

Differential Calculus and the Application of Derivatives

Yadeta

Adama Science and Technology University
School of Applied Natural Sciences
Applied Mathematics Program
Adama, ETHIOPIA

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Outline I

1 Derivatives

- Tangent Line
- Definition of Derivative
- Techniques of Differentiation
- Derivatives of Trigonometric Functions
- The Chain Rule
- Implicit Differentiation
- Inverse Trigonometric Functions
- Hyperbolic Functions
- Inverse Hyperbolic Functions
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2 Application of the Derivative

- Maxima and Minima of a Function
- The Mean Value Theorem
- Monotonicity
- Concavity
- Curve Sketching



Outline II

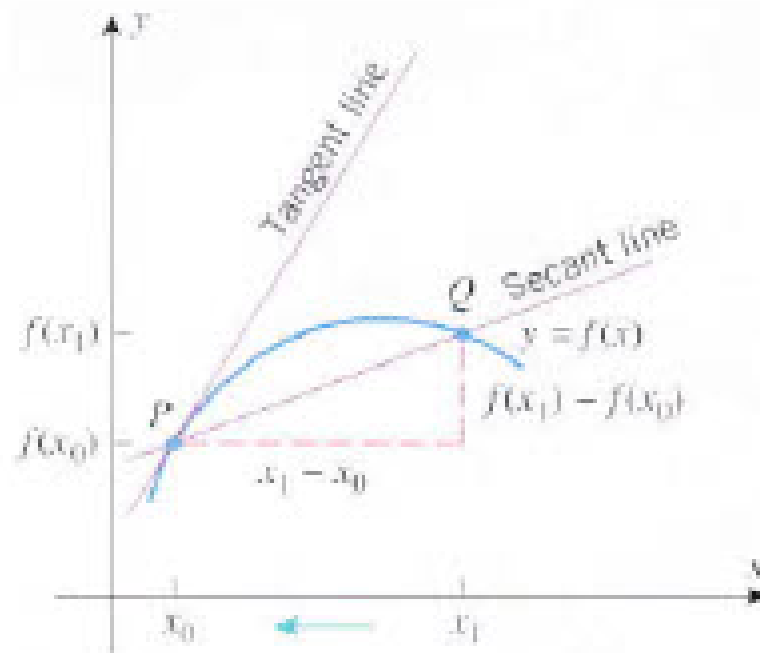
- Indeterminate Forms and L'Hospital's Rule
- Relative Rates
- Optimization Problem



Tangent Line

Definition (Tangent Line)

The tangent line to the graph of $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line through P with the slope m . $y - f(x_0) = m(x - x_0)$ is the point slope form of equation of the tangent line.



Where,

$$m = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \text{ OR}$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that this limit exists.



Tangent line

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Cont'd

Example

- ❶ Find the Slope of the Tangent line to the graph of $f(x)$ at the point $x = x_0$
 - (a) $f(x) = x^2 + 1; x_0 = 2$
 - (b) $f(x) = \sqrt{x + 9}; x_0 = 1$
 - (c) $f(x) = \frac{1}{\sqrt{x}}; x_0 = 4$
- ❷ Find the equation of the tangent line to the graph of $f(x)$ in the above example at the given points.



Definition of Derivative

Definition

The function f' defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

*is called the **Derivative of f with respect to x** .*

Note:

- ❶ *The derivative f' of a function f is a function whose value at x is the slope of the tangent line to the graph of $y = f(x)$ at x .*
- ❷ *Alternatively, it is a function whose value at x is the instantaneous rate of change of y with respect to x at the point x .*



Definition of Derivative

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Cont'd

Example

❶ Find the derivative of f w.r.t x of

(a) $f(x) = x^2 + 1$

(b) $f(t) = \sqrt{t + 9}$

(c) $f(x) = \frac{1}{\sqrt{x}}$

❷ Show that

$$f(x) = |x|$$

is not differentiable at $x = 0$



Definition fo Derivative

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Cont'd

Theorem

- ② *If $f(x)$ differentiable at x , then it is continuous at x .*

Note:

The converse of the above theorem is not true.

Example

Show that $f(x) = |x|$ is continuous but not differentiable at $x = 0$.



Derivative of Constant and Power Function

Theorem

- ❶ *The Derivative of a constant function is zero. i.e.*
$$\frac{d}{dx}[c] = 0; \forall c \in \mathcal{R}$$
- ❷ *If n is any positive integer, then $\frac{d}{dx}[x^n] = nx^{n-1}$*

Example

Let $f(x) = x^7$, then $f'(x) = \frac{d}{dx}[x^7] = 7x^6$

Note:

For any real number n , the above formula holds true.



Sum, Product and Quotient Rule

Theorem

⑧ If f and g are differentiable, then

(a) $\frac{d}{dx} [cf(x)] = cf'(x)$ Where $c \in \mathcal{R}$

(b) $\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$

(c) $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

(d) $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$



Sum, Product and Quotient Rule ... Cont'd

Example

Find the derivative of f , if

❶ $f(x) = \frac{\pi}{x^2}$

❷ $f(x) = (x^2 - 1) \sqrt{x}$

❸ $f(x) = \frac{x}{x^2 - 1}$

❹ $f(x) = (x^3 - 5)(1 - 3x^2)$



Derivatives of Exponential Functions

Definition

e is a number such that

$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a \implies \ln a = \log_e^a$ Where $e = 2.71828 \dots$ is an irrational number.

Theorem

$$\frac{d}{dx} [a^x] = a^x \ln a$$

Example (Derivate f , if)

① $f(x) = e^x$

② $f(x) = x^2 - 3^x$

Derivative of Logarithmic Function

Recall that

$$\lim_{x \rightarrow 0} [1 + x]^{\frac{1}{x}} = e$$

Theorem

$$\frac{d}{dx} [\log_b x] = \frac{1}{x \ln b}$$

Example (Find the derivative of)

① $\frac{d}{dx} [\log_3 x]$



Derivatives of Trigonometric Function

Show that

$$\textcircled{1} \quad \frac{d}{dx} [\sin(x)] = \cos(x)$$

$$\textcircled{2} \quad \frac{d}{dx} [\cos(x)] = -\sin(x)$$

$$\textcircled{3} \quad \frac{d}{dx} [\tan(x)] = \sec^2(x)$$

$$\textcircled{4} \quad \frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x)$$

$$\textcircled{5} \quad \frac{d}{dx} [\sec(x)] = \sec(x) \tan(x)$$

$$\textcircled{6} \quad \frac{d}{dx} [\cot(x)] = -\csc^2(x)$$

Example (Derivate)

$$\textcircled{1} \quad f(x) = \frac{\sec(x)}{1 + \tan(x)}$$

$$\textcircled{2} \quad f(x) = x^2 \sec(x) - x$$



The Chain Rule

Theorem (The Chain Rule)

If $g(x)$ is differentiable at x and $f(x)$ is differentiable at the point $g(x)$ then $\frac{d}{dx} [(f \circ g)(x)]$ is given as

$$\frac{d}{dx} [(f \circ g)(x)] = f'(g(x))g'(x)$$

OR

If $y = f(g(x))$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



The Chain Rule

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Cont'd

Example

① *Differentiate*

a) $f(x) = (x^2 - 1)^3$

b) $f(x) = e^{\cos(x)}$

c) $f(x) = \sin(\sqrt{1-x})$

d) $f(x) = \frac{(x^2 + 1)^{10} (\sin^3(\sqrt{x}))}{\sqrt{1 + \csc(x)}}$

e) $f(x) = \ln\left(\frac{x^2 \sin(x)}{\sqrt{1+x}}\right)$

② *Find $\frac{dw}{dt}$ if $w = \csc(x)$ and $x = 4t^3 - 1$* 

Implicit Function

Definition

A given equation in x and y is said to defines the function f implicitly if the graph of $f(x)$ coincides with some segment of the graph of the equation.

