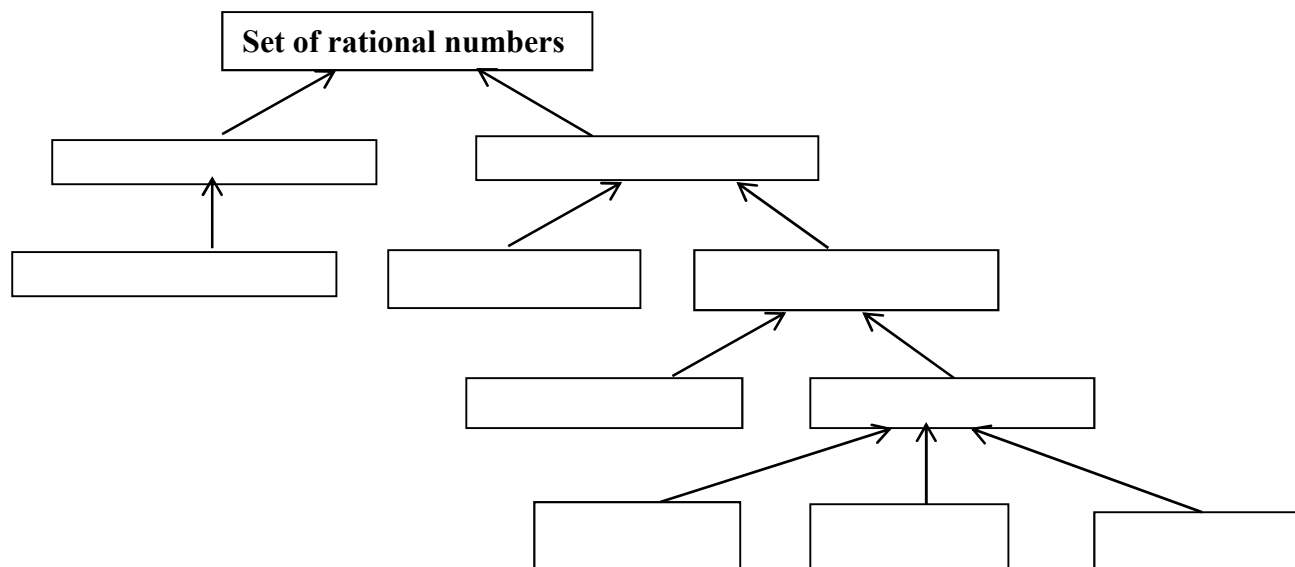


Figure 1.1

2. Define the set of:
 - a. Natural numbers
 - b. Rational numbers
 - c. Whole numbers
 - d. Integers
 - e. Prime numbers
 - f. Composite numbers
3. Starting from the bottom and moving upward, read the diagram in figure 1.1 for your group in the class in the form “..... is the subset of”.

The diagram in figure 1.1 shows the relationships between the sets of Natural numbers, Whole numbers, Integers and Rational numbers.

1.1.1 Natural Numbers, Whole Numbers, Integers, Prime Numbers and Composite Numbers

In this subsection, you will revise important facts about the sets of natural numbers, prime numbers, composite numbers and integers. You have learnt several facts about these sets in previous grades.

Natural Numbers

The set natural numbers, denoted by the symbol $\mathbb{N}$, is described by $\mathbb{N} = \{1, 2, 3, \dots\}$

Natural numbers are also known as **Counting numbers** or **Positive integers**.

Whole Numbers

If we add the number zero to the set of natural numbers, we have another set of numbers which we call the **set of whole numbers**.

The set of whole numbers, denoted by the symbol $\mathbb{W}$, is described by

$$\mathbb{W} = \{0, 1, 2, 3, \dots\}$$

Note : The set of natural numbers and the set of whole numbers are closed under addition and multiplication. That is

- i. For any natural numbers x and y ; $x + y$ and $x \cdot y$ are also natural numbers.
- ii. For any whole numbers x and y ; $x + y$ and $x \cdot y$ are also whole numbers.

Integers

As human experience and knowledge increases, they found that the set of whole numbers and the set of whole numbers are not adequate to meet their need. For instance, they found that these two sets are not able to solve equations of the type $x + 5 = 1$. Thus, to avoid such limitation they extend the existing sets to **the set of integers**.

The set of integers, denoted by the symbol $\mathbb{Z}$, described by

$$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- Those integers which are greater than zero are called **positive integers** and those which are less than zero are called **negative integers**.

Zero is an integer which is neither positive nor negative integer.

Note: The set of integers is closed under addition, subtraction and multiplication.

That is, if $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, then $x + y \in \mathbb{Z}$, $x - y \in \mathbb{Z}$ and $x \cdot y \in \mathbb{Z}$.

Factors and Multiples of Numbers

If you consider the product of two numbers, say 8 and 6, you will get $8 \times 6 = 48$. Then we say that 6 and 8 are both factors of 48. The product 48 is said to be a multiple of both 6 and 8.

More generally, given two natural numbers m and p , then m is said to be a **MULTIPLE** of p if there is a natural number q such that $m = p \times q$.

In this case we say that p is called a **FACTOR** or **DIVISOR** of m .

Similarly, if such q exists, q is also a **factor** or **divisor** of m , and m is a **multiple** of q .

Note that if m is **MULTIPLE of** p , then we also say that ' m is **DIVISIBLE by** p '.

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As seen above, 6 and 8 are both the factors of 48.

- i. What are the other factors of 48?
- ii. Do you agree that 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48 are all factors of 48?

Prime numbers

We see that some natural numbers have only two factors, the number itself and 1, while some others have more than two distinct factors. These classifications of natural numbers are defined as follow.

DEFINITION 1.1. (Prime numbers and composite numbers)

1. A natural number that has exactly two distinct factors, 1 and the number itself, is called a **PRIME NUMBER**.
2. A natural number that has more than two factors is called a **COMPOSITE NUMBER**.

For example,

- The set of factors of 2 is $\{1, 2\}$. Hence, 2 is prime number.
- The set of factors of 3 is $\{1, 3\}$. Hence, 3 is prime number.
- The set of factors of 4 is $\{1, 2, 4\}$. Hence, 4 is composite number.
- The set of factors of 5 is $\{1, 5\}$. Hence, 5 is prime number.
- The set of factors of 8 is $\{1, 2, 4, 8\}$. Hence, 8 is composite number.
- The set of factor of 1 is $\{1\}$. Is 1 prime number? Is 1 composite number?

Note: 1 is neither prime nor composite.

ADDITIONAL REMARK

THE SIEVE OF ERATOSTHENES

A method of finding the prime numbers less than any given number is first suggested by a Greek Mathematician Eratosthenes, 2nd Century. This method consists of sifting out those numbers which are not prime and it is called '**The Sieve of Eratosthenes**'. One way of describing the sieve is as follow.

1. Write all the natural numbers from 2 to the given number.
2. Cross out all the numbers that have 2 as a factor, except 2 itself.
3. Cross out all the remaining numbers that have 3 as a factor, except 3 itself.

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4. Take the next number which is not cross out, possibly 5, and cross out all the remaining numbers that have 5 as a factor, except 5 itself, and so on.

After this process is completed, the numbers remaining in the list are all prime.

Example 1

Find all prime numbers less than 30.

Solution

- First generate the list of integers from 2 to 30. (see list 1 below)
- The first number is 2. Cross out all multiples of 2, except 2 itself. (see list 2 below)
- The next number after 2 is 3. Cross out all multiples of 3, except 3 itself. (see list 3 below)
- The next number which is not crossed out after 3 is 5. Cross out all multiples of 5, except 5 itself. (see list 4 below)
- The next number which is not crossed out after 5 is 7. Cross out all multiples of 7, except 7 itself. These numbers are 14, 21, 28. But they are all crossed out already. Lastly, the numbers that are not crossed out at this point are all prime numbers less than 30. (see list 5 below)
- Thus, the set of prime numbers less than 30 is $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$.

List 1	List 2	List 3
2 3 4 5 6 7 8 9	2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9	2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9
10 11 12 13 14 15	~~10~~ 11 ~~12~~ 13 ~~14~~ 15	~~10~~ 11 ~~12~~ 13 ~~14~~ 15
16 17 18 19 20 21	~~16~~ 17 ~~18~~ 19 ~~20~~ 21	~~16~~ 17 ~~18~~ 19 ~~20~~ 21
22 23 24 25 26 27	~~22~~ 23 ~~24~~ 25 ~~26~~ 27	~~22~~ 23 ~~24~~ 25 ~~26~~ 27
28 29 30	~~28~~ 29 ~~30~~	~~28~~ 29 ~~30~~

List 4	List 5
2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9	2 3 7
~~10~~ 11 ~~12~~ 13 ~~14~~ 15	11 13
~~16~~ 17 ~~18~~ 19 ~~20~~ 21	17 19
~~22~~ 23 ~~24~~ ~~25~~ ~~26~~ 27	23
~~28~~ 29 ~~30~~	29

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Divisibility Test

The following rules can help you to determine whether a number is divisible by 2, 3, 4, 5, 6, 8, 9 or 10.

- 1) A number is divisible by 2, if its unit digit is divisible by 2.

i.e. the ones digit is either 0, 2, 4, 6 or 8.

For example; 92,656 is divisible by 2, since the last digit is 6 (even).

- 2) A number is divisible by 3 if the sum of the digits is divisible by 3.

For example; 954 is divisible by 3 since $9 + 5 + 4 = 18$ which is divisible by 3.

- 3) A number is divisible by 4 if the number formed by the last two digits is divisible by 4.

For example; 5,632 is divisible by 4, since 32 is divisible by 4.

- 4) A number is divisible by 5 if its unit digit is either 0 or 5.

For example; 705 is divisible by 5, since its last digit is 5.

12,370 is divisible by 5, since its last digit is 0.

- 5) A number is divisible by 6 if its divisible by 2 and 3

For example; 954 is divisible by 6, since it is divisible by both 2 and 3.

- 6) A number is divisible by 8 if the number formed by its last three digits is divisible by 8.

For example; 34,848 divisible by 8, since 848 is divisible by 8.

- 7) A number is divisible by 9 if the sum of the digits is divisible by 9.

For example; 954 is divisible by 9, since $9 + 5 + 4 = 18$ is divisible by 9.

- 8) A number is divisible by 10 if its unit digit is 0.

For example; 480 is divisible by 10, because its unit digit is 0.

Example 2: 12,468 is divisible by

- a) 2, because its last digit 8 is divisible by 2.
- b) 3, because the sum of all its digits i.e $1 + 2 + 4 + 6 + 8 = 21$ is divisible by 3.
- c) 4, because the number formed by the last two digit i.e 68 is divisible by 4.
- d) 6, because it is divisible by both 2 and 3.

Example 3: 12,468 is not divisible by

- a) 5, because its units digit is neither 0 nor 5.
- b) 8, because the number formed by the last three digits i.e 468 is not divisible by 8.
- c) 9, because the sum of all its digits i.e $1 + 2 + 4 + 6 + 8 = 21$ is not divisible by 9.

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d) 10, because the unit digit is not 0.

ADDITIONAL REMARK

Test for divisibility by 7

To check whether a given natural number is divisible by 7, we may use the following procedure.

- ✓ Double the last digit and subtract the result from the number resulted after truncated the last digit.
- ✓ Repeat the process for larger numbers until the result is a 2-digit number divisible by 7.
- ✓ If the resulting difference is divisible by 7, then so is the original number.
- ✓ Apply this rule over and over again when necessary.

Example 4: 133 is divisible by 7 since $13 - 2 \times 3 = 13 - 6 = 7$ is divisible by 7.

Application of Divisibility Test

We can use Divisibility Tests to obtain all factors of a composite number.

Example 5. Find all factors of 48 using divisibility test.

Solution : $1 \times 48 = 48 = 48 \times 1$. Hence, 1 and 48 divide 48.

$2 \times 24 = 48 = 24 \times 2$. Hence, 2 and 24 divide 48.

$3 \times 16 = 48 = 16 \times 3$. Hence, 3 and 16 divide 48.

$4 \times 12 = 48 = 12 \times 4$. Hence, 4 and 12 divide 48.

$6 \times 8 = 48 = 8 \times 6$. Hence, 6 and 8 divide 48.

Therefore, 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48 are factors of 48. Thus, 48 has ten factors.

The prime factors of 48 are only 2 and 3.

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More Divisibility Tests

Divisible by:	Test	Example
11	1. Alternately subtract then add the digits from L to R; result divisible by 11 2. Subtract last digit from the rest	918,082: 1. $9 - 1 + 8 - 0 + 8 - 2 = 22$ $22 = 2 \times \mathbf{11}$. 2. 627: $62 - 7 = 55$ $55 = 5 \times \mathbf{11}$
12	Divisible by 3 and 4	324: $3 + 2 + 4 = 9$, divisible by 3 24 divisible by 4
13	Iteratively add 4 times the last digit to the rest	637: $63 + (7 \times 4) = 91$, $9 + (1 \times 4) = \mathbf{13}$
14	Divisible by 2 and 7	
15	Divisible by 3 and 5	
16	If the thousands digit is even, take the last three digits If the thousands digit is odd, add 8 to the last three digits With the 3-digit #, multiply hundreds digit by 4, then add the last two digits.	254,176: Thousands digit = 4, so take 176 $(1 \times 4) + 76 = 80$ $80 = 5 \times \mathbf{16}$ 693,408: Thousands digit = 3, $408 + 8 = 416$ $(4 \times 4) + 16 = 32$ $32 = 2 \times \mathbf{16}$
17	Subtract 5 times the last digit from the rest.	221: $22 - (1 \times 5) = \mathbf{17}$
18	Divisible by 2 and 9	
19	Add twice the last digit to the rest	437: $43 + (7 \times 2) = 57$ $5 + (7 \times 2) = \mathbf{19}$
20	Divisible by 10 and the tens digit is even	360: last digit = 0 and 6 is even
25	Last 2 digits = 25, 50 or 75	
50	Last 2 digits = 50 or 00	

Prime Factorization

A natural number may be written as a product of numbers in different forms.

For instance, the natural number 48 can be written in the form of product as follow.

$$48 = 1 \times 48$$

$$48 = 2 \times 24$$

$$48 = 2 \times 4 \times 6$$

$$48 = 3 \times 4 \times 4$$

$$48 = 2 \times 3 \times 8$$

$$48 = 3 \times 16$$

$$48 = 2 \times 2 \times 2 \times 3$$

Above, the only product whose all factors are prime is the last one $48 = 2 \times 2 \times 2 \times 3$. We call this product the **Prime Factorization** of 48. The numbers 2 and 3 are the prime factors of 48.

No other prime factors of 48 exist.

In general, when a natural number is expressed as a product of prime factors, then we say that we have found the **PRIME FACTORIZATION** of that number.

For instance, the prime factorization of 3825 is

$$3825 = 3 \times 3 \times 5 \times 5 \times 17 = 3^2 \times 5^2 \times 17$$

Thus, the set of prime factors of 3825 is $\{3, 5, 17\}$

Note: No matter how we carry out the factorization, every composite number can be factorized as the product of prime factors in exactly one way, except for the order of the factors.

FUNDAMENTAL THEOREM OF ARITHMETIC : The prime factorization of any composite number is unique, except for the order of the factors.

Methods of Prime Factorization

The prime factorization of a composite number can be determined using different familiar techniques. Some of these are listed and illustrated as follow.

1. Repeated division starting with the smallest prime

- First divide the number by its smallest prime factor.
- If the resulting quotient is a composite number, find its smallest prime factor of the quotient and divide it by this prime factor in the same manner.
- Repeated division starting with the smallest prime that
- Continue the same process until the quotient is prime.

For example, to find the prime factors of 168, we proceed as follows.

The smallest prime factor of 168 is 2.

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$$168 \div 2 = 84$$

$$84 \div 2 = 42$$

$$42 \div 2 = 21$$

$$21 \div 3 = 7$$

Now the quotient is 7 which is prime. Then the product of all the prime factors and the last prime quotient so obtained is the prime factorization of 168, which is given by

$$168 = 2 \times 2 \times 2 \times 3 \times 7 = 2^3 \times 3 \times 7$$

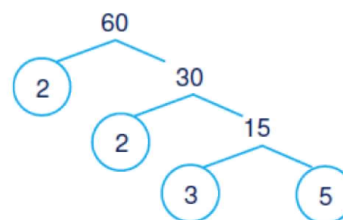
Thus, 168 has only three prime factors : 2, 3 and 7.

2. Using Factoring Trees

- You are familiar with factorizing natural numbers using Factoring-Tree.
- For example, Factoring-Tree is employed to find the prime factorization of 60 as follow.

From the factoring tree, the prime Factorization of 60 is

$$60 = 2^2 \times 3 \times 5$$



3. Using Factorization Table

You may use a two-column table to factorize a composite number into its prime factors. The method is done in the same manner as the first method, as illustrated bellow.

For example, let us illustrate the method to find the prime factorization of 6615. The smallest prime factor of 6615 is 3.

Divide 6615 by 3 gives quotient 2205 (not prime)

Repeat the process until you obtain prime quotient, and label the factors and the quotients in the table as shown in the table.

The last quotient is 7, which is prime. The process is complete.

Thus, the prime factorization of 6615 is

$$6615 = 3^3 \times 5 \times 7^2$$

From its prime factorization, 6615 has exactly 3 prime factors: 3, 5 and 7.

How many factors does 6615 have?

6615	3
2205	3
735	3
245	5
49	7
7	7
1	1

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Additional Remark

You may use prime factorization to find the total number of factors of a composite number.

Example 6. Find all factors of 54.

