

15, Dec.-12]

8

Vector Integral
Calculus and Application

8.1 : Line Integral

I) Work done by $\vec{F}$

If $\vec{F}$ represents the force field, then $\vec{F} \cdot \hat{T}$ represents a force along the tangent to the curve C , under this force say the particle is displaced by distance say ds along C .

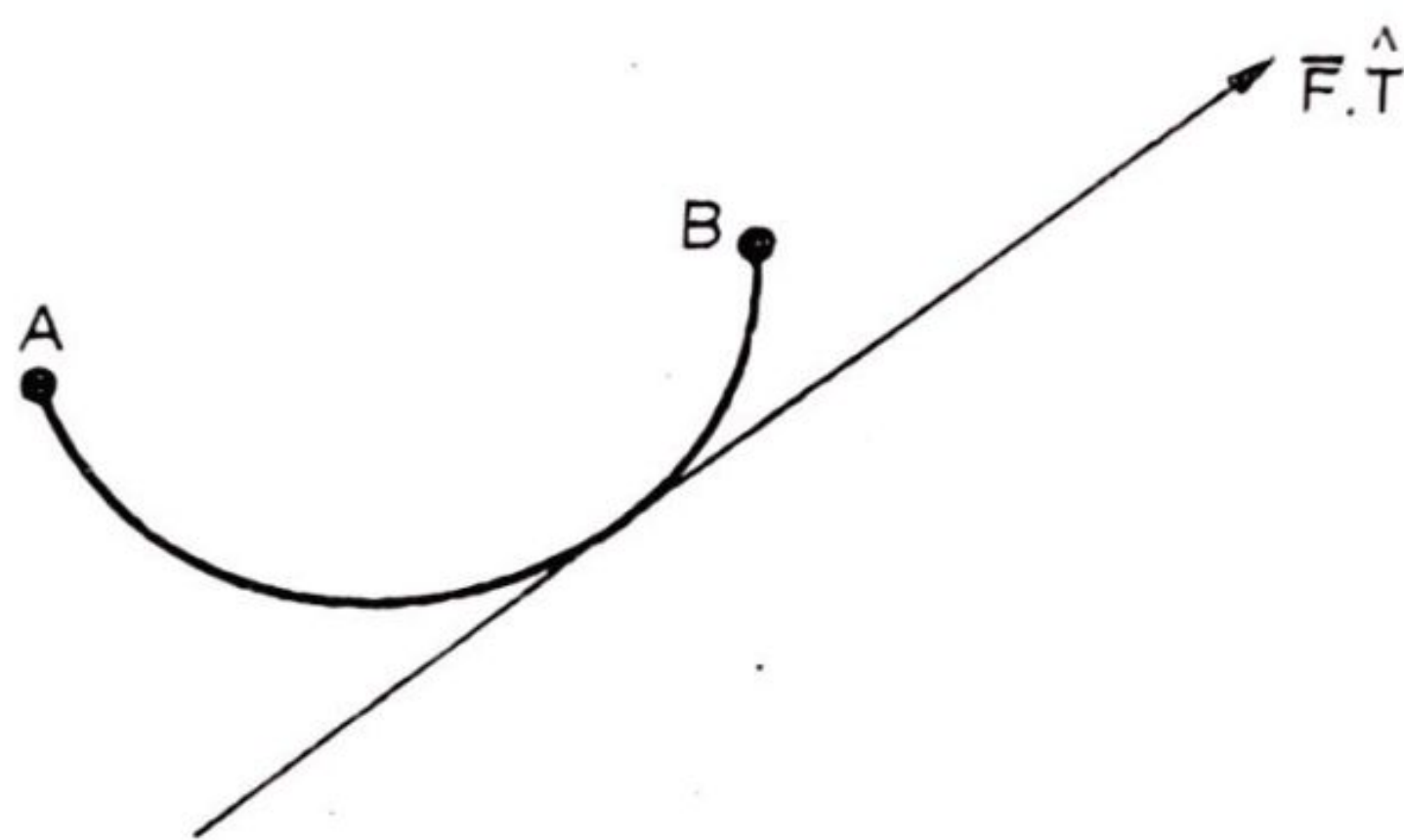


Fig. 8.1

$$\therefore (\vec{F} \cdot \hat{T}) ds = \text{Force} \times \text{Displacement} = \text{Work done}$$

$\therefore$ Total work done in moving a particle along C from A to B is given by

$$\text{Work done} = \int_A^B \vec{F} \cdot \hat{T} ds = \int_A^B \vec{F} \cdot d\vec{r}$$

II)

Note : **Circulation** In fluid dynamics, if $\vec{V}$ represents the velocity of the fluid particle and C is the closed curve, then the line integral $\oint_C \vec{V} \cdot d\vec{R}$ is called circulation of $\vec{V}$ round the curve C .

Note : In evaluation of line integrals we have to express the line integral $\int_C \vec{F} \cdot d\vec{r}$ in terms of one variable either x or y or z or t or θ and then integrate within corresponding limits.

Reduction Formulae

1) $\int_0^{2\pi} \sin^m \theta \cos^n \theta d\theta = 0$ if both are not even.

$\int_0^{2\pi} \sin^m \theta \cos^n \theta d\theta = 4 \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta$
if m, n both even.

2) $\int_0^{\pi} \sin^m \theta \cos \theta d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos \theta d\theta$
if n is even and any m.

$\int_0^{\pi} \sin^m \theta \cos \theta d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos \theta d\theta = 0$
if n is odd.

3) $\int_0^{\pi/2} \sin^m \theta d\theta = \int_0^{\pi/2} \cos^m \theta d\theta$

$\int_0^{\pi/2} \sin^m \theta d\theta = \left[\frac{(m-1) \dots 2 \text{ or } 1}{(m) \dots (2)} \right] \times \frac{\pi}{2}$ if m even

4) $\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{[(m-1) \dots] [(n-1) \dots]}{[(m+n) \dots]} \times \left[\frac{\pi}{2} \right]$

if both even
 $\int_0^{\pi/2} \sin \theta \cos \theta d\theta = \int_0^{\pi/2} \sin \theta \cos^m \theta d\theta = \frac{1}{m+1}$

Note :

1) For single integral over the circle $x^2 + y^2 = a^2$

Put $x = a \cos \theta$, $y = a \sin \theta$

$\therefore dx = -a \sin \theta d\theta$, $dy = a \cos \theta d\theta$

and the limits of θ are 0 to 2π .

2) For single integral over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Put $x = a \cos \theta$, $y = b \sin \theta$

$\therefore dx = -a \sin \theta d\theta$, $dy = b \cos \theta d\theta$

and the limits of θ are 0 to 2π .

Q.1 If $\vec{F} = x^2 \hat{i} + (x-y) \hat{j} + (y+z) \hat{k}$ displaces a particle from A(1, 0, 1) to B(2, 1, 2) along the straight line AB, find the work done.

[SPPU : Dec.-15]

Ans. : We have $\vec{F} \cdot d\vec{r} = x^2 dx + (x-y) dy + (y+z) dz$

The equation of line joining (1, 0, 1) to (2, 1, 2) is

$\frac{x-1}{2-1} = \frac{y-0}{1-0} = \frac{z-1}{2-1} \Rightarrow \frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{1} = t$

$\therefore x = 1+t$, $y = t$, $z = 1+t$

$dx = dt$, $dy = dt$, $dz = dt$

Limits are from $t = 0$ to $t = 1$

$\therefore$ Work done $= \int_0^1 \vec{F} \cdot d\vec{r}$
 $= \int_0^1 [(1+t)^2 dt + (1+t-t)dt + (t+1+t)dt]$
 $= \int_0^1 [t^2 + 2t + 1 + 1 + 2t + 1] dt$
 $= \int_0^1 [t^2 + 4t + 3] dt$

$$= \left[\frac{1^3}{3} + 4 \frac{1^2}{2} + 3 \cdot 1 \right]_0^1$$

$$= \left(\frac{1}{3} + \frac{4}{2} + 3 \right) - 0$$

$$= \frac{2+12+18}{6} = \frac{32}{6} = \frac{16}{3}$$

∴ Work done = $\frac{16}{3}$

Q.2 If $\vec{F} = (2x + y^2) \hat{i} + (3y - 4x) \hat{j}$ then evaluate $\int_C \vec{F} \cdot d\vec{r}$ around the parabolic arc $y^2 = x$ joining $(0, 0)$ to $(1, 1)$.

Ans. : We have $\vec{F} \cdot d\vec{r} = (2x + y^2) dx + (3y - 4x) dy$ [SPPU : Dec.-11, May-15]

Put $x = y^2 \Rightarrow dx = 2y dy$

$$\vec{F} \cdot d\vec{r} = (2y^2 + y^2) (2y dy) + (3y - 4y^2) dy$$

$$= 6y^3 dy + (3y - 4y^2) dy$$

$$\vec{F} \cdot d\vec{r} = (6y^3 - 4y^2 + 3y) dy$$

As the curve is parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$. Thus limits of y are 0 to 1.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6y^3 - 4y^2 + 3y) dy = \left[6 \frac{y^4}{4} - 4 \frac{y^3}{3} + 3 \frac{y^2}{2} \right]_0^1$$

$$= \frac{6}{4} - \frac{4}{3} + \frac{3}{2} - 0$$

$$= \frac{18-16+18}{12} = \frac{20}{12} = \frac{5}{3}$$

Q.3 Find the work done in moving a particle from $(0, 1, -1)$ to $\left(\frac{\pi}{2}, -1, 2\right)$ in a force field $\vec{F} = (y^2 \cos x + z^3) \hat{i} + (2y \sin x - 4) \hat{j} + (3xz^2 + 2) \hat{k}$.

[SPPU : Dec.-14]

Decode

A Guide for Engineering Students

Ans. : We have

$$\vec{F} \cdot d\vec{r} = (y^2 \cos x + z^3) dx + (2y \sin x - 4) dy + (3xz^2 + 2) dz$$

$$= d(y^2 \sin x + xz^3 - 4y) + 2dz$$

Consider

$$\text{Work done} = \int \vec{F} \cdot d\vec{r}$$

$$= \int_{(0,1,-1)}^{\left(\frac{\pi}{2}, -1, 2\right)} [d(y^2 \sin x + xz^3 - 4y) + 2dz]$$

$$= [y^2 \sin x + xz^3 - 4y + 2z]_{(0,1,-1)}^{(\pi/2, -1, 2)}$$

$$= \left[(-1)^2 \sin \frac{\pi}{2} + \left(\frac{\pi}{2}\right) (-1)^3 - 4(-1) + 2(2) \right]$$

$$- [1^2 \sin 0 + 0(-1)^3 - 4(1) + 2(-1)]$$

$$= [1 + 4\pi - 4 + 4] - [-6]$$

$$= 15 + 4\pi$$

Q.4 Find work done in moving a particle once around $\frac{x^2}{16} + \frac{y^2}{9} = 1$ in plane $z = 0$, where $\vec{F} = (3x - 2y) \hat{i} + (2x + 8y) \hat{j} + y^2 \hat{k}$.

[SPPU : Dec.-08, May-09]

Ans. : Consider

$$\vec{F} = (3x - 2y) \hat{i} + (2x + 8y) \hat{j} + y^2 \hat{k}$$

Consider

$$\int_C \vec{F} \cdot d\vec{r} = \int (3x - 2y) dx + (2x + 8y) dy + (y^2) dz$$

$a = 4$ and $b = 3$ for ellipse.

$$x = 4 \cos \theta \therefore dx = -4 \sin \theta d\theta$$

$$y = 3 \sin \theta \therefore dy = 3 \cos \theta d\theta$$

Decode

A Guide for Engineering Students

$$\begin{aligned} \therefore \int \vec{F} \cdot d\vec{r} &= \int [3(4 \cos \theta) - 2(3 \sin \theta)](-4 \sin \theta d\theta) \\ &\quad + [2(4 \cos \theta) + 8(3 \sin \theta)](3 \cos \theta d\theta) + [9 \sin^2 \theta] (0) \\ &= \int_0^{2\pi} (-48 \sin \theta \cos \theta d\theta + 24 \sin^2 \theta d\theta + 24 \cos^2 \theta d\theta) \\ &\quad + 72 \sin \theta \cos \theta d\theta \\ &= \int_0^{2\pi} 24(\sin^2 \theta + \cos^2 \theta) d\theta + 24 \sin \theta \cos \theta d\theta \\ &= \int_0^{2\pi} 24(1) d\theta + 0 \quad \text{(using conversion formula)} \\ &= 48\pi \end{aligned}$$

2.5 Find the work done in moving a particle along $x = a \cos \theta$, $y = a \sin \theta$, $z = b \theta$ from $\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$ under a field of force given by

$$\vec{F} = -3a \sin^2 \theta \cos \theta \hat{i} + a(2 \sin \theta - 3 \sin^3 \theta) \hat{j} + b \sin 2\theta \hat{k}$$

Ans. : We have $x = a \cos \theta$, $y = a \sin \theta$, $z = b\theta$ [SPPU : Dec.-13, May-14]

$$dy = a \cos \theta d\theta, dz = b d\theta$$

$$\begin{aligned} \int \vec{F} \cdot d\vec{r} &= \int_{\pi/4}^{\pi/2} \{-3a \sin^2 \theta \cos \theta (-a \sin \theta d\theta) \\ &\quad + (a(2 \sin \theta - 3 \sin^3 \theta))(a \cos \theta d\theta) + b \sin 2\theta (b d\theta)\} \\ &= \int_{\pi/4}^{\pi/2} \{3a^2 \sin^3 \theta \cos \theta d\theta + 2a^2 \sin \theta \cos \theta \\ &\quad - 3a^2 \sin^3 \theta \cos \theta d\theta + b^2 \sin 2\theta d\theta\} \\ &= \int_{\pi/4}^{\pi/2} \{2a^2 \sin \theta \cos \theta + b^2 \sin 2\theta\} d\theta \end{aligned}$$

$$\begin{aligned} &= \int_{\pi/4}^{\pi/2} (a^2 + b^2) \sin 2\theta d\theta = (a^2 + b^2) \left[-\frac{\cos 2\theta}{2} \right]_{\pi/4}^{\pi/2} \\ &= (a^2 + b^2) \frac{1}{2} \left[-\cos \pi + \cos \frac{\pi}{2} \right] \\ &= \frac{a^2 + b^2}{2} [-(-1) + 0] = \frac{a^2 + b^2}{2} \end{aligned}$$

Q.6 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for

$\vec{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$ along the path straight line joining $(0, 0, 0)$ and $(2, 1, 3)$. [SPPU : May-10, Dec.-12]

Ans. : Given that $\vec{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$

Consider

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y) dy + z dz$$

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C [3x^2 dx + (2xz - y) dy + z dz] \quad \dots (Q.6.1)$$

The equation of line joining $(0, 0, 0)$ to $(2, 1, 3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$\therefore x = 2t, y = t, z = 3t \quad t = 0 \text{ to } t = 1$$

Equation (Q.6.1) becomes

$$\begin{aligned} \text{Work done} &= \int_0^1 [3(4t^2)2 dt + (12t^2 - t)dt + 3t3dt] \\ &= \int_0^1 [36t^2 + 8t] dt = \left[36 \frac{t^3}{3} + 8 \frac{t^2}{2} \right]_0^1 \\ &= 36 \left(\frac{1}{3} \right) + \frac{8}{2} (1) = 16 \end{aligned}$$

Q.7 Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $z = 0$ under the force field $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$.

Ans. : Step 1 : We have [SPPU : Dec.-05, 11, 12, 15, May-10]

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

Step 2 : Let $x = 5 \cos \theta$, $y = 4 \sin \theta$ and $z = 0$

$$\therefore \vec{F} = (10 \cos \theta - 4 \sin \theta)\hat{i} + (5 \cos \theta + 4 \sin \theta)\hat{j} + (15 \cos \theta - 8 \sin \theta)\hat{k}$$

$$\text{and } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 5 \cos \theta \hat{i} + 4 \sin \theta \hat{j} + 0 \hat{k}$$

$$d\vec{r} = (-5 \sin \theta \hat{i} + 4 \cos \theta \hat{j}) d\theta$$

$$\text{Step 3 : Work done} = \int_0^{2\pi} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} [-50 \sin \theta \cos \theta + 20 \sin^2 \theta + 16 \cos \theta \sin \theta + 2 \cos^2 \theta] d\theta$$

$$= \int_0^{2\pi} [-34 \sin \theta \cos \theta + 20] d\theta = \int_0^{2\pi} [-17 \sin 2\theta + 20] d\theta$$

$$= \left[17 \frac{\cos 2\theta}{2} + 20\theta \right]_0^{2\pi} = 40\pi$$

8.2 : Green's Theorem

1) Cartesian Form :

If R is a closed region of the XY -plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivation in R , then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where C is traversed in the positive (counter clockwise) direction.

Decode

A Guide for Engineering Students

2) Green's Theorem in Plane in Vector Notation

Green's theorem in the XOY plane can be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} dx dy$$

Q.8 Verify Green's theorem in plane $\oint_C [(3x^2 - 8y^2) dx + (4y - 6xy) dy]$ where C is the boundary

the region defined by $y = \sqrt{x}$, $y = x^2$. [SPPU : Dec.-05, 11, 12, 15, May-10]

Ans. : $y^2 = x$, $y = x^2$, intersects at $O(0, 0)$, $A(1, 1)$ we have

$$M = 3x^2 - 8y^2, N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y$$

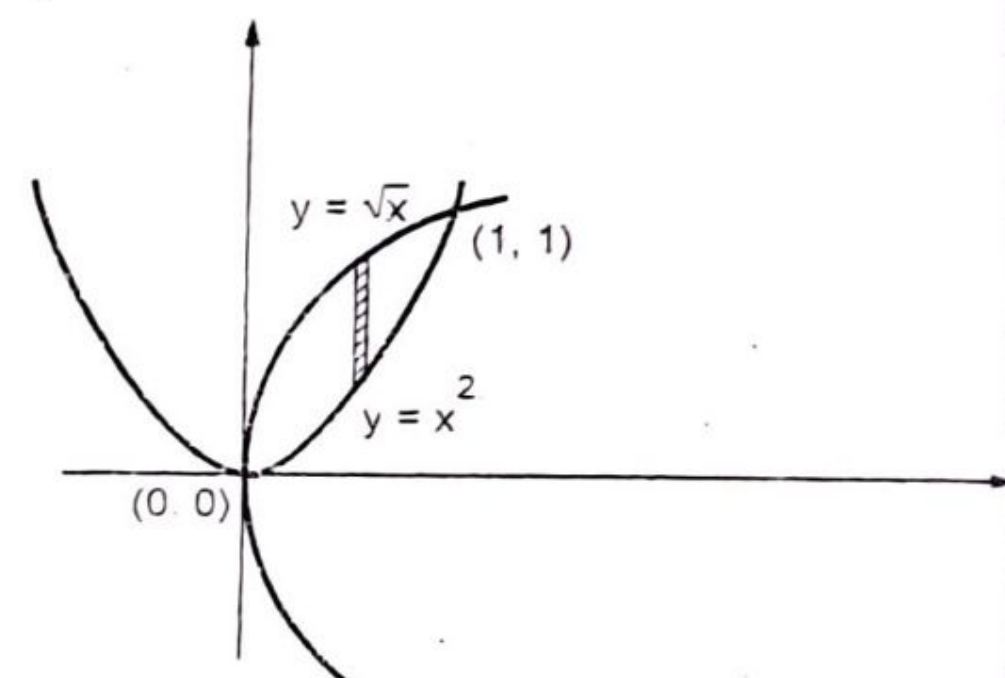


Fig. Q.8.1

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

Let R be the region bounded by two parabolas.