Example

$x^2 + y^2 = 1$ define the function $f_1(x) = \sqrt{1 - x^2}$ and $f_2(x) = -\sqrt{1 - x^2}$ implicitly.



Implicit Differentiation

Implicit Differentiation

Let $F(x, y) = c$; $c \in \mathcal{R}$ define the function $y = f(x)$, then to find $\frac{dy}{dx}$ follow the following steps

- 1 Take the derivative w.r.t x both sides of the equation. i.e.

$$\frac{d}{dx} [F(x, y)] = \frac{dc}{dx}$$

- 2 Solve the result in step 1 for $\frac{dy}{dx}$.



Implicit Differentiation

Example

Find $\frac{dy}{dx}$

❶ $5x^2 + \sin(xy) = yx$

❷ $y(xy + 2^y) = 2$



Definition of Inverse Trigonometric Functions

Definition

- ① *The inverse sine function, denoted by $\sin^{-1}$ is defined as*

$$y = \sin^{-1}(x) \iff \sin(y) = x; \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\implies \frac{d}{dx} [\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

- ② *The inverse cosine function, denoted by $\cos^{-1}$ is defined as*

$$y = \cos^{-1}(x) \iff \cos(y) = x; \forall y \in [0, \pi]$$
$$\implies \frac{d}{dx} [\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}}$$



Definition of Inverse Trigonometric Functions

Definition

- ③ The inverse tangent function, denoted by $\tan^{-1}$ is defined as

$$y = \tan^{-1}(x) \iff \tan(y) = x; \forall y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$\implies \frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{1+x^2}$$

- ④ The inverse cotangent function, denoted by $\cot^{-1}$ is defined as

$$y = \cot^{-1}(x) \iff \cot(y) = x; \forall y \in (0, \pi)$$
$$\implies \frac{d}{dx} [\cot^{-1}(x)] = -\frac{1}{1+x^2}$$



Definition & Derivative of Inverse Trig. Functions

Definition

⑤ The inverse co-secant function, denoted by $\csc^{-1}$ is defined as

$$y = \csc^{-1}(x) \iff \csc(y) = x; \forall y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } y \neq 0$$
$$\implies \frac{d}{dx} [\csc^{-1}(x)] = -\frac{1}{x\sqrt{x^2 - 1}}$$

⑥ The inverse secant function, denoted by $\sec^{-1}$ is defined as

$$y = \sec^{-1}(x) \iff \sec(y) = x; \forall y \in (0, \pi)$$
$$\implies \frac{d}{dx} [\sec^{-1}(x)] = \frac{1}{x\sqrt{x^2 - 1}}$$



Definition & Derivative of Inverse Trig. Functions

Example

- ❶ Show that $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$
- ❷ Differentiate
 - a) $y = \sin^{-1}(x^3)$
 - b) $y = \sec^{-1}(1 - 2^x)$



Definition of Hyperbolic Functions

Definition

- ① *The sine hyperbolic function denoted by $\sinh x$ is defined as*

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

- ② *The cosine hyperbolic function denoted by $\cosh x$ is defined as*

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- ③ *similarly, the other hyperbolic functions are defined as*

a) $\tanh x = \frac{\sinh x}{\cosh x}$

④ $\coth x = \frac{\cosh x}{\sinh x}$

b) $\operatorname{csch} x = \frac{1}{\sinh x}$

d) $\operatorname{sech} x = \frac{1}{\cosh x}$

Derivatives of Hyperbolic Functions

Example (Show that)

$$① \quad \frac{d}{dx} [\sinh x] = \cosh x$$

$$② \quad \frac{d}{dx} [\cosh x] = \sinh x$$

$$③ \quad \frac{d}{dx} [\tanh x] = \operatorname{sech}^2 x$$

$$④ \quad \frac{d}{dx} [\operatorname{csch} x] = -\operatorname{csch} x \coth x$$

$$⑤ \quad \frac{d}{dx} [\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

$$⑥ \quad \frac{d}{dx} [\coth x] = -\operatorname{csch}^2 x$$



Inverse Hyperbolic Functions

Definition

Inverse hyperbolic functions are defined as

<i>Function</i>	<i>Domain</i>	<i>Range</i>
$y = \sinh^{-1} x \iff \sinh y = x$	$\mathcal{R}$	$\mathcal{R}$
$y = \cosh^{-1} x \iff \cosh y = x$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1} x \iff \tanh y = x$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1} x \iff \coth y = x$	$(-\infty, \infty)/[-1, 1]$	$(-\infty, \infty)/\{0\}$
$y = \operatorname{sech}^{-1} x \iff \operatorname{sech} y = x$	$(0, 1]$	$[0, \infty)$
$y = \operatorname{csch}^{-1} x \iff \operatorname{csch} y = x$	$(-\infty, \infty)/\{0\}$	$(-\infty, \infty)/\{0\}$



Logarithmic Forms of Inverse Hyperbolic Function

The following relationships hold for all x in the domain of the stated hyperbolic function.

$$1. \sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)$$

$$2. \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$3. \operatorname{sech}^{-1} x = \ln \left(\frac{1+\sqrt{1-x^2}}{x} \right)$$

$$4. \cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right)$$

$$5. \operatorname{coth}^{-1} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

$$6. \operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|} \right)$$



Derivative of Inverse Hyperbolic Function

Theorem

$$\begin{aligned}
 1. \quad \frac{d}{dx} [\sinh^{-1} x] &= \frac{1}{\sqrt{1+x^2}} \\
 2. \quad \frac{d}{dx} [\cosh^{-1} x] &= \frac{1}{\sqrt{x^2-1}}; x > 1 \\
 3. \quad \frac{d}{dx} [\tanh^{-1} x] &= \frac{1}{1-x^2}; |x| < 1 \\
 4. \quad \frac{d}{dx} [\coth^{-1} x] &= \frac{1}{1-x^2}; |x| > 1 \\
 5. \quad \frac{d}{dx} [\operatorname{sech}^{-1} x] &= -\frac{1}{x\sqrt{1-x^2}}; 0 < x < 1 \\
 6. \quad \frac{d}{dx} [\operatorname{csch}^{-1} x] &= -\frac{1}{|x|\sqrt{1+x^2}}; x \neq 0
 \end{aligned}$$