Solution. The prime factorization of 72 is $72 = 2^3 \times 3^2$

Hence all factors of 72 consist of atmost three 2's and two 3's. Thus, the possible factors of 72 are obtained by multiplying powers of the form 2^n and 3^m , where $n = 0, 1, 2$ or 3 and $m = 0, 1$ or

2. Then the possible combinations are

$2^0 \times 3^0 = 1$	$2^1 \times 3^0 = 2$	$2^2 \times 3^0 = 4$	$2^3 \times 3^0 = 8$
$2^0 \times 3^1 = 3$	$2^1 \times 3^1 = 6$	$2^2 \times 3^1 = 12$	$2^3 \times 3^1 = 24$
$2^0 \times 3^2 = 9$	$2^1 \times 3^2 = 18$	$2^2 \times 3^2 = 36$	$2^3 \times 3^2 = 72$

Therefore, 72 has $4 \times 3 = (3 + 1) \times (2 + 1) = 12$ factors. These are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72.

We summarize the above method as follow.

Suppose the natural number n is expressed as a product of distinct prime factors

$p_1, p_2, \dots, p_m$ with their respective exponents, $k_1, k_2, \dots, k_m$, as

$$n = p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \times \dots \times p_m^{k_m},$$

then the number of factors of n is the product $(k_1 + 1)(k_2 + 1)(k_3 + 1) \dots (k_m + 1)$.

Thus, the number of prime factors of $2^5 \times 3^2 \times 5 \times 7^3$ is $(5+1)(2+1)(1+1)(3+1) = 144$.

Examples 7

- Which of the following natural numbers are prime, composite or neither? Give reason.
a) 13 b) 27 c) 89
- a) List all factors of 60. How many factors did you find?
b) List all prime factors of 60.
- Find the prime factorization of 792.

Solution

- 1) a) $1 \times 13 = 13 = 13 \times 1$.

There is no any other pair of natural numbers whose product is 13. That is the only positive factors of 13 are 1 and 13. Hence, 13 is prime number.

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b) $1 \times 27 = 27$, and $3 \times 9 = 27$.

1, 3, 9 and 27 are the positive factors of 27.

That is 27 has more than three positive factors. Hence, 27 is **NOT** prime number. 27 is composite number.

c) The positive factors of 89 are 1 and itself 89, no other factor. Hence, 89 is prime number.

2) a) $1 \times 60 = 60 = 60 \times 1$. Hence, 1 and 60 divides 60.

$2 \times 30 = 60 = 30 \times 2$. Hence, 2 and 30 divides 60.

$3 \times 20 = 60 = 20 \times 3$. Hence, 3 and 20 divides 60.

In a similar manner, $4 \times 15 = 5 \times 12 = 6 \times 10 = 60$.

Therefore, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60 are factors of 60. Thus, 60 has 12 factors.

b) 2, 3, and 5 are the only prime factors of 60.

3) Recall that the prime factorization of a natural number can be obtained using different method.

$$792 = 2 \times 396$$

$$= 2 \times 2 \times 198$$

$$= 2 \times 2 \times 2 \times 99$$

$$= 2 \times 2 \times 2 \times 3 \times 33$$

$$= 2 \times 2 \times 2 \times 3 \times 3 \times 11$$

(OR)

792	2
396	2
198	2
99	3
33	3
11	11
1	1

In the last product we obtain prime factors 2, 3 and 11.

Therefore, the prime factorization of 792 is

$$792 = 2^3 \times 3^2 \times 11$$

DEFINITION 1.2. A pair of prime numbers that differ by two are called **TWIN PRIMES**.

Example 8

- a) The pairs of prime numbers 3 and 5, 5 and 7, 29 and 31, 41 and 43, and 71 and 73 are twin prime numbers. Because each number in the pair is prime, and they differ by 2.
- b) 19 and 23 are consecutive prime numbers but $23 - 19 \neq 2$. Hence, the pair of prime numbers 19 and 23 are **NOT** twin primes.
- c) 13 and 15 differ by 2, but 15 is not prime.

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Exercise 1.1

1. For each of the following statements write TRUE if the statement is correct or FALSE otherwise. if your answer is false justify by giving a counter example or reason.
 - a. The set $\{1, 2, 3, \dots\}$ describes the set of natural numbers.
 - b. The set $\{1, 2, 3, \dots\} \cup \{\dots, -3, -2, -1\}$ describes the set of integers.
 - c. 57 is a composite number.
 - d. $\{1\} \cap \{\text{prim numbers}\} = \emptyset$
 - e. $\{\text{prime numbers}\} \cup \{\text{prime numbers}\} = \{1, 2, 3, \dots\}$
 - f. $\{\text{odd numbers}\} \cap \{\text{composite numbers}\} \neq \emptyset$
 - g. 5 is a factor of 72.
 - h. 48 is a multiple of 12.
 - i. 621 is divisible by 3.
 - j. $\{\text{factor of } 24\} \cap \{\text{factor of } 87\} = \{1, 2, 3\}$
 - k. $\{\text{multiple of } 6\} \cap \{\text{multiple of } 4\} = \{12, 24\}$
 - l. $2^2 \times 3^2 \times 5$ is the prime factorization of 180
2. Given two natural numbers a and b what is meant by
 - a. a is a factor of b
 - b. a is divisible by b
 - c. a is a multiple of b
3. List all prime natural numbers less than 100.
4. Find the largest natural number less than 300.
5. Find the prime factorization of each of the following number
 - a) 324 b) 432 c) 117 d) 360 e) 1530
6. List all pairs of twin primes less than 200.
7. Use the divisibility test to determine whether 18,832 is divisible 2, 3, 4, 5, 6, 7, 8, 9 and 10.

1.1.2 Common Factors and Common Multiples

1.1.2A. Common Factors and the Greatest Common Factor

Given two or more natural numbers, a number which is a factor of all of them is called a **Common Factor**.

Numbers may have more than one common factor.

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Notation: In this teaching material, the set of factors of a natural number n may be denoted by F_n .

Example 9: Find the common factors of 18 and 30.

Solution: The set of factors of 18 and 30 are

$$F_{18} = \{1, 2, 3, 6, 9, 18\} \text{ and}$$

$$F_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

Therefore, the set of common factors of 18 and 30 = $\{1, 2, 3, 6\}$.

Of these common factors, the largest is 6, which is known as the **Greatest Common Factor** 18 and 30.

DEFINITION 1.3. The **GREATEST or HIGHEST COMMON FACTOR (GCF or HCF)** of two or more natural numbers is the greatest natural number that is a factor of all of the given numbers.

For instance, as shown in the above example, the GCF of 18 and 30 is 6.

Notation: For $a, b \in \mathbb{N}$, the GCF or HCF of a and b is denoted by **GCF(a, b)** or **HCF(a, b)**.

Note: Two or more natural numbers that have a G.C.F of 1 are called **RELATIVELY PRIME**.

Example 10: Find the GCF of

a) 30 and 45

b) 36, 42 and 48

c) 50 and 69

Solution

a) First find the common factors of 30 and 45.

$$F_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\} \text{ and } F_{45} = \{1, 3, 5, 9, 15, 45\}.$$

$$\text{Then the set of common factors of 30 and 45 is } F_{30} \cap F_{45} = \{1, 3, 5, 15\}$$

$$\text{Therefore, GCF} = 15$$

b) The factors of 36, 42 and 48 are

$$F_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

$$F_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\} \text{ and } F_{48} = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}.$$

$$\text{Then the set of all common factors of 36, 42 and 48} = \{1, 2, 3, 6\}$$

$$\text{Thus, the greatest common factor of the three numbers is } \text{GCF}(36, 42, 48) = 6$$

c) $F_{50} = \{1, 2, 5, 10, 25, 50\}$ and $F_{69} = \{1, 3, 23, 69\}$

$$\text{Thus, the only common factor of 50 and 69 is 1. Therefore, } \text{GCF}(50, 69) = 1.$$

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That is 50 and 69 are relatively prime numbers.

In the previous examples, we use **listing the factors method** to find GCF of two or more natural numbers. Another simpler but most applicable method to find GCF is **Prime Factorization Method**. To apply this method we may use the following steps.

Steps to use Prime Factorization Method

Step 1: Write the **prime factorization** of each number.

Step 2: Identify all of the **common prime** factors.

Step 3: The **product of the common prime factors** with the smallest respective exponents in any of the numbers is the GCF.

Note: If the numbers have no common prime factors, then the GCF of the numbers is taken to be 1, and the numbers are said to be **RELATIVELY PRIME**.

Example 11: Using prime factorization method, find the G.C.F of the following numbers.

a) 120 and 180

b) 60, 72 and 96

c) 28 and 135

Solution

a) The prime factorizations of 120 and 180 are

$$120 = 2^3 \times 3 \times 5 \quad \text{and} \quad 180 = 2^2 \times 3^2 \times 5$$

$\Rightarrow$ The common prime factors are all 2, 3 and 5.

$\Rightarrow$ The common prime factors with their least exponent are 2^2 , 3^1 , and 5^1

Then the GCF of 120 and 180 is the product these common prime factors with their least exponent.

$$\text{Therefore, } \text{GCF}(120, 180) = 2^2 \times 3^1 \times 5^1 = 4 \times 3 \times 5 = 60$$

b) In a similar manner the prime factorizations of 60, 72 and 96 are

$$60 = 2^2 \times 3 \times 5, \quad 72 = 2^3 \times 3^2 \quad \text{and} \quad 96 = 2^5 \times 3$$

$$\text{Then } \text{GCF}(60, 72, 96) = 2^2 \times 3 = 12$$

c) The prime factorizations of 28 and 135 are

$$28 = 2^2 \times 7 \quad \text{and} \quad 135 = 3^3 \times 5$$

The numbers have no any common prime factor. Therefore, $\text{GCF}(28, 135) = 1$

Note: If two natural numbers are consecutive, then they are relatively prime.

For instance, 50 and 51 are consecutive. Hence, $\text{GCF}(50, 51) = 1$

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1.1.2B. Common Multiples and the Least Common Multiple

Given two or more natural numbers, a number which is a multiple of all of them is called a **Common Multiple** of the numbers.

Numbers may have more than one common multiple.

Notation: The set of all multiples of a natural number n may be denoted by M_n .

For instance, the set of multiples of 7 is $M_7 = \{7, 14, 21, 28, 35, \dots\}$

Example 12: Find the common multiples of 6 and 9.

Solution: The set of multiple of 9 and 12 are

$$M_6 = \{6, 12, \underline{18}, 24, 30, \underline{36}, 42, 48, \underline{54}, 60, \dots\} \text{ and}$$

$$M_9 = \{9, \underline{18}, 27, \underline{36}, 45, \underline{54}, 63, \dots\}$$

Therefore, the set of common multiples of 6 and 9 is $M_6 \cap M_9 = \{18, 36, 54, \dots\}$.

Of these common multiples, the least is 18, which is known as **Least Common Multiple** of 6 and 9.

DEFINITION 1.4. For any two natural numbers a and b , the **LEAST COMMON MULTIPLE** of a and b is the smallest multiple of both a and b .

Notation: For $a, b \in \mathbb{N}$, the LCM of a and b is denoted by $\text{LCM}(a, b)$.

There are two methods to find the L.C.M of two or more natural numbers.

- i) Listing Method , and ii) Prime Factorization Method

METHOD 1. Listing Method

To find LCM using listing method we follow the following steps:

Step 1. List all multiples of each number.

Step 2. Identify the common multiples.

Step 3. Then the LCM of the numbers is the smallest of the common multiples of the numbers.

Example 13: Find the LCM of the following numbers.

a) 3 and 5

b) 10 and 15

Solution

a) First find the multiples 3 and 5.

$$M_3 = \{3, 6, 9, 12, 15, 18, 21, \dots\} \quad \text{and} \quad M_5 = \{5, 10, 15, 20, 25, \dots\}$$

The set of common multiples of 3 and 5 is $M_3 \cap M_5 = \{15, 30, 45, \dots\}$

Thus, the smallest of the common multiples of 3 and 5 is 15.

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Therefore, L.C.M (3, 5) = 15

b) The multiples 10 and 15 are

$$M_{10} = \{10, 20, \underline{30}, 40, \dots\} \text{ and } M_{15} = \{15, \underline{30}, 45, 60, \dots\}$$

Therefore, the least common multiples of 10 and 15 is $\text{LCM}(10, 15) = 30$.

METHOD 2. Prime Factorization Method

To find LCM using prime factorization method we follow the following steps:

Step 1. Write the prime factorization of each number.

Step 2. Identify all the prime factors in each number.

Step 3. Then the product of each of these prime factors with the greatest number of times it appears in the prime factorization is the LCM of the numbers.

Example 14. Find the L.C.M of each of the following group of numbers.

a) 48 and 120

b) 36, 150 and 280

Solution

a) First write the prime factorizations of 48 and 120.

$$48 = 2^4 \times 3 \quad \text{and} \quad 120 = 2^3 \times 3 \times 5$$

The prime factors in the prime factorizations with their highest exponents are

$$2^4, 3^1 \text{ and } 5^1. \text{ Therefore, } \text{L.C.M}(48, 120) = 2^4 \times 3^1 \times 5^1 = 240$$

b) The prime factorizations of 36, 150 and 280 are

$$36 = 2^2 \times 3^2, \quad 150 = 2 \times 3 \times 5^2 \quad \text{and} \quad 280 = 2^3 \times 5 \times 7$$

All the prime factors that appear in these prime factorizations are 2, 3, 5 and 7.

The prime factors that appear with the greatest exponent are 2^3 , 3^2 , 5^2 and 7.

Then taking the product of these powers, we obtain the LCM:

$$\text{LCM}(36, 150, 280) = 2^3 \times 3^2 \times 5^2 \times 7 = 12,600$$

Remark

1. For two natural numbers a and b , $\text{GCF}(a, b) \times \text{LCM}(a, b) = a \times b$.

$$\text{For instance, } 72 = 2^3 \times 3^2 \text{ and } 300 = 2^2 \times 3 \times 5^2$$

$$\text{Then } \text{GCF}(72, 300) = 2^2 \times 3 = 12 \text{ and } \text{LCM}(72, 300) = 2^3 \times 3^2 \times 5^2 = 1800$$

$$\text{Now, } 72 \times 300 = 21600 \text{ and } \text{GCF}(72, 300) \times \text{LCM}(72, 300) = 12 \times 1800 = 21600$$

2. For two natural numbers a and b , if $\text{GCF}(a, b) = 1$, then $\text{LCM}(a, b) = a \times b$

$$\text{For instance, } 54 = 2 \times 3^3 \text{ and } 275 = 5^2 \times 11$$

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This implies that $\text{GCF}(54, 275) = 1$ and $\text{LCM}(54, 275) = 2 \times 3^2 \times 5^2 \times 11 = 54 \times 275$

3. If two natural numbers a and b are consecutive, then $\text{GCF}(a, b) = 1$ and $\text{LCM}(a, b) = a b$

Exercise 1.2

1. Check whether each of the following number is prime or composite.
a. 45 b. 23 c. 91 d. 153 e. 109 f. 111,111
2. Is the value of $2a + 3b$ prime or composite, when $a = 11$ and $b = 7$.
3. List the first six multiples of each of the following numbers
a. 7 b. 5 c. 14 d. 25 e. 150
3. Determine whether each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9 or 10:
a. 48 b. 153 c. 2,470 d. 144 e. 12,357
4. Write all the common factors of 36 and 84.
4. When will the LCM of two numbers be the product of the numbers?
5. How many factors does each of the following number have?
a. 12 b. 18 c. 96 d. 1080
6. The GCF of two numbers is 3 and their LCM is 180. If one of the numbers is 45, what is the other number?
7. The product of two numbers is 96. If their GCF is 4, then
a. what is their LCM, and b. find the two numbers.
8. Find two rational numbers between $\frac{1}{3}$ and $\frac{3}{4}$
9. Find the GCF and LCM of each of the following numbers
a. 36 and 48 c. 48, 60 and 72 e. 60, 200 and 420
b. 35, 49 and 84 d. $2 \times 3^3 \times 5^2$ and $2^3 \times 3 \times 5^2$
10. There are between 50 and 60 eggs in a basket. When Oliyad counts by 3s' there are 2 eggs left over and when he counts by 5s' there are 4 left over. How many eggs are there in the basket?
11. If $\text{LCM}(60, x) = 1260$ and x is even, then what is the smallest possible value of x ?

1.1.3. Rational numbers**DEFINITION 1.5 (RATIONAL NUMBERS)**

Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, is called a **RATIONAL NUMBER**.

For instance; $\frac{3}{5}$, $-\frac{27}{19}$, $4 = \frac{32}{8}$, $0 = \frac{0}{5}$, $-1 = \frac{-1}{1}$ are rational numbers.

Notation: The set of rational numbers, denoted by $\mathbb{Q}$, is the set described by

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \text{ and } b \text{ are integers and } b \neq 0 \right\}$$

Remark:

1. Any integer is a rational number.

That is if $a \in \mathbb{Z}$, then a can be expressed as $\frac{a}{1} \in \mathbb{Q}$.

2. In the fraction $\frac{a}{b}$, a is called the **NUMERATOR** and b is called **DENOMINATOR**.

3. A rational number $\frac{a}{b}$ is said to be in **its lowest term** or **in lowest form** if $\text{GCF}(a, b) = 1$.

There are three types of fraction.

Given a fraction $\frac{a}{b}$, where a and b are positive integers;

- i) If $a < b$, then $\frac{a}{b}$ is called **PROPER FRACTION**.

For instance; $\frac{1}{2}$, $-\frac{11}{27}$, $\frac{121}{211}$ are proper fractions.

- ii) If $a > b$, then $\frac{a}{b}$ is called **IMPROPER FRACTION**.

For instance; $\frac{3}{2}$, $-\frac{29}{12}$, $\frac{2157}{211}$ are improper fractions.

- iii) If c is an integer and $a < b$, the $c\frac{a}{b}$ is called **MIXED FRACTION**.

For instance; $-3\frac{2}{5}$, $5\frac{12}{27}$, $11\frac{1}{21}$ are mixed fractions.