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Decode

A Guide for Engineering Students

$$\begin{aligned}
 &= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \, dy \, dx \\
 &= 5 \int_0^1 [y^2]_{x^2}^{\sqrt{x}} \, dx = 5 \int_0^1 (x - x^4) \, dx = 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\
 &= 5 \left[\frac{1}{2} - \frac{1}{5} \right] = \frac{3}{2} \quad \dots (Q.8.1)
 \end{aligned}$$

Next $\oint_C M \, dx + N \, dy = \oint_{C_1} (M \, dx + N \, dy) + \oint_{C_2} (M \, dx + N \, dy)$

Along C_1 ; $x^2 = y \Rightarrow dy = 2x \, dx$, x varies from 0 to 1.

$$\begin{aligned}
 M \, dx + N \, dy &= (3x^2 - 8y^2) \, dx + (4y - 6xy) \, dy \\
 &= (3x^2 - 8y^4) \, dx + (4x^2 - 6x^3) \, 2x \, dx \\
 &= (3x^2 + 8x^3 - 20x^4) \, dx
 \end{aligned}$$

$$\begin{aligned}
 M \, dx + N \, dy &= \int_0^1 (3x^2 + 8x^3 - 20x^4) \, dx \\
 &= [x^3 + 2x^2 - 4x^4]_0^1 \\
 &= -1
 \end{aligned}$$

Along C_2 ; $y^2 = x \Rightarrow dx = 2y \, dy$

$$\begin{aligned}
 M \, dx + N \, dy &= (3y^4 - 8y^2) \, 2y \, dy + (4y - 6y^3) \, dy \\
 &= (4y - 22y^3 + 6y^5) \, dy
 \end{aligned}$$

$$\begin{aligned}
 M \, dx + N \, dy &= \int_1^0 (4y - 22y^3 + 6y^5) \, dy \\
 &= \left[2y^2 - \frac{22}{4}y^4 + y^6 \right]_1^0 = \frac{5}{2}
 \end{aligned}$$

$$M \, dx + N \, dy = -1 + \frac{5}{2} = \frac{3}{2} \quad \dots (Q.8.2)$$

From equations (Q.8.1) and (Q.8.2), Green's theorem is verified.

Q.9 Apply Green's theorem to evaluate

$$\oint_C (2x^2 - y^2) \, dx + (x^2 + y^2) \, dy$$

where C is the boundary of the area enclosed by the axis and the upper half of the circle. $x^2 + y^2 = 16$.

[SPPU : May-15]

Ans. : Green's theorem states that

$$\oint_C M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \quad \dots (Q.9.1)$$

Let

$$M = 2x^2 - y^2 \text{ and } N = x^2 + y^2$$

$$\frac{\partial N}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x + y)$$

$\therefore$ Equation (Q.9.1) becomes

$$I = \oint_C M \, dx + N \, dy = \iint_R 2(x + y) \, dx \, dy$$

Given that $x^2 + y^2 = 16 \therefore x = r \cos \theta$

$$y = r \sin \theta \quad dx \, dy = r \, dr \, d\theta$$

$$I = \int_0^\pi \int_0^4 2(r \cos \theta + r \sin \theta) r \, dr \, d\theta$$

$$= \int_0^\pi 2(\cos \theta + \sin \theta) \left[\frac{r^3}{3} \right]_0^4 \, d\theta$$

$$= 2 \left(\frac{4^3}{3} \right) \int_0^\pi [\cos \theta + \sin \theta] \, d\theta$$

$$= \frac{2(4)^3}{3} [\sin \theta - \cos \theta]_0^\pi$$

$$= 2 \times \frac{64}{3} [0 - (-1) - 0 + 1] = \frac{256}{3}$$

8.3 : Stoke's Theorem

1) If S be an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds, \text{ where}$$

$\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C .

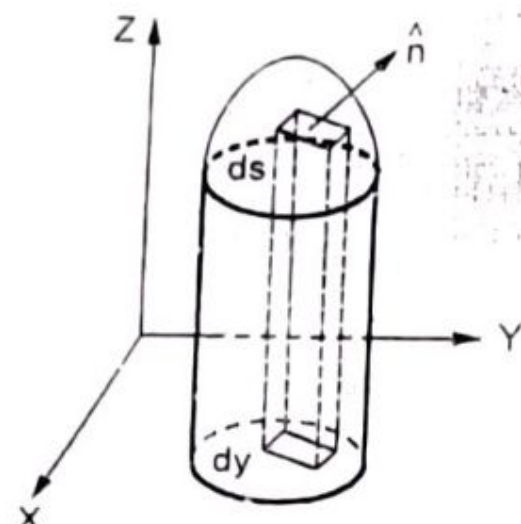


Fig. 8.2

Note

- 1) The abbreviation of $\hat{N} \, ds$ or $\hat{n} \, ds$ is $d\vec{s}$.
- 2) If the surface is in XOY plane then $\hat{N} = \hat{k}$, $ds = dx \, dy$.
- 3) If the surface is closed then bounding curve does not exist therefore we cannot evaluate Stoke's theorem.
- 4) If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$.
 $\therefore \oint_C \vec{F} \cdot d\vec{r} = 0$.
- 5) If S_1 and S_2 are two surfaces with the same bounding curve C then

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{s}_1 = \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{s}_2$$

Decode

A Guide for Engineering Students

6) If the surface is not in XOY plane then $\hat{N} = \frac{\nabla \phi}{|\nabla \phi|}$ the projection formula $ds = \frac{dx \, dy}{|\hat{N} \cdot \hat{k}|}$ (where $\phi = C$ is the for the projection of surface element ds on XOY plane

Q.10 Verify Stokes' theorem for $\vec{F} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$ for the surface of rectangular bounded by $x = 0, y = 0, x = 1, y = 2, z = 0$.

[SPPU : May-10, 13, Dec-12]

Ans. : Step 1 : Given that

$$\vec{F} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$$

Step 2 :

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_C [xy^2 dx + y dy] \\ &= \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \\ &= 0 + \int_0^2 y \, dy + \int_1^0 4x \, dx + \int_0^2 y \, dy \\ &= \int_0^2 y \, dy + \left[\frac{4x^2}{2} \right]_1^0 - \int_0^2 y \, dy = -2 \end{aligned}$$

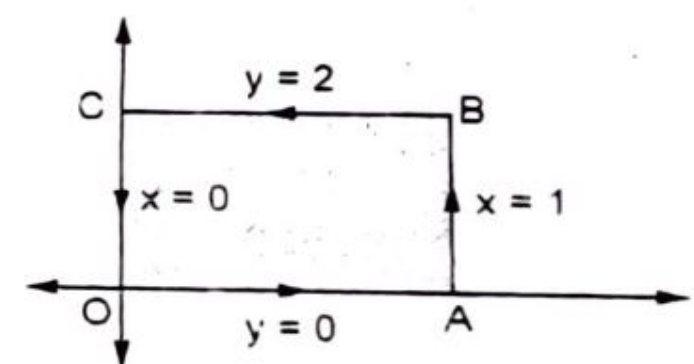


Fig. Q.10.1

Decode

A Guide for Engineering Students

Step 3 : Now, to obtain surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y & 0 \end{vmatrix} = -2xy \hat{k}$$

$\hat{n} = \hat{k}$ and $dS = dx dy$

Step 4 :

$$\begin{aligned} (\nabla \times \vec{F}) \cdot \hat{n} dS &= \iint_S (-2xy) \hat{k} \cdot \hat{k} dx dy \\ &= \int_{x=0}^1 \int_0^2 (-2xy) dy dx \\ &= -2 \int_0^1 \left[x \frac{y^2}{2} \right]_0^2 dx \\ &= -4 \int_0^1 x dx = -4 \left[\frac{x^2}{2} \right]_0^1 \\ &= -2 \end{aligned}$$

$$\text{Step 5 : Thus } \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} dS = -2$$

Hence Stoke's theorem is verified.

11 Verify Stoke's theorem $\vec{F} = -y^3 \hat{i} + x^3 \hat{j}$ over ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[SPPU : Dec.-14, for $a = b = 1$]

as. : Step 1 : Consider

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} \\ &= \hat{i} (0-0) - \hat{j} (0-0) + \hat{k} (3x^2 + 3y^2) \end{aligned}$$

Step 2 : As the surface is in X-O-Y plane.

$$\therefore \hat{N} = \hat{k}, ds = dx dy$$

Step 3 : Consider $\iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds$

Step 4 : Substitute the values.

$$= \iint 3(x^2 + y^2) dx dy.$$

Step 5 : Consider

$$\iint x^2 dx dy \text{ over } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Step 6 : Put $x = a r \cos \theta$, $y = b r \sin \theta$

$$dx dy = a b r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 a^2 r^2 \cos^2 \theta \cdot a b r dr d\theta$$

Step 7 : Integrate

$$= \int_0^{2\pi} a^3 b \left(\frac{r^4}{4} \right)_0^1 \cdot \cos^2 \theta d\theta$$

Using conversion formula

$$= \frac{a^3 b}{4} \cdot 4 \int_0^{\pi/2} \cos^2 \theta d\theta$$

Using reduction formula

$$= a^3 b \left(\frac{1}{2} \cdot \frac{\pi}{2} \right) = a^3 b \frac{\pi}{4}$$

Step 8 : Similarly we can show

$$\therefore \iint y^2 dx dy = ab^3 \frac{\pi}{4}$$

Step 9 : Thus $\iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds$

$$= 3 \left[a^3 b \frac{\pi}{4} + ab^3 \frac{\pi}{4} \right] = 3 ab \frac{\pi}{4} [a^2 + b^2] \quad \dots (Q.11.1)$$

Step 10 : To find $\oint_C \vec{F} \cdot d\vec{r}$

Consider

$$\int_C \vec{F} \cdot d\vec{r} = \int_C -y^3 dx + x^3 dy$$

over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Step 11 : The parametric equations of ellipse are

$$x = a \cos \theta, y = b \sin \theta, z = 0$$

$$\therefore dx = -a \sin \theta d\theta, dy = b \cos \theta d\theta, dz = 0$$

Step 12 : Substituting we get

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} -b^3 \sin^3 \theta (-a \sin \theta) d\theta \\ &\quad + a^3 \cos^3 \theta b \cos \theta d\theta \\ &= \int_0^{2\pi} (ab^3 \sin^4 \theta + a^3 b \cos^4 \theta) d\theta \end{aligned}$$

Step 13 : Using conversion formula

$$= 4 \int_0^{\pi/2} (ab^3 \sin^4 \theta + a^3 b \cos^4 \theta) d\theta$$

Step 14 : Using reduction formula

$$\begin{aligned} &= 4 \left[ab^3 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) + a^3 b \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) \right] \\ &= 3ab \frac{\pi}{4} (a^2 + b^2) \quad \dots (Q.11.2) \end{aligned}$$

From equations (Q.11.1) and (Q.11.2) Stoke's theorem is verified.

Q.12 Evaluate $\oint_C (e^x dx + 2y dy - dz)$ where C is the curve $x^2 + y^2 = 4$,

$$z = 2.$$

[SPPU : Dec.-15]

Ans. : Let $I = \int (e^x dx + 2y dy - dz)$ and $x^2 + y^2 = 2, z = 2$

$\therefore$ By stokes theorem

Decode

A Guide for Engineering Students

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds$$

where $\vec{F} = e^x \hat{i} + 2y \hat{j} - \hat{k}$

$$\begin{aligned} \therefore \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} \\ &= \hat{i} (0 - 0) - \hat{j} (0 - 0) + \hat{k} (0 - 0) \\ &= 0 \end{aligned}$$

$\therefore$ Equations (Q.12.1) becomes

$$I = \oint_C \vec{F} \cdot d\vec{r} = \iint_S 0 d\vec{s} = 0$$

Q.13 Using Stokes's theorem, evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{s}$ where

$\vec{F} = 3y\hat{i} - xz^2\hat{j} + yz^2\hat{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$. [SPPU : May]

Ans. : By Stoke's theorem

$$I = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} d\vec{s} = \oint_C \vec{F} \cdot d\vec{r}$$

The equation of paraboloid is

$$x^2 + y^2 = 2z \text{ and } z = 2$$

$$\therefore x^2 + y^2 = 2 \times 2 = 4$$

$$\therefore \text{Put } x = 2 \cos \theta, y = 2 \sin \theta$$

$$dx = -2 \sin \theta d\theta, dy = 2 \cos \theta d\theta$$

$\therefore$ Equation (Q.13.1) becomes

$$\begin{aligned} I &= \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3y dx - xz^2 dy + yz^2 dz) \\ &= \int_0^{2\pi} (3 \times 2 \sin \theta) (-2 \sin \theta d\theta) - 2 (2 \cos \theta) (2 \cos \theta d\theta) \end{aligned}$$

Decode

A Guide for Engineering Students

$$\begin{aligned}
 &= \int_0^{2\pi} [-12\sin^2 \theta - 8\cos^2 \theta] d\theta \\
 &= 4 \int_0^{\pi/2} [-12\sin^2 \theta - 8\cos^2 \theta] d\theta \\
 &= 4 \left(-12 \left(\frac{1}{2} \cdot \frac{\pi}{2} \right) - 8 \left(\frac{1}{2} \cdot \frac{\pi}{2} \right) \right) = -12\pi - 8\pi = -20\pi
 \end{aligned}$$

Q.14 Using Stokes theorem, evaluate $\int_C (x+y) dx + (2x-z) dy + (y+z) dz$ where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and $x+y=2a$.

Ans. : We have $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$ [SPPU : May-15]

$$\begin{aligned}
 \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} \\
 &= \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(2-1) = 2\hat{i} + \hat{k}
 \end{aligned}$$

Here the surface S is the intersection of $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and $x+y=2a$

$\hat{N}$ is the unit normal to the plane $x+y=2a$

$$\hat{N} = \frac{\nabla(x+y-2a)}{|\nabla(x+y-2a)|} = \frac{\hat{i}+\hat{j}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$\nabla \times \vec{F} \cdot \hat{N} = \frac{1}{\sqrt{2}}(2+0+0) = \sqrt{2}$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{N} ds = \sqrt{2} \iint_S ds$$

$$= \sqrt{2} [\text{Area of } S]$$

$$= \sqrt{2} [\text{Area of given circle}]$$

We have $x^2 + y^2 + z^2 - 2ax - 2ay = 0$

$$(x-a)^2 + (y-0)^2 + (z-0)^2 = 2a^2$$

...(Q.14.1)

Decode

A Guide for Engineering Students

Centre = $(a, a, 0)$ and radius = $\sqrt{2}a$

The plane $x+y=2a$ passes through $(a, a, 0)$

Radius of circle = Radius of sphere

Equation (Q.14.1) becomes $= a\sqrt{2}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \hat{n} ds = \sqrt{2}\pi [a\sqrt{2}]^2 = 2\sqrt{2}\pi a^2$$

Q.15 Use Stoke's theorem for

$\int_C (4yi + 2zj + 6yk) \cdot d\vec{r}$ where C is intersection of $x^2 + y^2 + z^2 = 2z$ and $x = z - 1$.

[SPPU : Dec.-04, 06, 12, 13, 14, May-14]

Ans. : Step 1 : Consider $\vec{F}$ and find $\nabla \times \vec{F}$

$$\begin{aligned}
 \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix} \\
 &= \hat{i}(6-2) + 0\hat{j} + (0-4)\hat{k} \\
 &= 4\hat{i} + 0\hat{j} - 4\hat{k}
 \end{aligned}$$

Step 2 : Here surface 'S' is the intersection of circle

$$x^2 + y^2 + z^2 - 2z = 0 \text{ and } x = z - 1$$

$\hat{N}$ is unit normal to the plane $x - z + 1 = 0$

$$\hat{N} = \frac{\nabla(x-z+1)}{|\nabla(x-z+1)|} = \frac{\hat{i}-\hat{k}}{\sqrt{1^2+(-1)^2}} = \frac{\hat{i}-\hat{k}}{\sqrt{2}}$$

$$\nabla \times \vec{F} \cdot \hat{N} = (4\hat{i} - 4\hat{k}) \cdot \frac{(\hat{i}-\hat{k})}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = \frac{8}{\sqrt{2}}$$

Step 3 :

$$\begin{aligned}
 \iint_S \nabla \times \vec{F} \cdot \hat{N} ds &= \frac{8}{\sqrt{2}} \iint_S ds \\
 &= \frac{8}{\sqrt{2}} (\text{Area of } S)
 \end{aligned}$$

Decode

A Guide for Engineering Students

$$= \frac{8}{\sqrt{2}} (\text{Area of given circle})$$

$$= \frac{8}{\sqrt{2}} (\pi r^2) \quad r \text{ is radius of given circle} \quad \dots (Q.15.1)$$

Step 4 : The given sphere is

$$x^2 + y^2 + z^2 - 2z = 0$$

$$\text{i.e. } x^2 + y^2 + (z-1)^2 = 1$$

$\therefore$ Centre = (0, 0, 1), Radius = 1

As the centre satisfies the equation $x = z - 1$.

$\therefore$ The plane $x = z - 1$ passes through the centre of the given sphere. $\therefore$ The circle obtained is of maximum radius.