Higher Order derivative

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $f'' = (f')'$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f . Similarly, $f''' = (f'')'$. Thus, the n^{th} derivative of f , denoted by $f^{(n)}$, is given as

$$f^{(n)} = (f^{(n-1)})'$$

Example

Find the second and third derivative of $f(x) = x \cos(x)$



Higher Order derivative

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Cont'd

Example

Find a formula for n^{th} derivative of f

❶ $f(x) = xe^x$

❷ $f(x) = \frac{2}{2x+1}$

❸ $f(x) = \ln(x+1)$



Absolute Maxima and Minima

Definition

A function f is said to have an absolute maximum on an interval I at the point x_0 if

$$f(x_0) \geq f(x); \forall x \in I$$

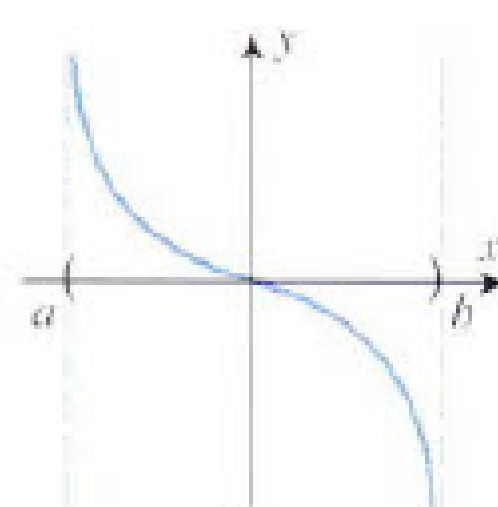
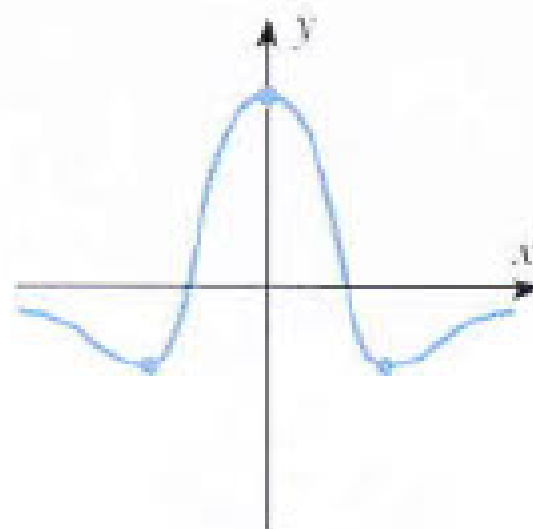
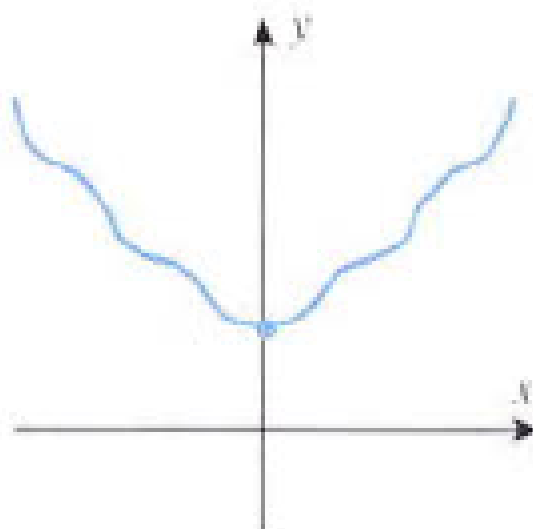
Similarly, f is said to have an absolute minimum on I at the point x_0 if

$$f(x_0) \leq f(x); \forall x \in I$$

If f has either an absolute maximum or absolute minimum on I at x_0 , then f is said to have an absolute extremum on I on I at x_0 .



Absolute Maxima and Minima



Example

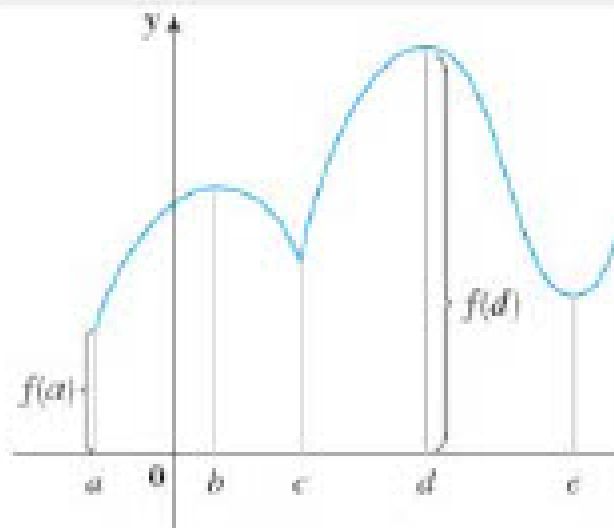
Since $f(x) = x^2 \geq 0$, $\forall x \in \mathcal{R}$, $f(0) = 0$ is the absolute minimum value of f .



Extreme Value Theorem

Definition

A function f is said to have a local maximum (or local maximum) at $x = c$ if $f(c) \geq f(x)$, $\forall x$ in some open interval containing c . Similarly, f is said to have a local minimum at $x = c$ if $f(c) \leq f(x)$, $\forall x$ in some open interval containing c .



$\Rightarrow f(c)$ and $f(e)$ are local minimum values of f .

$\Rightarrow f(b)$ and $f(d)$ is a local maximum values of f .



Critical Number of a function

Definition

A number c in the domain of f is said to be a critical number of f if $f'(c) = 0$ or $f'(c)$ does not exist.

Example

Find the critical number of

❶ $f(x) = x^{\frac{3}{4}}(4 - x)$

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c , then $f'(c) = 0$.



The Closed Interval Method

Theorem (Extreme Value Theorem)

If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and absolute minimum value on $[a, b]$

To find the absolute maximum and minimum value of a continuous function f on a closed interval $[a, b]$:

- ① Find the values of f at the critical numbers of f in (a, b) .
- ② Find the values of f at the endpoints of the interval.
- ③ The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value



The Closed Interval Method ... Cont'd

Example

Find the absolute maximum and minimum values of the function on the given interval

❶ $f(x) = x^3 - 3x^2 + 1, \left[-\frac{1}{2}, 4\right]$

❷ $f(x) = 2x^3 - 15x^2 + 36x, [1, 5]$

❸ $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}, [-1, 1]$



Rolle's Theorem

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three hypotheses:

- ❶ *f is continuous on the closed interval $[a, b]$*
- ❷ *f is differentiable on the open interval (a, b)*
- ❸ *$f(a) = f(b)$*

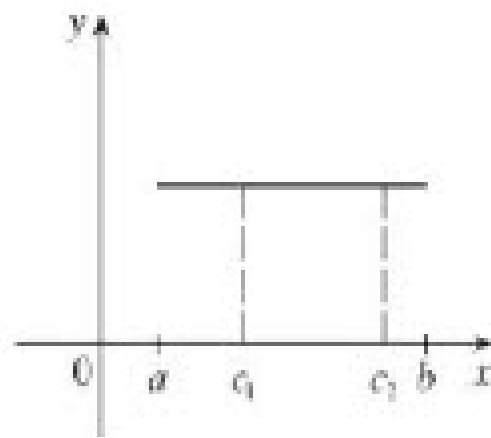
Then, there is a number $c \in (a, b)$ such that $f'(c) = 0$