Note: Any mixed fraction $c\frac{a}{b}$ can be expressed as improper fraction, and vice versa.

$$c\frac{a}{b} = \frac{(b \times c) + a}{b} = c + \frac{a}{b}$$

Example 15. Change the following mixed fractions into improper fraction.

a) $5\frac{3}{8}$

b) $-11\frac{2}{7}$

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Solution

$$\text{a) } 5\frac{3}{8} = 5 + \frac{3}{8} = \frac{8 \times 5 + 3}{8} = \frac{40 + 3}{8} = \frac{43}{8}$$

$$\text{b) } -11\frac{2}{7} = -\left(11\frac{2}{7}\right) = -\left(\frac{7 \times 11 + 2}{7}\right) = -\frac{77 + 2}{7} = -\frac{79}{7}$$

Example 16. Change the following improper fraction into mixed fractions.

$$\text{a) } \frac{23}{4}$$

$$\text{b) } -\frac{52}{11}$$

Solution

a) To change $\frac{23}{4}$ into mixed fraction, divide 23 by 4, and then the quotient is the integer part, and the remainder is the numerator of the mixed fraction.

When 23 is divided by 4, we obtain quotient 5 and remainder 3. That is

$$23 \div 4 = 5 + \frac{3}{4}$$

$$\text{Therefore, } \frac{23}{4} = 5\frac{3}{4}$$

$$\text{b) } -\frac{52}{11} = -\left(\frac{52}{11}\right) = -\left(4 + \frac{8}{11}\right) = -4\frac{8}{11}$$

Equivalent Rational Numbers

Given a rational number $\frac{a}{b}$, you can find out infinitely many rational numbers which are the same as $\frac{a}{b}$. These numbers are said to be equivalent rational numbers.

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are said to be equivalent, denoted by $\frac{a}{b} = \frac{c}{d}$, if $ad = bc$.

For instance, $\frac{2}{5} = \frac{8}{20}$, that is $\frac{2}{5}$ and $\frac{8}{20}$ are equivalent, since $2 \times 20 = 40 = 5 \times 8$.

Remark: Given a rational number $\frac{a}{b}$ and a natural number c . Then $\frac{a}{b} = \frac{a \times c}{b \times c}$

That is $\frac{a}{b}$ and $\frac{a \times c}{b \times c}$ are equivalent rational numbers.

For instance, $\frac{3}{7} = \frac{6}{14} = \frac{15}{35} = \frac{30}{70} = \frac{21}{49}$

Note: If a and b are two positive integers, then

$$\text{i. } \frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

$$\text{ii. } \frac{-a}{-b} = \frac{a}{b}$$

Operation on Rational Numbers

As we can perform the four basic operations: addition, subtraction, multiplication and division by non-zero numbers on $\mathbb{W}$ and $\mathbb{Z}$, we can perform the same operations on the set of rational numbers as follow.

Rule: Let a, b, c and $d \in \mathbb{Z}$, where $c, d \neq 0$.

1. Addition of Rational Numbers

$$\text{i. } \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \text{ii. } \frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$$

Since, $\mathbb{Z}$ is closed under addition and multiplication, $a+b, ad, bc, cd \in \mathbb{Z}$.

Hence, $\frac{a+b}{c}$ and $\frac{ad+bc}{cd}$ are rational numbers.

2. Subtraction of Rational Numbers

$$\text{i. } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \qquad \text{ii. } \frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}$$

Since $\mathbb{Z}$ is closed under subtraction and multiplication, $a-b, ad, bc, cd \in \mathbb{Z}$.

Hence, $\frac{a-b}{c}$ and $\frac{ad-bc}{cd}$ are rational numbers.

3. Multiplication of Rational Numbers

$$\frac{a}{c} \times \frac{b}{d} = \frac{a \times b}{c \times d} = \frac{ab}{cd}$$

Since, $\mathbb{Z}$ is closed under multiplication, $ab, cd \in \mathbb{Z}$ and $cd \neq 0$.

Hence, $\frac{ad}{cd}$ is rational number.

4. Division of Rational Numbers

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{cb}, \text{ where } b, c, d \neq 0$$

Since, $\mathbb{Z}$ is closed under multiplication and $b, c, d \neq 0$, we have ad and $bc \in \mathbb{Z}, bc \neq 0$.

Hence, $\frac{ad}{cb}$ is rational number.

Conclusion :

1. The set of rational numbers is closed under addition, subtraction and multiplication.
2. The set of non-zero rational numbers is closed under division.

RULE : Let $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers with $b > 0$ and $d > 0$. Then

- i. $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.
- ii. $\frac{a}{b} > \frac{c}{d}$ if and only if $ad > bc$.

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Example 17.

- a) $\frac{2}{7} < \frac{5}{9}$, because $2 \times 9 < 5 \times 7$
- b) $\frac{-4}{11} < \frac{3}{8}$, because $-4 \times 8 < 3 \times 11$
- c) $\frac{-12}{5} > \frac{9}{-7}$, because $\frac{-9}{7}$ and $-12 \times (-7) > 5 \times 9$

Density of Rational Numbers

You know that for any integer, there is always a predecessor and a successor. That is, for any integer there is the next smaller integer and the next larger integer.

For instance,

- a. The predecessor and successor of 27 are 26 and 28, respectively.
- b. The predecessor and successor of -9 are -10 and -8 , respectively.

In general, if k is an integer, then $k - 1$ is its predecessor and $k + 1$ is its successor.

- ? Does a rational number has a predecessor and a successor?
- ? What are the predecessor and the successor of $\frac{3}{5}$? of $-\frac{1}{2}$?
- ? What are the predecessor and the successor of a rational number 6?

Obviously, we cannot find out the predecessors and the successors of the above rational numbers. You cannot apply the terms predecessor and successor on any rational numbers. For this reason, we say that the set of rational numbers is **DENSE**.

This means that between any two rational numbers, you can find out another rational number.

But, this is no true for the sets of integers, whole numbers and natural numbers.

In general, for every pair of rational numbers a and b such that $a < b$, there exists atleast one rational number c and consequently infinitely many rational numbers c such that $a < c < b$.

Suppose $a, b \in \mathbb{Q}$ such that

$$a < b \quad \dots\dots\dots (1)$$

Add a to both sides of (1). We obtain

$$2a < a + b \quad \dots\dots\dots (2)$$

Dividing both sides of (2) by 2, gives

$$a < \frac{a+b}{2} \quad \dots\dots\dots (3)$$

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In a similar manner,

Add b to both sides of (1). We obtain

$$a + b < 2b \quad \dots\dots\dots (4)$$

Dividing both sides of (4) by 2, gives

$$\frac{a + b}{2} < b \quad \dots\dots\dots (5)$$

Combining equations (3) and (4), we obtain $a < \frac{a + b}{2} < b$ and $\frac{a + b}{2} \in \mathbb{Q}$.

Hence, $c = \frac{a + b}{2}$ is a rational number between a and b . That is $a < c < b$ or $a < \frac{a + b}{2} < b$

Moreover, in the similar manner, since $a < c$ and $c < b$, we have

$$a < \frac{a + c}{2} < c \text{ and } c < \frac{c + b}{2} < b$$

Replacing $c = \frac{a + b}{2}$ and simplify, gives $\frac{a + c}{2} = \frac{a + \frac{a + b}{2}}{2} = \frac{3a + b}{4}$ and $\frac{c + b}{2} = \frac{\frac{a + b}{2} + b}{2} = \frac{a + 3b}{4}$

Collecting these together, we obtain

$$a < \frac{3a + b}{4} < \frac{a + b}{2} < \frac{a + 3b}{4} < b$$

These shows that $\frac{3a + b}{4}$, $\frac{a + b}{2}$ and $\frac{a + 3b}{4}$ are three rational numbers between a and b .

Continuing in this manner, we obtain infinitely many rational numbers between a and b .

Example 18.

Find three rational numbers between $\frac{2}{3}$ and $\frac{5}{8}$.

Solution

Observe that $\frac{5}{8} < \frac{2}{3}$, because $5 \times 3 < 2 \times 8$. This implies that

$$\frac{\frac{5}{8} + \frac{2}{3}}{2} = \frac{\frac{15 + 16}{24}}{2} = \frac{31}{48} \text{ is between } \frac{2}{3} \text{ and } \frac{5}{8}. \text{ That is } \frac{2}{3} < \frac{31}{48} < \frac{5}{8}.$$

Similarly, $\frac{2}{3} < \frac{\frac{2}{3} + \frac{31}{48}}{2} < \frac{31}{48}$ and $\frac{31}{48} < \frac{\frac{31}{48} + \frac{5}{8}}{2} < \frac{5}{8}$. Simplifying the middle fraction, we have

$$\frac{2}{3} < \frac{21}{32} < \frac{31}{48} \text{ and } \frac{31}{48} < \frac{61}{98} < \frac{5}{8}. \text{ Thus, we have } \frac{2}{3} < \frac{21}{32} < \frac{31}{48} < \frac{61}{98} < \frac{5}{8}$$

Therefore, $\frac{21}{32}$, $\frac{31}{48}$ and $\frac{61}{98}$ are three prime numbers between $\frac{2}{3}$ and $\frac{5}{8}$.

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ADDITIONAL REMARK

Additional method to Find Rational Numbers Between Two Rational Numbers

Method 2.

If $\frac{a}{b}$ and $\frac{c}{d}$ are two distinct rational numbers then $\frac{a+c}{b+d}$ is also a rational numbers

lying between $\frac{a}{b}$ and $\frac{c}{d}$, There, fore $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

Method 3.

Finding n rational numbers between any two rational numbers a and b when $a < b$, use the following steps:

Step 1: Find the gap between a and b this is equal to $(b - a)$

Step 2: Divide the gap into $n+1$ smaller equal gaps by dividing $(b - a)$ by $(n+1)$,

thus we have $d = \frac{b-a}{n+1}$

Step 3: Multiply d by 1, 2, 3.....until n and add a to each product to get new rational numbers. Thus, the new rational numbers are $a+1 \times d$, $a+2 \times d$,
..... $a+n \times d$

i.e. $a+d, a+2d, \dots, a+nd$

There fore $a < a+d < a+2d < \dots < b$

1.2 The Real Number System

Competencies

At the end of this sub unit, students will be able to:

- *show that repeating decimals are also rational numbers.*
- *identify irrational numbers.*
- *locate some irrational numbers on a number line.*
- *define real numbers.*

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- describe the correspondence between real numbers and points on a numbers line.
- realize the relationship between a power with fractional exponent and a radical form.
- convert powers with fractional exponent to radical form and vice-versa.
- perform any one of the four operations on the set of real numbers.
- use the laws of exponents to simplify expression.
- give appropriate upper and lower bounds for a given data to a specified accuracy (e.g. rounding off).
- express any positive rational number in its standard form.
- explain the notion of rationalization.
- identify a rationalizing factor for a given expression.
- use the Euclid's division algorithm to express given quotients of the form p/q , $p > q$.

1.2.1. Representation of Rational Numbers by Decimals

Although considerably more useful than whole numbers, fractions are not always adequate and many times lead to awkward manipulative procedures. For example, suppose you were asked to $\frac{10230}{78893} + \frac{2061}{37891}$ with a hand calculator. These fractions can be easily converted to decimals and the answer will be given to the accuracy of the calculator.

This shows that decimals are a convenient numeration system for fraction. Hence, it is important to study about decimal numbers.

Revision on Decimals

A **DECIMAL NUMBER** is a way of writing a fraction.

Decimals are one of the types of numbers, which has a whole number part and the fractional part separated by a dot ‘.’. The dot present between the whole number and fractions part is called the **decimal point**. The decimal point goes between Ones and Tenths.

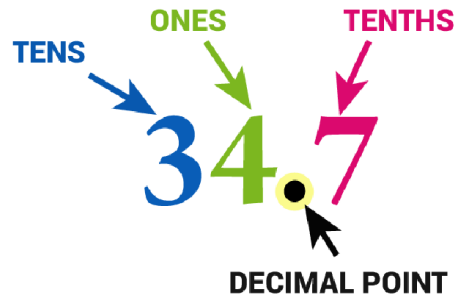
Thus, 0.105; 123.5 ; 7,000,956 ; 2.333..... ; 17.252252225 ; 3.14178032651..... are decimal numbers.

For example, 34.7 is a decimal number.

Here, **34 is a whole number part** and **7 is the fractional part**.

Here is the number “**thirty-four and seven-tenths**” written as a decimal number:

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34.7 has 3 Tens, 4 Ones and 7 Tenths.

Another example, consider the decimal 17.591. Here 17 is the whole number part and 591 is the fraction part.

As we move further left, every number place gets 10 times bigger. As we move further right, every number place gets 10 times smaller.

Hence, in 17.591;

The first digit on the right of the decimal point means tenths ($\frac{1}{10}$)

The second digit on the right means hundredths ($\frac{1}{100}$)

The third digit on the right means thousandths ($\frac{1}{1000}$).

We could re-write 17.591 as $17.591 = 17 + \frac{5}{10} + \frac{9}{100} + \frac{1}{1000}$

So 17.591 is **17** and **5 tenths, 9 hundredths** and **1 thousandth**.

- We refer to the number of digits after the decimal point as the **decimal places**.

So 17.591 has 3 decimal places.

There are three different types of decimal numbers. These are

- i) Terminating decimals,

For example, 0.154 and 78.111111 are terminating decimals.

- ii) Repeating decimals (or Non-terminating repeating decimals);

For example, $2.5555 \dots = 2.\bar{5}$ and $0.31712712 \dots = 0.31\overline{712}$ are repeating decimal numbers.

- iii) Non-terminating decimals (or Non-terminating non-repeating decimals)

For example, 0.1251718905 ... and 0.313313331 ... are repeating decimal.

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Expressing Fractions as Terminating or Repeating Decimals

REMARK: Any fraction of natural numbers can be expressed as a decimal by dividing the numerator by the denominator.

Example 19. Show that rational numbers $\frac{3}{8}$ and $\frac{7}{12}$ can each be expressed as a decimal.

Solution

$$\frac{3}{8} \text{ means } 3 \div 8$$

$$\begin{array}{r} 0.375 \\ 8 \overline{) 3.000} \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

$$\therefore \frac{3}{8} = 0.375$$

$$\frac{7}{12} \text{ means } 7 \div 12$$

$$\begin{array}{r} 0.5833 \dots \\ 12 \overline{) 7.0000} \\ \underline{60} \\ 100 \\ \underline{96} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

$$\therefore \frac{7}{12} = 0.5833 \dots$$

- As shown in the above example, the fraction (rational number) $\frac{3}{8}$ can be expressed as the decimal 0.375. Decimals like 0.375 are called **TERMINATING DECIMALS**, because the division ends or terminates, when the remainder is zero.
- The fraction $\frac{7}{12}$ can be expressed as the decimal 0.58333..... (Here, the digit 3 repeats and the division does not terminate.)

Decimals like 0.58333..... are called **REPEATING DECIMALS** or **NON-TERMINATING REPEATING DECIMALS**.

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REMARK: To show a **repeating digit** or a **block of repeating digits** in a repeating decimal number, we put a **bar** above the repeating digit (or block of digits).

For instance, $0.58333\ldots$ can be written as $0.58\bar{3}$, and $0.0818181\ldots$ can be written as $0.0\bar{81}$.

This method of writing a repeating decimal is known as **BAR NOTATION**.

The portion of a decimal that repeats is called the **REPETEND**.

For example, in the decimal representation

- i) $0.58333\ldots = 0.58\bar{3}$, the repetend is 3.
- ii) $-7.2818181\ldots = -7.2\bar{81}$, the repetend is 81.

In general, any rational number $\frac{a}{b}$ can be expressed as a decimal by dividing the numerator a by the denominator b

- When you divide ‘ a ’ by ‘ b ’ one of the following two cases will occur.

Case 1. The division process ends or terminates when a remainder of zero is obtained. In this case the decimal is called a **TERMINATING DECIMAL**.

Case 2. The division process does not terminate as the remainder never becomes zero such a decimal is called a **REPEATING DECIMAL**.

Expressing Terminating and Repeating Decimals as Fractions

- i. Every terminating decimal can be expressed as a fraction (ratio of two integers) with a denominator of 10; 100 ; 1000 and so on.

For instance,

i) $0.7 = \frac{7}{10}$

ii) $0.135 = \frac{135}{1000}$

iii) $25.02 = \frac{2502}{100}$

In general, if ‘ d ’ is a terminating decimal number that has ‘ n ’ digits after a decimal point, then we rewrite d as

$$d = d \times 1 = d \times \frac{10^n}{10^n} = \frac{d \times 10^n}{10^n}$$
 The last right side gives the fractional form of ‘ d ’.

Example 20. Convert each of the following decimals as fraction.

a) 2.66

b) 3.2

c) 0.0625

Solution. Using the above equation,

- a) In $d = 2.66$, we have $n = 2$. Hence,

$$d = \frac{d \times 10^n}{10^n} = \frac{2.66 \times 10^2}{10^2} = \frac{2.66 \times 100}{100} = \frac{266}{100} = \frac{133}{50}$$

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$$\text{Hence, } 2.66 = \frac{133}{50}$$

b) In $d = 3.2$, we have $n = 1$. Hence,

$$d = \frac{d \times 10^n}{10^n} = \frac{3.2 \times 10^1}{10^1} = \frac{3.2 \times 10}{10} = \frac{32}{10} = \frac{16}{5}. \quad \text{Hence, } 3.2 = \frac{16}{5}$$

c) In $d = 0.0625$, we have $n = 4$. Hence,

$$d = \frac{d \times 10^n}{10^n} = \frac{0.0625 \times 10^4}{10^4} = \frac{0.0625 \times 10000}{10000} = \frac{625}{10000} = \frac{1}{16}$$

$$\text{Hence, } 0.0625 = \frac{1}{16}$$

ii. Repeating decimals can also be expressed as fraction (ratio of two integers), as shown in the following example.