Step 5 : i.e. Radius of circle = Radius of sphere = 1

Step 6 : Hence from equation (Q.15.1) the required integral is

$$= \frac{8}{\sqrt{2}} \pi (1)^2 = \frac{8\pi}{\sqrt{2}}$$

Q.16 Evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{s}$

where $\vec{F} = (x^3 - y^3)\hat{i} - xyz\hat{j} + y^2\hat{k}$ and S is the surface $x^2 + 4y^2 + z^2 - 2x = 4$ above the plane $x = 0$

[SPPU : May-11, 12, Dec.-12, 13, 15]

Ans. : By Stoke's theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{s} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C [(x^3 - y^3) dx - xyz dy + y^2 dz]$$

where C is the curve $4y^2 + z^2 = 4$

$$\therefore \frac{y^2}{1^2} + \frac{z^2}{2^2} = 1 \text{ which is an ellipse.}$$

$$\therefore \text{Put } y = \cos \theta, z = 2 \sin \theta, x = 0$$

Decode

A Guide for Engineering Students

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} y^3 dz = \int_{\theta=0}^{2\pi} \cos^2 \theta (2 \cos \theta) d\theta$$

$$= 2 \int_{\theta=0}^{2\pi} \cos^3 \theta d\theta$$

$$= 2 \times 0 = 0$$

8.4 : The Divergence Theorem (or Gauss Divergence Theorem)

If $\vec{F}$ is a continuously differentiable vector function in the volume V bounded by the closed surface S , then

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \text{div } \vec{F} dv$$

where $\hat{n}$ is the unit outward drawn normal to the surface S .

$$\text{i.e. } \iiint_V \nabla \cdot \vec{F} dv = \iint_S \vec{F} \cdot \hat{n} ds$$

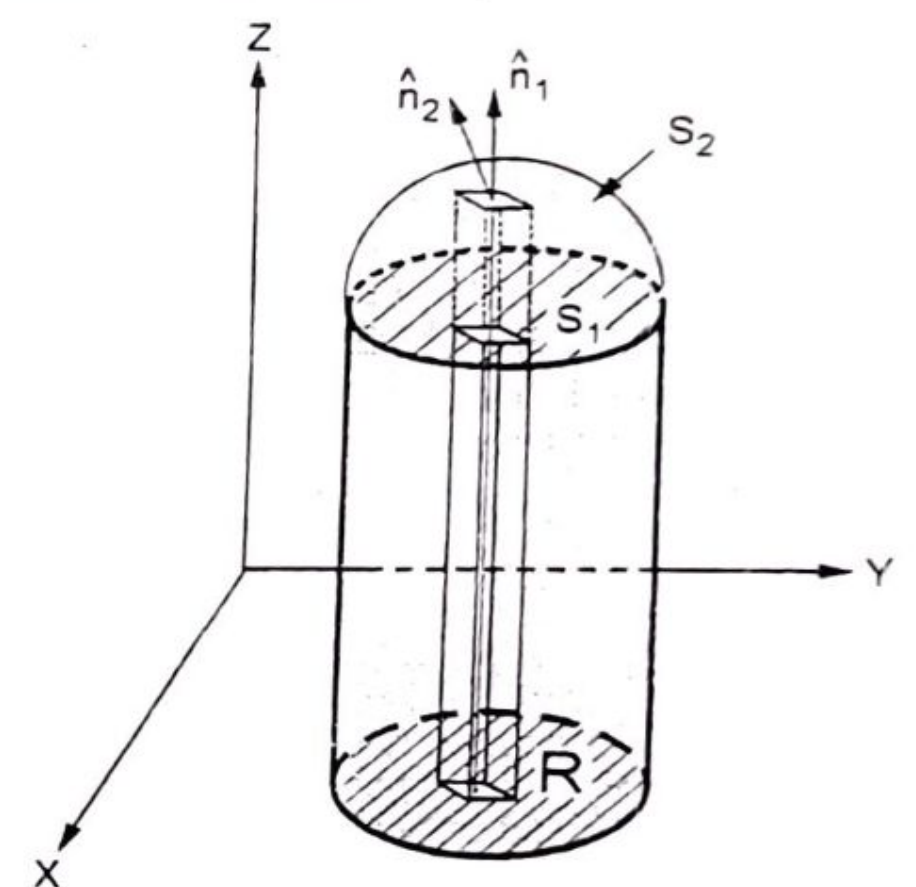


Fig. 8.3

Decode

A Guide for Engineering Students

17 Verify Gauss Divergence theorem $\vec{F} = (x+y^2)\hat{i} - 2xz\hat{j} + 2yz\hat{k}$ over the volume of tetrahedron bounded by co-ordinate planes and plane $2x+y+2z=6$. [SPPU : Dec.-14, 15]

Ans. : Step 1 : Consider $\vec{F}$ and find $\nabla \cdot \vec{F}$

$$\nabla \cdot \vec{F} = 1 - 0 + 2y$$

$$= 1 + 2y$$

Step 2 : Consider $\iiint \nabla \cdot \vec{F} \, dv = \iiint (1+2y) \, dx \, dy \, dz$

over the volume bounded by $x=0, y=0, z=0, 2x+y+2z=6$

Now consider $2x+y+2z=6$

$$+ \frac{x}{3} + \frac{z}{3} = 1$$

Put $x = \frac{x}{3}, y = \frac{y}{6}, z = \frac{z}{3}$ then

$dx \, dy \, dz = 3 \cdot 6 \cdot 3 \, dx \, dy \, dz$ and the volume is bounded by $x=0, y=0, z=0, x+y+z=1$.

Thus the integral becomes

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+2 \cdot 6y) \cdot 3 \cdot 6 \cdot 3 \, dz \, dy \, dx$$

$$= 54 \int_0^1 \int_0^{1-x} (1+12y)(1-x-y) \, dy \, dx$$

Put $1-x=h$ for convenience

$$= 54 \int_0^1 \int_0^h (1+12y)(h-y) \, dy \, dx$$

$$= 54 \int_0^1 \int_0^h [(h-y) + 12(hy-y^2)] \, dy \, dx$$

$$= 54 \int_0^1 \left[\frac{(h-y)^2}{-2} + 12 \left[h \frac{y^2}{2} - \frac{y^3}{3} \right] \right]_0^h \, dx$$

$$= 54 \int_0^1 \left\{ 0 + 12 \left[\frac{h^3}{2} - \frac{h^3}{3} \right] \right\} - \left[\frac{h^2}{-2} + 0 \right] \, dx$$

Put $h = 1-x$

$$= 54 \int_0^1 2(1-x)^3 + \frac{(1-x)^2}{2} \, dx$$

$$= 54 \left[\frac{2(1-x)^4}{-4} + \frac{(1-x)^3}{-3 \cdot 2} \right]_0^1$$

$$= 54 \left\{ [0-0] - \left[-\frac{2}{4} - \frac{1}{6} \right] \right\}$$

$$= 54 \left(\frac{1}{2} + \frac{1}{6} \right) = 54 \cdot \frac{4}{6} = 36$$

Step 3 : For double integral draw the Fig. Q.17.1.

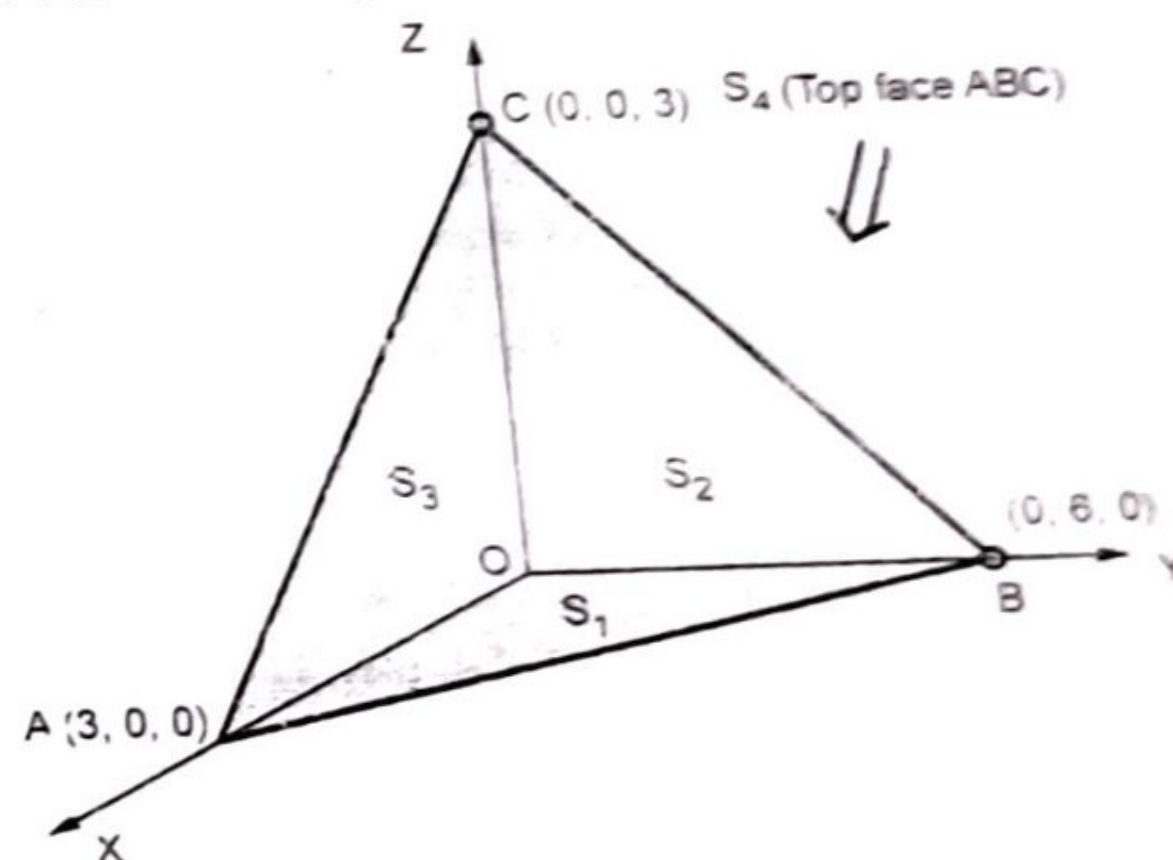


Fig. Q.17.1

Step 4 : For S_1 OAB $\vec{N} = -\hat{k}$, $ds = dx \, dy$, $z=0$, $2x+y=6$, $\vec{F} \cdot \vec{N} = -2yz = 0$

For S_2 OBC $\hat{N} = -\hat{i}$, $ds = dy dz$, $x = 0$,
 $y + 2z = 6$, $\vec{F} \cdot \hat{N} = -(x + y^2) = -y^2$

For S_3 OAC $\hat{N} = -\hat{j}$, $ds = dx dz$,

$y = 0$, $2x + 2z = 6$, $x + z = 3$, $\vec{F} \cdot \hat{N} = 2x$

$$\text{Step 5 : } \iint_{S_1} \vec{F} \cdot \hat{N} ds = \iint 0 dx dy = 0$$

$$\text{Step 6 : } \iint_{S_2} \vec{F} \cdot \hat{N} ds = \iint -y^2 dy dz$$

over the triangle bounded by $y = 0$, $z = 0$, $y + 2z = 6$.

$$= \int_0^3 \int_0^{6-2z} -y^2 dy dz = \int_0^3 -\frac{(6-2z)^3}{3} dz$$

$$= \left[-\frac{(6-2z)^4}{3 \cdot 4(-2)} \right]_0^3$$

$$= 0 - \frac{6^4}{3 \cdot 4 \cdot 2} = -\frac{6^3}{3} = -54$$

$$\text{Step 7 : } \iint_{S_3} \vec{F} \cdot \hat{N} ds = \iint 2x dx dz$$

over the triangle bounded by $x = 0$, $z = 0$, $x + z = 3$

$$= \int_0^3 \int_0^{3-z} 2x dx dz = 2 \int_0^3 \frac{(3-z)^2}{2} dz$$

$$= \left[\frac{(3-z)^3}{-3} \right]_0^3 = -\frac{3^3}{3} = -9$$

Step 8 : Now S_4 : ABC

Let $0 = 2x + y + 2z - 6 = 0$

$$\nabla 0 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{N} = \frac{\nabla 0}{|\nabla 0|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

$$\vec{F} \cdot \hat{N} = \frac{2(x + y^2) - 2x + 4yz}{3} = \frac{2}{3} (y^2 + 2yz)$$

The projection of S_4 on XOY plane is triangle OAB.

$$ds = \frac{dx dy}{|\hat{N} \cdot \hat{k}|} = \frac{dx dy}{|2/3|}$$

$$\therefore \iint_{S_4} \vec{F} \cdot \hat{N} ds = \iint \frac{2}{3} (y^2 + 2yz) \frac{dx dy}{2/3}$$

$$= \iint (y^2 + 2yz) dx dy$$

On S_4 $2z = 6 - 2x - y$ $\therefore \iint y^2 + y(6 - 2x - y) dx dy$

$$= \iint y(6 - 2x) dx dy$$

Over the triangle OAB bounded by

$x = 0$, $y = 0$, $2x + y = 6$

$$= \int_0^3 \int_0^{6-2x} y(6-2x) dy dx$$

$$= \int_0^3 \frac{(6-2x)^2}{2} (6-2x) dx$$

$$= \left[\frac{1}{2} \frac{(6-2x)^4}{4(-2)} \right]_0^3 = 0 + \frac{6^4}{2 \cdot 4 \cdot 2} = 81$$

Step 9 : Adding on the integrals we get

$$\iint_S \vec{F} \cdot \hat{N} ds = 0 + (-54) + 9 + 81$$

$$= 36$$

G.D.T. is verified.

Q.18 Evaluate $\iint (2xy\hat{i} + yz^2\hat{j} + xz\hat{k}) \cdot d\vec{s}$ over the total surface of region bounded by $x = 0, y = 0, z = 0, y = 3$ and $x + 2z = 6$.

Ans. : Step 1 : Consider [SPPU : Dec.-10, 15, May-14]

$$\nabla \cdot \vec{F} = 2y + z^2 + x$$

Step 2 : Now by G.D.T.

$$\iint_S \vec{F} \cdot \hat{N} ds = \iiint \nabla \cdot \vec{F} dv$$

$$= \iiint (2y + z^2 + x) dx dy dz$$

Step 3 : Over the volume bounded by $x = 0, y = 0, z = 0, y = 3, x + 2z = 6$

$$= \int_0^3 \int_0^3 \int_0^{6-2z} (2y + z^2 + x) dx dy dz$$

$$= \int_0^3 \int_0^3 \left[2y(6-2z) + z^2(6-2z) + \frac{(6-2z)^2}{2} \right] dy dz$$

$$= \int_0^3 \left[\frac{(3^2)(6-2z) + (6z^2 - 2z^3)(3)}{2} + \frac{(6-2z)^2}{2} (3) \right] dz$$

$$= \left[9 \frac{(6-2z)^2}{2(-2)} + 3 \left(6 \cdot \frac{z^3}{3} - 2 \frac{z^4}{4} \right) + \frac{(6-2z)^3}{2 \cdot 3(-2)} (3) \right]_0^3$$

$$= \left[0 + 3 \left(2 \cdot 3^3 - \frac{3^4}{2} \right) + 0 \right] - \left[9 \cdot \frac{6^2}{-4} + 0 - \frac{6^3}{2 \cdot 3 \cdot 2} (3) \right]$$

$$= 3^4 \left(2 - \frac{3}{2} \right) + \frac{6^2 \cdot 9}{4} + \frac{6^3}{4}$$

$$= \frac{81}{2} + 82 + 54 = \frac{351}{2}$$

Q.19 Evaluate $\iint \vec{r} \cdot d\vec{s}$ over the surface of sphere $x^2 + y^2 + z^2 = 1$.

[SPPU : May-12]

Ans. : By G.D.T.

$$\iint \vec{r} \cdot d\vec{s} = \iiint \nabla \cdot \vec{r} dv$$

$$= \iiint 3 dv$$

$$= 3 (\text{vol})$$

$$= 3 \cdot \frac{4}{3} \pi a^3 = a = 1$$

$$= 4\pi$$

Q.20 Evaluate $\iint (x^3\hat{i} + y^3\hat{j} + z^3\hat{k}) \cdot d\vec{s}$ over the surface of sphere $x^2 + y^2 + z^2 = a^2$.

[SPPU : May-06, 10, 13, 15, Dec.-11, 12, 13]

Ans. : Step 1 : By G.D.T.

$$\iint \vec{F} \cdot \hat{N} ds = \iiint \nabla \cdot \vec{F} dv$$

$$= \iiint 3(x^2 + y^2 + z^2) dx dy dz$$

Over $x^2 + y^2 + z^2 = a^2$

Step 2 : Use spherical polar co-ordinate for sphere

Put $x = r \sin \theta \cos \phi$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$dx dy dz = r^2 \sin \theta dr d\phi d\theta$$

Here the value of $x^2 + y^2 + z^2$ is r^2 .

Step 3 : Use the standard limits for full sphere of radius.

The integral becomes

$$= \int_0^\pi \int_0^{2\pi} \int_0^a 3r^2 r^2 \sin \theta dr d\phi d\theta$$

$$= 3 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^a r^4 dr$$

$$= 3 [-\cos \theta]_0^\pi (2\pi) \left[\frac{r^5}{5} \right]_0^a$$

$$= 3(2) \cdot 2\pi \cdot \frac{a^5}{5} = \frac{12\pi}{5} a^5$$

Q.21 Use G.D.T. to evaluate

$\iiint_S \vec{F} \cdot \hat{N} ds$ where $\vec{F} = y^2 z^2 \hat{i} + x^2 z^2 \hat{j} + x^2 y^2 \hat{k}$ where S is the surface of hemisphere $x^2 + y^2 + z^2 = 9$ above XOY plane.

[SPPU : May-05, 15, Dec.-14]

Ans. : Consider the hemisphere

S_1 = Surface of hemisphere

S_2 = Surface of circle

To find the integral over S_1 by G.D.T.

$$\iiint_S \vec{F} \cdot \hat{N} ds = \iiint_V \nabla \cdot \vec{F} dv$$

But S is combination of S_1 and S_2

$$\iiint_{S_1} \vec{F} \cdot \hat{N} ds + \iiint_{S_2} \vec{F} \cdot \hat{N} ds = \iiint_V \nabla \cdot \vec{F} dv \quad \dots (Q.21.1)$$

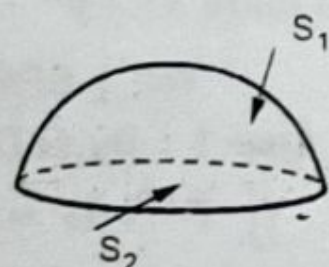


Fig. Q.21.1

$$\text{Now } \nabla \cdot \vec{F} = \frac{\partial}{\partial x} (y^2 z^2) + \frac{\partial}{\partial y} (x^2 z^2) + \frac{\partial}{\partial z} (x^2 y^2) = 0 + 0 + 0 = 0$$

Thus equation (Q.21.1) becomes

$$\iiint_{S_1} \vec{F} \cdot \hat{N} ds + \iiint_{S_2} \vec{F} \cdot \hat{N} ds = 0$$

$$\therefore \iiint_{S_1} \vec{F} \cdot \hat{N} ds = - \iiint_{S_2} \vec{F} \cdot \hat{N} ds$$

For S_2 $\hat{N} = -\hat{k}$, $z = 0$, $ds = dx dy$ and $x^2 + y^2 = 9$

$$\therefore \vec{F} \cdot \hat{N} = -x^2 y^2$$

Consider the surface integral over S_2

$$\therefore \iint_{S_2} \vec{F} \cdot \hat{N} ds = \iint_{S_2} -x^2 y^2 dx dy$$

over $x^2 + y^2 = 9$

Put $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$ and substitute the limits for circle of radius 3.

$$= \int_0^{2\pi} \int_0^3 -r^2 \cos^2 \theta r^2 \sin^2 \theta r dr d\theta$$

Integrate w.r.t. r and use conversion formula for θ .

$$= -4 \int_0^{2\pi} \sin^2 \theta \cos^2 \theta d\theta \left[\frac{r^6}{6} \right]_0^3$$

Use reduction formula for θ .

$$= -4 \left[\frac{1}{4} \right] \left[\frac{1}{2} \right] \frac{\pi}{2} \left(\frac{3}{6} \right) = -\frac{\pi}{4} \cdot \frac{3^6}{6}$$

$$= -\frac{\pi 3^6}{24}$$

Thus from equation (Q.21.2)

$$\iint_{S_1} \vec{F} \cdot \hat{N} ds = \frac{\pi 3^6}{24}$$

Q.22 Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\vec{s}$, where S is the curved surface of the cone $x^2 + y^2 = z^2$, $z = 4$

[SPPU : May-14, Marks 4]

Ans. : To evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\vec{s}$, using divergence theorem.

$$\iiint_V \vec{F} \cdot \vec{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Consider $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$

$$\iiint_V \nabla \cdot \vec{F} \, dv = 0$$

Q.23 Prove that $\iiint_V \frac{1}{r^2} \, dv = \iint_S \frac{1}{r^2} \vec{r} \cdot d\vec{s}$, where S is closed surface enclosing the volume V .