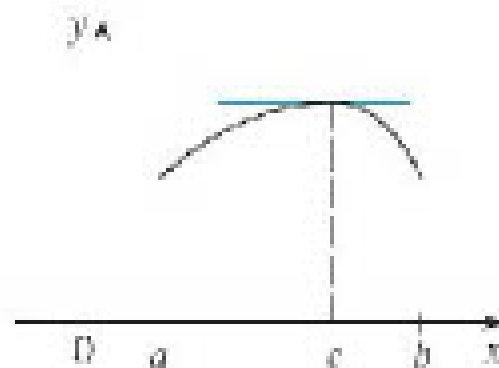
Rolle's Theorem

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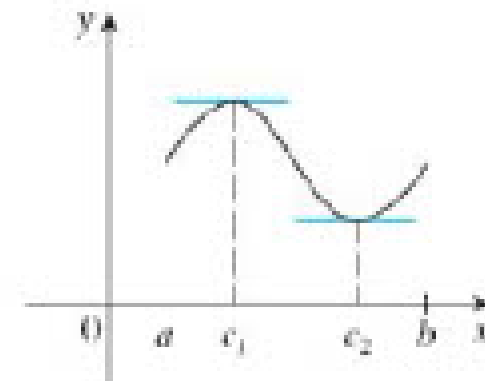
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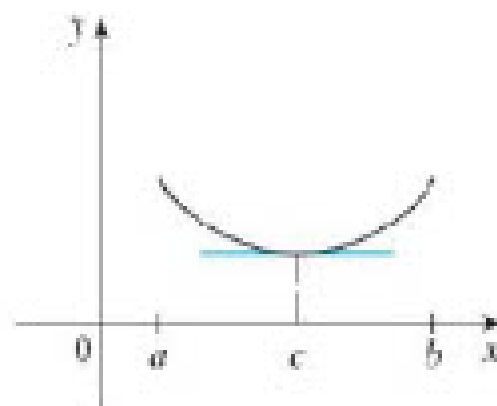
(a)



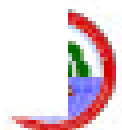
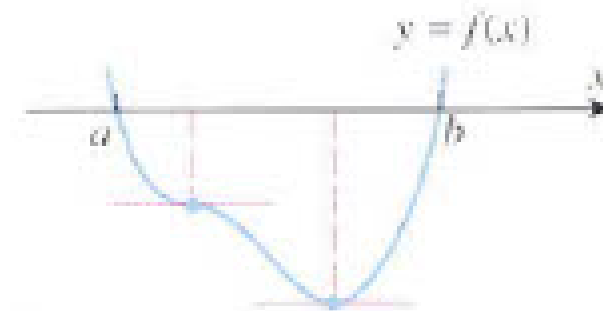
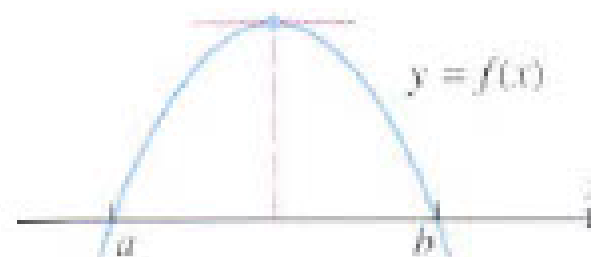
(b)



(c)



(d)



Rolle's Theorem

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Cont'd

Example

- ❶ *Verify Rolle's Theorem for $f(x) = \sin(x)$ on $[0, 2\pi]$*
- ❷ *Show that the equation $x^3 + x - 1 = 0$ has exactly one real root.*



The Mean Value Theorem (MVT)

Theorem (The Mean Value Theorem)

Let f be a function that satisfies the following hypotheses:

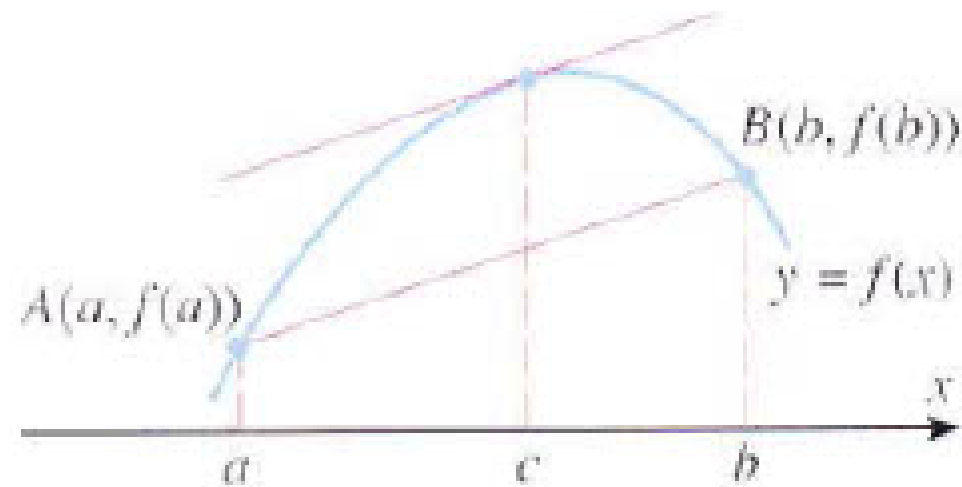
- ❶ *f is continuous on the closed interval $[a, b]$*
- ❷ *f is differentiable on the open interval (a, b)*

Then, there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



The Mean Value Theorem (MVT) ... Cont'd



Example

Verify MVT by

- ❶ $f(x) = x^3 - x$ on $[0, 2]$
- ❷ $f(x) = \frac{x^3}{4} + 1$ on $[0, 2]$

Increasing and Decreasing Test

Theorem

Let a function f be a function that is continuous on $[a, b]$ and differentiable on (a, b)

- ① *If $f'(x) > 0; \forall x \in (a, b)$, then f is increasing on $[a, b]$*
- ② *If $f'(x) < 0; \forall x \in (a, b)$, then f is decreasing on $[a, b]$*
- ③ *If $f'(x) = 0; \forall x \in (a, b)$, then f is constant on $[a, b]$*

Example

Find the interval in which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing/decreasing.