Example 21. Express each of the following repeating decimals as fraction.

a) $0.\overline{7}$

b) $0.\overline{25}$

c) $0.8\overline{39}$

Solution

a) $0.\overline{7} = 0.7777\dots$

$$\text{Let } d = 0.\overline{7} = 0.7777\dots \rightarrow (1)$$

Here, there is only one repetend, which is 7.

Then, multiply both sides by $10^1 = 10$.

$$10d = 7.777\dots \rightarrow (2)$$

Subtract equation (1) from equation (2) :

$$10d - d = 7.777\dots$$

$$- \underline{0.777\dots}$$

$$\Rightarrow 9d = 7$$

$$\Rightarrow \frac{9d}{9} = \frac{7}{9} \Rightarrow d = \frac{7}{9}$$

$$\text{Hence, } 0.\overline{7} = \frac{7}{9} = 0.777\dots$$

b) $0.\overline{25} = 0.252525\dots$

$$\text{Let } d = 0.\overline{25} = 0.252525\dots \rightarrow (3)$$

Here, the repetends are 2 and 5.

Then, multiply both sides by $10^2 = 100$.

$$100d = 25.2525\dots \rightarrow (4)$$

Subtract equation (3) from (4) :

$$100d - d = 25.2525\dots$$

$$- \underline{0.2525\dots}$$

$$\Rightarrow 99d = 25$$

$$\Rightarrow \frac{99d}{99} = \frac{25}{99} \Rightarrow d = \frac{25}{99}$$

$$\text{Hence, } 0.\overline{25} = \frac{25}{99}$$

c) Let $d = 0.8\overline{39} = 0.839393939\dots \rightarrow (5)$

Here, there is one non-repeating digit (which is 8), and there is **two repeating digits**: 3 and 9.

First, multiply both sides of (5) by $10^1 = 10$ which gives

$$10d = 10 \times 0.8\overline{39} = 8.39393939\dots \rightarrow (6)$$

Again, multiply both sides of (5) by $10^{1+2} = 10^3 = 1000$, which gives

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$$1000d = 1000 \times 0.\overline{839} = 839.393939 \dots \rightarrow (7)$$

Then, subtract (6) from (7) gives

$$1000d - 10d = 839.393939 \dots - 8.393939 \dots$$

$$\Rightarrow 990d = 831 \Rightarrow d = \frac{831}{990}$$

$$\text{Therefore, } 0.\overline{839} = \frac{831}{990}$$

- In general, if 'd' is a repeating decimal with 'k' non - repeating and 'p' repeating digits after the decimal point, then d can be expressed as fraction using the formula

$$d = d \times 1 = d \times \frac{10^{k+p}-10^k}{10^{k+p}-10^k} = \frac{d \times (10^{k+p}-10^k)}{10^{k+p}-10^k} = \frac{(d \times 10^{k+p}) - (d \times 10^k)}{10^{k+p}-10^k}$$

Example 22. Convert each of the following decimal as fraction

a) $0.\overline{23}$

b) $3.\overline{46}$

c) $4.\overline{812}$

Solution

a) In $d = 0.\overline{23} = 0.232323 \dots$, we have $k = 0$ and $p = 2$. Then

$$\begin{aligned} d &= d \left(\frac{10^{k+p}-10^k}{10^{k+p}-10^k} \right) = 0.\overline{23} \left(\frac{10^{0+2}-10^0}{10^{0+2}-10^0} \right) = \frac{0.\overline{23} \times 10^2 - 0.\overline{23} \times 10^0}{100-1} \\ &= \frac{0.\overline{23} \times 100 - 0.\overline{23} \times 1}{100-1} = \frac{23.2323 \dots - 0.2323 \dots}{99} = \frac{23}{99} \end{aligned}$$

$$\text{Therefore, } 0.\overline{23} = \frac{23}{99}$$

b) In $d = 3.\overline{46} = 3.46666 \dots$, we have $k = 1$ and $p = 1$. Then

$$\begin{aligned} d &= d \left(\frac{10^{k+p}-10^k}{10^{k+p}-10^k} \right) = \frac{d \times 10^{k+p} - d \times 10^k}{10^{k+p}-10^k} \\ &= \frac{3.\overline{46} \times 10^{1+1} - 3.\overline{46} \times 10^1}{10^{1+1}-10^1} = \frac{3.\overline{46} \times 100 - 3.\overline{46} \times 10}{10^2-10^1} \\ &= \frac{346.666 \dots - 34.666 \dots}{100-10} = \frac{321}{90} = \frac{52}{15}. \text{ Therefore, } 3.\overline{46} = \frac{52}{15} \end{aligned}$$

c) $-4.\overline{812} = -4 + (-0.\overline{812})$. Let $d = -0.\overline{812}$

In $d = -0.\overline{812} = -0.8121212 \dots$, we have $k = 1$ and $p = 2$. Then

$$\begin{aligned} d &= d \left(\frac{10^{k+p}-10^k}{10^{k+p}-10^k} \right) = \frac{d \times 10^{k+p} - d \times 10^k}{10^{k+p}-10^k} = \frac{(-0.\overline{812} \times 10^{1+2}) - (-0.\overline{812} \times 10^1)}{10^{1+2}-10^1} \\ &= -\frac{(0.\overline{812} \times 1000) - (0.\overline{812} \times 10)}{1000-10} = -\frac{812.\overline{12} - 8.\overline{12}}{1000-10} = -\frac{804}{990} = -\frac{134}{165} \end{aligned}$$

$$\text{Therefore, } -4.\overline{812} = -4 + (-0.\overline{812}) = -4 + \left(-\frac{134}{165} \right) = -\frac{794}{165}$$

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Note: From the above discussions, we have the following conclusion.

1. Every rational number can be expressed as either a terminating decimals or repeating decimal.
2. Every terminating or repeating decimal represents a rational number.

1.2.2. Irrational Numbers

In the previous discussion, we have shown that every rational number can be expressed as either a terminating decimal or repeating decimal, and every terminating or repeating decimal represents a rational number.

However, you have left with the question, “Can every number be represented by a terminating or repeating decimal numeral?”

Consider the following decimal:

0.101001000100001. ----->. (i)

You can see that this decimal has a definite pattern and it can be written indefinitely, and there is no block of digits which is repeating.

A similar decimal is given as under:

0.12345678910111213. -----> (ii)

Can you write the next group of digits in (i) and (ii)? The next six digits in (i) are 000001..... and in (ii) they are 14 15 16

Decimal numbers as in (i) and (ii) are examples of decimal numbers which are neither terminating nor repeating. These decimals are non-terminating and non-repeating decimals.

Such decimals as in (i) and (ii) represent a type of number known as **irrational numbers**.

Thus, *a decimal expansion which is neither terminating nor is repeating represents an **IRRATIONAL NUMBER**.*

Example 23 :

a) 0.318327911 ;

b) 2.131131113..... and

c) 7.868868886. are non – repeating decimals. Hence, they are irrational numbers.

- An **IRRATIONAL NUMBER** is a number that cannot be expressed as $\frac{a}{b}$ such that a and b are integers and $b \neq 0$.

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By using scientific calculators, you can see that

$$\sqrt{5} = 2.2360679774997896964091736687313 \dots \dots$$

which is non terminating and non-repeating. Thus, it is irrational number.

Note: Roots that are not perfect are irrational numbers.

For example;

a. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{8}, \sqrt{7}, \sqrt{\frac{13}{27}}, \sqrt{2.05}, \sqrt[3]{4}, \sqrt[4]{32}, \dots \dots$ are irrational numbers.

b. $\pi = 3.1415926535897932384626433832795 \dots \dots$ is irrational number, since the decimal the decimal does not repeat or terminate.

Note that the decimal 3.14 and the fraction $\frac{22}{7}$ are not exactly equal to π , they are the rational approximations to the irrational number π .

Example 24. Show that $\sqrt{2}$ is not a rational number.

Solution :

Suppose $\sqrt{2}$ is rational number. Since any rational number can be written in the form of fraction of integers in lowest term, there are $m, n \in \mathbb{N}$, $\text{GCF}(m, n) = 1$ such that $\frac{m}{n} = \sqrt{2}$

$$\Rightarrow \frac{m^2}{n^2} = 2 \quad \text{From this we have} \quad m^2 = 2n^2$$

$$\Rightarrow m^2 \text{ is even, since } m^2 \text{ is a multiple of 2.}$$

$$\Rightarrow m \text{ is even, since if the square of a number is even, the number itself is even.}$$

$$\Rightarrow \text{There exist } k \in \mathbb{Z} \text{ such that } m = 2k.$$

$$\Rightarrow (2k)^2 = 2n^2, \text{ since } m^2 = 2n^2$$

$$\Rightarrow n^2 = 2k^2$$

$$\Rightarrow n^2 \text{ is even, since } n^2 \text{ is a multiple of 2.}$$

$$\Rightarrow n \text{ is even, since if the square of a number is even, the number itself is even.}$$

This contradicts the assumption that $\text{GCF}(m, n) = 1$.

Therefore, $\sqrt{2}$ is not a rational number. Hence, $\sqrt{2}$ is irrational number.

Note: In the similar manner as above, you can show that the square root of prime numbers such as $\sqrt{3}, \sqrt{5}, \sqrt{7}, \dots \dots$ are irrational numbers.

Example 25. Show that $5 + \sqrt{2}$ is irrational number.

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Solution : Assume that $5 + \sqrt{2}$ is a rational number.

$$\Rightarrow \text{There exist } m, n \in \mathbb{Z}, \text{GCF}(m, n) = 1 \text{ such that } \frac{m}{n} = 5 + \sqrt{2}$$

$$\Rightarrow \sqrt{2} = \frac{m}{n} - 5$$

$$\Rightarrow \sqrt{2} = \frac{m - 5n}{n}$$

This implies that $\sqrt{2}$ is a rational number, since $m - 5n$ and n are both integers, by the assumption.

This contradicts the fact that $\sqrt{2}$ is an irrational number. This shows that the assumption that $5 + \sqrt{2}$ is a rational number.

Therefore, $5 + \sqrt{2}$ is not a rational number.

Example 26. Show that $\sqrt[3]{2}$ is an irrational number.

Solution :

Suppose $\sqrt[3]{2}$ is a rational number. There are $a, b \in \mathbb{Z}$, $\text{GCF}(a, b) = 1$ such that $\frac{a}{b} = \sqrt[3]{2}$

Cubing both sides gives $\frac{a^3}{b^3} = 2$. From this we have $a^3 = 2b^3$

$$\Rightarrow a^3 \text{ is even.}$$

$$\Rightarrow a \text{ is even, since if the cube of a number is even, the number itself is even.}$$

$$\Rightarrow \text{There exist } t \in \mathbb{Z} \text{ such that } a = 2t.$$

$$\Rightarrow (2t)^3 = 2b^3, \text{ which results in } b^3 = 4t^3$$

$$\Rightarrow b^3 \text{ is even, since } b^3 \text{ is a multiple of 2.}$$

$$\Rightarrow b \text{ is even, since if the cube of a number is even, the number itself is even.}$$

$$\Rightarrow a \text{ and } b \text{ are even implies that } \text{GCF}(a, b) \neq 1.$$

This contradicts the assumption that $\text{GCF}(a, b) = 1$.

Therefore, $\sqrt[3]{2}$ is an irrational number.

Locating Irrational Numbers on the Number Line

From your lower grade mathematics course, you know that corresponding to every rational number, there corresponds a point on the number line. That is there is a definite point on the number line which corresponds to each rational number. How about the converse of this statement.

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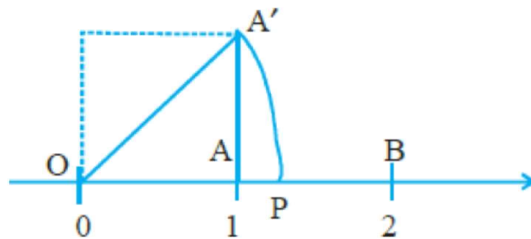
Given a point on the number line, will it always correspond to a rational number? The answer to this question is also “**NO**”.

For clarifying this, we take the following example.

Example 27. Using geometrical construction, show that there is a point which corresponds to the irrational number $\sqrt{2}$.

Solution :

On the number line given below, take points O, A and B representing rational numbers 0, 1 and 2, respectively. At A draw $\overline{AA'}$ such that $\overline{OA} \perp \overline{AA'}$ and $AA' = 1$ unit. Hence, triangle $\triangle OAA'$ is isosceles triangle with $AA' = OA = 1$ unit



By Pythagoras's Theorem, $(OA')^2 = (OA)^2 + (AA')^2$

This implies that $OA' = \sqrt{2}$ unit

Taking O as centre and radius $\overline{OA'}$, if we draw an arc, we reach the point P on the number line, so that $OA' = OP$. Hence, the point P on the number line represents the number $\sqrt{2}$.

As $\sqrt{2}$ is irrational, we conclude that there are points on the number line (like P) which are not represented by a rational number.

Similarly, we can show that we can have points like $\sqrt{3}$, $\sqrt{5}$, $2 + \sqrt{3}$, $5\sqrt{2}$ etc, on a number line which are not represented by rationals.

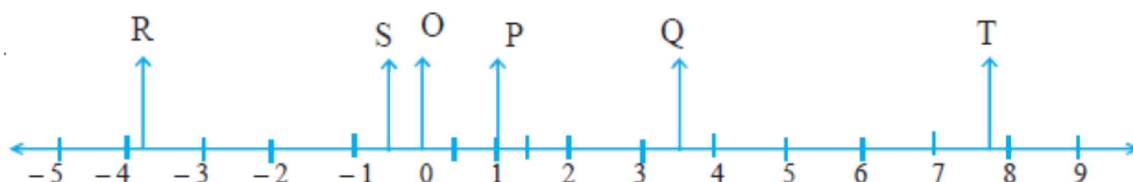
Therefore, the number line consists of points corresponding to rational numbers and irrational numbers both.

We have thus extended the system of rational numbers to include irrational numbers also. The system containing rationals and irrationals both is called the real number system.

The system of numbers consisting of all rational and irrational numbers is called the **SYSTEM OF REAL NUMBERS**.

ACTIVITY 1.2

1. Show that the following numbers are irrational.
a. $\sqrt{5}$ b. $-\sqrt{3}$ c. $3 + \sqrt{2}$ d. $1 - \sqrt{3}$
2. Locate each of the following irrational numbers on the number line, by using geometrical construction:
a. $\sqrt{5}$ b. $-\sqrt{3}$ c. $3 + \sqrt{2}$ d. $1 - \sqrt{3}$
3. Write the rational numbers corresponding to the points O, P, Q, R, S and T on the number line in the following figure:



4. Evaluate the following. Identify whether the results are rational or irrational.
a. $0.3030030003 \dots + 0.1414414441 \dots$ d. $(2 - \sqrt{3})(2 + \sqrt{3})$
b. $0.5757757775 \dots - 0.242442444 \dots$ e. $\sqrt{5} \div \sqrt{10}$
c. $2 \times 3.13113111311113 \dots$ f. $\sqrt{8} \div \sqrt{2}$

From **Example 25** and **Activity 1.2**, you can generalize the following facts:

1. The sum and difference of a rational number and an irrational number is always irrational number. That is if x is any rational and y is any irrational numbers, then

$$x + y \text{ and } x - y \text{ are both always irrationals.}$$

2. The product and ratio of non-zero rational number and an irrational number is always irrational. That is if x is rational, $x \neq 0$ and y is any irrational numbers, then

$$x \times y, x \div y \text{ and } y \div x \text{ are always irrationals.}$$

But, if $x = 0$ and y is any irrational numbers, then

$$x \times y = 0 \text{ and } x \div y = 0 \text{ which are both rational and } y \div x \text{ does not exist.}$$

3. The set of irrational number is not closed under addition, subtraction, multiplication and division. That is if x and y are two irrational numbers, then

$$x + y, x - y, x \times y \text{ and } x \div y \text{ may not be irrational.}$$

4. If x and y are two integers and $y \neq 0$, and p is a positive integer that is not a perfect square, then $y\sqrt{p}$, $x + y\sqrt{p}$ and $x - y\sqrt{p}$ are irrational numbers.

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For example, $7\sqrt{3}$, $3 + 2\sqrt{5}$ and $5 - 3\sqrt{13}$ are irrational numbers.

Example 28

- $\sqrt{7} + \sqrt{8}$ and $4\sqrt{3} \times \sqrt{8}$ are irrational numbers.
- $1.535535553 \dots - 0.4040040004 \dots = 1.131131113 \dots$ is irrational number.
- $0.727727772 \dots + 0.1616616661 \dots = 0.8888 \dots = 0.\overline{8}$ is rational number.
- $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$ and $\frac{\sqrt{27}}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} = 3$ are rational number.

Exercise 1.4

- Identify each of the following numbers as rational or irrational
 - $\frac{5}{6}$
 - $2.\overline{34}$
 - $-0.1213141516 \dots$
 - $\sqrt{0.81}$
 - $0.121121112 \dots$
 - 1.232232223
 - $\sqrt{5} - \sqrt{2}$
 - $\sqrt[3]{72}$
 - $1 + \sqrt{3}$
- Verify each of the following computations by converting the decimals to fractions:
 - $0.\overline{275} + 0.\overline{714} = 0.\overline{989}$
 - $0.\overline{6} - 1.\overline{142875} = -0.\overline{476190}$
- Express each of the following repeating decimals as fraction.
 - $0.\overline{9}$
 - $2.\overline{501}$
 - $0.1\overline{32}$
- For each of the following, decide whether the statement is **TRUE** or **FALSE**. If your answer is false, give a counter example to justify
 - The sum of any two irrational numbers is an irrational number
 - The sum of any two rational numbers is a rational number
 - The sum of any two terminating decimal is a terminating decimal
 - The product of a rational number and an irrational number is irrational
- If $x = 2.3515515551 \dots$ and $y = 0.021373373337 \dots$, then find the value of $10x + 10^3y$ in the form of $\frac{a}{b}$.
- Given $x = 1.31331333 \dots$ and $y = 0.595595559 \dots$ find $2x + y =$ _____
- If $a = 1.323323332 \dots$, $b = 0.101101110 \dots$ and $\frac{1}{ac - bc} = 27$, then find c.
- If x and y are twin prime numbers such that $x > y$ and $x + y = 84$, then $x =$ _____ and $y =$ _____

1.2.3 Real Numbers

You have learnt that decimals which terminate or repeat correspond to rational numbers, and those decimals which do not terminate and do not repeat correspond to irrational numbers.

