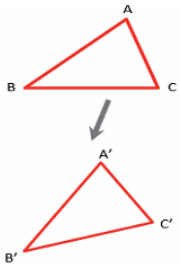
UNIT 6

TRANSFORMATION OF THE PLANE

A rigid motion is a motion which preserves distance. That means, for A and B, AB = A'B' where A' and B' are the images of A and B respectively as shown



- Rigid motion carries any plane figures to a congruent plane figure. That means rigid motion carries triangles to congruent triangles, parallelograms to congruent parallelograms, etc.
- Within the rigid motion, there are three main types of transformations that you will learn in this unit. These are translation, reflection and rotations.

TRANSLATION

- A translation is a transformation that occurs when every point of a figure is moved from one location to another location along the same direction through the same distance.
- If point P is translated to point P', then the vector PP' is called the translation vector.

You can state a translation formula in terms of coordinates as follows:

Let T = ***(a, b)*** be a translation vector, then

a. The origin is translated to ***(a, b)*** That means (0, 0) $\xrightarrow{T(a,b)}$ ***(a, b)***

b. The image of the point $P\,(x, y)$ under the translation vector T will be the point $P'\,(x + a, y + b)$.

EXAMPLE

Suppose you have a vector $\mathbf{v} = (2, 3)$, and you want to translate it by another vector $\mathbf{t} = (4, -1)$.

Translation:

Add the two vectors:

$$\mathbf{v'} = \mathbf{v} + \mathbf{t} = (2, 3) + (4, -1) = (2 + 4, 3 + (-1)) = (6, 2)$$

So, the translated vector is $\mathbf{v'} = (6, 2)$.

EXAMPLE 2

If A (−4, −4), B (−2, −1) and C (−1, −5) are the vertices of a triangle ABC, find the coordinates of the image of ΔABC under the translation T = (6, 5). Draw ΔABC and its image on the same plane.

ANSWER

As A (−4, −4), B (−2, −1) and C (−1, −5) are the vertices of ΔABC, the coordinates of the vertices of image of ΔABC can be obtained by using the formula below:

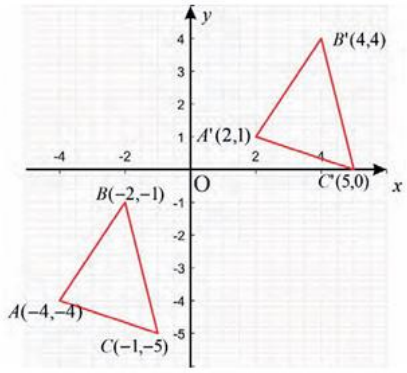
you have, P(x, y)=(x + a,y + b) =(x + 6, y + 5).

Hence, A': (−4 + 6, −4 + 5) = (2, 1).

B': (−2 + 6 , −1 + 5) = (4, 4).

C': (−1 + 6, −5 + 5) = (5, 0).

Graphically, you can show these points as shown in Figure 6.4.



LINE TRANSLATION

PROCEDURE

1. **Original Line Equation**: Start with the equation of the line, say $y = mx + c$.
2. **Translation Vector**: Suppose the translation vector is (a, b), where a is the horizontal shift and b is the vertical shift.
3. **Translate**:
   - Replace x with $(x - a)$.
   - Replace y with $(y - b)$.
4. **New Equation**: The new translated line equation becomes:

$$(y - b) = m(x - a)$$

EXAMPLE

If a translation T takes the point $(-2, 3)$ to the point $(1, 2)$, then what are the images of the following line

a. L: $3x - y + 4 = 0$

ANSWER

Let T be the translation vector, A $= (-2, 3)$ and $B = (1, 2)$. Then,

T $= \overrightarrow{AB}$ =B – A = (1−(−2), 2−3) = (3, −1). Thus, T:(3, −1).

The point P (x, y) is translated to the point P′ $= (x', y')$.

Here, $x' = x + 3, y' = y - 1$.

Therefore, $x = x' - 3, y = y' + 1$.