The First Derivative Test

Theorem

Suppose that c is a critical number of a continuous function of f

- ① *If f' changes sign from +ve to -ve at c , then f has a local maximum at c*
- ② *If f' changes sign from -ve to +ve at c , then f has a local minimum at c*
- ③ *If f' does not change sign at c , then f has no local maximum and minimum at c*

Example

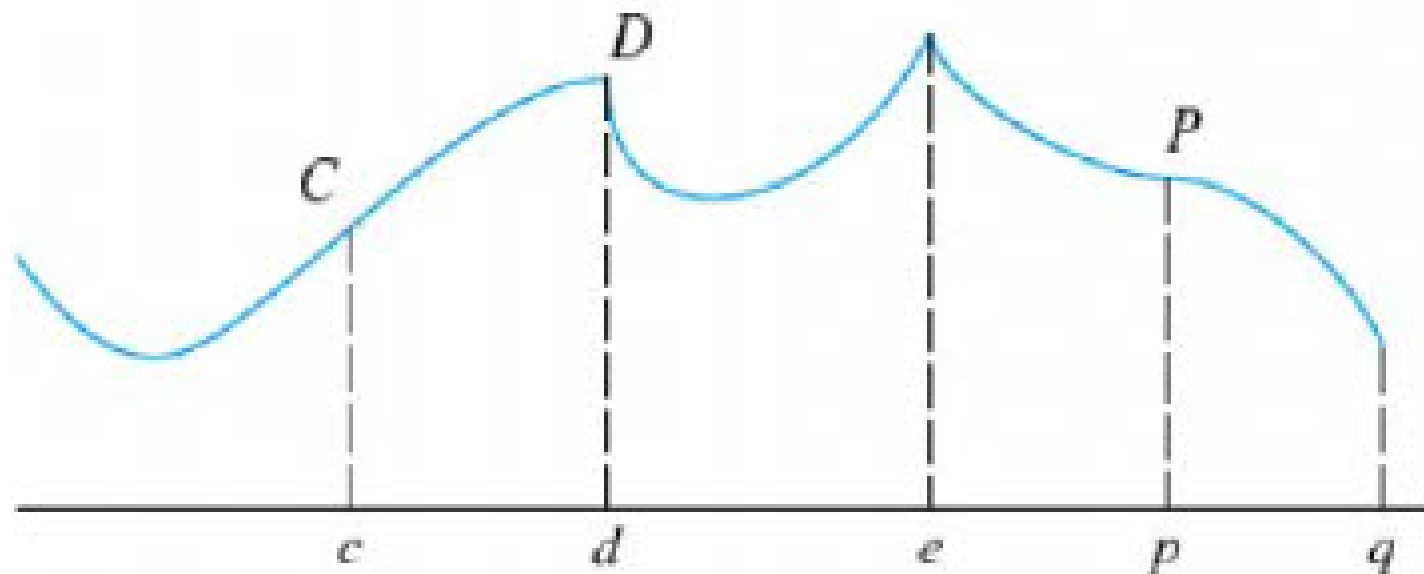
Find local extremes of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.



Concavity Test

Definition

If the graph of f lies above all of its tangents on an interval I , then it is called concave upward on I . If the graph of f lies below all of its tangents on an interval I , then it is called concave downward on I .



Concavity Test

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Cont'd

Theorem (Concavity Test)

Suppose f is twice differentiable on I

- ❶ *If $f''(x) > 0, \forall x \in I$, then f is concave upward on I*
- ❷ *If $f''(x) < 0, \forall x \in I$, then f is concave downward on I*

Definition

A point P on the graph of $y = f(x)$ is called an inflection point if f changes concavity at P



Second Derivative Test

Theorem (Second Derivative)

Suppose $f''(x)$ is continuous near c , then

- ❶ If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c
- ❷ If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c

Example

Given $y = x^4 - 4x^3$, find

- ❶ Interval of concavity
- ❷ Inflection points
- ❸ Extreme value of f



Guideline for Sketching a Curve

All the information needed to sketch the graph of $y = f(x)$ are:

- ➊ Domain
- ➋ Intercepts
- ➌ Symmetry
- ➍ Asymptotes
- ➎ Interval of Monotonicity
- ➏ Local Extreme values
- ➐ Concavity and Inflection Points

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$



Indeterminate Forms of $\frac{0}{0}$

Theorem (L'Hospital's Rule)

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a itself). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

OR

$$\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty$$

Then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$



Indeterminate Forms of $\frac{0}{0}$... Cont'd

Note:

L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

Example

Evaluate

① $\lim_{x \rightarrow 1} \frac{\ln(x)}{x-1}$

② $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$



Indeterminate Forms of $0 \cdot \infty$

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a itself). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = \infty$$

Then, it is not clear what the value of

$$\lim_{x \rightarrow a} (f(x)g(x))$$

if any will be. This limit is called an indeterminate form of $0 \cdot \infty$.



Indeterminate Forms of $0 \cdot \infty$... Cont'd

Note:

Indeterminate products can be converted to the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ writing as a quotient.

Example

Evaluate

- ① $\lim_{x \rightarrow 0^+} (x \ln(x))$
- ② $\lim_{x \rightarrow \frac{\pi}{4}} (1 - \tan x) \sec(2x)$



Indeterminate Forms of $\infty - \infty$

Definition

A limit problem that leads to one of the expressions

$$\infty - \infty, \infty + (-\infty), -\infty + \infty, -\infty - (-\infty)$$

is called an indeterminate form of type $\infty - \infty$, because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction.

Example

$$\textcircled{1} \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right)$$



Indeterminate Forms of 0^∞ , ∞^0 and 1^∞

Definition

Limits of the form

$$\lim_{x \rightarrow a} f(x)^{g(x)}$$

that give rise to 0^∞ , ∞^0 and 1^∞ is called indeterminate form of type 0^∞ , ∞^0 and 1^∞ .

Example

- ❶ Show that $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$
- ❷ Evaluate $\lim_{x \rightarrow 0^+} (1 + \sin(4x))^{\cot(x)}$



Relative Rates

Definition

The first interpretation/application of a derivative is rate of change. If $f(x)$ represents a quantity at any x then the derivative $f'(x)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Example

- 1 A rectangular water tank is being filled at a constant rate of 20 liters/sec. The base of the tank has dimensions $w = 1\text{m}$ and $L = 2\text{m}$. What is the rate of change of the height of water in the tank?



Relative Rates

...

Cont'd

Example

- ② *An airplane is flying in a straight direction at a constant height of 500km. The angle of elevation of the airplane from a fixed point of observation is α . The speed of the airplane is 500km/hr. What is the rate of change of angle α when it is 25° ?*

Relative Rates

...

Cont'd

Example

- ② *An airplane is flying in a straight direction at a constant height of 500km. The angle of elevation of the airplane from a fixed point of observation is α . The speed of the airplane is 500km/hr. What is the rate of change of angle α when it is 25° ?*
- ③ *If two resistors with R_1 and R_2 are connected in parallel as shown in the figure below, their electrical behavior is equivalent to a resistor of resistance R such that*

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 changes at a rate of $r = \frac{dR_1}{dt}$ and R_2 is constant, find the rate of change of R

Relative Rates

...