The set of decimals of all kinds corresponds to the union of the set of rational numbers and the set of irrational numbers. the union of these two sets of numbers can be considered as a new field of numbers and defined as follow:

Definition 1.6 (REAL NUMBERS)

The union of the set of rational numbers and the set of irrational numbers is called the **SET REAL NUMBER**, denoted by $\mathbb{R}$.

The set of real numbers can be described as

$$\mathbb{R} = \{x : x \text{ is a rational number or an irrational number}\}$$

$$\mathbb{R} = \{x : x \text{ is a rational number}\} \cup \{x : x \text{ is an irrational number}\}$$

From definition 1.6, it is clear that every decimal number corresponds to a real number.

Thus, in general, we have the following:

- i. If a number is rational, then it is real number.
- ii. If a number is irrational, then it is real number.

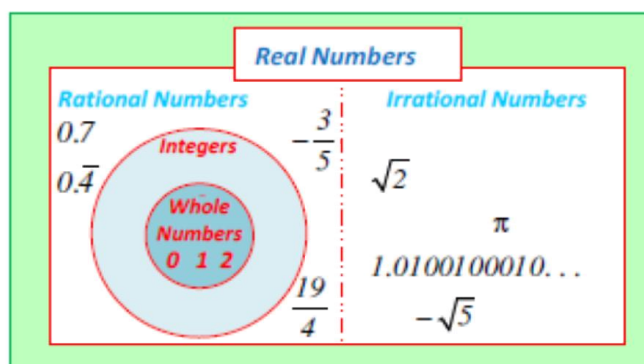
Note that the converses of these two statements are not true. Why?

Another important concept here is that stated as follow:

To every real number there corresponds a single point on the number line, and to every point on the number line there corresponds a real number.

That is there is a one-to-one correspondence between the set of real numbers and the set of points on the number line.

The set of real numbers and its subsets are pictorially shown in the adjacent diagram.



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Since every real number can be located on the number line, the number line can be used to compare and order all real numbers.

Order and Comparison of Set of Real Numbers

Suppose that we have two unequal positive real numbers a and b written as decimals as

$$a = a_0.a_1a_2a_3\dots \text{ and } b = b_0.b_1b_2b_3\dots$$

- i. $a = b$ if and only if $a_0 = b_0, a_1 = b_1, a_2 = b_2, \dots$
- ii. If k is the smallest number for which $a_k \neq b_k$, then $a < b$ if and only if $a_k < b_k$.

This is the algebraic method of comparing two decimal numbers.

For example; if $a = 4.5275\textbf{8}31\dots$ and $b = 4.5275\textbf{4}39\dots$, then $a \neq b$, because $a_5 \neq b_5$.

Moreover, since $a_5 > b_5$, we have $a > b$.

There are algebraic methods of comparing and ordering real numbers. Here are the two important order properties of set of real numbers.

1. Trichotomy Property of Order

For any two real numbers ' a ' and ' b ', one and only one of the following holds true:

$$a < b \text{ or } a = b \text{ or } a > b.$$

2. Transitive Property of Order

For any three real numbers a , b and c , if $a < b$ and $b < c$, then $a < c$.

A third property, stated below, can be derived from the Trichotomy Property and the Transitive Property of Order.

For any two non- negative real numbers a and b , if $a^2 < b^2$, then $a < b$.

For example,

a. $\sqrt{7} < \sqrt{13}$, since $\sqrt{7}^2 = 7 < 13 = \sqrt{13}^2$

b. $\frac{5\sqrt{3}}{3} > \frac{11}{5}$, since $(\frac{5\sqrt{3}}{3})^2 = \frac{25}{3} > (\frac{11}{5})^2 = \frac{121}{25}$ and $25 \times 25 = 625 > 3 \times 121 = 363$

Exercise 1.5

1) Compare the numbers a and b using the symbol $<$, $=$ or $>$.

a) $a = 1.575777$; $b = 1.757555$ d) $a = 0.\overline{857}$; $b = 0.\overline{857}$

b) $a = -0.6667$; $b = -\frac{2}{3}$ e) $a = 0.\overline{41}$; $b = 0.4141141114\dots$

c) $a = \frac{\sqrt{6}}{4}$; $b = 0.\overline{6}$ f) $a = -\frac{5}{6}$; $b = -\frac{\sqrt{3}}{2}$

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d) $a = \frac{22}{7}$; and $b = \pi$

h) $a = 5.112112112 \dots$; $b = 5.11211121112 \dots$

e) $a = 3.1415555 \dots$; $b = \pi$

2) State whether each of the following sets is closed under each of the following operations:

i) addition

ii) subtraction

iii) multiplication

iv) division

a) The set of natural numbers $\mathbb{N}$.

h) The set of even integers $\mathbb{E}$

b) The set of integers $\mathbb{Z}$.

i) The set of odd natural numbers $\mathbb{O}$

c) The set of rational numbers $\mathbb{Q}$.

j) The set of negative integers $\mathbb{Z}^-$.

d) The set of irrational numbers.

k) $\{-1, 0, 1\}$

e) The set of real numbers $\mathbb{R}$.

l) $\{-1, 1\}$

f) The set of all positive multiples of 3.

m) The set of prime numbers

g) The set of integral multiples of 3

2.2.4. Exponents and Radicals

A. Roots and Radicals

Recall that a square root of a number is a number that when multiplied by itself yields the original number. For example, 5 is a square root of 25, because $5^2 = 25$. Since $(-5)^2 = 25$, we can say that -5 is a square root of 25 as well.

Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign $\sqrt{\quad}$ to denote the **PRINCIPAL (NON-NEGATIVE) SQUARE ROOT** and a negative sign in front of the radical $-\sqrt{\quad}$ is to denote the negative square root of the number.

Thus, $\sqrt{25} = 5$ -----> Positive square root of 25

$-\sqrt{25} = -5$ -----> Negative square root of 25

Thus, we have the following definition.

DEFINITION 1.7 (SQUARE ROOT)

1. For two real numbers a and b ; if $a^2 = b$, then a is called the **SQUARE ROOT OF b.**

2. For two non-negative real numbers a and b ; if $a^2 = b$, then $a = \sqrt{b}$ is the **SQUARE ROOT OF b.**

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For example,

a. $5^2 = 25$. Therefore, $5 = \sqrt{25}$

b. $\left(\frac{7}{3}\right)^2 = \frac{49}{9}$. Therefore, $\frac{7}{3} = \sqrt{\frac{49}{9}}$

Note:

1. If 'b' is a positive real number, then $\sqrt{b}$ is a positive real number.
2. Negative real numbers do not have square roots in the set of real numbers, since $a^2 \geq 0$ for any real number a.
3. Square root of zero is zero. That is $\sqrt{0} = 0$

Cube Root:

A cube root of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called **the index**.

For example,

$\sqrt[3]{64} = 4$, read as '**the principal cube root of 64**' or simply '**the cube root 64**' because $4^3 = 64$

Remark. The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply here.

For example,

a. $\sqrt[3]{-1000} = -10$, because $10^3 = 1000$.

That is -10 is the cube root of -1000 , since -1000 is the cube of -10 .

b. $\sqrt[3]{\frac{8}{125}} = \frac{2}{5}$, because $\frac{8}{125} = \left(\frac{2}{5}\right)^3$

In general, given any real number a , we have the following property:

If $\sqrt[3]{a} = b$, then $a = b^3$

Note:

1. When simplifying cube roots, look for factors that are perfect cubes.
2. Every real number has exactly one cube root.

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Example 29

a. $4 = \sqrt[3]{64}$. Because $4^3 = 64$.

c. $\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$

b. $0^3 = 0$. Hence, $0 = \sqrt[3]{0}$.

d. $\sqrt[3]{\frac{216}{1331}} = \sqrt[3]{\frac{6^3}{11^3}} = \sqrt[3]{\left(\frac{6}{11}\right)^3} = \frac{6}{11}$

Now we may generalize the above discussion as follow:

DEFINITION 1.8 (THE n^{th} ROOT)

For any , $b \in \mathbb{R}$ and positive integers n , if $a^n = b$, then we say that ' a ' is an n^{th} root of b .

Example 30

a) $(-3)^5 = -243$, hence -3 is the fifth root of -243 .

b) $2^{10} = 1024$, hence 2 is the tenth root of 1024 .

DEFINITION 1.9 (THE PRINCIPAL n^{th} ROOT)

If ' b ' is any real number and n is a positive integer greater than 1, then the **PRINCIPAL n^{th} ROOT** of b , denoted by $\sqrt[n]{b}$ is defined as

$$\sqrt[n]{b} = \begin{cases} \text{the positive } n^{\text{th}} \text{ root of } b & \text{if } b > 0 \\ \text{the negative } n^{\text{th}} \text{ root of } b & \text{if } b < 0 \text{ and } n \text{ is odd} \\ 0, & \text{if } b = 0 \end{cases}$$

Note

1. If $b < 0$ and n is even, there is no real n^{th} root of b , because an even power of any real number is a non-negative number.
2. The symbol $\sqrt[n]{}$ is called a **RADICAL SIGN**.
3. The expression $\sqrt[n]{b}$ is called a **RADICAL**.
4. n is called the **INDEX** and b is called the **RADICAND**.

Note: When simplifying an n^{th} root, look for factors that are perfect n^{th} power.

Thus, When simplifying an n^{th} root, we may use the following rule.

Let ' b ' any real number and n is a positive integer greater than 1. then the **PRINCIPAL n^{th} ROOT** of b^n is obtained as follow:

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$$\sqrt[n]{b^n} = \begin{cases} b, & \text{if } n \text{ is odd} \\ |b|, & \text{if } n \text{ is even} \\ 0, & \text{if } b = 0 \end{cases}$$

Example 31. Evaluate the following.

a. $\sqrt[3]{-125}$

c. $\sqrt[5]{(0.00032)^5}$

b. $\sqrt[4]{(-81)^4}$

d. $\sqrt[4]{\left(\frac{625}{16}\right)^4}$

Solution

a. $\sqrt[3]{-125} = \sqrt[3]{(-5)^3} = -5$

c. $\sqrt[5]{0.00032} = \sqrt[5]{\frac{32}{10,000}} = \sqrt[5]{\left(\frac{2}{10}\right)^5} = \frac{1}{5}$

b. $\sqrt[4]{(-3)^4} = |-3| = 3$

d. $\sqrt[4]{\frac{625}{16}} = \sqrt[4]{\left(\frac{5}{2}\right)^4} = \left|\frac{5}{2}\right| = \frac{5}{2}$

Rational Approximations of Irrationals

As we have discussed earlier, the square roots of non-perfect square numerals are irrationals. Thus; $\sqrt{2}$, $\sqrt{12}$, $\sqrt{31}$, $\sqrt{105}$ and so on are irrational numbers, and they cannot be expressed as terminating or repeating decimals. However, it is possible to approximate irrational numbers as closely as desired using decimals. Some of the methods of this approximations are discussed below.

1. Divide-and-Average Method

The rational approximations of irrational numbers can be found through successive trials, using a scientific calculator. The method of successive trials uses the following property:

For any three positive real numbers a , b and c and a positive integer n , if $a^n < b < c^n$, then $b < \sqrt[n]{b} < c$.

Example 32. Find a rational approximation of $\sqrt{59}$ to the nearest thousandth.

Solution: Use the above property, and divide-and-average.

Since, $49 < 59 < 64$, we have $7^2 < 59 < 8^2$

Then using the above property, we have

$$7 < \sqrt{59} < 8$$

Now, estimate $\sqrt{59}$ to the nearest tenth : take $\sqrt{59} \approx 7.5$

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Divide 59 by 7.5

$$\frac{59}{7.5} = \frac{590}{75}$$

From the adjacent

long division,

quotient = 7.8667

Then averaging the

Quotient and the

Divisor, gives

$$\frac{7.5 + 7.8667}{2} = 7.6834$$

Then, divide 59 by 7.6834.

From the second division, quotient = 7.6788

Now, you can see that

$$(7.6788)^2 < 59 < (7.6834)^2.$$

This shows that $\sqrt{59}$ is between 7.6788 and 7.6834, to the nearest 10,000th, one more digit than the required nearest.

Therefore, $\sqrt{59} \approx 7.679$, to the nearest thousandth.

$ \begin{array}{r} 7.8666..... \\ 75 \overline{) 590} \\ \underline{- 525} \\ 650 \\ \underline{- 600} \\ 500 \\ \underline{- 450} \\ 50 \end{array} $	$ \begin{array}{r} 7.6788..... \\ 76,834 \overline{) 590,000} \\ \underline{- 537,838} \\ 521,620 \\ \underline{- 461,004} \\ 606160 \\ \underline{- 537838} \\ 683220 \\ \underline{- 614672} \\ 68548 \end{array} $
---	--

2. Approximation Using Continued Fractions

Another important method of approximating a square root irrationals is given below.

Let N be any positive integer such that $\sqrt{N}$ is an irrational number and N_0 be a perfect square nearest to N . Then one can express $\sqrt{N}$ as:

$$\sqrt{N} = \sqrt{N_0} + \frac{N - N_0}{\sqrt{N} + \sqrt{N_0}}$$

It can also be expressed as a continued fraction as:

$$\sqrt{N} = \sqrt{N_0} + \frac{N - N_0}{2\sqrt{N_0} + \frac{N - N_0}{2\sqrt{N_0} + \frac{N - N_0}{2\sqrt{N_0} + \dots}}}$$

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Example 33:

- a. Approximate $\sqrt{2}$ rounded to two decimal places.

Solution: Here $N = 2$ and $N_0 = 1$.

Then, we have
$$\sqrt{2} \approx 1 + \frac{(2-1)}{2 + \frac{1}{2 + \frac{1}{2}}} = 1 + \frac{1}{2 + \frac{1}{5}} = 1 + \frac{5}{12} = 1.42$$

- b. Approximate $\sqrt{3}$ rounded to three decimal places.

Solution: Here $N = 3$ and $N_0 = 4$, thus, we have

$$\sqrt{3} \approx 2 + \frac{(3-4)}{4 + \frac{-1}{4 + \frac{-1}{4}}} = 2 + \frac{-1}{4 + \frac{-4}{15}} = 2 - \frac{13}{48} = 1.729.$$

Fractional Exponents

Previously, you are aware of squares and square roots. The way these roots can be expressed will drive us into fractional exponents.

Definition 1.10 (The n^{th} Power)

1. If $b \in \mathbb{R}$ and n is a positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$
2. If $b \in \mathbb{R}$, $b \neq 0$ and $n \in \mathbb{R}$, then $\frac{1}{b^n} = b^{-n}$

Example 34. Write the following in exponential form.

a. $\sqrt{5}$

b. $\frac{1}{\sqrt[3]{17}}$

c. $\sqrt[3]{-23}$

Solution

a. $\sqrt{5} = 5^{\frac{1}{2}}$

b. $\frac{1}{\sqrt[3]{17}} = \frac{1}{17^{\frac{1}{3}}} = 17^{-\frac{1}{3}}$

c. $\sqrt[3]{-23} = (-23)^{\frac{1}{3}}$

Example 35. Simplify each of the following numbers

a. $(-125)^{\frac{1}{3}}$

b. $\left(\frac{256}{81}\right)^{\frac{1}{4}}$

c. $(-1)^{\frac{1}{11}} \times (-216)^{\frac{1}{3}}$

d. $(0.00032)^{\frac{1}{5}}$

Solution

a. $(-125)^{\frac{1}{3}} = \sqrt[3]{-125} = \sqrt[3]{(-5)^3} = -5$

b. $\left(\frac{256}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{256}{81}} = \sqrt[4]{\frac{5^4}{3^4}} = \sqrt[4]{\left(\frac{5}{3}\right)^4} = \left|\frac{5}{3}\right|$

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c. $(-1)^{\frac{1}{11}} \times (-216)^{\frac{1}{3}} = {}^{11}\sqrt{-1} \times {}^3\sqrt{-216} = {}^{11}\sqrt{(-1)^{11}} \times {}^3\sqrt{(-6)^3} = (-1) \times (-6)$

d. $(0.00032)^{\frac{1}{5}} = ((0.2)^5)^{\frac{1}{5}} = 0.2$

ACTIVITY 1.3

1. In each of the following, check whether x and y are equal.

a. $x = 27^{\frac{1}{3}} \times 8^{\frac{1}{3}} \quad ; \quad y = (27 \times 8)^{\frac{1}{3}}$

c. $x = 16^{\frac{1}{2}} \times 81^{\frac{1}{2}} \quad ; \quad y = (16 \times 81)^{\frac{1}{2}}$

b. $x = {}^3\sqrt{27} \times {}^3\sqrt{8} \quad ; \quad y = {}^3\sqrt{27 \times 8}$

d. $x = \sqrt{16} \times \sqrt{81} \quad ; \quad y = \sqrt{16 \times 81}$

2. In each of the following, check whether x and y are equal.

a. $x = 27^{\frac{1}{3}} \div 8^{\frac{1}{3}} \quad ; \quad y = (27 \div 8)^{\frac{1}{3}}$

b. $x = \sqrt{16} \div \sqrt{81} \quad ; \quad y = \sqrt{16 \div 81}$

Theorem 1.2

1. For any two real numbers a and b and for all integers $n \geq 2$,

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$$

2. For any two real numbers a and b where $b \neq 0$ and for all integers $n \geq 2$

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$$

Remark: From Theorem 1.2 and Definition 1.2, we can conclude that

1. For any two real numbers a and b and for all integers $n \geq 2$,

$${}^n\sqrt{a} \times {}^n\sqrt{b} = {}^n\sqrt{a \times b}$$

2. For any two real numbers a and b where $b \neq 0$ and for all integers $n \geq 2$

$$\frac{{}^n\sqrt{a}}{{}^n\sqrt{b}} = {}^n\sqrt{\frac{a}{b}}$$

Example 36. Simplify each of the following numbers

a. $(625 \times 16)^{\frac{1}{4}}$

c. $\sqrt{36x^2}$

e. $(1024 \div 243)^{\frac{1}{5}}$

b. $9^{\frac{1}{3}} \times 12^{\frac{1}{3}} \times 4^{\frac{1}{3}}$

d. $(-\frac{27}{125}a^3b^6)^{\frac{1}{3}}$

f. $\frac{\sqrt{54x^3y}}{\sqrt{24xy}}$

Solution

a. $(625 \times 16)^{\frac{1}{4}} = 625^{\frac{1}{4}} \times 16^{\frac{1}{4}} = {}^4\sqrt{625} \times {}^4\sqrt{16} = {}^4\sqrt{5^4} \times {}^4\sqrt{2^4} = 5 \times 2 = 10$