Hence evaluate $\iint_S \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} \cdot d\vec{s}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

[SPPU : May-14, Marks 4]

$$\vec{F} = \frac{\vec{r}}{r^2}$$

$$\nabla \cdot \vec{F} = \nabla \cdot \frac{\vec{r}}{r^2} = (-2 + 3) r^{-2} = \frac{1}{r^2}$$

Thus by G.D.T.

$$\iint_S \frac{\vec{r}}{r^2} \cdot \vec{n} \, ds = \iiint_V \frac{1}{r^2} \, dv$$

To evaluate

$$\iint_S \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} \cdot d\vec{s} \quad S \text{ is the surface of the sphere } x^2 + y^2 + z^2 = a^2$$

From above

$$\Rightarrow \iint_S \frac{\vec{r}}{r^2} \cdot d\vec{s} = \iiint_V \frac{1}{r^2} \, dv$$

Decode

A Guide for Engineering Students

$$\therefore \iint_S \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} \cdot d\vec{s} = \iiint_V \frac{1}{r^2} \, dv$$

over sphere $x^2 + y^2 + z^2 = a^2$

$$x = a \sin \theta \cos \phi, \quad y = a \sin \theta \sin \phi, \quad z = a \cos \theta$$

$$dv = r^2 \sin \theta \, dr \, d\theta \, d\phi, \quad r = 0 \text{ to } a, \quad \theta = 0 \text{ to } \pi/2, \quad \phi = 0 \text{ to } 2\pi$$

$$\phi = 0 \text{ to } 2\pi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \frac{1}{r^2} r^2 \sin \theta \, dr \, d\theta \, d\phi = a \int_0^{2\pi} \int_0^{\pi/2} [-\cos \theta]_0^{\pi/2} d\theta \, d\phi$$

$$= -a \int_0^{2\pi} (0 - 1) d\phi = +a (\phi)_0^{2\pi} = 2\pi a$$

Q.24 Show that $\iint_S \frac{\vec{r}}{r^2} \cdot \vec{n} \, ds = \iiint_V \frac{dv}{r^2}$

[SPPU : May-08, 13, Dec.-13]

Ans. : Let $\vec{F} = \frac{\vec{r}}{r^2}$

$$\therefore \nabla \cdot \vec{F} = \nabla \cdot \frac{\vec{r}}{r^2} = (-2 + 3) r^{-2} = \frac{1}{r^2}$$

Thus by G.D.T.

$$\iint_S \frac{\vec{r}}{r^2} \cdot \vec{n} \, ds = \iiint_V \frac{1}{r^2} \, dv$$

Q.25 Use divergence theorem to evaluate $\iint_S (x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot d\vec{s}$

where S is the surface bounded by $x^2 + y^2 = 1$ and $z = 0$, $z = 1$.

[SPPU : May-15]

Ans. : We have $\vec{F} = x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 1 - 4y + 2z$$

Decode

A Guide for Engineering Students

By Gauss divergence theorem

$$\begin{aligned} I &= \iint_S \vec{F} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{F} \, dv \\ &= \int_0^1 \int_0^1 \int_0^1 (1+4y+2z) \, dx \, dy \, dz \\ &= \int_R \int (z+4yz+z^2)_0^1 \, dx \, dy \\ &= \int_R [(1+4y+1)-0] \, dx \, dy \\ &= \int_R (2+4y) \, dx \, dy \end{aligned}$$

Put $x = r \cos \theta$, $y = r \sin \theta$, $dx \, dy = r \, dr \, d\theta$ and $r = 0$ to 1 and $\theta = 0$ to 2π

$$\begin{aligned} \therefore I &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 [2 + 4r \sin \theta] r \, dr \, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[2 \frac{r^2}{2} + \frac{4r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left[1 + \frac{4}{3} \sin \theta \right] d\theta \\ &= \left[\theta - \frac{4}{3} (-\cos \theta) \right]_0^{2\pi} \\ &= 2\pi - \frac{4}{3} \cos 2\pi - 0 - \frac{4}{3} \cos 0 \\ &= 2\pi - \frac{4}{3} + \frac{4}{3} = 2\pi \end{aligned}$$

Q.26 Show that $\iiint_V \frac{2}{r} \, dv = \iint_S \frac{\vec{r} \cdot \hat{n}}{r} \, ds$

Ans. : By divergence theorem

$$\iint_S \frac{\vec{r} \cdot \hat{n}}{r} \, ds = \iiint_V \nabla \cdot \left(\frac{\vec{r}}{r} \right) \, dv \quad \dots(Q.26.1)$$

Decode

A Guide for Engineering Students

Now $\nabla \cdot \frac{\vec{r}}{r} = \nabla \cdot (\vec{r} r^{-1})$

$$\begin{aligned} \therefore \nabla \cdot \frac{\vec{r}}{r} &= (\nabla \cdot \vec{r}) r^{-1} + \nabla(r^{-1}) \cdot \vec{r} \\ &= \frac{3}{r} + \left(-\frac{1}{r^2} \right) \frac{1}{r} \vec{r} \cdot \vec{r} = \frac{3}{r} + \left(-\frac{1}{r^3} \right) \vec{r} \cdot \vec{r} \\ &= \frac{3}{r} - \frac{1}{r^3} r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r} \end{aligned}$$

$\therefore$ Equation (Q.26.1) becomes

$$\iint_S \frac{\vec{r} \cdot \hat{n}}{r} \, ds = \iiint_V \frac{2}{r} \, dv$$

Decode

A Guide for Engineering Students

Applications of Vectors to Electromagnetic Fields

9.1 : Basic Concepts

- 1) **Coulomb's Law** : If two electric charges q_1 and q_2 are separated by a distance r , then the force $\vec{F}$ between them is given by

$$\vec{F} \propto \frac{q_1 q_2}{r^2} \Rightarrow \vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \text{ in free space.}$$

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3} \vec{r} \quad \because \hat{r} = \frac{\vec{r}}{r}$$

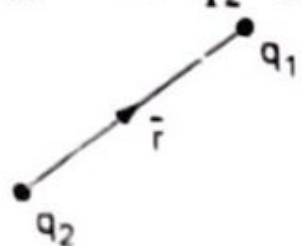


Fig. 9.1

ϵ_0 is called the permittivity of free space and it is given by 8.854×10^{-12} F/m.

where F/m is farads per meter

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ Nm}^2/\text{C}$$

Note : The force $\vec{F}$ always acts along the line joining the charges.

- 2) **The Electric Field Intensity** : The electric field intensity $\vec{E}$ at a point is the force experienced by a unit charge placed at that point and it is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$$

- 3) **Charge Density ρ** : It is the total charge contained in unit volume in space and it is given by

$$\rho = \frac{dq}{dv} \quad \therefore q = \int \rho dv$$

$$Q = \int \rho dv$$

- 4) **Electrostatic Potential (V)** : The electrostatic potential at point p located at a distance r from a charge q is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

- 5) **Electric Flux** : The electric flux originates on positive charge and terminates on negative charge. In the absence of negative charge it terminates at ∞ .

The electric flux ϕ through surface ds is given by

$$d\phi = \vec{E} \cdot d\vec{s} \\ = \vec{E} \cdot \hat{n} ds,$$

where $\hat{n}$ is unit normal vector at ds .

Thus the total electric flux through the entire surfaces is

$$\phi = \oint_S \vec{E} \cdot d\vec{s}$$

6) Gauss Law in Electrostatics

Statement : The total flux out of a closed surface S is equal to the net charge within the surface.

$$Q = \oint_S \vec{D} \cdot d\vec{s}$$

- Q.1** Find the charge in the volume defined by $1 \leq r \leq 3$ in spherical co-ordinates if

$$\rho = \frac{5 \cos^2 \phi}{r^4} \text{ C/m}^3$$

Ans. : We have $1 \leq r \leq 3$, $0 \leq \theta \leq \pi$

$$0 \leq \phi \leq 2\pi$$

$$\therefore Q = \int_0^{2\pi} \int_0^\pi \int_1^3 \rho r^2 \sin \theta dr d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \int_1^3 \frac{5 \cos^2 \phi}{r^4} r^2 \sin \theta dr d\theta d\phi$$

$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^{\pi} 5 \cos^2 \phi \sin \theta \left[-\frac{1}{r} \right]_{r=1}^{r=3} d\theta d\phi \\
 &= 5 \cdot \frac{2}{3} \int_0^{2\pi} \cos^2 \phi [-\cos \theta]_0^{\pi} d\phi \\
 &= \frac{10}{3} \int_0^{2\pi} \cos^2 \phi [-(-1) - 1] d\phi \\
 &= \frac{10}{3} \int_0^{2\pi} \cos^2 \phi [+1 - 1] d\phi
 \end{aligned}$$

$$Q = 0$$

9.2 : Work Done in Moving a Point Charge

We have $\vec{F} = q \vec{E}$ where $\vec{F}$ is a force $\vec{E}$ is electric field and q is the point charge.

The work done in the electric field is defined as

$$W = \int_C -q \vec{E} \cdot d\vec{r}$$

Q.2 Find the work done in moving a charge of $+2C$ from $(2, 0, 0)$ m to $(0, 2, 0)$ m along the straight line path joining the two points, if the electric field is $\vec{E} = 2x\hat{i} - 4y\hat{j}$.

Ans. : We have

$$\vec{E} \cdot d\vec{r} = 2x dx - 4y dy$$

$$W = \int_{AB} -2(2x dx - 4y dy)$$

The equation of AB is

$$\frac{x}{2} + \frac{y}{2} = 1 \text{ i.e. } y = 2 - x$$

$$\therefore dy = -dx$$

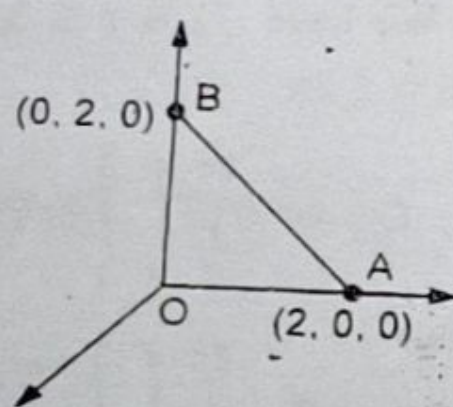


Fig. Q.2.1

$$\begin{aligned}
 \therefore W &= \int_2^0 -2[2x dx - 4(2-x)(-dx)] \\
 &= \int_2^0 (-4x - 16 + 8x) dx \\
 &= \int_2^0 (4x - 16) dx \\
 &= \left[4 \frac{x^2}{2} - 16x \right]_2^0 = 24
 \end{aligned}$$

9.3 : Magnetic Field

It originates from either a constant current or a permanent magnet. A current carrying conductor produces a magnetic field around it.

The magnetic field induction $d\vec{H}$ at point P where $\vec{r}$ is displacement vector from a current element $d\vec{l}$ of a conductor carrying current i is given by

$$d\vec{H} \propto \frac{id\vec{l} \times \vec{r}}{r^3}$$

$$\therefore d\vec{H} = \frac{\mu_0}{4\pi} \frac{id\vec{l} \times \vec{r}}{r^3}$$

Thus the total resultant field at point P is

$$\vec{H} = \oint d\vec{H}$$

9.4 : Maxwell's Equations

[SPPU : May-10, 11, 12, 13, Dec.-10, 11, 12]

When charges are in motion the electric and magnetic fields associated with this motion which have space and time variation. This is the phenomenon of electromagnetism. The Maxwell's equations represent the law of electromagnetism and are very useful in the analysis of electromagnetic fields.

The equations given below are called Maxwell's equations in most general forms, where charges and the conduction current may be present in the region.

i) Maxwell's equations in point form are

$$\nabla \times \vec{H} = \vec{J}_C + \frac{\partial \vec{D}}{\partial t} \quad \dots (i)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \dots (ii)$$

$$\nabla \cdot \vec{D} = \rho \quad \dots (iii)$$

$$\nabla \cdot \vec{B} = 0 \quad \dots (iv)$$

ii) Maxwell's equations in integral form are

$$\oint_C \vec{H} \cdot d\vec{r} = \int_S \left(\vec{J}_C + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s} \quad \dots (9.1)$$

$$\oint_C \vec{E} \cdot d\vec{r} = \int_S \left(-\frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{s} \quad \dots (9.2)$$

$$\int_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv \quad \dots (9.3)$$

$$\int_S \vec{B} \cdot d\vec{s} = 0 \quad \dots (9.4)$$

Note

1) Equations (i), (ii) and equations (9.1), (9.2) are equivalent under Stokes theorem.

2) Equations (iii), (iv) and equations (9.3), (9.4) are equivalent under Gauss divergence theorem.

For free space, Maxwell's equations take the form [there are no charges i.e. $\rho = 0$ and no conduction currents $\vec{J}_C = 0$].

Note

A) Maxwell's equations in point form are

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \vec{B} = 0$$

B) Maxwell's equations in integral form are

$$\oint_C \vec{H} \cdot d\vec{r} = \int_S \left(\frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

$$\oint_C \vec{E} \cdot d\vec{r} = \int_S \left(-\frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{s}$$

$$\int_S \vec{D} \cdot d\vec{s} = 0$$

$$\int_S \vec{B} \cdot d\vec{s} = 0$$

Q.3 Find $\vec{D}$, $\vec{B}$, $\vec{H}$ and $\vec{J}_D$ for
 $\vec{E} = E_m \sin(\omega t + \beta z) \hat{j}$

Ans. : We have

$$\vec{D} = \epsilon_0 \vec{E} = \epsilon_0 E_m \sin(\omega t + \beta z) \hat{j}$$

$$\text{Consider } \nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_m \sin(\omega t + \beta z) & 0 \end{vmatrix}$$

$$= -\beta E_m \cos(\omega t + \beta z) \hat{i}$$

By Maxwell's Equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\therefore -\frac{\partial \vec{B}}{\partial t} = -\beta E_m \cos(\omega t + \beta z) \hat{i}$$

Integrating we get

$$\vec{B} = \beta \frac{E_m}{\omega} \sin(\omega t + \beta z) \hat{i}$$

We neglect constant of integration in this static field.

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{\beta E_m}{\mu \omega} \sin(\omega t + \beta z) \hat{i}$$

Q.4 Two of Maxwell's electromagnetic equations are $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Given $\vec{B} = \text{curl } \vec{A}$ then deduce that $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\text{grad } V$ where V is a scalar point function.

[SPPU : May-10, 11, Dec.-12, 13, 14, 15]

Ans. : Given $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

and $\vec{B} = \text{curl } \vec{A} = \nabla \times \vec{A}$

$$\therefore \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} (\nabla \times \vec{A}) = \nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\text{As } \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\therefore \nabla \times \frac{\partial \vec{A}}{\partial t} = -\nabla \times \vec{E}$$

$$\therefore \nabla \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t} \right] = 0$$

But $\nabla \times \vec{F} = 0$ and $\vec{F} = -\nabla V$

$$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\text{grad } V$$

Q.5 If $\vec{E} = \nabla \phi$, $\nabla^2 \phi = -4\pi\rho$ prove that $\iint_S \vec{E} \cdot d\vec{s} = -4\pi \iiint_V \rho dV$

where V is volume enclosed by closed surface S .

[SPPU : May-14]

Ans. : We have

$$\begin{aligned} \iint_S \vec{E} \cdot d\vec{s} &= \iiint_V \nabla \cdot \vec{E} dV \quad (\text{by Gauss divergence theorem}) \\ &= \iiint_V \nabla \cdot \nabla \phi dV \\ &= \iiint_V \nabla^2 \phi dV \\ &= \iiint_V (-4\pi\rho) dV \\ &= -4\pi \iiint_V \rho dV \end{aligned}$$

Q.6 Show that $\vec{E} = -\nabla\phi - \frac{1}{C} \frac{\partial \vec{A}}{\partial t}$ and $\vec{H} = \nabla \times \vec{A}$ are solutions of Maxwell's equations.

$$\text{i) } \nabla \times \vec{H} = \frac{1}{C} \frac{\partial \vec{E}}{\partial t} \quad \text{ii) } \nabla \times \vec{E} = -\frac{1}{C} \frac{\partial \vec{H}}{\partial t}$$

$$\text{iii) } \nabla \cdot \vec{H} = 0 \quad \text{iv) } \nabla \cdot \vec{E} = 4\pi\rho$$

$$\text{if a) } \nabla \cdot \vec{A} + \frac{1}{C} \frac{\partial \phi}{\partial t} = 0 \quad \text{b) } \nabla^2 \phi - \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho$$

$$\text{c) } \nabla^2 \vec{A} = \frac{1}{C^2} \frac{\partial^2 \vec{A}}{\partial t^2}$$

where C is the velocity of light, $\vec{A}$ is the vector potential and ϕ is the scalar potential.