Cont'd

Example

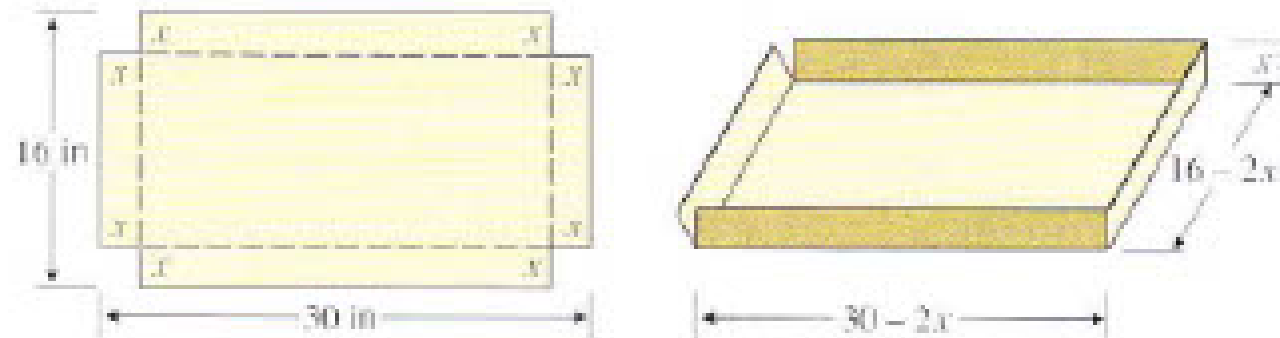
- ④ *Two cars start moving from the same point in two directions that makes 90° at a constant speed of v_1 and v_2 . Find the rate of change of the distance D between the two cars.*



Optimization Problem

Example

- ① An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume.



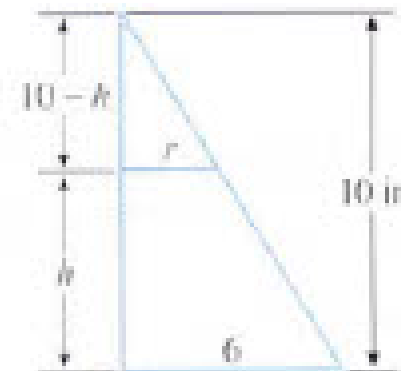
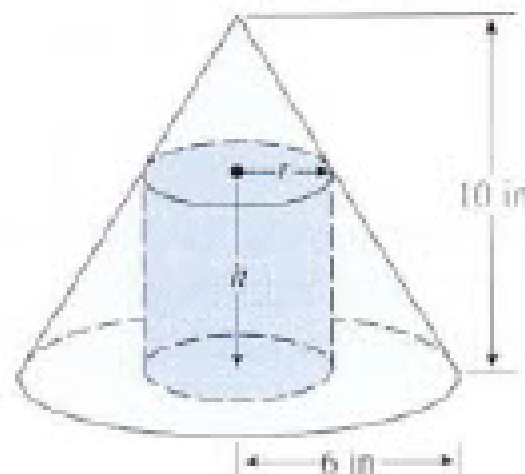
Optimization Problem

...

Cont'd

Example

- ① Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.



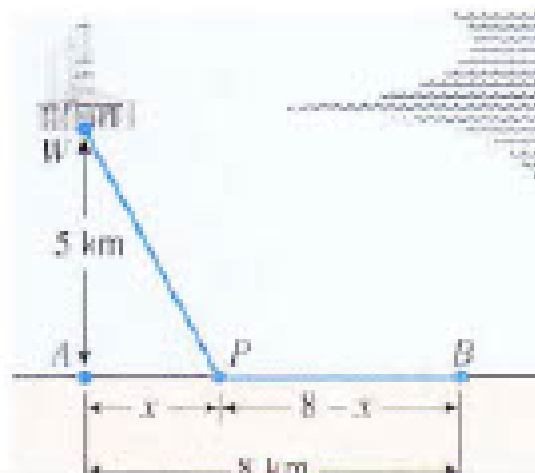
Optimization Problem

...

Cont'd

Example

- ① The following figure shows an offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A by piping it, on a straight line under water W to some shore point P between A and B and then on to B via pipe along the shoreline. if the cost of laying pipe is $\$1,000,000/\text{km}$ under water and $\$500,000/\text{km}$ over land. where should the point P be located to minimize the cost of laying the pipe line?



Integral Calculus and Its Applications

Yadeta

Adama Science and Technology University
School of Applied Natural Sciences
Applied Mathematics Program
Adama, ETHIOPIA

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Outline I

- 1 Integration
 - Anti-derivative; Indefinite Integrals
 - Properties of the Indefinite Integral
 - Techniques of Integration
 - Fundamental Theorem of Calculus
 - Improper Integrals
 - Application of The Integral
 - Area Under a Curve
 - Area Between Two Curves
 - Volume of a Solid
 - Volume of Revolution



Anti-derivative; Indefinite Integrals

Definition

Let $f(x)$ be continuous on $[a, b]$. If $F(x)$ is continuous on $[a, b]$ and $F'(x) = f(x) \quad \forall x \in (a, b)$, then F is called an anti-derivative of f .

Note:

- The process of finding anti-derivatives is called anti-differentiation or integration.*
- The collection of all anti-derivatives of f , denoted by $\int f(x) dx$ is called the indefinite integral of f , and it can be constructed by integrating.*



Anti-derivative; Indefinite Integrals ... Cont'd

Note:

- Every anti-derivative of $f(x)$ can be written in the form $F(x) + C$, for some C . Thus,

$$\begin{aligned}\frac{d}{dx}[F(x)] = f(x) &\iff \int f(x) dx = F(x) + C. \\ \frac{d}{dx}[g(x)] = g'(x) &\iff \int g'(x) dx = g(x) + C.\end{aligned}$$



Properties of the Indefinite Integral

$$\textcircled{1} \quad \frac{d}{dx} \left[\int f(x) dx \right] = f(x).$$

$$\textcircled{2} \quad \int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx \quad \forall \alpha, \beta \in \mathcal{R}.$$

(Linearity)

Example

$$\textcircled{1} \quad \int x^2 dx = \frac{1}{3}x^3 + C, \text{ since } \frac{d}{dx} \left[\frac{1}{3}x^3 + C \right] = x^2.$$

$$\textcircled{2} \quad \frac{d}{dx} \left[\int x^5 dx \right] = x^5.$$

$$\textcircled{3} \quad \int \cos x dx = \sin x + C$$

Integration by Substitution

Remark

- *Some Integrations become easy with appropriate variable change.*
- *Integration by variable change is called **integration by substitution** .*

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$I = \int f(g(x)) \cdot g'(x) dx = \int f(u) du$$



Integration by Substitution ... Cont'd

Example

Evaluate each of the following integrals

① $\int x e^{x^2} dx$

② $\int x^3 \cos(x^4 + 2) dx,$

③ $\int e^{\sin x} \cos x dx,$



Integration by Parts

We know from the product rule of differentiation that:

$$\begin{aligned}\frac{d}{dx} [f(x)g(x)] &= f'(x)g(x) + f(x)g'(x) \\ \int [f'(x)g(x) + f(x)g'(x)] dx &= f(x)g(x) \\ \Leftrightarrow \int f(x)g'(x) dx &= f(x)g(x) - \int f'(x)g(x) dx\end{aligned}$$



Integration by Parts

Remark

Let $u = f(x)$ and $v = g(x) \implies du = f'(x)dx$ and $dv = g'(x)dx$

$$\implies \int u dv = uv - \int v du$$

Example (Evaluate)

① $\int x \sin x dx$

② $\int e^x \sin x dx$



Method of the Partial Fraction

MPF

Integrals $\int \frac{f(x)}{g(x)} dx$, where f and g are polynomials, be integrated by the Method of the Partial Fraction.