A translation map lines onto parallel lines. Let L' be the image of L under T. Then,

a. $L: 3x - y + 4 = 0$

$\Rightarrow \quad 3(x' - 3) - (y' + 1) + 4 = 0$

$\Rightarrow \quad 3x' - 9 - y' - 1 + 4 = 0$

$\Rightarrow \quad 3x' - y' - 6 = 0$

$\Rightarrow L': 3x - y - 6 = 0.$

CIRCLE TRANSLATION

PROCEDURE

1. **Original Circle Equation**: Start with the equation of the circle $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius.
2. **Translation Vector**: Suppose the translation vector is (a, b), where a is the horizontal shift and b is the vertical shift.
3. **Translate**:
   - The new center becomes $(h + a, k + b)$.
4. **New Circle Equation**: The translated equation becomes:

$$(x - (h + a))^2 + (y - (k + b))^2 = r^2$$

REFLECTIONS

- Let L be a fixed line in the plane. A reflection A about a line L is a transformation of the plane onto itself which carries point A of the plane into the point A' of the plane such that L is the perpendicular bisector of AA'. The line L is called the line of reflection or the axis of reflection

BASIC PROPERTIES OF REFLECTION

1. A reflection about a line L has the property that, for two points **A** and **B** in the plane, if **A** = **B**, then M(**A**) = M(**B**). Hence, **reflection** is a function mapping the set of points in the plane into the set of points in the plane except reflection of points about the axis of reflection.
2. A reflection about a line L maps distinct points to distinct points, i.e., if **A** ≠ **B**, then M(**A**) ≠ M(**B**). Equivalently, it has the property that, for points **A** and **B** in the plane, M(**A**) = M(**B**), then **A** = **B**. Thus, reflection is **a one-to-one mapping.**
3. For every point **A'** in the plane, there exists a point **A** such that M(**A**) = **A'**. If the point **A'** is on L, then there exists **A** = **A'** such that M(**A**) = **A'**. Thus, reflection is **an onto mapping.**

TUTORIAL

A REFLECTION IN LINE Y=MX

Steps for Reflection Across y=mx

1. **Given Line**: $y = mx$ is the line of reflection, where m is the slope.
2. **Point to Reflect**: Let the point to be reflected be (x_1, y_1).
3. **Slope of Perpendicular Line**: The slope of the line perpendicular to $y = mx$ is $-\frac{1}{m}$.
4. **Equation of Perpendicular Line**: Find the equation of the line passing through (x_1, y_1) and perpendicular to $y = mx$:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

5. **Intersection Point**: Solve the system of equations (the original line and the perpendicular line) to find the intersection point between the two lines.

6. **Reflect the Point**: Use the midpoint formula to find the reflected point. The reflection of the point will lie symmetrically on the other side of the line.

   The reflected point (x', y') is the mirror image of (x_1, y_1) across the line $y = mx$.

1. If *(x, y)* is reflected in the *x*-axis *(y = 0)*, then its image is the point *(x, −y)*.
2. If *(x, y)* is reflected in the *y*-axis *(x = 0)*, then its image is the point *(−x, y)*.
3. If *(x, y)* is reflected in the line $y = x$, then its image is the point *(y, x)*.
4. If *(x, y)* is reflected in the line $y = -x$, then its image is the point *(−y, −x)*.

GENERAL FORMULA

$$x' = x\cos 2\theta + y\sin 2\theta$$

$$y' = x\sin 2\theta - y\cos 2\theta$$ where θ is the angle of inclination of the line L: $y = mx$.

B REFLACTION IN THE LINE Y= MX+B

PROCEDURE

1. Find the slope of the line L, say *m*.
2. 2. Find the equation of the line $\overline{AA'}$, which passes through the point A (*x*, *y*) and has slope $-\frac{1}{m}$.
3. Find the point of intersection B of L and $\overline{AA'}$ which serves as the midpoint of $\overline{AA'}$
4. Using B as the mid-point of $\overline{AA'}$, find the coordinates of A' .