[SPPU : May-14]

Ans. : We have

$$\vec{E} = -\nabla\phi - \frac{1}{C} \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{H} = \nabla \times \vec{A}$$

$$\begin{aligned} \text{i) } \nabla \times \vec{H} &= \nabla \times (\nabla \times \vec{A}) \\ &= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \end{aligned}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t} (-\nabla\phi) - \frac{1}{C} \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\begin{aligned}\frac{1}{C} \frac{\partial \bar{E}}{\partial t} &= -\frac{1}{C} \frac{\partial}{\partial t} \nabla \phi - \frac{1}{C^2} \frac{\partial^2 \bar{A}}{\partial t^2} \\ &= -\frac{1}{C} \nabla (-C \nabla \cdot \bar{A}) - \nabla^2 \bar{A} \\ &= \nabla (\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\end{aligned}$$

From equation (Q.6.1) and (Q.6.2) we get,

$$\nabla \times \bar{H} = \frac{1}{C} \frac{\partial \bar{E}}{\partial t}$$

$$\begin{aligned}\nabla \times \bar{E} &= \nabla \times \left[-\nabla \phi - \frac{1}{C} \frac{\partial \bar{A}}{\partial t} \right] \\ &= -\frac{1}{C} \frac{\partial}{\partial t} [\nabla \times \bar{A}]\end{aligned}$$

$$\nabla \times \bar{E} = -\frac{1}{C} \frac{\partial \bar{H}}{\partial t}$$

$$\nabla \cdot \bar{H} = \nabla \cdot (\nabla \times \bar{A}) = 0$$

$$\begin{aligned}\nabla \cdot \bar{E} &= \nabla \cdot \left\{ -\nabla \phi - \frac{1}{C} \frac{\partial \bar{A}}{\partial t} \right\} \\ &= -\nabla^2 \phi - \frac{1}{C} \frac{\partial}{\partial t} (\nabla \cdot \bar{A}) \\ &= -\nabla^2 \phi + \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} \\ &= 4\pi\rho\end{aligned}$$

Maxwell's equations are given by

$$\nabla \cdot \bar{E} = 0, \nabla \cdot \bar{H} = 0, \nabla \times \bar{E} = -\frac{\partial \bar{H}}{\partial t}, \nabla \times \bar{H} = \frac{\partial \bar{E}}{\partial t}. \text{ Show that } \bar{E} \text{ and } \bar{H} \text{ satisfy } \nabla^2 u = \frac{\partial^2 u}{\partial t^2}.$$

$$\bar{H} \text{ satisfy } \nabla^2 u = \frac{\partial^2 u}{\partial t^2}.$$

[SPPU : May-10, 12, 13, 15, Dec.-11]

... (Q.8) We have

$$\begin{aligned}\nabla \times (\nabla \times \bar{E}) &= \nabla (\nabla \cdot \bar{E}) - (\nabla \cdot \nabla) \bar{E} \\ &= 0 - \nabla^2 \bar{E} \quad (\because \nabla \cdot \bar{E} = 0)\end{aligned}$$

$$\begin{aligned}\text{Consider } \nabla \times (\nabla \times \bar{E}) &= \nabla \times \left(-\frac{\partial \bar{H}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \bar{H}) \\ &= -\frac{\partial}{\partial t} \left(\frac{\partial \bar{E}}{\partial t} \right)\end{aligned}$$

$$\therefore -\nabla^2 \bar{E} = -\frac{\partial}{\partial t} \left(\frac{\partial \bar{E}}{\partial t} \right)$$

$$\therefore \nabla^2 \bar{E} = \frac{\partial^2 \bar{E}}{\partial t^2}$$

i.e. $\bar{E}$ satisfies the equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$

$$\text{Now, } \nabla \times (\nabla \times \bar{H}) = \nabla (\nabla \cdot \bar{H}) - (\nabla \cdot \nabla) \bar{H} = -\nabla^2 \bar{H}$$

$$\begin{aligned}\nabla \times (\nabla \times \bar{H}) &= \nabla \times \left(\frac{\partial \bar{E}}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \times \bar{E}) \\ &= \frac{\partial}{\partial t} \left(-\frac{\partial \bar{H}}{\partial t} \right)\end{aligned}$$

$$-\nabla^2 \bar{H} = -\frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\nabla^2 \bar{H} = \frac{\partial^2 \bar{H}}{\partial t^2}$$

Thus $\bar{H}$ satisfies the equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$

Q.8 If i) $\nabla \cdot \bar{D} = \rho$ ii) $\nabla \cdot \bar{H} = 0$

$$\text{iii) } \nabla \times \bar{H} = \frac{1}{C} \left(\frac{\partial \bar{D}}{\partial t} + \rho \bar{v} \right) \quad \text{iv) } \nabla \times \bar{D} = -\frac{1}{C} \frac{\partial \bar{H}}{\partial t} \text{ where } C \text{ is}$$

constant then prove that

$$\nabla^2 \bar{D} - \frac{1}{C^2} \frac{\partial^2 \bar{D}}{\partial t^2} = \nabla \rho + \frac{1}{C^2} \frac{\partial}{\partial t} (\rho \bar{v})$$

$$\nabla^2 \bar{H} - \frac{1}{C^2} \frac{\partial^2 \bar{H}}{\partial t^2} = -\frac{1}{C} \nabla \times (\rho \bar{v})$$

[SPPU : Dec.-10, May-12]

Ans. : We have

$$\begin{aligned}\nabla \times (\nabla \times \vec{H}) &= \nabla(\nabla \cdot \vec{H}) - (\nabla \cdot \nabla) \vec{H} \\ &= 0 - \nabla^2 \vec{H} = -\nabla^2 \vec{H}\end{aligned}$$

Consider

$$\begin{aligned}\nabla \times (\nabla \times \vec{H}) &= \nabla \times \left\{ \frac{1}{C} \left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{v} \right) \right\} \\ -\nabla^2 \vec{H} &= \frac{1}{C} \frac{\partial}{\partial t} \left(-\frac{1}{C} \frac{\partial \vec{H}}{\partial t} \right) + \frac{1}{C} \nabla \times (\rho \vec{v}) \\ &= -\frac{1}{C^2} \frac{\partial^2 \vec{H}}{\partial t^2} + \frac{1}{C} \nabla \times (\rho \vec{v})\end{aligned}$$

$$\therefore \nabla^2 \vec{H} - \frac{1}{C^2} \frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{C} \nabla \times (\rho \vec{v})$$

To prove that the first result, consider

$$\begin{aligned}\nabla \times (\nabla \times \vec{D}) &= \nabla(\nabla \cdot \vec{D}) - (\nabla \cdot \nabla) \vec{D} \\ &= \nabla \rho - \nabla^2 \vec{D}\end{aligned} \quad \dots (Q.8.1)$$

$$\begin{aligned}\text{Now } \nabla \times (\nabla \times \vec{D}) &= \nabla \times \left(-\frac{1}{C} \frac{\partial \vec{H}}{\partial t} \right) \\ &= -\frac{1}{C} \frac{\partial}{\partial t} (\nabla \times \vec{H}) \\ &= -\frac{1}{C} \frac{\partial}{\partial t} \left\{ \frac{1}{C} \left(\frac{\partial \vec{D}}{\partial t} + \rho \vec{v} \right) \right\}.\end{aligned} \quad \dots (Q.8.2)$$

From equation (Q.8.1) and equation (Q.8.2), we get

$$\begin{aligned}\nabla \rho - \nabla^2 \vec{D} &= -\frac{1}{C^2} \frac{\partial^2 \vec{D}}{\partial t^2} - \frac{1}{C^2} \frac{\partial}{\partial t} (\rho \vec{v}) \\ \therefore \nabla^2 \vec{D} - \frac{1}{C^2} \frac{\partial^2 \vec{D}}{\partial t^2} &= \nabla \rho + \frac{1}{C^2} \frac{\partial}{\partial t} (\rho \vec{v})\end{aligned}$$

□□□

10

Complex Differentiation

10.1 : Analytic Function

1) A function $f(z)$ is said to be analytic in a region R if it is analytic at every point of a region R . Analytic function is also known as holomorphic or regular or monogenic.

A point at which the function ceases to be analytic is called a singular point of function.

2) **Entire function** : A function which is analytic everywhere in the complex plane is known as entire function.

3) Necessary and sufficient conditions for $f(z) = u + iv$ to be analytic in a region R are

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in region R .

$$\text{ii) } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ at all points of } R.$$

The conditions in (ii) are known as **Cauchy - Riemann equations** in briefly C.R equations.

iii) Cauchy-Riemann equations in polar form :

$$\text{Let } f(z) = u(x, y) + iv(x, y) = u + iv$$

$$\text{and } z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

$$\therefore f(z) = f(r e^{i\theta}) = u + iv$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

which are C.R equations in polar form.

iv) If $f(z) = u + iv$ is analytic in a region R, then u and v satisfy C.R equations in R.

$$\text{i.e. } u_x = v_y \text{ and } v_x = -u_y$$

v) C.R equations are necessary but not sufficient conditions for function to be an analytic. These C.R equations are sufficient conditions for analytic function if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions.

$$\text{i.e. } f \text{ analytic} \Rightarrow \text{C-R equations}$$

But C.R equations may or may not $\Rightarrow$ Analytic function.

C.R equations + Continuous partial derivatives $\Rightarrow$ Analyticity.

4) **Harmonic Function** : A function $\phi(x, y)$ is said to be harmonic function if it is continuous and has continuous first and second order partial derivatives and satisfies Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Note : If $f(z) = u + iv$ is analytic then both u and v are harmonic functions.

5) Conjugate Harmonic Function

[SPPU : Dec.-05, 06, 07, 08, 09, 10, 11, 12]

If $f(z) = u(x, y) + iv(x, y)$ is analytic function, then the real part u of $f(z)$ is known as conjugate harmonic function of imaginary part v of $f(z)$ and vice versa.

6) Construction of Analytic Function :

A) **Theorem** : If $f(z) = u + iv$ is an analytic function of z then $f(z)$ is independent of $\bar{z}$.

Proof : Let $f(z) = u(x, y) + iv(x, y)$

be an analytic function of $z = x + iy$

$$\text{Now } \bar{z} = x - iy$$

$$z + \bar{z} = 2x \Rightarrow x = \frac{z + \bar{z}}{2}$$

Decode

$$\text{And } z - \bar{z} = 2iy \Rightarrow y = \frac{z - \bar{z}}{2i}$$

Substituting x and y in equation (10.1) we get,

$$f(z) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + i v\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)$$

Here $f(z)$ is a function of two complex variables z and $\bar{z}$. Now u, v are functions of x, y . And x and y are functions of z and $\bar{z}$. We have

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} (u + iv) \\ &= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial z} \right) + i \left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial z} \right) \\ &= \left[\frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial u}{\partial y} \left(-\frac{1}{2i} \right) \right] + i \left[\frac{\partial v}{\partial x} \cdot \frac{1}{2} + \frac{\partial v}{\partial y} \left(-\frac{1}{2i} \right) \right] \\ &= \frac{1}{2} \frac{\partial u}{\partial x} + \frac{i}{2} \frac{\partial u}{\partial y} + \frac{i}{2} \frac{\partial v}{\partial x} - \frac{1}{2} \frac{\partial v}{\partial y} \end{aligned}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad \dots (10.2)$$

As $f(z) = u + iv$ is analytic function, u and v satisfy C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ substituting these relations in equation (10.2), we}$$

get $\frac{\partial f}{\partial \bar{z}} = 0 + i0 = 0$ which means that $f(z)$ is independent of $\bar{z}$.

Remarks : i) If $f(z)$ is analytic function of z , then it consist of expressions in z only. i.e. if $f(z)$ is analytic function of z then x and y occurs in $f(z)$ in the combination of $x + iy$.

ii) If $f(z) = u(x, y) + iv(x, y)$ is analytic function then substitute $x = z$ and $y = 0$ in $f(z)$, we get $f(z)$ in terms of z .

B) Milne - Thompson Method

i) To find analytic function $f(z) = u + iv$ whose real part $u(x, y)$ is given :

$\Rightarrow$ If u is known $\therefore u_x$ and u_y are obtained.

Decode

$$\begin{aligned}\text{Now } f'(z) &= u_x + i v_x \\ &= u_x + i(-u_y) \text{ by C-R equations}\end{aligned}$$

$$f'(z) = u_x(x, y) - i u_y(x, y) \quad \dots (10.3)$$

As $f(z)$ is analytic, $f'(z)$ is also analytic.

By substituting $x = z$ and $y = 0$ in equation (10.3), we get,

$$f'(z) = u_x(z, 0) - i u_y(z, 0)$$

Integrating both sides with respect to z we get

$$f(z) = \int u_x(z, 0) dz - i \int u_y(z, 0) dz + C$$

which is analytic function.

Here we get $f(z)$ in terms of z . To find $v(x, y)$ if $u(x, y)$ is given, substitute $z = x + iy$ in $f(z)$ and then find imaginary part of $f(z)$.

ii) To find analytic function $f(z) = u + iv$ whose imaginary part $v(x, y)$ is given :

$$\begin{aligned}f(z) &= u_x + i v_x = v_y + i v_x \\ &= v_y(x, y) + i v_x(x, y)\end{aligned}$$

Substituting $x = z$ and $y = 0$, and then integrating we get,

$$f(z) = \int v_y(z, 0) dz + i \int v_x(z, 0) dz + C$$

To find $u(x, y)$ substitute $z = x + iy$ and then find real part of $f(z)$.

Note In polar form

$$f'(z) = e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = \frac{1}{i r} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right)$$

C) i) Suppose that $f(z) = u + iv$ is an analytic function and $u(x, y)$ is known.

Claim : To determine $v(x, y)$

Since u and v satisfy C-R equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots (10.4)$$

We have

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

But u is known $\therefore \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be calculated and hence

obtained using equation (10.5).

We write $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$\therefore$ Equation (10.5) becomes,

$$dv = M dx + N dy$$

$$\text{Now } \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0$$

$$\text{i.e. } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (10.6) is exact

$$\therefore v = \int_{y \text{ constant}} M dx + \int \left[\begin{array}{l} \text{terms in } N \text{ not} \\ \text{containing } x \end{array} \right] dy + C$$

$$\text{i.e. } v = \int_{y \text{ constant}} \left(-\frac{\partial u}{\partial y} \right) dx + \int \left[\begin{array}{l} \text{terms in } \frac{\partial u}{\partial x} \text{ not} \\ \text{containing } x \end{array} \right] dy + C$$

ii) Suppose that $f(z) = u + iv$ is an analytic function and v is known.

Claim : To determine $u(x, y)$

We have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

which is exact differential equation.

$\therefore$ Its general solution is

$$u = \int_{y \text{ constant}} \frac{\partial v}{\partial y} dx + \int \left[\begin{array}{l} \text{terms in } \left(-\frac{\partial v}{\partial x} \right) \text{ not} \\ \text{containing } x \end{array} \right] dy + C$$

Q.1 If $f(z)$ is analytic function then show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$$

[SPPU : Dec.-10, 13, 15, May-06, 12]

Ans. : Let

$$f(z) = u + iv \therefore |f(z)|^2 = u^2 + v^2 = g$$

$$\frac{\partial}{\partial x} g = \frac{\partial}{\partial x} (u^2 + v^2) = 2uu_x + 2vv_x$$

$$\frac{\partial^2 g}{\partial x^2} = 2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2$$

$$\text{Similarly, } \frac{\partial^2 g}{\partial y^2} = 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2$$

Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g$$

$$= 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \quad \dots (Q.1.1)$$

Since $f(z)$ is analytic, u, v satisfy Laplace equation

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

$$\text{and } f'(z) = u_x + iv_x = u_y + iv_y$$

$$\therefore |f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$$

Equation (Q.1.1) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 2[2|f'(z)|^2] = 4|f'(z)|^2$$

Q.2 Find the condition under which

$$u = ax^3 + bx^2y + cxy^2 + dy^3 \text{ is harmonic.}$$

[SPPU : Dec.-14, May-11]

Ans. : We have $u = ax^3 + bx^2y + cxy^2 + dy^3$

$$\therefore \frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by$$

$$\frac{\partial u}{\partial y} = bx^2 + 2cx + 3dy^2$$

$$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

Now if u is harmonic, it must satisfy Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore 6ax + 2by + 2cx + 6dy = 0$$

$$\therefore (6a + 2c)x + (2b + 6d)y = 0$$

$$\Rightarrow 6a + 2c = 0 \text{ and } 2b + 6d = 0$$

$$\Rightarrow 3a + c = 0 \text{ and } b + 3d = 0$$

which are required conditions

Q.3 If $u = \frac{1}{2} \log(x^2 + y^2)$ and $f(z) = u + iv$ is analytic then find

$f(z)$ in terms of z and hence find v . [SPPU : May-12]

Ans. : Let $f(z) = u + iv$ be an analytic function and

$$u = \frac{1}{2} \log(x^2 + y^2)$$

$\therefore f(z)$ satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \dots (Q.3.1)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot (2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

Now

$$f'(z) = u_x + iv_x = u_x - iu_y$$

$$f'(z) = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

As $f(z)$ is analytic, $f'(z)$ is also analytic.

By substituting $x = z$ and $y = 0$, we get

$$f'(z) = \frac{z}{z^2} + i0 = \frac{1}{z}$$

Integrating w.r.t z we get

$$f(z) = \int \frac{1}{z} dz + C = \log z + C$$

where

$$C = C_1 + iC_2$$

To find v substitute $z = x + iy$ in $f(z)$, we get,

$$\begin{aligned} f(z) &= \log(x + iy) + C_1 + iC_2 \\ &= \log\sqrt{x^2 + y^2} + i \tan^{-1} \frac{y}{x} + C_1 + iC_2 \\ &= \log\sqrt{x^2 + y^2} + i \left(\tan^{-1} \frac{y}{x} + C_2 \right) \end{aligned}$$

$$\text{where } C_1 = 0 = \frac{1}{2} \log(x^2 + y^2) + i \left(\tan^{-1} \frac{y}{x} + C_2 \right)$$

$$v(x, y) = \tan^{-1} \frac{y}{x} - C_2$$

Q.4 If $v = 3x^2y - y^3$. Find analytic function $f(z) = u + iv$ in terms of z and hence find $u(x, y)$. [SPPU : Dec.-07, 11, 12]

Ans. : Let $f(z) = u + iv$ be analytic function and

$$v = 3x^2y - y^3$$

$\therefore f(z)$ satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \dots (Q.4.1)$$

$$\text{Now } f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$f'(z) = (3x^2 - 3y^2) + i(6xy)$$

Decode

A Guide for Engineering Students

As $f(z)$ is analytic, $f'(z)$ is also analytic function

$\therefore$ Substituting $x = z$ and $y = 0$ in $f'(z)$,

we get

$$f'(z) = 3z^2$$

$$f(z) = \int 3z^2 dz + C$$

$$f(z) = z^3 + C$$

To find $u(x, y)$ substitute $z = x + iy$ in $f(z)$, we get

$$\begin{aligned} f(z) &= (x + iy)^3 + C_1 + iC_2 \\ &= x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3 + C_1 + iC_2 \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 + C_1 + iC_2 \\ &= (x^3 - 3xy^2 + C_1) + i(3x^2y - y^3 + C_2) \end{aligned}$$

where

$$C_2 = 0$$

$$f(z) = (x^3 - 3xy^2 - C_1) + i(3x^2y - y^3)$$

$$u(x, y) = x^3 - 3xy^2 + C_1$$

Q.5 Determine the analytic function

$$f(z) = u + iv \text{ in terms of } z \text{ if } u = \cosh(x) \cos y$$

[SPPU : Dec.-12, May]

Ans. : Given that $f(z) = u + iv$ is an analytic function and $u = \cosh(x) \cos y$

$\therefore f(z)$ satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\text{Now } f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

$$= \sinh(x) \cos y - i \cosh(x) (-\sin y)$$

$$f'(z) = \sinh(x) \cos y + i \cosh(x) \sin y$$

As $f(z)$ is an analytic function, $f'(z)$ is also analytic.

$\therefore$ Substituting $x = z$ and $y = 0$ in $f'(z)$ we get,

$$f'(z) = \sinh(z) + i0 = \sinh(z)$$

Decode

A Guide for Engineering Students

Integrating w.r.t. z , we get

$$f(z) = \cosh(z) + C$$

Q.6 Find the imaginary part of the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$. [SPPU : May-15]

Ans. : Let $f(z) = u + iv$ be an analytic function.

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

and $f(z)$ must satisfy C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = 3x^2 - 3y^2 + 6x \text{ and } u_y = -6xy - 6y$$

$$f'(z) = u_x + i v_x = u_x - i u_y$$

$$f'(z) = (3x^2 - 3y^2 + 6x) - i(-6xy - 6y) \quad \dots(Q.6.1)$$

As $f(z)$ is analytic, $f'(z)$ is also analytic.