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- b. Here, since 9, 12 and 2 are not perfect cubes, it is better to combine the powers based on definition 1.2

$$\begin{aligned}9^{\frac{1}{3}} \times 12^{\frac{1}{3}} \times 2^{\frac{1}{3}} &= \sqrt[3]{9} \cdot \sqrt[3]{12} \cdot \sqrt[3]{2} = \sqrt[3]{9 \times 12 \times 2} \\&= \sqrt[3]{3^2 \times (2^2 \times 3) \times 2} = \sqrt[3]{3^3 \times 2^3} = \sqrt[3]{3^3} \times \sqrt[3]{2^3} = 3 \times 2 = 6\end{aligned}$$

c. $\sqrt{36x^2} = \sqrt{6^2x^2} = \sqrt{6^2} \cdot \sqrt{x^2} = |6| \cdot |x| = 6|x|$

d. $\left(-\frac{27}{125}a^3b^6\right)^{\frac{1}{3}} = \sqrt[3]{-\frac{27}{125}a^3b^6} = \sqrt[3]{-\frac{3^3}{5^3}a^3(b^3)} = \sqrt[3]{\left(-\frac{3}{5}\right)^3 a^3(b^3)} = -\frac{3}{5}ab^2$

e. $(1024 \div 243)^{\frac{1}{5}} = \left(\frac{1024}{243}\right)^{\frac{1}{5}} = \frac{(1024)^{\frac{1}{5}}}{(243)^{\frac{1}{5}}} = \frac{(4^5)^{\frac{1}{5}}}{(3^5)^{\frac{1}{5}}} = \frac{\sqrt[5]{4^5}}{\sqrt[5]{3^5}} = \frac{4}{3}$

f. $\frac{\sqrt{54x^3y}}{\sqrt{24xy}} = \sqrt{\frac{54x^3y}{24xy}} = \sqrt{\frac{54}{24} \cdot \frac{x^3}{x} \cdot \frac{y}{y}} = \sqrt{\frac{9}{4} \cdot x^2 \cdot 1} = \sqrt{\left(\frac{3}{2}\right)^2 \cdot x^2} = \frac{3}{2}|x|$

Rule of Exponents

Rules of exponents will make simplifications of radicals and powers more easier.

Theorem 1.3 (Rule of Exponents)

If a and b are two non-zero real numbers and m and n are real numbers, then we have the following laws of exponents; provided that each powers are defined:

- | | |
|---|---|
| 1. $a^0 = 1, a \neq 0$ and $a^1 = a$ | 5. $a^n \times b^n = (ab)^n$ |
| 2. $a^m \times a^n = a^{m+n}$ | 6. $\frac{a^m}{a^n} = a^{m-n}$ |
| 3. $(a^m)^n = a^{mn} = (a^n)^m$ | 7. $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$ and $\frac{1}{a^{-n}} = a^n$ |
| 4. $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$ | 8. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ |

For example;

- $5^3 \times 5^7 = 5^{3+7} = 5^{10}$
- $8^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}} = \sqrt{8 \times 2} = \sqrt{16} = 4$

Definition 1.11. For $a \geq 0$ and p and q any two positive integers,

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}$$

Example 37. Simplify each of the following

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a. $\sqrt[3]{5} \times \sqrt[5]{8} \times \sqrt[3]{\frac{1}{5}} \times \sqrt[5]{4}$

c. $\frac{5^{\frac{1}{8}} \times 27^{\frac{1}{5}}}{(5)^{-\frac{1}{8}} \times 9^{\frac{1}{5}}}$

b. $\frac{1}{3}\sqrt{27x^4y^2}$

e. $\frac{\sqrt{98x^3y^2}}{14\sqrt{xy}}, x > 0, y > 0$

Solution

a. $\sqrt[3]{5} \times \sqrt[5]{8} \times \sqrt[3]{\frac{1}{5}} \times \sqrt[5]{4} = \sqrt[3]{5 \times \frac{1}{5}} \times \sqrt[5]{8 \times 4} = \sqrt[3]{1} \times \sqrt[5]{32} = 1(2) = 2$

b. $\frac{1}{3}\sqrt{27x^4y^2} = \frac{1}{3}\sqrt{3^2(3)x^2 \cdot x^2 \cdot y^2} = \frac{1}{3} \times 3|x| \cdot |x| |y| \sqrt{3} = 1 \cdot |x|^2 |y| \sqrt{3} = x^2 |y| \sqrt{3}$

c. $\frac{5^{\frac{1}{8}} \times 27^{\frac{1}{5}}}{(5)^{-\frac{1}{8}} \times 9^{\frac{1}{5}}} = 5^{\frac{1}{8} - (-\frac{1}{8})} \times (\frac{27}{9})^{\frac{1}{5}} = 5^{\frac{1}{4}} \times 3^{\frac{1}{5}} = \sqrt[4]{5} \cdot \sqrt[5]{3}$

d. $\frac{\sqrt{98x^3y^2}}{14\sqrt{xy}} = \frac{\sqrt{2(7^2)x^3y^2}}{14\sqrt{xy}} = \frac{7}{14} \sqrt{\frac{2x^3y^2}{xy}} = \frac{1}{2} \sqrt{2x^2y} = \frac{1}{2} |x| \sqrt{2y} = \frac{1}{2} x \sqrt{2y}$, since $x > 0$.

Addition and Subtraction of Radicals

Theorems 1.2 and 1.3, and the rules of exponent mainly deals with multiplication and division of radicals as well as multiplication and division of powers. In this sub-topic we shall discuss about addition and subtraction of radicals.

Definition 1.12 Radicals that have the same index and the same radicand are called **LIKE RADICALS**. But they may have different coefficients.

For example;

i. $2\sqrt{2}, \sqrt{2}, \frac{3}{4}\sqrt{2}, \dots$ are like radicals.

ii. $3\sqrt[3]{2}, \sqrt[5]{2}, \sqrt[3]{3}, \dots$ are not like radicals.

Rule:

For $a \in \mathbb{R}, n \in \mathbb{N}$ and p and q any two real numbers;

i. $p\sqrt[n]{a} + q\sqrt[n]{a} = (p + q)\sqrt[n]{a}$, and

ii. $p\sqrt[n]{a} - q\sqrt[n]{a} = (p - q)\sqrt[n]{a}$

Example 38. Perform the operations and simplify each of the following

a. $\sqrt{6}(\sqrt{12} - \sqrt{3})$

d. $(5\sqrt[3]{54} - 2\sqrt[3]{2}) \times \frac{1}{\sqrt[3]{16}}$

b. $\sqrt{75} - \sqrt{48}$

e. $8\sqrt{24} + \frac{2}{3}\sqrt{54} - 2\sqrt{96}$

c. $\sqrt{2x^2} - \sqrt{50x^2}$

Solution

$$\begin{aligned}
\text{a. } \sqrt{6}(\sqrt{12} - \sqrt{3}) &= \sqrt{6} \cdot \sqrt{12} - \sqrt{6} \cdot \sqrt{3} = \sqrt{72} - \sqrt{18} \\
&= \sqrt{2 \times 6^2} - \sqrt{2 \times 3^2} = 6\sqrt{2} - 3\sqrt{2} = (6 - 3)\sqrt{2} = 3\sqrt{2} \\
\text{b. } \sqrt{75} - \sqrt{48} &= \sqrt{3(5^2)} - \sqrt{3(4^2)} = 5\sqrt{3} - 4\sqrt{3} = \sqrt{3} \\
\text{c. } \sqrt{2x^2} - \sqrt{50x^2} &= \sqrt{2}|x| - 5\sqrt{2}|x| = (1 - 5)\sqrt{2}|x| = -4\sqrt{2}|x| \\
\text{d. } 5\sqrt[3]{54} - 2\sqrt[3]{2} &= 5\sqrt[3]{2 \times 27} - 2\sqrt[3]{2} = 5\sqrt[3]{2} \times \sqrt[3]{27} - 2\sqrt[3]{2} \\
&= 5(3)\sqrt[3]{2} - 2\sqrt[3]{2} = 15\sqrt[3]{2} - 2\sqrt[3]{2} = 13\sqrt[3]{2} \\
\text{e. } 8\sqrt{24} + \frac{2}{3}\sqrt{54} - 2\sqrt{96} &= 8\sqrt{4 \times 6} + \frac{2}{3}\sqrt{9 \times 6} - 2\sqrt{16 \times 6} \\
&= 8 \times 2\sqrt{6} + \frac{2}{3} \times 3\sqrt{6} - 2 \times 4\sqrt{6} = 16\sqrt{6} + 2\sqrt{6} - 8\sqrt{6} \\
&= (16 + 2 - 8)\sqrt{6} = 10\sqrt{6}
\end{aligned}$$

1.2.5 The Four Operations on Real Numbers

In this section, you will discuss operations on the set of real numbers. The properties you have studied so far will help you to investigate many other properties of the set of real numbers. Basic properties that govern addition and multiplication of real numbers are stated below.

1. Closure Properties

The set real numbers is closed under addition and multiplication. This means that the sum and product of two real numbers is always a real number. That is

For any $a, b \in \mathbb{R}$, we have always that $a + b \in \mathbb{R}$ and $a \times b \in \mathbb{R}$.

2. Commutative Properties

Addition and multiplication of real numbers are commutative in $\mathbb{R}$.

That is, for all $a, b \in \mathbb{R}$,

$$\text{i. } a + b = b + a \quad \text{and} \quad \text{ii. } a \times b = b \times a$$

3. Associative Properties

Addition and multiplication of real numbers are associative in $\mathbb{R}$.

That is, for all a, b and $c \in \mathbb{R}$,

$$\text{i. } (a + b) + c = a + (b + c) \quad \text{and} \quad \text{ii. } (a \times b) \times c = (a \times b) \times c$$

4. Existence of Additive and Multiplicative Identities

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There are real additive and multiplicative identities 0 and 1, respectively, on the set of real numbers such that

- i. $a + 0 = a = 0 + a$, for all $a \in \mathbb{R}$, and 0 is called **ADDITIVE IDENTITY** on $\mathbb{R}$.
- ii. $a \times 1 = a = 1 \times a$, for all $a \in \mathbb{R}$, and 1 is called **MULTIPLICATIVE IDENTITY** on $\mathbb{R}$.

5. Existence of Additive and Multiplicative Inverses

- i. Every real number $a \in \mathbb{R}$, has an **ADDITIVE INVERSE** $-a \in \mathbb{R}$ such that
 $a + (-a) = 0 = (-a) + a$ and $-a$ is called the **additive inverse** of a on $\mathbb{R}$.
- ii. Every non-zero real number $a \in \mathbb{R}$, has a **MULTIPLICATIVE INVERSE** $\frac{1}{a} \in \mathbb{R}$ such that $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$ and $\frac{1}{a}$ is called the **multiplicative inverse** of a on $\mathbb{R}$.

6. Distributive Property

Multiplication is distributive over addition on $\mathbb{R}$. That is if $a, b, c \in \mathbb{R}$, then

- i. $a \times (b + c) = a \times b + a \times c$
- ii. $(b + c) \times a = b \times a + c \times a$

Exercise 1.6

1. Approximate the following irrational numbers as decimals rounded to three decimal places.

- a. $\sqrt{5}$
- b. $\sqrt{11}$
- c. $\sqrt{7}$
- d. $\sqrt{10}$

2. Find the numerical values of each of the following

- a. $(4^{-1})^4 \times 2^5 \times \left(\frac{1}{16}\right)^3 \times (8^{-2})^5 \times (64^2)^3$
- b. $\sqrt{176} - 2\sqrt{275} + \sqrt{1584} - \sqrt{891}$
- c. $\sqrt[4]{0.0001} - \sqrt[5]{0.00032}$
- d. $2\sqrt[3]{0.125} + \sqrt[4]{0.0016}$
- e. $5\sqrt{18} - 3\sqrt{72} + 4\sqrt{50}$

3. If $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$, then what is the values of $\frac{2x + 3y}{z - 4x}$?

4. If $\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right) - \left(\frac{\sqrt{2}}{\sqrt{2}-1}\right) = 2m$, then find the value of m .

5. Find the simplest value of $\left(\sqrt{3 \times 2^2 + \sqrt{2^3 + \sqrt{4^3}}}\right)^{-\frac{1}{2}}$.

6. a) $\frac{3\sqrt{24} - 2\sqrt{18}}{-\sqrt{2}} =$ _____

b) The simplest form of $\frac{\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) - \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)}{(\sqrt{2}+1)^2 - 3} =$ _____

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7. Let x be a multiplicative inverse of 0.6, Then what is the additive inverse of x .

8. Find the additive inverse of each of the following real numbers.

a. 5 b. $-\frac{1}{2}$ c. $\sqrt{2} + 1$ d. $-\sqrt{3} + 2$

9. Find the multiplicative inverse of each of the following real numbers

a. 3 b. $\sqrt{5}$ c. $\frac{2}{3}$ d. $1 - \sqrt{3}$

10. Simplify each of the following expressions.

a. $\sqrt{180} + 2\sqrt{80} - \frac{2}{3}\sqrt{45}$ b. $\sqrt[3]{-0.027} - 2^{\frac{1}{2}} \times (1.28)^{0.5}$

c. $\left(\sqrt[3]{54 \sqrt[5]{\frac{1}{32}}} \right)^{-1}$ d. $\sqrt[5]{\sqrt{32}} \times \sqrt[3]{16} \times \sqrt[3]{\sqrt{2}}$

e. $\frac{\left(\frac{1}{2^2} - 2^{-\frac{1}{2}} \right)^2}{2^{-2}}$ f. $\frac{\sqrt{48}}{2\sqrt{27} - \sqrt{75}}$

g. $(0.04)^{-1.5} \times (0.125)^{\frac{-2}{3}} - \left(\frac{1}{125} \right)^{\frac{-2}{3}}$ h. $\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}$

i. $\frac{\sqrt{7 + \sqrt{24}}}{2} + \frac{\sqrt{7 - \sqrt{24}}}{2}$ j. $\frac{\sqrt{a + 2\sqrt{ab}} + b}{\sqrt{a} + \sqrt{b}}$

1.2.6 Limits of Accuracy

In this sub-section, you shall discuss certain concepts such as **approximation, accuracy in measurements, significant figures (s.f), decimal places (d.p)** and **rounding off numbers**.

In addition to this, you shall discuss how to give appropriate upper and lower bounds for data to a specified accuracy (for example measured lengths).

1. Counting and Measuring

Counting and measuring are an integral part of our daily life. Most of us do so for various reasons and at various occasions.

For example, you can count the money you receive from someone, a tailor measures the length of the shirt he/she makes for us, and a carpenter counts the number of screws required to make a desk.

Counting:

Counting is the process of counting involves finding out the exact number of things.

For example, you do counting to find out the number of students in a class. The answer

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is an exact number and is either correct or, if you have made a mistake, incorrect. On many occasions, just an estimate is sufficient and the exact number is not required or important.

Measuring:

If you are finding the length of a football field, the weight of a person or the time it takes to walk down to school, you are measuring. The answers are not exact numbers because there could be errors in measurements.

2. Estimation

In many instances, exact numbers are not necessary or even desirable. In those conditions, approximations are given. The approximations can take several forms. Here you shall deal with the common types of approximations.

A. Rounding

If 38,518 people attend a football game this figure can be reported to various levels of accuracy.

To the nearest 10,000 this figure would be rounded up to 40,000.

To the nearest 1000 this figure would be rounded up to 39,000.

To the nearest 100 this figure would be rounded down to 38,500

In this type of situation, it is unlikely that the exact number would be reported.

B. Decimal Places (d.p)

A number can also be approximated to a given number of decimal places (d.p). This refers to the number of figures written after a decimal point.

Example 39. Round off the number 28,617 to nearest

a. 10,000

b. 1000

c. 100

d. 10

Solution

a. $28,617 = 20,000 + \underline{8617} \rightarrow 10,000$

Therefore, 28,617 rounded to the nearest 10,000s is 30,000

b. $28,617 = 28,000 + \underline{617} \rightarrow 1000$

Therefore, 28,617 rounded to the nearest 1000s gives 29,000

c. $28,617 = 28,600 + \underline{17} \rightarrow 0$

Therefore, 28,617 rounded to the nearest 100s is 28,600

d. $28,617 = 28,610 + \underline{7} \rightarrow 10$

Therefore, 28,617 rounded to the nearest 10 is 28,620.

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A number can also be approximated to a give number of decimal places (d.p). This refers to the **number of figures written after a decimal point.**

Example 40.