This method consists of two steps:

- ① Finding the partial fractions of the integrand
- ② Integrating the resulting partial fractions

Success in writing $\frac{f(x)}{g(x)}$ as a sum of partial fractions depends on:

- ① The degree of $f(x)$ must be less than the degree of $g(x)$.
- ② We must know the factors of $g(x)$.

Method of the Partial Fraction

Let $\frac{f(x)}{g(x)}$ is a proper fraction and all the factors of $g(x)$ are known.

1. [Linear Factor Rule]

Let $(x - a)^m$ is the highest power of $x - a$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_m}{(x - a)^m}$$

2. [Quadratic Factor Rule]

Let $(x^2 + px + q)^n$ is the highest power of quadratic factor of $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}$$

Where B_i 's and C_i 's are constants to be determined.



Method of the Partial Fraction

3. Set the original fraction, $\frac{f(x)}{g(x)}$, equal to the sum of all these partial fractions.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.
5. Back substitute the values of the constants and integrate.

Example (Evaluate)

① $\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx$

② $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$



Trigonometric Integrals

Theorem (The Reduction Formula)

The sine and cosine integrals are given by the formula:

$$\textcircled{1} \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\textcircled{2} \int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Example (Evaluate)

$$\textcircled{1} \int \sin^3 x \, dx$$

$$\textcircled{2} \int \cos^4 x \, dx$$



Integration by Trigonometric Substitution

Expression	Substitution	Identities required
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \pi < \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example (Integrate)

❶ $\int \sqrt{9 - x^2} dx.$

❷ $\int \frac{dx}{2x^2 - 12x + 26}$

❸ $\int \frac{dx}{\sqrt{21 - 4x - x^2}}$

Fundamental Theorem of Calculus Part I

Theorem (Fundamental Theorem of Calculus Part I)

Suppose f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) and the derivative is

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$



Fundamental Theorem of Calculus Part I ... Cont'd

Example

Find the derivative of

❶ $g(x) = \int_a^x \sqrt{t^2 + 1} \, dt$

❷ $g(x) = \int_a^{x^4} \sec t \, dt$



Fundamental Theorem of Calculus Part II

Theorem (Fundamental theorem of Calculus Part II)

If f is continuous at every point of $[a, b]$ and F is any anti-derivative of f on $[a, b]$, i.e. $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$



Fundamental Theorem of Calculus Part II

Example

Find the definite integral

① $\int_{\pi/6}^{\pi/3} \sec t \, dt$

② $\int_1^4 \left(\frac{3}{2} \sqrt{x} - \frac{1}{x^4} \right) dx$



Improper Integrals

Definition (Improper Integrals)

Integrals with infinite limits of integration are improper integrals.

- ① If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$
- ② If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$



Improper Integrals

...

Cont'd

Definition (Improper Integrals

...

Cont'd)

⑧ If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx \text{ for } c \in \mathcal{R}$$

Note:

If the above limit is finite, the improper integral converges and its value is the value of the limit. Otherwise, the improper integral diverges.



Improper Integrals

...

Cont'd

Example

Evaluate the improper integral

$$\textcircled{1} \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx$$

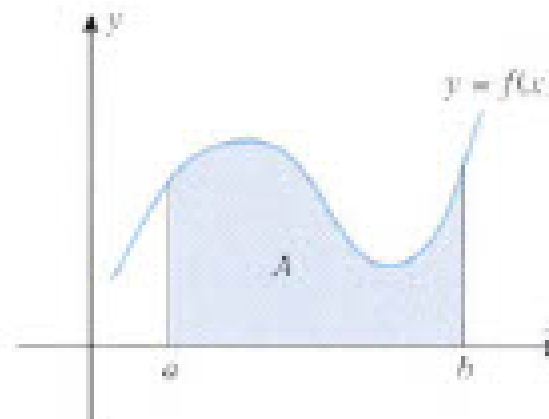
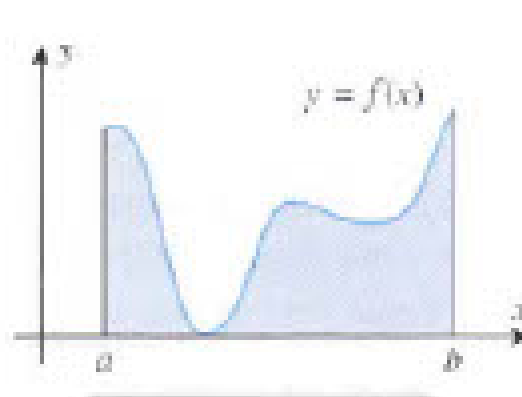
$$\textcircled{2} \int_1^{\infty} \frac{\ln x}{x^2} dx$$



Area Under a Curve

Definition (Area Under a Curve)

If a function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for $\forall x \in [a, b]$, then the area A under a curve $y = f(x)$ over the interval $[a, b]$ is defined by



$$A = \int_a^b f(x) dx$$



Area Under a Curve

...

Cont'd

Example

Find the area under the curve

① $y = 9 - x^2$ over $[0, 3]$

② $y = \cos x$ over $[0, \pi]$



Area Under a Curve

...

Cont'd

Example

Find the area under the curve

③ $y = 1 - x^2$ over $[0, 2]$

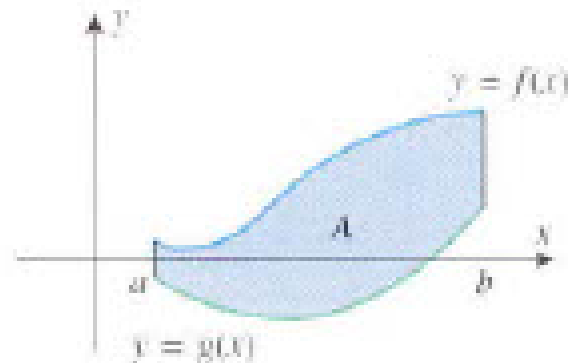
④ $y = \sqrt{x}$ over $[1, 2]$



Area Between Two Curves

Definition

If f and g are continuous on the interval $[a, b]$, and if $f(x) \geq g(x)$, $\forall x \in [a, b]$, then the area of the region bounded above by $y = f(x)$ and below by $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$ is



$$A = \int_a^b [f(x) - g(x)] dx$$



Area Between Two Curve

...