By putting $x = z, y = 0$, we get

$$f'(z) = 3z^2 + 6z + i(0) = 3z^2 + 6z$$

Integrating we get

$$f(z) = 3 \frac{z^3}{3} + 6 \frac{z^2}{2} + C \text{ where } C = C_1 + iC_2$$

$$= z^3 + 3z^2 + C$$

5] To find v put $z = x + iy$, we get

$$f(z) = (x + iy)^3 + 3(x + iy)^2 + C_1 + iC_2$$

$$u + iv = [x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3]$$

$$+ 3[x^2 - y^2 + 2i xy] + C_1 + iC_2$$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + C_1)$$

$$+ i(3x^2y - y^3 + 6xy + C_2)$$

$$v = 3x^2y - y^3 + 6xy + C_2 \text{ which is the required imaginary part of } z.$$

Q.7 Show that $u = y^3 - 3x^2y$ is harmonic function. Find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z . [SPPU : Dec.-15]

[SPPU : Dec.-15]

Ans. : We have $u = y^3 - 3x^2y \therefore u_x = -6xy$

$$u_{xx} = -6y \text{ and } v_y = 3y^2 - 3x^2, u_{yy} = 6y$$

$$\therefore u_{xx} + u_{yy} = -6y + 6y = 0$$

$$\therefore u = y^3 - 3x^2y \text{ is harmonic function.}$$

Let $f(z) = u + iv$ be an analytic function

$\therefore f(z)$ satisfies C.R. equations.

$$u_x = v_y \text{ and } v_x = -u_y$$

$$f'(z) = u_x + i v_x = u_x - i v_y$$

$$f'(z) = -6xy - i(3y^2 - 3x^2) \quad \dots(Q.7.1)$$

As $f(z)$ is analytic, $f'(z)$ is also analytic

$\therefore$ By substituting $x = z$ and $y = 0$ in (Q.7.1)

$$\text{We get } f'(z) = 0 - i(0 - 3z^2)$$

$$\therefore f'(z) = 3iz^2$$

Integrating w.r.t. z , we get

$$f(z) = 3i \frac{z^3}{3} + c = iz^3 + C \text{ where } C = C_1 + iC_2$$

$\therefore$ To find v put $z = x + iy$ in $f(z)$

$$\therefore f(z) = i(x + iy)^3 + C_1 + iC_2$$

$$u + iv = i(x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3) + C_1 + iC_2$$

$$= i(x^3 - 3xy^2 - iy^3 + 3x^2y) + C_1 + iC_2$$

$$\therefore v = x^3 - 3xy^2 + C_2$$

Q.8 Find analytic function

$$f(z) = u + iv \text{ where } u = r^3 \cos 3\theta + r \sin \theta.$$

[SPPU : May-15]

Ans. : We have C.R. equations in polar form

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$

$$u_r = 3r^2 \cos 3\theta + \sin \theta$$

$$\text{Here } u_\theta = -3r^2 \sin 3\theta + r \cos \theta$$

Now $\frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r} = 3r^3 \cos 3\theta + r \sin \theta$

Integrating w.r.t. θ and treating r as a constant,

$$v = 3r^3 \frac{\sin 3\theta}{3} - r \cos \theta + \phi(r) \\ = r^3 \sin 3\theta - r \cos \theta + \phi(r)$$

$$\frac{\partial v}{\partial r} = 3r^2 \sin 3\theta - \cos \theta + \phi'(r) = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\Rightarrow 3r^2 \sin 3\theta - \cos \theta + \phi'(r) = 3r^2 \sin 3\theta - \cos \theta$$

$$\Rightarrow \phi'(r) = 0 \Rightarrow \phi(r) = \text{Constant} = C$$

$$\therefore v = r^3 \sin 3\theta - r \cos \theta + C$$

and

$$f(z) = u + iv = r^3 \cos 3\theta + r \sin \theta + i(r^3 \sin 3\theta - r \cos \theta + C) \\ = r^3 (\cos 3\theta + i \sin 3\theta) - ir (\cos \theta + i \sin \theta) + iC \\ = r^3 e^{3i\theta} - i r e^{i\theta} + iC \\ = z^3 - iz + C_1 \quad (\because z = re^{i\theta})$$

Q.9 Let $f(z) = u + iv$ be an analytic function and $u = v^2$ then show that $f(z)$ is constant. [SPPU : Dec.-11, 15]

Ans. : Let $f(z) = u + iv$ be an analytic function

$\therefore f(z)$ satisfies C.R. equations

$$\therefore u_x = v_y \text{ and } v_x = -u_y$$

We have

$$u = v^2 \Rightarrow u - v^2 = 0$$

Differentiating w.r.t. x and y , we get

$$\frac{\partial u}{\partial x} - 2v \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} - 2v \frac{\partial v}{\partial y} = 0$$

By C.R. equations, we get

$$\frac{\partial v}{\partial y} - 2v \frac{\partial v}{\partial x} = 0 \quad \dots (Q.9.1)$$

and $-\frac{\partial v}{\partial x} - 2v \frac{\partial v}{\partial y} = 0$

Multiplying equation (Q.9.1) by $2v$, we get

$$2v \frac{\partial v}{\partial y} - 4v^2 \frac{\partial v}{\partial x} = 0$$

Equation (Q.9.3) + Equation (Q.9.2) $\Rightarrow$

$$\Rightarrow -\frac{\partial v}{\partial x} (1 + 4v^2) = 0$$

$$\Rightarrow \frac{\partial v}{\partial x} = 0 \text{ or } 1 + 4v^2 = 0$$

$$\Rightarrow v^2 = -\frac{1}{4} \text{ which is impossible as } v \text{ is a real function.}$$

$$\Rightarrow \frac{\partial v}{\partial x} = 0 \text{ i.e. } v \text{ is independent of } x$$

$$\therefore v = \phi(y)$$

$$\text{Now from } \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow u = \psi(x)$$

Now substituting u and v in equation $u = v^2$ we get

$$\psi(x) = \phi^2(y)$$

which is possible only when

$$\psi(x) = \phi^2(y) = \text{Constant}$$

$$\Rightarrow u \text{ and } v \text{ are constant}$$

$$\Rightarrow f(z) = u + iv \text{ is a constant function.}$$

Q.10 Prove that analytic function with constant argument is constant. [SPPU : Dec.-08, 12, May-12, 13, 14]

Ans. : Let $f(z) = u + iv$ be analytic function with constant argument let $\arg(f(z)) = k = \text{constant}$.

$$\therefore \arg(u + iv) = k$$

$$\tan^{-1}\left(\frac{v}{u}\right) = k$$

$$\therefore v = u \tan k \quad \dots (Q.10.1)$$

Case i : Suppose $k = 0$, then $v = 0 \Rightarrow v_x = v_y = 0$

By C.R. equations we get $u_x = u_y = 0$

$\therefore u$ is a constant function.

$\therefore f(z) = u + iv = \text{constant function.}$

Case ii : Differentiating equation (Q.10.1) w.r.t. x , we get

$$v_x = u_x \tan k$$

$$\therefore u_x \tan k - v_x = 0$$

Differentiating equation (Q.10.1) w.r.t. y , we get, ... (Q.10.2)

$$v_y = u_y \tan k$$

By C.R. equations, we get,

$$u_x = -v_x \tan k$$

$$\therefore -u_x + v_x \tan k = 0$$

Multiplying equation (Q.10.2) by $\tan k$, we get, ... (Q.10.3)

$$u_x \tan^2 k - v_x \tan k = 0$$

Adding equations (Q.10.3) and (Q.10.4), we get, ... (Q.10.4)

$$u_x (1 + \tan^2 k) = 0$$

$$u_x = 0$$

Similarly, we get $v_x = 0$

$$\therefore f'(z) = u_x + i v_x = 0$$

$\therefore f(z)$ is a constant function.

10.2 : Bilinear Transformation

The transformation $w = \frac{az+b}{cz+d}$

where a, b, c, d are complex constants and $ad-bc \neq 0$ is known as the bilinear transformation or Mobius transformation or linear fractional transformation.

$$\text{Now } \frac{dw}{dz} = \frac{(cz+d)a - (az+b)c}{(az+d)^2} = \frac{ad-bc}{(az+d)^2} \neq 0$$

$$(\because ad-bc \neq 0)$$

Hence bilinear transformation is conformal.

Note 1) There is 1-1 correspondence between all points in z -plane and w -plane.

2) If z maps into itself in the w -plane i.e.

$w = z$ then

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0 \quad \dots (10.7)$$

The roots of equation (10.7) are the invariant or fixed points of the bilinear transformation. But equation (10.7) has exactly two roots, so bilinear transformation has at most two invariant points. If two roots of equation (10.7) are equal then the corresponding bilinear transformation is said to be parabolic.

3) The bilinear transformation is a combination of translation, rotation and inversion.

$$4) \text{ We have } w = \frac{az+b}{cz+d} = \left(\frac{a}{d}\right)z + \frac{b}{d} = \frac{Az+B}{Cz+D}$$

Thus to define bilinear transformation, we require 3 pairs points only.

5) The cross ratio of four points z_1, z_2, z_3, z_4 is denoted by (z_1, z_2, z_3, z_4) and defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

The cross ratio is invariant under bilinear transformation.

$$\text{i.e. } \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

This property is useful to find bilinear transformation $w = f(z)$ if $z_i \rightarrow \infty$ or $w_i \rightarrow \infty, i=1,2,3$.

Note For simplicity, we use the following formula.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

If $z_1 = \infty$ then $\lim_{|z_1| \rightarrow \infty} \frac{z - z_1}{z_2 - z_1} = \lim_{|z_1| \rightarrow \infty} \left(\frac{\frac{z}{z_1} - 1}{\frac{z_2}{z_1} - 1} \right) = 1$

Similarly if z_2 or $z_3 \rightarrow \infty$ then $\frac{z_2 - z_3}{z_2 - z_1}$ or $\frac{z_2 - z_3}{z - z_3} \rightarrow 1$

6) Matrix method : Let $w = T(z) = \frac{az + b}{cz + d}$ the associated matrix is

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the inverse of the linear fractional or bilinear

transformation is $T^{-1}(w) = \frac{dw - b}{-cw + a}$ and the associated matrix is

$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \text{adj } A$ where $\text{adj } A$ is the adjoint matrix of A .

Q.11 Find bilinear transformation which maps the points $z = 1, i, -1$ to the points $0, 1, \infty$ respectively.

[SPPU : May-12, Dec.-11, 15]

Ans. : Using the cross ratio property

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\therefore \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \quad \dots (Q.11.1)$$

Here $z_1 = 1, z_2 = i, z_3 = -1, w_1 = 0, w_2 = 1, w_3 = \infty$ as $w_3 = \infty, \frac{w_2 - w_3}{w - w_3} = 1$

$\therefore$ Equation (Q.11.1) becomes

$$\frac{w - 0}{(1 - 0)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$\therefore w = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)} \times \frac{(-i - 1)}{(-i - 1)} = \frac{(z - 1)(1 - 2i - 1)}{(z + 1)(1 + 1)} = \frac{-iz + i}{z - 1}$$

$$\therefore w = \frac{-iz + i}{z + 1}$$

Q.12 Find the bilinear transformation, which maps the points $0, 1$ onto the points $0, i, 3i$ respectively.

[SPPU : May-06, 11, 15, Dec.-14,]

Ans. : By using cross ratio property, we get

$$\frac{(w - 0)(i - 3i)}{(w - 3i)(i - 0)} = \frac{(z + 1)(0 - 1)}{(z - 1)(0 + 1)}$$

$$\therefore \frac{w(-2i)}{(w - 3i)i} = \frac{-z - 1}{z - 1}$$

$$\therefore \frac{-2w}{w - 3i} = \frac{-z - 1}{z - 1}$$

$$\text{i.e. } S(w) = T(z)$$

$$\therefore w = S^{-1}(T(z))$$

Let $w = \frac{az + b}{cz + d}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj} \begin{bmatrix} -2 & 0 \\ 1 & -3i \end{bmatrix} \right\} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3i & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3i & 3i \\ -1 & 3 \end{bmatrix}$$

$$\therefore w = \frac{3i(z + 1)}{-z + 3}$$

Q.13 Find the bilinear transformation which maps the points 1, 0, i of the z-plane onto the points $\infty, -2, -\frac{1}{2}(1+i)$ of the w-plane.

[SPPU : Dec.-14, 15, Marks 4]

Ans. : Consider $w = \frac{az+b}{cz+d}$

When $z = 1, w = \infty \Rightarrow \infty = \frac{a+b}{c+d} \Rightarrow c+d=0 \Rightarrow c=-d$

When $z = 0, w = -2 \Rightarrow -2 = \frac{b}{d} \Rightarrow b = -2d$

When $z = i, w = -\frac{1}{2}(1+i) \Rightarrow -\frac{1}{2}(1+i) = \frac{ia+b}{ic+d}$
 $-\frac{1}{2}(1+i) = \frac{ia-2b}{-id+d}$

$$-\frac{1}{2}(1+i)(1-i)d = ai - 2d$$

$$-(1-1)d = 2(ai - 2d)$$

$$-d = ai - 2d$$

$$d = ai \Rightarrow a = -di$$

$$w = \frac{-2-iz}{1-z} = \frac{2+iz}{z-1}$$

Q.14 Find the bilinear transformation which maps the points 1, i, 2i on the points -2i, 0, 1, in w-plane.

[SPPU : Dec.-10, May-14]

Ans. : By using cross property, we have

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{where } z_1=1, z_2=i, z_3=2i, w_1=-2i, w_2=0$$

$$w_3=1$$

We get

$$\frac{(w+2i)(0-1)}{(w-1)(0+2i)} = \frac{(z-i)(i-2i)}{(z-2i)(i-1)}$$

$$\frac{-w-2i}{2iw-2i} = \frac{-iz+i}{(i-1)z+2+2i}$$

Decode

A Guide for Engineering Students

i.e. $S(w) = T(z)$

$$w = S^{-1}[T(z)]$$

Let $w = \frac{az+b}{cz+d}$

Then the associated matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj} \begin{bmatrix} -1 & -2i \\ 2i & -2i \end{bmatrix} \right\} \begin{bmatrix} -i & : \\ i-1 & 2+2i \end{bmatrix}$$

$$= \begin{bmatrix} -2i & 2i \\ -2i & -1 \end{bmatrix} \begin{bmatrix} -i & i \\ i-1 & 2+2i \end{bmatrix}$$

$$= \begin{bmatrix} -2-2-2i & 2-4+4i \\ -2-i+1 & 2-2-2i \end{bmatrix} = \begin{bmatrix} -4-2i & -2+4i \\ -1-i & -2i \end{bmatrix}$$

$$w = \frac{(-4-2i)z + (-2+4i)}{(-1-i)z + (-2i)}$$

which is the required transformation.

10.3 : Special Conformal Transformations

[SPPU : May-05, 06, 07, 08, 09, 10, 11, 13,

Dec.-05, 06, 07, 08, 09, 10, 11, 12]

1) The transformation $w = z^2$

[SPPU : May-08]

We have $u+iv = z^2 = (x+iy)^2 = (x^2-y^2) + 2ixy$

Equating real and imaginary parts, we get:

$$u = x^2 - y^2 \text{ and } v = 2xy$$

If $u = \text{Constant}$ say a , then $x^2 - y^2 = a$ which represents a rectangular hyperbola. If $v = \text{Constant}$ say b , then $2xy = b \therefore xy = \frac{b}{2}$ which is also

represents a rectangular hyperbola. Differentiating equations $x^2 - y^2 = a$

and $xy = \frac{b}{2}$ w.r.t. x , we get

$$2x - 2y \frac{dy}{dx} = 0 \text{ and } x \frac{dy}{dx} + y = 0$$

Decode

A Guide for Engineering Students

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} \text{ and } \frac{dy}{dx} = -\frac{y}{x}$$

$\therefore$ Slope of tangent of $x^2 - y^2 = a$ is $\frac{x}{y}$ and $xy = \frac{b}{2}$ is $-\frac{y}{x}$.

$\therefore$ The product of slopes of tangents $= \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right) = -1$

$\therefore$ These two hyperbolas are also orthogonal to each other.

Thus a pair of lines $u = a$ and $v = b$ which are orthogonal to each other in w -plane are mapped into a pair of orthogonal rectangular hyperbolas in the z -plane.

Again if $x = c$ then $y = \frac{v}{2}$ and $y^2 = c^2 - u$ eliminating y , we get $v^2 = 4c^2(c^2 - u)$ which represents a parabola.

Similarly if $y = d$ then we get parabola

$$v^2 = 4d^2(d^2 + u)$$

Hence pair of orthogonal lines $x = c$ and $y = d$ are mapped into orthogonal parabolas in w -plane.

Now $\frac{dw}{dz} = 2z = 0$ if $z = 0$

$\therefore z = 0$ is the critical point of the transformation.

If $z = re^{i\theta}$ and $w = z^2 = r^2 e^{i2\theta} = Re^{i\phi}$

Thus upper half of the z -plane $0 < \theta < \pi$ transforms into the entire w -plane $0 \leq \phi \leq 2\pi$ same for lower half z -plane.

2) The transformation $w = e^z$

We have $w = Re^{i\phi}$ and $z = x + iy$

$$\therefore w = e^z \Rightarrow Re^{i\phi} = e^x e^{iy}$$

$$\Rightarrow R = e^x \text{ and } \phi = y$$

If x is a constant say $x = a$ $\therefore R = e^a = \text{constant}$ which represents a circle with centre at $(0, 0)$ and radius R . Thus the lines parallel to Y -axis in the z -plane are mapped into concentric circles with centre at $(0, 0)$ in the w -plane.

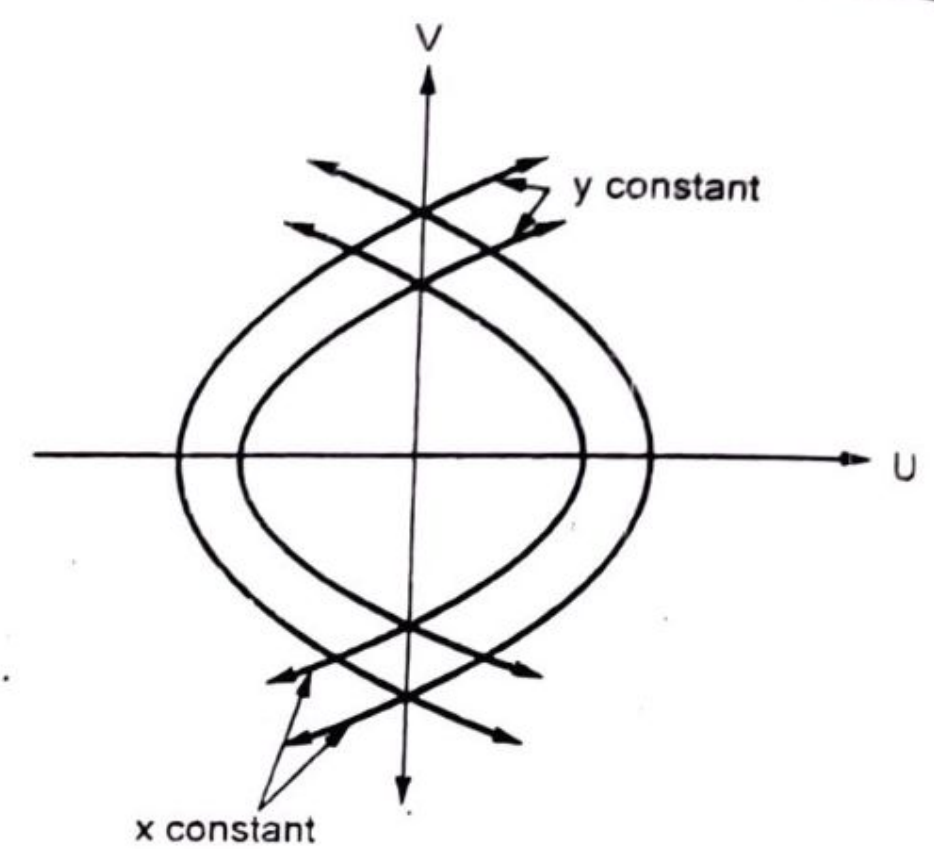


Fig. 10.1

If $y = b = \text{constant}$ then we have $\phi = b = \text{constant}$ which represents a straight line passing through origin makes angle b with positive U -axis in the UV plane. Thus the lines parallel to X -axis in the z -plane are mapped into radial lines in the w -plane. If $0 \leq y \leq 2\pi$, we have $0 \leq \phi \leq 2\pi$ in the w -plane. A horizontal strip of width 2π in z -plane will cover the entire w -plane.