REFLECTION OF A LINE IN THE LINE

To find the images of a line s when reflected about the line L, take the following way.

1. If a line $\overline{AA'}$ is perpendicular to the axis of reflection L, then $\overline{AA'}$ is its own image.
2. If $\overline{AA'}$ is a line parallel to the line of reflection **L**, to find the image of $\overline{AA'}$ when reflected about L, we follow the following steps.

a. Choose any point B on $\overline{AA'}$.
b. Find the image of B, M(B) = B'
c. Find the equation of $\overline{AA'}$ which is the line passing through B' with slope equal to the slope of L.

ROTATION

- A rotation R about a point O through an angle 0 is a transformation of the plane onto itself which carries every point A of the plane into the point A' of the plane such that OA = OA' and m (< AOA') = θ. O is called the centre of rotation and θ is called the angle of rotation.

A ROTATION WHEN THE CENTER OF RAOTATION IS ABOUT THE ORIGN

PROCEDURE

1. **Given Point**: Let the point be (x, y).
2. **Angle of Rotation**: Let the angle of rotation be θ.
3. **Rotation Formulas**:
   - New x': $x' = x\cos\theta - y\sin\theta$
   - New y': $y' = x\sin\theta + y\cos\theta$
4. **New Coordinates**: The new coordinates after rotation are (x', y').

NOTE

1. $\theta = \frac{\pi}{2}$ X' = -Y AND Y' =-X
2. $\theta = \pi \Rightarrow R(x, y) = (-x, -y)$.
3. $\theta = \frac{3\pi}{2} \Rightarrow R(x, y) = (y, -x)$.
4. $\theta = 2n\pi$ for $n \in Z \Rightarrow$ R is the identity transformation.
5. Every circle with centre at the centre of rotation is fixed.

B ROTATION WHEN THE CENTER OF ROTATION IS THE POINT (a ,b)

Procedure

1. **Given Point**: Let the point to rotate be (x, y), and the center of rotation be (a, b).
2. **Translate to Origin**:
   - Subtract the center (a, b) from (x, y): $(x - a, y - b)$.
3. **Rotation by θ**:
   - Use rotation formulas:

$$x' = (x - a)\cos\theta - (y - b)\sin\theta$$

$$y' = (x - a)\sin\theta + (y - b)\cos\theta$$

Note

One can also obtain the image of a line under a given rotation as follows:

1. Choose two points on the line.
2. Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

Note

As in the case of translation and reflection, to find the image of a circle under a given rotation we follow the following steps:

1. Find the centre and radius of the given circle.
2. Find the image of the centre of the circle under the given rotation.
3. Equation of the image circle will be an equation of the circle centred at the image of the centre of the given circle with radius the same as the radius of the given circle.

Applications

Example 1

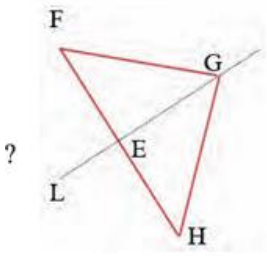


Figure 6.29

Figure 6.29 shows a reflection in line L.

a. Which two points are invariant?
b. If the length of FG is 14cm, what is the length of GH?
c. If the length of FH is 16 cm, what is the length of EH?

Solution

a. The invariant points are E and G.
b. Let $\overrightarrow{FG} = 14$cm then $\overrightarrow{GH} = 14$cm. Since, $\overrightarrow{GH} = \overrightarrow{FG} = 14$cm.
c. If the length of FH = 16cm then the length of EH $= \frac{1}{2}$(length of FH) $= \frac{1}{2} \times 16\text{cm} = 8\text{cm}$.

Example 2

Given the translation $(x + 6, y - 2)$, find the image of the point $(2, 6)$.

Solution

The given translation is, $(x, y) \rightarrow (x + 6, y - 2)$.

It is given that $(x, y) = (2, 6)$. Hence, the translated point is,

$(x + 6, y - 2) = (2 + 6, 6 - 2) = (8, 4)$.