- a. When we write 7.864 to 1 d.p we obtain 7.9
- b. When we write 5.574 to 2 d.p, 5.57
- c. When we write $6.\overline{437}$ to 1d.p., we obtain 6.4
- d. When we write 4.56556555. . . to 1 d.p., we obtain 4.6.

C. Significant Figures (s.f)

Significant figures may be more useful than decimal places as a means of specifying accuracy because they are independent of the unit used.

For example $49\text{m} = 4900\text{ cm} = 0.049\text{ km}$. In each case, it is implied that the length is accurate to two significant figures.

Numbers can be approximated to a given number of significant figures. When we desire to use significant figures to indicate the accurate of approximation, we count the number of digits in the number from left to right beginning at the first non - zero digit, which is known as the number of significant figures

Rules to determine whether a zero is significant or not

1. Zero appearing anywhere between two non- zero digits are significant.

For example;

- | | |
|-----------------------|------------------------|
| a) 101.1203 has 7 s.f | d) 0.6205004 has 7 s.f |
| b) 1.072 has 4 s.f | e) 300 has 1 s.f |
| c) 20121 has 5 s.f | f) 20 has 1s.f |

2. Trailing zero in a number containing a decimal point are significant

For example;

- a) 12.2300 has 6 s.f, the underline zeros are Trailing zero.
- b) 0.000122300 has 6 s.f, the underline zeros are Trailing zero.
- c) 21.5320000 has 9 s.f, the underline zeros are Trailing zero
- d) 0.05001000 has 7 s.f ,the underline zeros are Trailing zero

3. Leading zeros are not significant.

For example;

- a) 0.00052 has 2 s.f the underline zeros are leading zeros

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- b) 0.00682 has 3 s.f the underline zeros are leading zeros
- c) 0.00510002 has 6 s.f the underline zeros are leading zeros
- d) 0.000147 has 3 s.f the underline zeros are leading zeros

4. All non-zero digits are significant.

For example;

- a) 91 has 2 s.f
- b) 123.517 has 6 s.f

Accuracy in Measurement

Upper Bound (u.b) and Lower Bound (l.b)

In measurement, sometimes it is necessary to give appropriate **upper** and **lower bounds** for data to a specified accuracy (for example: numbers rounded off or numbers expressed to a given number of significant figures).

Numbers can be written to different degrees of accuracy. Suppose a data has measure x .

1. If x is a whole number, then
the lower bound of x is $x - 0.5$ and its upper bound of x is $x + 0.5$.
2. If x is correct to 1 d.p, then
the lower bound of x is $x - 0.05$ and its upper bound of x is $x + 0.05$.
3. If x is correct to 2 d.p, then
the lower bound of x is $x - 0.005$ and its upper bound of x is $x + 0.005$, and so on.

Example 41. Find the upper bound and lower bound of each of the following

- a. 6
- b. 4.7
- c. 0.83

Solution:

- a. $x = 6$ implies l.b of $x = 6 - 0.5 = 5.5$ and u.b of $x = 6 + 0.5 = 6.5$
- b. $x = 4.7$ implies l.b of $x = 4.7 - 0.05 = 4.65$ and u.b of $x = 4.7 + 0.05 = 4.75$
- c. $x = 0.83$ implies l.b of $x = 0.83 - 0.005 = 0.825$ and u.b of $x = 0.83 + 0.005 = 0.835$

Note: A number can be approximated by

- i. by rounding
- ii. by writing a given number of decimal place and
- iii. by expressing to a given number of significant figure

1.2.7 Scientific Notation (Standard Form)

In science and technology, it is usual to see very large and very small numbers. Very large and very small numbers may sometimes be difficult to work with or write. Hence you often write very large or very small numbers in scientific notation also called **Standard Form** or **Scientific Notation**.

Note: If n is a positive integer, multiplying a number by 10^n moves its decimal point n - places to the right, and multiplying it by 10^{-n} moves the decimal point n - places to the left.

Definition 1.13 A number is said to be in **SCIENTIFIC NOTATION (or STANDARD FORM)**, if it is written as a product of the form

$$a \times 10^k, \text{ where } 1 \leq a < 10 \text{ and } k \text{ is an integer}$$

For example; the scientific notation form of

a. 243,900,000 is 2.439×10^8

b. 0.000000595 is 5.95×10^{-7}

Example 42. Find the scientific notation form of the following numerals:

a. 0.00767

c. $(0.00012)^2$

e. 0.067×10^{-3}

b. 400,400

d. $(0.002)^{-5}$

Solution

a. $0.00767 = 7.67 \times 10^{-3}$

b. $400,400 = 4.004 \times 10^5$

c. $(0.00012)^2 = (1.2 \times 10^{-4})^2 = (1.2)^2 \times (10^{-4})^2 = 1.44 \times 10^{-8}$

d. $(0.002)^{-5} = (2 \times 10^{-3})^5 = (2)^5 \times (10^{-3})^5 = 32 \times 10^{-15} = 3.2 \times 10^{-14}$

e. $0.067 \times 10^{-3} = 6.7 \times 10^{-2} \times 10^{-3} = 6.7 \times 10^{-2+(-3)} = 6.7 \times 10^{-5}$

Example 43. Express each of the following numbers in ordinary decimal notation.

a. 4.882×10^5

b. 1.19×10^{-5}

c. 2.021×10^2

Solution

a. $4.882 \times 10^5 = 488,200$

b. $1.19 \times 10^{-5} = 0.0000119$

c. $2.021 \times 10^2 = 202.1$

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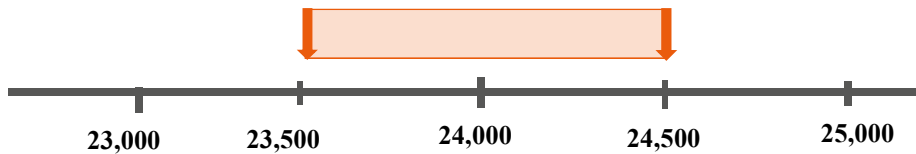
Example 44. 24,000 kg of cheese produced by local factory this year.

- a. If the amount of cheese was rounded to the nearest 1000 kg, what is the smallest amount that could have been produced? What is the largest amount? 24,499kg

Solution. 24 000 kg to the nearest 1000 kg.

The smallest amount = lower bound = 23,500 kg

The largest amount = upper bound = 24,499kg

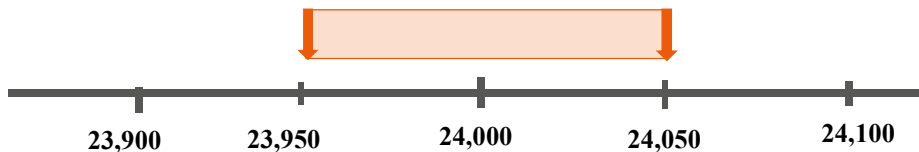


- b. If the amount of cheese was rounded to the nearest 100 kg, what is the smallest amount that could have been produced? 23,950 kg

Solution. 24 000 kg to the nearest 100 kg.

The smallest amount = lower bound = 23,950 kg

The largest amount = upper bound = 24,049 kg



Example 45. A team cycled a total distance of 360 km to raise funds for sports equipment.

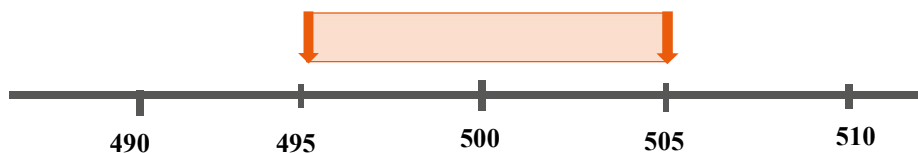
This number was rounded to the nearest 10.

- a. What is the smallest possible total distance?
b. What is the largest possible total distance?

Solution: a. 355 km b. 364 km

Example 46. 500 is rounded to 1 significant figure.

- a. What is the lower bound? b. What is the upper bound?

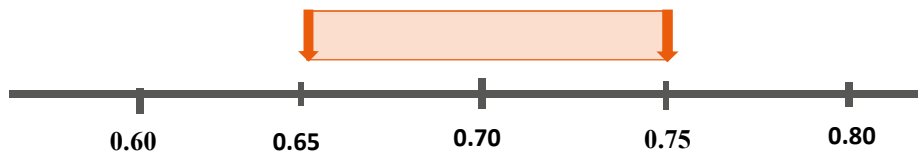


Solution. a. 495 b. 505

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Example 47. 0.7 is rounded to 1 significant figure.

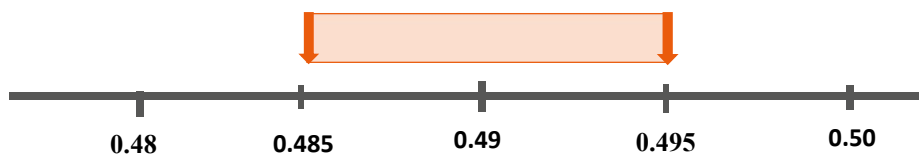
- a. What is the lower bound? b. What is the upper bound?



Solution. a. 0.65 b. 0.75

Example 48. 0.49 is rounded to 2 significant figures.

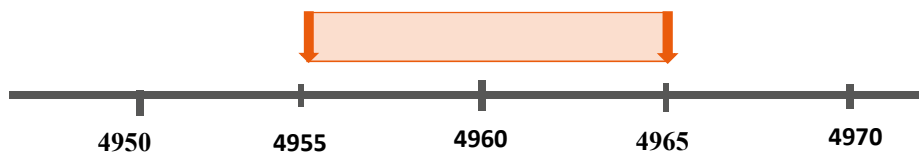
- a. What is the lower bound? b. What is the upper bound?



Solution. a. 0.485 b. 0.495

Example 49. 4960 is rounded to 3 significant figures.

- a. What is the lower bound? b. What is the upper bound?



Solution. a. 4955 b. 4965

Exercise 1.7

- Round the following numbers to the nearest 10,000.
 - 78,540
 - 950
 - 14,099
 - 2,984
- Round the following numbers of the nearest 100.
 - 78,540
 - 950
 - 14,099
 - 2,984
- Round the following numbers of the nearest integer.
 - 157.39
 - 12.049
 - 0.98
 - 2.95
- The distance around the world is approximately 40 000 km. This number was rounded to the nearest 1000.
 - What is the smallest possible distance?
 - What is the largest possible distance?

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5. Find the scientific notation form of the following numerals:

a. 0.00000108

c. $\sqrt{0.000064}$

e. $\frac{1}{0.0000002}$

b. 7,300,450

d. $10,531 \times 10^{-13}$

1.2.8 Rationalization

ACTIVITY 1.4

1. Find an approximate values of the following to two decimal places.

a. $\frac{1}{\sqrt{2}}$

b. $\frac{\sqrt{2}}{2}$

2. Compare the results of the approximations obtained in 1a and 1b. Are they equivalent?

3. Both $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ are irrational numbers. Which number will be more appropriate to handle in calculation work?

4. By what simplest number would you multiply each of the following expressions (irrational numbers) to obtain rational number? Discuss with your friends.

a. $\sqrt{2}$

b. $5\sqrt{3}$

c. $\sqrt[3]{5}$

Thus, you see that in order to evaluate an expression with a radical in the denominator, first you should transform the expression into an equivalent expression with a rational number in the denominator. The technique of transferring the irrational radical expression from the denominator to the numerator is called **RATIONALIZING THE DENOMINATOR**. This process changes the denominator from irrational into a rational number.

The number that can be used as a multiplier to rationalize the denominator is called **THE RATIONALIZING FACTOR**.

A rationalizing factor is always has value equal to one.

For instance, to change $\sqrt{2}$ into rational, multiply $\sqrt{2}$ by $\sqrt{2}$ itself, and obtain $\sqrt{2} \times \sqrt{2} = 2$.

Hence, to rationalize the denominator of the irrational number $\frac{1}{\sqrt{2}}$, we multiply the number by $\frac{\sqrt{2}}{\sqrt{2}}$ as follow.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}. \text{ (Here the denominator becomes rational).}$$

Thus, the rationalizing factor is $\frac{\sqrt{2}}{\sqrt{2}}$ which is equal to one.

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This shows that multiplying a number by rationalizing factor doesn't change the value of the number. It changes only the form of the number.

In the above example, we can have other numbers to rationalize the denominator : like $\frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{2}}{2\sqrt{2}}$,

$\frac{\sqrt{18}}{\sqrt{18}} = \frac{3\sqrt{2}}{3\sqrt{2}}$, $\frac{10\sqrt{2}}{10\sqrt{2}}$ and so on. But $\frac{\sqrt{2}}{\sqrt{2}}$ is the simplest.

Note: Before you start rationalization, first you should write the number to be rationalized in its simplest form, without one.

Note:

i. For $b \geq 0$, if $\sqrt{b}$ is irrational number, then $\sqrt{b} \times \sqrt{b} = b$ is rational.

ii. For any $b \in \mathbb{R}$, if $\sqrt[3]{b}$ is irrational number, then

$$\sqrt[3]{b} \times \sqrt[3]{b^2} = \sqrt[3]{b \times b^2} = \sqrt[3]{b^3} = b \text{ is rational.}$$

iii. For any positive real number a and b , $b \neq 0$, we have

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b}$$

There are different forms of rationalizing factors, depending upon the form of the number to be rationalized. Some of these are given below:

RULE 1.

a. If $b \in \mathbb{Z}$, $\sqrt{b}$ is an irrational number and c is any non-zero real number, then the

rationalizing factor of $\frac{c}{\sqrt{b}}$ is $\frac{\sqrt{b}}{\sqrt{b}}$, and $\frac{c}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{c\sqrt{b}}{b}$.

b. If $\sqrt[3]{b}$ is an irrational number and c is any non-zero real number, then the rationalizing factor

of $\frac{c}{\sqrt[3]{b}}$ is $\frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}}$, and $\frac{c}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{c\sqrt[3]{b^2}}{\sqrt[3]{b \times b^2}} = \frac{c\sqrt[3]{b^2}}{\sqrt[3]{b^3}} = \frac{c\sqrt[3]{b^2}}{b}$.

Example 50. In each of the following, state the rationalizing factor and rationalize the denominator. Write your answer in simplest form.

a. $\frac{6}{\sqrt{3}}$

b. $-2\sqrt{\frac{3}{7}}$

c. $\frac{5\sqrt{3}}{8\sqrt{5}}$

d. c. $\frac{3}{\sqrt[3]{2}}$

Solution

a. $\frac{6}{\sqrt{3}}$: The rationalizing factor is $\frac{\sqrt{3}}{\sqrt{3}}$, and

$$\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

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b. $-2\sqrt{\frac{3}{7}} = \frac{-2\sqrt{3}}{\sqrt{7}}$: The rationalizing factor is $\frac{\sqrt{7}}{\sqrt{7}}$, and

$$-2\sqrt{\frac{3}{7}} = \frac{-2\sqrt{3}}{\sqrt{7}} = \frac{-2\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = -\frac{2\sqrt{3 \times 7}}{\sqrt{7 \times 7}} = -\frac{2\sqrt{21}}{\sqrt{49}} = -\frac{2\sqrt{21}}{7}$$

c. $\frac{5\sqrt{3}}{8\sqrt{5}}$; the rationalizing factor is $\frac{\sqrt{5}}{\sqrt{5}}$, and

$$\frac{5\sqrt{3}}{8\sqrt{5}} = \frac{5\sqrt{3}}{8\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{3 \times 5}}{8\sqrt{5 \times 5}} = \frac{5\sqrt{15}}{8\sqrt{25}} = \frac{5\sqrt{15}}{8 \times 5} = \frac{\sqrt{15}}{8}$$

d. $\frac{3}{\sqrt[3]{2}}$, the rationalizing factor is $\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$, and

$$\frac{3}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}} \times \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{3 \times \sqrt[3]{4}}{\sqrt[3]{2 \times 4}} = \frac{3\sqrt[3]{4}}{\sqrt[3]{8}} = \frac{3\sqrt[3]{4}}{2}$$

ACTIVITY 1.5

1. By what simplest number would you multiply each of the following expressions (irrational numbers) to obtain rational number? Discuss with your friends.

a. $\sqrt{3} + 1$

c. $2\sqrt{3} - 1$

b. $2 - \sqrt{5}$

d. $3\sqrt{5} - 7\sqrt{2}$

2. Simplify the following and identify which of these numbers are rational and which are irrational.

a. $(1 + \sqrt{2})(1 - \sqrt{2})$

c. $(\frac{3}{2}\sqrt{5} - \frac{1}{3}\sqrt{3})(\frac{3}{2}\sqrt{5} + \frac{1}{3}\sqrt{3})$

b. $(\sqrt{2} + 2\sqrt{5})(\sqrt{2} - 2\sqrt{5})$

d. $(1 + \sqrt{2})(1 + \sqrt{2})$

Using the fact that $(a - b)(a + b) = a^2 - b^2$.