The rectangular region $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$ in the z -plane transforms into the region $e^{a_1} \leq R \leq e^{a_2}$ and $b_1 \leq \phi \leq b_2$ in the w -plane.

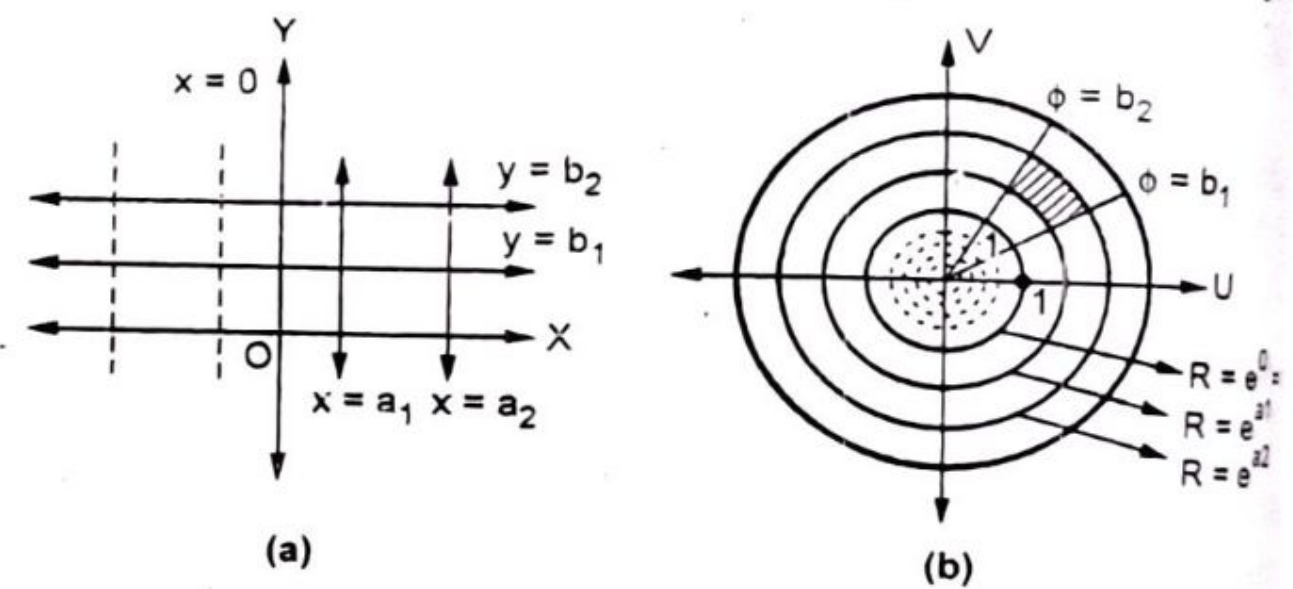


Fig. 10.2

- The left half of XOY plane is mapped into the interior of the unit circle and the right half is mapped into the exterior of the unit circle in w -plane.

$x = a$ and $y = b$ are mutually orthogonal. Also their map circles and radial lines are also mutually orthogonal. Thus proves the conformal nature of $w = e^z$ everywhere.

3) The transformation $w = z + \frac{1}{z}$ is known as Joukowski's transformation.

[SPPU : Dec.-05]

Let $z = r(\cos \theta + i \sin \theta)$ and $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$

We have $w = u + iv = z + \frac{1}{z}$

$$= r(\cos \theta + i \sin \theta) + \frac{1}{r}(\cos \theta - i \sin \theta)$$

$$u + iv = \left(r + \frac{1}{r}\right) \cos \theta + i \left(r - \frac{1}{r}\right) \sin \theta$$

$$u = \left(r + \frac{1}{r}\right) \cos \theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right) \sin \theta$$

$$\text{As } \cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1$$

... (10.8)

$$\frac{u^2}{\cos^2 \theta} - \frac{v^2}{\sin^2 \theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$$

$$\frac{u^2}{4 \cos^2 \theta} - \frac{v^2}{4 \sin^2 \theta} = 1$$

... (10.9)

From equation (10.8) it follows that the circles $r = \text{Constant}$ in z -plane map into a family of ellipses in the w -plane. These ellipses are confocal for $\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$ i.e. a constant.

In particular if $r = 1$ then we get $u = 2 \cos \theta$ and $v = 0$ i.e. the segment $-2 \leq u \leq 2$ of real axis in w -plane. Thus exterior of the circle $|z| = 1$ maps into the entire w -plane.

Decode

- The radial line $\theta = \text{constant} = b$ in z -plane maps into a hyperbola $\frac{u^2}{4 \cos^2 b} - \frac{v^2}{4 \sin^2 b} = 1$ which is a confocal in the w -plane.

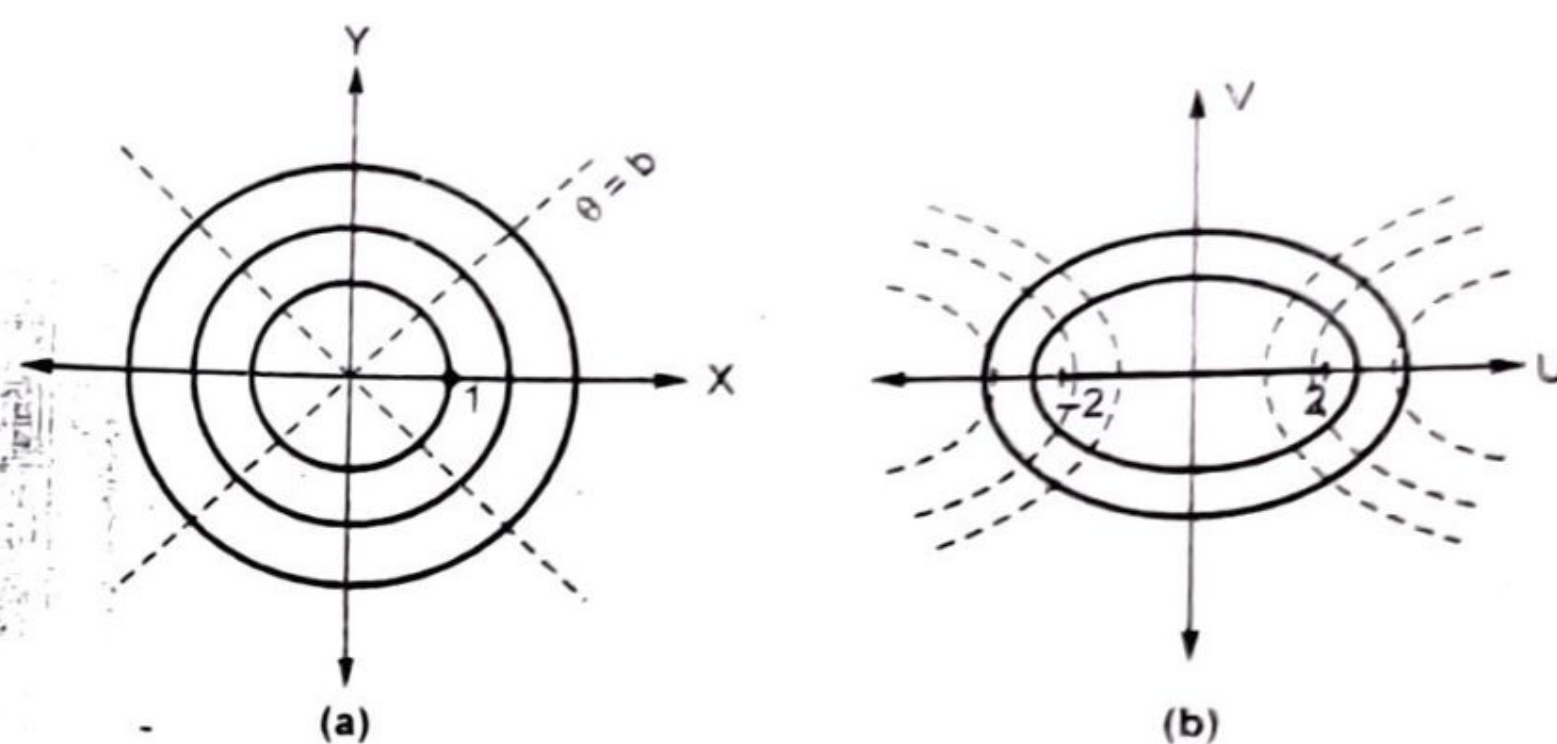


Fig. 10.3

Here $r = a$ and $\theta = b$ are orthogonal to each other in z -plane. Also their maps ellipses and hyperbolas are orthogonal. This proves the conformal nature of $w = z + \frac{1}{z} = \frac{z^2 + 1}{z}$ except at points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ in w -plane.

$$\therefore \frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2} = 0 \text{ at } z = 1 \text{ and } z = -1$$

4) The transformation $w = \sinh z$

[SPPU : May-05, Dec.-11]

We have $w = \sinh z = \sinh(x + iy)$

$$\therefore u + iv = \sinh x \cos y + i \cosh x \sin y$$

$$\therefore u = \sinh x \cos y, \quad v = \cosh x \sin y$$

$$\therefore \cos y = \frac{u}{\sinh x}, \quad \sin y = \frac{v}{\cosh x}$$

$$\text{Now } \cos^2 y + \sin^2 y = 1$$

$$\Rightarrow \frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1$$

... (10.10)

$$\text{And } \cosh^2 x - \sinh^2 x = 1$$

Decode

$$\Rightarrow \frac{v^2}{\sin^2 y} - \frac{u^2}{\cos^2 y} = 1 \quad \dots (10.11)$$

- The straight line parallel to Y-axis i.e. $x = a$ maps to ellipse $\frac{u^2}{\sinh^2 a} + \frac{v^2}{\cosh^2 a} = 1$ in the w-plane.

Thus family of straight lines $x = a$ in z-plane map into the family of ellipses in the w-plane.

- The family of straight lines $y = b$ in z-plane map into the family of hyperbola $\frac{v^2}{\sin^2 b} - \frac{u^2}{\cos^2 b} = 1$ in the w-plane.
- The rectangular region $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$ in the z-plane maps into the shaded area bounded by corresponding ellipse and hyperbola in the w-plane.
- The lines $x = a$ and $y = b$ are mutually orthogonal in the z-plane. Also the corresponding mapped family of ellipses and hyperbola in w-plane are mutually orthogonal.

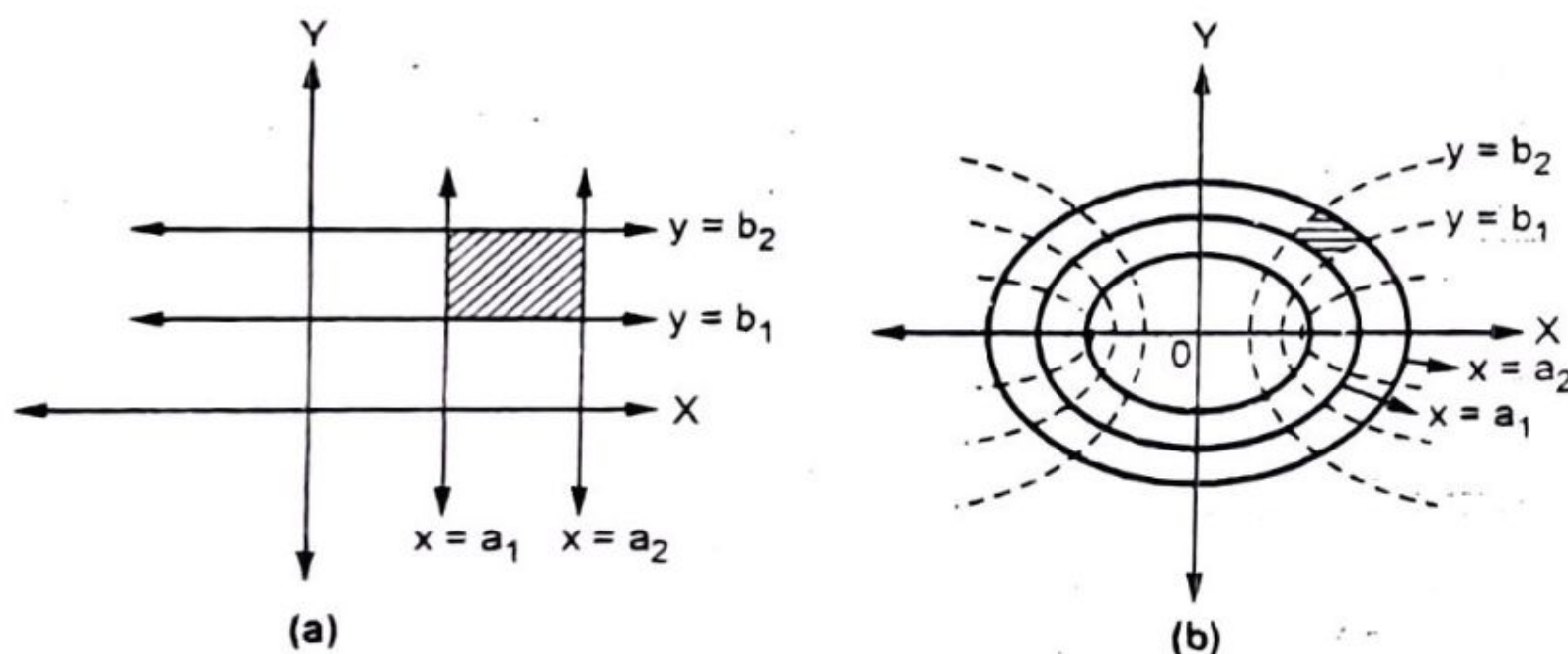


Fig. 10.4

This proves the conformality of $w = \sinh z$.

5) The transformation $w = \cosh z$ [SPPU : May-09]

We have

$$u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$$

$$\therefore u = \cosh x \cos y, \quad v = \sinh x \sin y$$

$$\text{Eliminating } x, \text{ we get } \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

$$\text{while eliminating } y, \text{ we get } \frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$$

Thus

- The family of straight lines $x = a$ map into ellipse $\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} = 1$ in the w-plane.
- The family of straight lines $y = b$ map into hyperbola $\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1$ in the w-plane.
- The rectangular region $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$ in the z-plane maps into the region in the w-plane bounded by the corresponding hyperbolae and ellipses. Thus if the lines $x = a$ and $y = b$ are mutually orthogonal, also the corresponding mapped family of ellipses and hyperbolae are mutually orthogonal. This proves the conformality of $w = \cosh z$.

Q.15 Find the map of straight line $y = 3x$ under the transformation

$$w = \frac{z-1}{z+1}$$

[SPPU : Dec.-15, May-11, 13]

Ans. : We have

$$w = \frac{z-1}{z+1}$$

$$zw + w = z - 1 \text{ or } z(w-1) = -1-w \Rightarrow z = \frac{w+1}{1-w}$$

$$\text{Let } z = x + iy \text{ and } w = u + iv$$

$\therefore$ We get

$$x + iy = \frac{(u+iv)+1}{1-(u+iv)} = \frac{(u+1)+iv}{(1-u)-iv} \times \frac{(1-u)+iv}{(1-u)+iv}$$

$$x + iy = \frac{(1-u^2-v^2) + i[v(1-u) + v(u+1)]}{(1-u)^2 + v^2}$$

$$x = \frac{1-u^2-v^2}{(1-u)^2+v^2}, \quad y = \frac{2v}{(1-u)^2+v^2}$$

The map of $y = 3x$ is

$$\frac{2v}{(1-u)^2+v^2} = \frac{3(1-u^2-v^2)}{(1-u)^2+v^2}$$

$$2v = 3(1-u^2-v^2)$$

$$3u^2 + 3v^2 + 2v - 3 = 0$$

$$u^2 + v^2 + \frac{2}{3}v - 1 = 0$$

which is the required equation of circle.

Q.16 Show that the bilinear transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$

[SPPU : Dec.-09, 10, 11, May-11]

Ans. : The given transformation is $w = \frac{2z+3}{z-4}$... (Q.16.1)

The inverse transformation of (Q.16.1) is

$$z = \frac{4w+3}{w-2}$$

$$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$$

The equation of the given circle is

$$x^2 + y^2 - 4x = 0 \text{ can be written as}$$

$$z\bar{z} - 2(z + \bar{z}) = 0$$

Substituting the values of z and $\bar{z}$, we get

$$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$16w\bar{w} + 12w + 12\bar{w} - 9$$

$$- 2(4w\bar{w} - 8w + 3\bar{w} - 6)$$

$$- 2(4w\bar{w} - 8\bar{w} + 3w - 6) = 0$$

$$22(w + \bar{w}) + 33 = 0$$

$$22(2u) + 33 = 0$$

$$(\because w = u + iv, \therefore u = 2(w + \bar{w}))$$

$$4u + 3 = 0$$

Which is the required equation of the straight line.

Q.17 Show that $w = \frac{z-i}{1-iz}$ maps upper half of z -plane onto interior of unit circle in w -plane. [SPPU : Dec. 14, 15]

Ans. : We have $w = \frac{z-i}{1-iz}$. Let $z = x + iy$ and $w = u + iv$

$$\therefore w(1-iz) = z-i$$

$$-izw - z = -i - w$$

$$\Rightarrow z = \frac{w+i}{1+iw}$$

$$x + iy = \frac{u + iv + i}{1 + i(u + iv)} = \frac{u + (v+1)i}{(1-v) + iu}$$

$$= \frac{u + (v+1)i}{(1-v) + iu} \times \frac{(1-v) - iu}{(1-v) - iu}$$

$$= \frac{u(1-v) + (v+1)u + i[-u^2 + (1+v)(1-v)]}{(1-v)^2 + u^2}$$

$$x + iy = \frac{u + u + i(-u^2 - v^2 + 1)}{(1-v)^2 + u^2}$$

But upper half of z plane is $y > 0$

$$\therefore \frac{-u^2 - v^2 + 1}{(1-v)^2 + u^2} > 0$$

$$\Rightarrow -u^2 - v^2 + 1 > 0$$

$$u^2 + v^2 < 1$$

which is the interior of unit circle in w -plane.

Q.18 Show that the transformation $\omega = \sin z$ transforms the straight lines $x = C$ of z plane into hyperbolas in the ω plane.

[SPPU : May-14]

Ans. : Given transformation

$$\omega = \sin z$$

$$\omega = u + iv, \quad z = x + iy$$

$$u + iv = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

$$u = \sin x \cosh y, \quad v = \cos x \sinh y$$

$$\frac{u}{\sin x} = \cosh y, \quad \frac{v}{\cos x} = \sinh y$$

$$\therefore \cosh^2 y - \sinh^2 y = 1$$

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

$$\text{When } x = C$$

$$\Rightarrow \frac{u^2}{\sin^2 C} - \frac{v^2}{\cos^2 C} = 1 \quad \text{which is hyperbola}$$

$\therefore$ straight lines $x = C$ of z plane is mapped on hyperbola in the w -plane.