Cont'd

Example

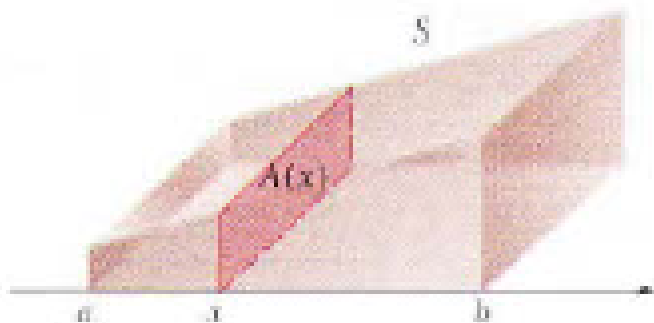
Find the area between the curves $x = y^2$ and $y = x - 2$



Volume

Definition

Let S be a solid bounded by two parallel planes perpendicular to the x -axis at $x = a$ and $x = b$. If for each $x \in [a, b]$, the cross-sectional area of S perpendicular to the x -axis is $A(x)$, then the volume of the solid is



$$V = \int_a^b A(x) dx$$



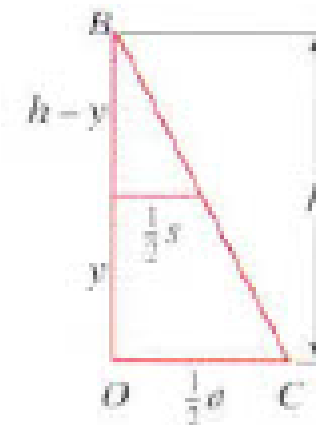
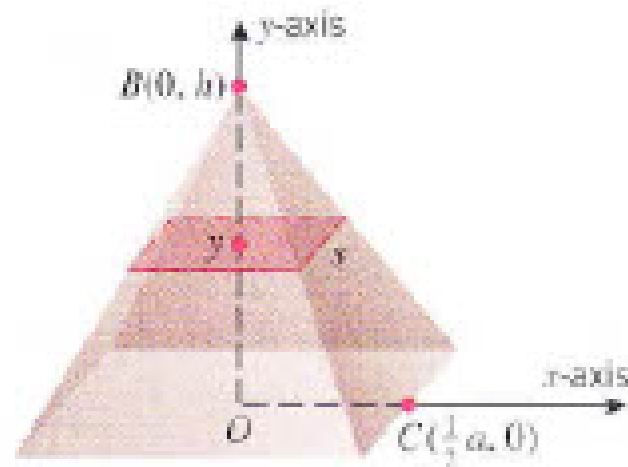
Volume of a Solid

...

Cont'd

Example

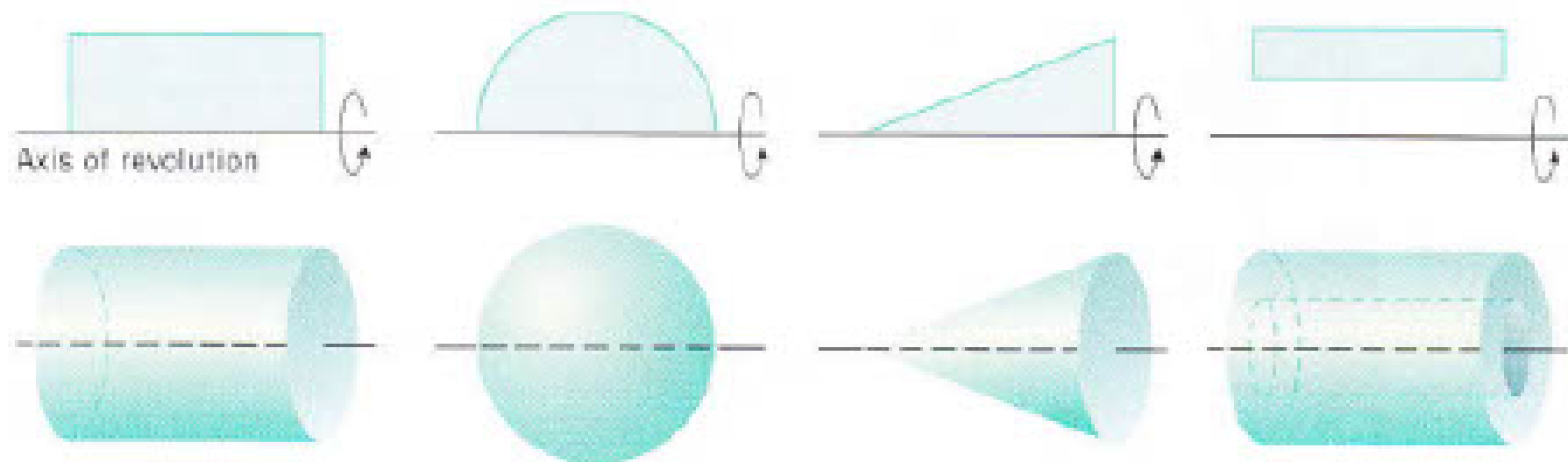
Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a .



Volume of Revolution

Definition

A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution.



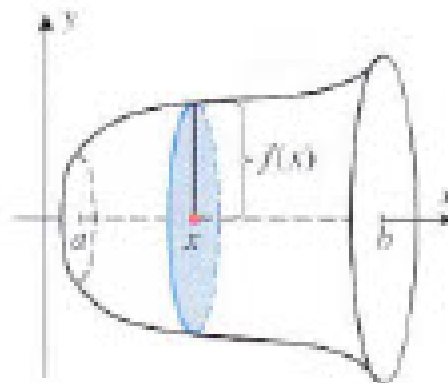
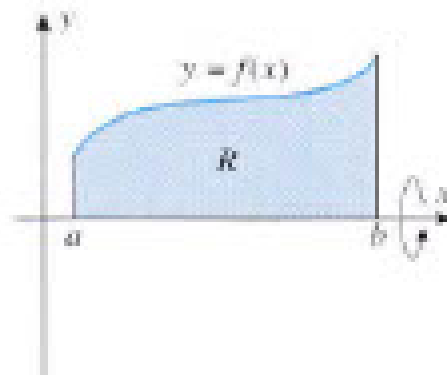
Volume of Revolution

...

Cont'd

Definition

Let f be continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x -axis, and on the sides by the lines $x = a$ and $x = b$. The volume of revolution that is generated by revolving the region R about the x -axis is given as



$$V = \int_a^b \pi [f(x)]^2 dx$$



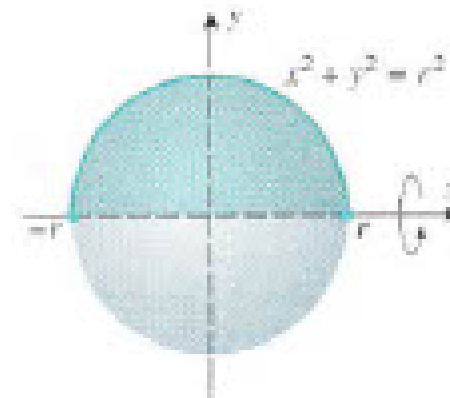
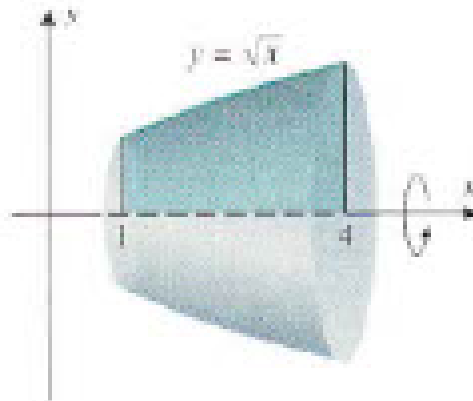
Volume

...

Cont'd

Example

- ❶ Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x -axis.



- ❷ Derive the formula for the volume of a sphere of radius r .