You can rationalize the denominator of expression such as $\frac{1}{a+\sqrt{b}}$, $\frac{1}{\sqrt{a}-b}$, $\frac{1}{\sqrt{a}-\sqrt{b}}$ and $\frac{1}{m\sqrt{a}-n\sqrt{b}}$,

where $\sqrt{a}, \sqrt{b}$ are irrational numbers, as follows:

RULE 2. If $a, b \in \mathbb{Z}$, $\sqrt{a}$ and $\sqrt{b}$ are irrational numbers and n and m are any non-zero real number, then

a. the rationalizing factor of $\frac{1}{a+\sqrt{b}}$ is $\frac{a-\sqrt{b}}{a-\sqrt{b}}$ and

$$\frac{1}{a+\sqrt{b}} = \frac{1}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}} = \frac{a-\sqrt{b}}{a^2-(\sqrt{b})^2} = \frac{a-\sqrt{b}}{a^2-b}$$

b. the rationalizing factor of $\frac{1}{\sqrt{a}-b}$ is $\frac{\sqrt{a}+b}{\sqrt{a}+b}$, and

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$$\frac{1}{\sqrt{a}-b} = \frac{1}{\sqrt{a}-b} \times \frac{\sqrt{a}+b}{\sqrt{a}+b} = \frac{\sqrt{a}+b}{(\sqrt{a})^2 - b^2} = \frac{\sqrt{a}+b}{a-b^2}$$

c. the rationalizing factor of $\frac{1}{\sqrt{a}-\sqrt{b}}$ is $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, and

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{1}{\sqrt{a}-\sqrt{b}} \times \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \right) = \frac{\sqrt{a}+\sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

d. the rationalizing factor of $\frac{1}{m\sqrt{a}-n\sqrt{b}}$ is $\frac{m\sqrt{a}+n\sqrt{b}}{m\sqrt{a}+n\sqrt{b}}$, and

$$\frac{1}{m\sqrt{a}-n\sqrt{b}} = \frac{1}{m\sqrt{a}-n\sqrt{b}} \times \frac{m\sqrt{a}+n\sqrt{b}}{m\sqrt{a}+n\sqrt{b}} = \frac{m\sqrt{a}+n\sqrt{b}}{(m\sqrt{a})^2 - (n\sqrt{b})^2} = \frac{\sqrt{a}+\sqrt{b}}{m^2a - n^2b}$$

Example 51. In each of the following, state the rationalizing factor and rationalize the denominator. Write your answer in simplest form.

a. $\frac{2}{\sqrt{5}-\sqrt{3}}$

b. $\frac{3}{\sqrt{6}+3\sqrt{2}}$

c. $\frac{\sqrt{3}+4}{\sqrt{3}-2}$

d. $\frac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

Solution

a. $\frac{2}{\sqrt{5}-\sqrt{3}}$, the rationalizing factor is $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, and hence

$$\frac{2}{\sqrt{5}-\sqrt{3}} = \frac{2}{\sqrt{5}-\sqrt{3}} \times \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} \right) = \frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} = \frac{2(\sqrt{5}+\sqrt{3})}{5-3} = \frac{2(\sqrt{5}+\sqrt{3})}{2} = \sqrt{5} + \sqrt{3}$$

b. $\frac{3}{\sqrt{6}+3\sqrt{2}}$, the rationalizing factor is $\frac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}}$, and

$$\frac{3}{\sqrt{6}+3\sqrt{2}} = \frac{3}{\sqrt{6}+3\sqrt{2}} \times \frac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}} = \frac{3(\sqrt{6}-3\sqrt{2})}{(\sqrt{6})^2 - (3\sqrt{2})^2} = \frac{3(\sqrt{6}-3\sqrt{2})}{6-9 \times 2} = \frac{3(\sqrt{6}-3\sqrt{2})}{-12} = \frac{(\sqrt{6}-3\sqrt{2})}{-4} = \frac{3\sqrt{2}-\sqrt{6}}{4}$$

c. $\frac{\sqrt{3}+4}{\sqrt{3}-2}$, the rationalizing factor is $\frac{\sqrt{3}+2}{\sqrt{3}+2}$ and hence,

$$\frac{\sqrt{3}+4}{\sqrt{3}-2} = \frac{\sqrt{3}+4}{\sqrt{3}-2} \times \frac{\sqrt{3}+2}{\sqrt{3}+2} = \frac{(\sqrt{3}+4)(\sqrt{3}+2)}{(\sqrt{3})^2 - (2)^2} = \frac{11+6\sqrt{3}}{3-4} = \frac{11+6\sqrt{3}}{-1} = -11 - 6\sqrt{3}$$

ADDITIONAL RATIONALIZATION

Note: For any $a, b \in \mathbb{R}$,

1. $a^3 - b^3 = (a - b)(a^2 + ab + a^2)$ and

2. $a^3 + b^3 = (a + b)(a^2 - ab + a^2)$.

Substituting $a = \sqrt[3]{x}$ and $b = \sqrt[3]{y}$ in the above factorization gives

3. $x - y = (\sqrt[3]{x} - \sqrt[3]{y})(\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2})$ and

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$$4. \quad x + y = (\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2}).$$

You can use the above result to rationalize denominators involving cube roots.

RULE 3. If $a, b \in \mathbb{Z}$, $\sqrt[3]{x}$ and $\sqrt[3]{y}$ are irrational numbers, then the rationalizing factor of $\frac{1}{\sqrt[3]{x} - \sqrt[3]{y}}$ is $\frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}$, and $\frac{1}{\sqrt[3]{x} - \sqrt[3]{y}} \times \frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} = \frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{(\sqrt[3]{x} - \sqrt[3]{y})(\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2})} = \frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{x + y}$

Example 52. State the rationalizing factor and rationalize the denominator. Write your answer in simplest form $\frac{3}{\sqrt[3]{5} + 2}$.

Solution Since $\sqrt[3]{5} + 2 = \sqrt[3]{5} + \sqrt[3]{8}$ the rationalizing factor is $\frac{\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2}}{\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2}}$ and

$$\begin{aligned} \frac{3}{\sqrt[3]{5} + 2} &= \frac{3}{\sqrt[3]{5} + \sqrt[3]{8}} = \frac{3}{\sqrt[3]{5} + \sqrt[3]{8}} \times \frac{\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2}}{\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2}} = \frac{3}{\sqrt[3]{5} + \sqrt[3]{8}} \times \frac{3(\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2})}{(\sqrt[3]{5} + \sqrt[3]{8})(\sqrt[3]{5^2} - \sqrt[3]{5.8} + \sqrt[3]{8^2})} \\ &= \frac{3(\sqrt[3]{25} - \sqrt[3]{40} + \sqrt[3]{64})}{\sqrt[3]{5^3} + \sqrt[3]{8^3}} = \frac{3(\sqrt[3]{25} - \sqrt[3]{40} + \sqrt[3]{64})}{5 + 8} = \frac{3(\sqrt[3]{25} - \sqrt[3]{40} + \sqrt[3]{64})}{13} \end{aligned}$$

Exercise 1.8

In each of the following, rationalize the denominator and simplify.

a. $\frac{\sqrt{5}}{\sqrt{7}}$

e. $\frac{4\sqrt{2}}{\sqrt[3]{2}}$

i. $\sqrt{\frac{2\sqrt{3}}{\sqrt{5}-2}}$

b. $\frac{-5}{3\sqrt{11}}$

f. $\frac{2}{\sqrt{\sqrt{5}}}$

j. $\frac{1-\sqrt{3}}{5\sqrt{3}+3\sqrt{2}}$

c. $\frac{3}{7}\sqrt{\frac{3}{7}}$

g. $\frac{\sqrt{3}-2}{\sqrt{3}+2}$

k. $\frac{\sqrt[3]{2}}{2\sqrt[3]{3}-1}$

d. $\frac{\sqrt{2}}{\sqrt{2}-\sqrt{8}}$

h. $\frac{12}{\sqrt{13}-1}$

l. $\frac{1}{\sqrt{5}-\sqrt{2}-1}$

1.2.8. Euclid's Division Algorithm

A. The division algorithm

Theorem 1.4 (Division Algorithm)

Let a and b be two non-negative integers and $b \neq 0$. Then there exist unique non-negative integers q and r such that $a = (q \times b) + r$, with $0 \leq r < b$

In Theorem 1.4 above

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‘a’ is called **DIVIDEND**,

‘b’ is called **DIVISOR**

‘q’ is called **QUOTIENT**

‘r’ is called **REMAINDER**.

Example 1. if $a = 17$ and $b = 7$ write a in the form $b \times q + r$ where $0 \leq r < b$

$$\frac{a}{b} = \frac{17}{7}, q = 2 \text{ and } r = 3, a = b \times q + r = 7 \times 2 + 3$$

1 if $a = 111$ and $b = 3$ write a in the form $b \times q + r$ where $0 \leq r < b$

$$\frac{a}{b} = \frac{111}{3}, q = 37 \text{ and } r = 0, a = b \times q + r = 3 \times 37 + 0$$

3. if $a = 5$ and $b = 8$ write a in the form $b \times q + r$ where $0 \leq r < b$

$$\frac{a}{b} = \frac{5}{8}, q = 0 \text{ and } r = 5, a = b \times q + r = 8 \times 0 + 5$$

B. The Euclidean Algorithm

Theorem 1.5 Euclidean Algorithm

If a, b, q , and r are positive integers such that $a = (q \times b) + r$, then $\text{G.C.F}(a, b) = \text{G.C.F}(b, r)$

Example 51. Find $\text{G.C.F}(224, 84)$

Solution: To find GCF (224, 84), you first divide 224 by 84. The divisor and remainder of this division are then used as dividend and divisor, respectively, in a succeeding division. The process is repeated until a remainder 0 is obtained.

The complete process to find GCF (224, 84) is shown below.

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Computation	Division algorithm form	Application of Euclidean Algorithm
$\begin{array}{r} 2 \\ 84 \overline{) 224} \\ \underline{168} \\ 56 \end{array}$	$224 = (2 \times 84) + 56$	$\text{GCF}(224, 84) = \text{GCF}(84, 56)$
$\begin{array}{r} 1 \\ 56 \overline{) 84} \\ \underline{56} \\ 28 \end{array}$	$84 = (1 \times 56) + 28$	$\text{GCF}(84, 56) = \text{GCF}(56, 28)$
$\begin{array}{r} 2 \\ 28 \overline{) 56} \\ \underline{56} \\ 0 \end{array}$	$56 = (2 \times 28) + 0$	$\text{GCF}(56, 28) = 28$ <i>(by inspection)</i>

Conclusion $\text{GCF}(224, 84) = 28$.

KEY TERMS

bar notation	principal n th root
composite number	principal square root
divisible	radical sign
division algorithm	radicand
factor	rational number
fundamental theorem of arithmetic	rationalization
greatest common factor (GCF)	real number
irrational number	repeating decimal
least common multiple (LCM)	repetend
multiple	scientific notation
perfect square	significant digits
prime factorization	significant figures
prime number	terminating decimal



UNIT SUMMARY

- The sets of Natural numbers, Whole numbers, Integers and Rational numbers denoted by $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, and $\mathbb{Q}$, respectively are described by

$$\mathbb{N} = \{1, 2, 3, \dots\}, \mathbb{W} = \{0, 1, 2, \dots\}, \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \text{ and}$$

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

- A composite number is a natural number that has more than two factors.
- A prime number is a natural number that has exactly two distinct factors, 1 and itself.
- Prime numbers that differ by two are called twin primes.
- When a natural number is expressed as a product of factors that are all prime, then the expression is called the prime factorization of the number.
- **Fundamental theorem of arithmetic:** Every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

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- The greatest common factor (GCF) of two or more numbers is the greatest factor that is common to all numbers.
- The least common multiple (LCM) of two or more numbers is the smallest or least of the common multiples of the numbers.
- The set of natural numbers, whole numbers, integers with four fundamental operations is done.
- A rational number is a number which can be put in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. When q is made positive and p and q have no other common factor, then a rational number $\frac{p}{q}$ is said to be in standard form or lowest form.
- Any rational number can be expressed as a repeating decimal or a terminating decimal.
- Any terminating decimal or repeating decimal is a rational number.
- Irrational numbers are decimal numbers that neither repeat nor terminate.
- The set of real numbers denoted by $\mathbb{R}$ is defined by

$$\mathbb{R} = \{x: x \text{ is rational or } x \text{ is irrational}\}$$

- The set of irrational numbers is not closed under addition, subtraction, multiplication and division.
- The sum of an irrational and a rational number is always an irrational number.
- For any two real numbers a and b and for all integers $n \geq 2$,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \times b} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0$$

- A number is said to be written in scientific notation (standard notation), if it is written in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.
- Let a and b be two non-negative integers and $b \neq 0$, then there exist unique nonnegative integers q and r such that $a = (q \times b) + r$ with $0 \leq r < b$.
- If a, b, q and r are positive integers such that $a = q \times b + r$, then $\text{GCF}(a, b) = \text{GCF}(b, r)$.

REVIEW EXERCISE ON UNIT ONE

1. Find i) the prime factorization, ii) the prime factors and iii) the numbers of factors of each of the following numbers.

a. 1240

b. 1436

c. 1190

d. 34,235

2. Find the GCF and LCM of each of the following pairs or triplets of numbers.

a. $x = 2^5 \times 3^2 \times 7$; $y = 2 \times 3 \times 5^2 \times 7^2$

b. $x = 2^3 \times 3^2 \times 5 \times 11$; $y = 2^4 \times 3 \times 7$ and $z = 2^3 \times 11$

c. $x = 2625$; $y = 3240$

d. $x = 360$, $y = 105$ and $z = 98$

3. If x is an odd natural numbers such that $L.C.M(40, x) = 1400$, then find the possible value of x .

4. The LCM of two numbers is 45 times their GCF. If one of the two numbers is 125 and the sum of their LCM and GCF is 1150, then the other numbers is _____

5. For $a, b \in \mathbb{N}$, if $a + b = 55$, $GCF(a, b) = 5$ and $LCM(a, b) = 120$, then $\frac{1}{a} + \frac{1}{b} =$ _____

6. If the $LCM(x, y) = p$ and $GCF(p, z) = q$, then find the LCM of x, y and z .

7. If $\frac{x}{y} = \frac{2}{3}$ and $\frac{y}{z} = \frac{3}{4}$, then $\frac{(2x+y)}{3z} =$ _____

8. Convert each of the following decimal numbers into a fraction form $\frac{a}{b}$.

a. $-2.4\overline{27}$

b. $0.43\overline{8}$

c. $4.6\overline{8}$

9. Simplify each of the following expression.

a. $\left(\frac{2}{3}\right)^{-2} \left(\frac{2}{\sqrt{3}}\right)^4 \div \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^6$

d. $\sqrt{-12^3 \sqrt{\frac{-1}{27}} a^6}$, for $a > 0$

b. $\frac{\left(\frac{1}{5}\right)^{-1} + \left(\frac{-1}{125}\right)^{\frac{1}{3}}}{\left[\left(\frac{1}{25}\right)\left(\frac{1}{100}\right)\right]^{-1}}$

e. $\left(\sqrt[5]{64^3 \sqrt{\frac{-a^{15}}{8}}}\right)^{-1}$, for any $a \in \mathbb{R}$

c. $\frac{\frac{x+1}{x-1} + \frac{x-1}{x+1}}{\frac{x}{x^2-1}}$, for $x \neq -1, 1, 0$

f. $\frac{2\sqrt{72}}{3} - \frac{3\sqrt{128}}{4} + 5\sqrt{\frac{1}{2}}$

g. $\frac{(\sqrt{5-2\sqrt{6}})(\sqrt{5+2\sqrt{6}})}{\sqrt[3]{-125}}$

h. $\frac{\frac{a}{1-a} + \frac{a+1}{a}}{\frac{1-a}{a} - \frac{a}{a+1}}$, for $a \neq 0, 1$

10. For each of the following expression, rationalize the denominator and simplify.

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a. $\frac{1}{1 - \sqrt{2} - \sqrt{5}}$

c. $\frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}}$

e. $\frac{2}{\sqrt{7} + \sqrt{5} - \sqrt{2}}$

b. $\frac{22}{7 + \sqrt{5}} - \frac{1}{\sqrt{7} + \sqrt{5}} + \frac{9}{5 - \sqrt{7}}$

d. $\frac{1}{\sqrt[3]{3} - \sqrt[3]{2}}$

f. $\frac{2}{\sqrt[4]{4} + 2}$

11. Simplify and express each of the following in scientific notation form.

a. $(0.00012)^2$

c. $(0.002)^5$

e. $\frac{1.247 \times 10^{-3}}{1.1 \times 10^2}$

b. 0.0067×10^{-3}

d. $(2.6 \times 10^5)(9.2 \times 10^{-13})$

12. a) If $= \sqrt{\frac{5+2\sqrt{6}}{5-2\sqrt{6}}}$, then find the value of $\frac{1+x^2}{1-x^2}$.

b) If $x = \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ and $= \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$, then $\frac{1}{x} - \frac{1}{y} =$ _____

13. $1\frac{7}{13} + \left(1\frac{2}{5} - \left(1\frac{5}{11} \div 7\frac{1}{2} \times 1\frac{1}{4} + 1\frac{1}{2}\right)\right) =$ _____

14. The value of $0.\overline{2} + 0.\overline{3} + 0.\overline{32}$ in fraction form is _____

15. If $a : b = 2 : 3$ and $b : c = 4 : 5$, then $a : b : c =$ _____

16. a) Find three rational numbers between $\frac{1}{2}$ and $\frac{3}{5}$.

b) Insert three rational numbers between $2.535535553 \dots$ and $2.533553355533 \dots$

c) Find three irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

17. The simplest value of $\frac{3 \times 9^{n+1} + 9 \times 3^{2n-1}}{9 \times 3^{2n} - 6 \times 9^{n-1}}$ is equal to _____

18. Calculate the upper and lower bounds for the following calculations, if each of the numbers is given to 1 decimal place.

a. $19.7 + 73.5 - 25.9$

c. $\frac{21.4}{7.7}$

b. 21.4×12.3

d. $\frac{12.5 \times 23.3}{1.9}$

19. The mass of a sack of vegetables is given as 5.4 kg.

a. Illustrate the lower and upper bounds of the mass on a number line.

b. Using M kg for the mass, express the range of values in which it must lie, as an inequality.

20. The masses to the nearest 0.5 kg of two parcels are 1.5 kg and 2.5 kg. Calculate the lower and upper bounds of their combined mass.

21. **CHALLENGE PROBLEMS.** Show that $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$ is rational number.

22. **CHALLENGE PROBLEMS.** Show that $\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} = \sqrt{6}$

23. **CHALLENGE PROBLEMS.** Find the simplest numeral for $\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$