Q.19 Find the map of the circle $|z - i| = 1$ under the transformation $w = \frac{1}{z}$ into w -plane.

[SPPU : Dec.-14, May-12]

Ans. : Let $z = x + iy$ and $w = u + iv$.

$$\text{Given that } w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

To find the image of $|z - i| = 1$

$$|x + iy - i| = 1$$

$$\therefore |x + i(y - 1)| = 1$$

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + y^2 = 2y$$

$\therefore$ Putting values of x and y , we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = \frac{-2v}{u^2 + v^2}$$

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{-2v}{u^2 + v^2}$$

$$1 = -2v$$

$$\Rightarrow v = -\frac{1}{2} \quad \text{which is a straight line}$$

Thus the image of given circle is a straight line $v = -\frac{1}{2}$ in w -plane.

11

Complex Integration

11.1 : Line Integral

I) Let $f(z) = u + iv$ and $z = x + iy$ $\therefore dz = dx + i dy$ then we have,

$$I = \int_C f(z) dz = \int_C (u + iv)(dx + i dy)$$

$$= \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

This shows that the evaluation of the line integral of the complex function can be reduced to the evaluation of two line integrals of real function.

II) Basic Properties :

$$1. \int_C (k_1 f(z) + k_2 g(z)) dz = k_1 \int_C f(z) dz + k_2 \int_C g(z) dz$$

$$2. \int_A^B f(z) dz = - \int_B^A f(z) dz$$

3. If $A < B < C$ then

$$\int_A^C f(z) dz = \int_A^B f(z) dz + \int_B^C f(z) dz$$

4. Let $|f(z)| \leq M$ then

$$\left| \int_C f(z) dz \right| \leq M L \text{ where } L \text{ is the length of the curve } C.$$

Remark : Real definite integrals are interpreted as area, but no such interpretation is possible for complex definite integrals.

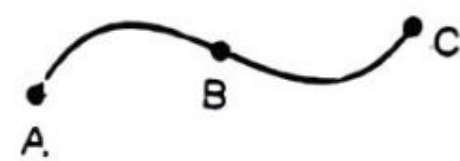


Fig. 11.1

Q.1 Evaluate $\int_{2+4i}^{5-5i} (z+1) dz$

i) Along the path $x = t^2 + 1$, $y = 3t + 1$

ii) Along the straight line joining $2 + 4i$ and $5 - 5i$.

[SPPU : May-11]

Ans. :

$$\text{Let } I = \int_{2+4i}^{5-5i} (z+1) dz$$

$$\text{Let } z = x + iy \therefore dz = dx + i dy$$

$$\therefore I = \int_{2+4i}^{5-5i} [(x+1) + iy] [dx + i dy]$$

$$= \int_{2+4i}^{5-5i} [(x+1) dx - y dy] + i \int_{2+4i}^{5-5i} [(x+1) dy - y dx] \quad \dots (Q.1.1)$$

i) Along the path $x = t^2 + 1$, $y = 3t + 1$

$$\therefore x = 2, y = 4 \Rightarrow t = 1$$

$$\text{and } x = 5 \text{ and } y = -5 \Rightarrow t = -2$$

$$\therefore dx = 2t dt, dy = 3 dt$$

Equation (Q.1.1) becomes

$$\begin{aligned} I &= \int_1^{-2} [(t^2 + 2) 2t dt - (3t + 1) 3 dt] \\ &\quad + i \int_1^{-2} [(t^2 + 2) 3 dt - (3t + 1) 2t dt] \\ &= \int_1^{-2} (2t^3 - 5t - 3) dt + i \int_1^{-2} (9t^2 + 2t + 6) dt \\ &= \left[2 \frac{t^4}{4} - 5 \frac{t^2}{2} - 3t \right]_1^{-2} + i \left[9 \frac{t^3}{3} + 2 \frac{t^2}{2} + 6t \right]_1^{-2} \\ &= 9 - 42i \end{aligned}$$

ii) Along the straight line joining $2 + 4i$ and $5 - 5i$

We have $\frac{x-2}{3} = \frac{y-4}{-9} = t$

$\therefore x = 3t + 2, \quad y = -9t + 4$

$\therefore dx = 3dt, \quad dy = -9dt$ Here $t = 0$ to $t = 1$

$\therefore I = \int_0^1 [(3t+2) 3dt + (9t-4)(-9dt)]$
 $+ i \int_0^1 [(3t+2)(-9dt) + (-9t+4) 3dt]$

$= \int_0^1 [-72t + 45] dt + i \int_0^1 [-54t - 15] dt$

$I = \left[-72 \frac{t^2}{2} + 45t \right]_0^1 + i \left[-54 \frac{t^2}{2} - 15t \right]_0^1$

$I = 9 - 42i$

11.2 : Cauchy's Theorem

1) **Cauchy's Theorem** : If $f(z)$ is an analytic function in a simply connected domain D , then for every simple closed curve C in D , $\oint_C f(z) dz = 0$.

Proof : Let R be region bounded by the closed curve C and

let $f(z) = u(x, y) + i v(x, y)$

and $z = x + iy \quad \therefore dz = dx + i dy$

then,

$\oint_C f(z) dz = \oint_C (u + iv)(dx + i dy)$

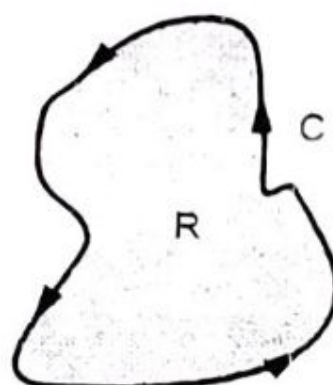


Fig. 11.2

$= \oint_C [u dx - v dy] + i \oint_C [u dy + v dx]$

As $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are

continuous in R .

$\therefore$ Applying Green's theorem on both the integrals of R.H.S. equation (11.1), we get

$\oint_C f(z) dz = \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] dx dy + i \iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dx dy$

Since $f(z)$ is analytic, C.R. equations must be satisfied

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore$ By using C.R. equations both the integrals of R.H.S. of equation (11.1) become zero.

$\therefore \oint_C f(z) dz = 0$

Note

- 1) Cauchy's theorem is also known as Cauchy's Integral theorem.
- 2) In a Cauchy's theorem the condition that f be analytic in D is sufficient rather than necessary.

e.g. $\oint_C \frac{dz}{z^2} = 0$ where C is a unit circle.

But $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$.

Corollary 1 :

Let $f(z)$ be an analytic function then $\int_{z_1}^{z_2} f(z) dz$ is independent of

the path joining the points z_1 and z_2 .

We can prove this corollary by using Cauchy's theorem.

Corollary 2 :

If $f(z)$ is analytic in the region R between two simple closed contours C_1 and C_2 then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

where C_1 and C_2 are described in a same direction.

Corollary 3 :

In a Cauchy's theorem the condition that the domain D be simply connected is quite essential. In other words by Cauchy's theorem, if $f(z)$ is analytic on and within a simple closed curve C then

$$\oint_C f(z) dz = 0$$

e.g. If C is the annulus $\frac{1}{2} < |Z| < \frac{3}{2}$ where

$f(z) = \frac{1}{z}$ is analytic but this domain is not simply connected, so we

cannot apply Cauchy's theorem, Here $\oint_C \frac{1}{z} dz = 2\pi i$.

11.3 : Cauchy's Integral Formula

[SPPU : May-05, 06, 08, 09, 10, 11, 13,
Dec.-05, 06, 08, 10, 11, 12]

Let $f(z)$ is analytic on and within a closed curve C and if a is any point within C then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Proof : Consider the function $\frac{f(z)}{z-a}$ which is analytic at all points within C except a .

Draw a small circle C_1 around point a with the point $z = a$ as a centre and radius r .

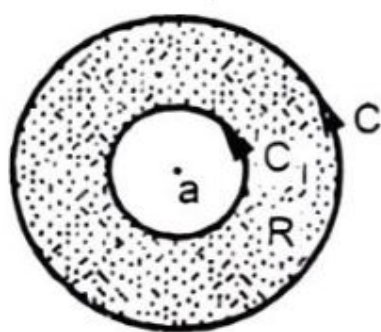


Fig. 11.3

$\therefore \frac{f(z)}{z-a}$ is analytic in the region R bounded by C_1 and C_2 (i.e. Dotted region).

$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz \quad \dots (11.3)$$

Consider R.H.S. = $\oint_{C_1} \frac{f(z)}{z-a} dz$, For any point on C_1 $z - a = re^{i\theta}$

$$\therefore dz = rie^{i\theta} d\theta$$

$$\therefore \text{R.H.S.} = \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} f(a + re^{i\theta}) d\theta \quad \dots (11.4)$$

In the limiting form, as C_1 shrinks to the point a i.e. as $r \rightarrow 0$ equation (11.4) becomes,

$$\oint_{C_1} \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta = i f(a) 2\pi = 2\pi i f(a)$$

$$\therefore \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$\therefore$ Equation (11.3) becomes

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

which is the desired Cauchy's Integral formula.

Corollary : We have $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$

Differentiating both sides w.r.t. a under integral sign, we get.

$$\begin{aligned}
 f'(a) &= \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a} \left(\frac{f(z)}{z-a} \right) dz \\
 &= \frac{1}{2\pi i} \oint_C -\frac{f(z)}{(z-a)^2} (-1) dz \\
 &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz
 \end{aligned}$$

Again differentiating w.r.t. a we get

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$$

In general

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

By above results, we observed that if $f(z)$ is analytic on the simple closed curve C then the values of function and all its derivatives can be determined at any point of C . Also it follows that an analytic function possesses analytic derivatives of all orders.

Q.2 Evaluate $\oint_C \frac{4z^2 + z}{z^2 - 1} dz$ where C is $|z - 1| = \frac{1}{2}$

[SPPU : May-05, 12, Dec.-10, 14]

Ans. : $I = \oint_C \frac{4z^2 + z}{z^2 - 1} dz$

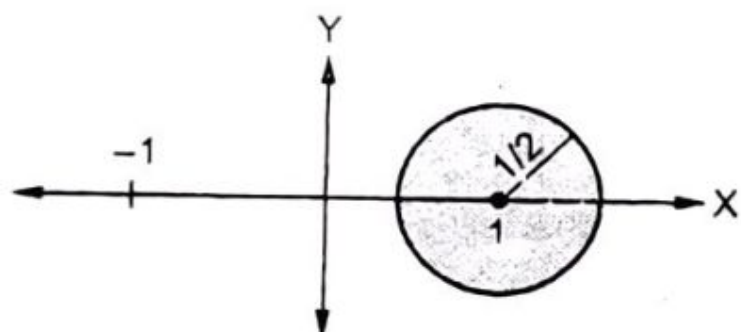


Fig. Q.2.1

$$\therefore z^2 - 1 = 0 \Rightarrow z = +1, -1$$

$\therefore z = 1, -1$ are two singularities and out of these singularities $z = 1$ lies inside and $z = -1$ lies outside of $|z - 1| = \frac{1}{2}$.

$\therefore f(z) = \frac{4z^2 + z}{z + 1}$ is analytic everywhere in $|z - 1| = \frac{1}{2}$

$\therefore$ By Cauchy's integral formula

$$I = 2\pi i f(1) = 2\pi i \left[\frac{4+1}{1+1} \right] = 5\pi i$$

Q.3 Evaluate $\oint \frac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4} dz$ where C is $|z| = 2$.

[SPPU : Dec.-08, 14 May-09, 11]

Ans. :

$$\text{Let } I = \oint \frac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4} dz$$

$$z = -\frac{\pi}{3} \text{ is pole of order 4}$$

$$z = -\frac{\pi}{3} \text{ lies inside of } |z| = 2$$

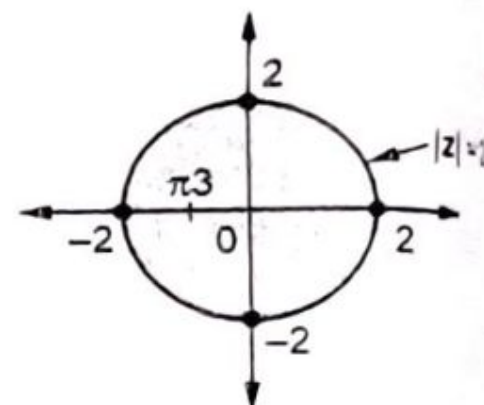


Fig. Q.3.1

$\therefore$ By Cauchy's integral theorem

$$I = \oint \frac{\sin 2z}{\left(z + \frac{\pi}{3}\right)^4} dz = \frac{2\pi i}{3!} f''' \left(-\frac{\pi}{3} \right)$$

where $f(z) = \sin 2z$

$$f'(z) = 2 \cos 2z$$

$$f''(z) = -4 \sin 2z$$

$$f'''(z) = -8 \cos 2z$$

$$\therefore f''' \left(-\frac{\pi}{3} \right) = -8 \cos \left(-\frac{\pi}{6} \right) = -8 \cos \frac{\pi}{6} = -8 \times \frac{1}{2} = -4$$

$\therefore$ Equation (Q.3.1) becomes.

$$I = \frac{2\pi i}{3 \times 2} \times (-4) = -\frac{4\pi i}{3}$$

Q.4 Evaluate $\oint_C \frac{z^2 + \cos^2 z}{\left(z - \frac{\pi}{4}\right)^3} dz$ where C is a circle $x^2 + y^2 = 1$

[SPPU : May-14]

$$I = \oint_C \frac{z^2 + \cos^2 z}{\left(z - \frac{\pi}{4}\right)^3} dz \quad \dots(Q.4.1)$$

$z = \frac{\pi}{4}$ is a pole of order 3.

$$I = \frac{2\pi i}{2!} f''\left(\frac{\pi}{4}\right) \quad \dots(Q.4.2)$$

$$f(z) = z^2 + \cos^2 z$$

$$f'(z) = 2z - 2 \cos z \sin z$$

$$= 2z - \sin 2z \Rightarrow f''(z) = 2 - 2 \cos 2z$$

$$f''\left(\frac{\pi}{4}\right) = 2 - 2 \cos 2 \cdot \frac{\pi}{4} = 2$$

$$I = \frac{2\pi i}{2} (2) = 2\pi i$$

Q.5 Evaluate $\oint_C \frac{z^2 + 1}{z^2 - 1} dz$ where C is the circle $|z - 1| = 1$.

[SPPU : May-15]

$$\text{Ans. : Let } I = \oint_C \frac{z^2 + 1}{z^2 - 1} dz$$

$$z^2 - 1 = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$$

$z = 1, -1$ are two singularities and out of these singularities and out of these singularities $z = 1$ lies inside of $|z - 1| = 1$ and $z = -1$ lies outside of $|z - 1| = 1$

$$f(z) = \frac{z^2 + 1}{z + 1} \text{ is analytic everywhere in } |z - 1| = 1$$

∴ By Cauchy's Integral formula

$$I = 2\pi i f(1) = 2\pi i \left[\frac{1+1}{1+1} \right] = 2\pi i$$

Q.6 Evaluate $\oint_C \frac{2z^2 + z + 5}{\left(z - \frac{3}{2}\right)^2} dz$ where C is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

[SPPU : May-10, Dec.-15]

Ans. : Let $f(z) = 2z^2 + z + 5$

$f(z)$ is analytic everywhere and $z = \frac{3}{2}$ lies within $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

∴ By Cauchy's integral formula

$$I = \oint_C \frac{f(z)}{\left(z - \frac{3}{2}\right)^2} dz = 2\pi i f'\left(\frac{3}{2}\right)$$

$$= 2\pi i [f'(z)]_{z=\frac{3}{2}} = 2\pi i [4z+1]_{z=\frac{3}{2}}$$

$$= 2\pi i [6+1] = 14\pi i$$

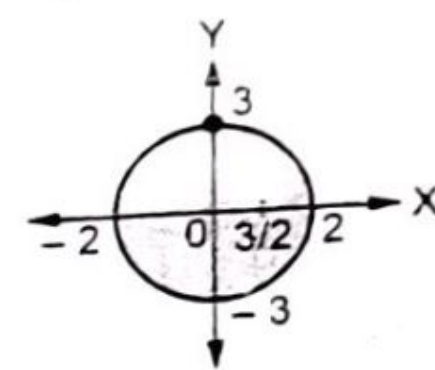


Fig. Q.6.1

Q.7 By using Cauchy's integral formula evaluate $\oint \frac{e^z}{z^2 + 1} dz$ over

$$|z - 1| = 1.$$

[SPPU : May-13]

Ans. :

$$\text{Let } f(z) = \frac{e^z}{z^2 + 1}, |z - 1| = 1$$

$$= \frac{e^z}{(z-i)(z+i)}$$

$f(z)$ is analytic within and on circle $|z - 1| = 1$. The singular points $z = \pm i$ lies outside of $|z - 1| = 1$

∴ By Cauchy's theorem

$$\oint \frac{e^z}{(z^2 + 1)} dz = 0$$

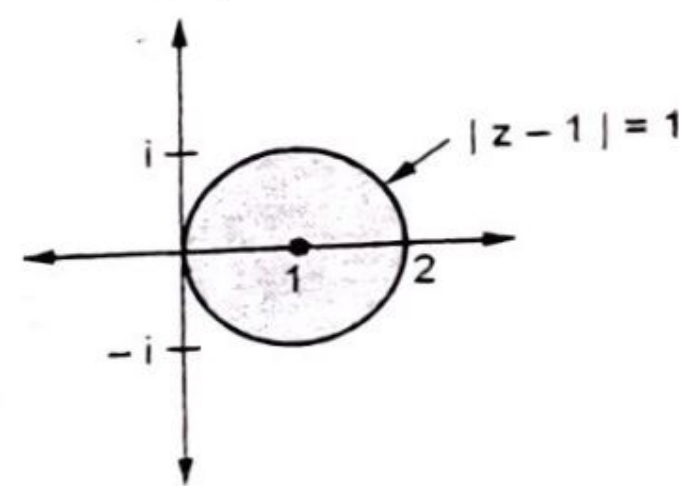


Fig. Q.7.1

11.4 : Zeroes, Singular Points, Poles and Residues

[SPPU : Dec.-06, 07, 09, 10, 11, 12, May-05, 07, 12, 13]

A] Zero : A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of a point

$z = a$, then by Taylor's series.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots$$

where $a_n = \frac{f^{(n)}(a)}{n!}$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a zero of order m at $z = a$. If $m = 1$ then zero is said to be simple.

B] Singularities of an analytic function :

A point at which function $f(z)$ ceases to be analytic is said to be singular point of $f(z)$.

C] Pole : If the principal part of the Laurent's series contain n terms

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \dots + \frac{a_{-n}}{(z-a)^n}$$

then the singular point $z = a$ is called the pole of $f(z)$ of order n .

If there is only one term $\frac{a_{-1}}{z-a}$ then $z = a$ is called the simple pole.

If $n = 2$ then $z = a$ is called double pole.

Note : If $f(z) = \frac{p(z)}{q(z)}$ then the poles of $f(z)$ are the zeroes of $q(z)$.

D] Residues :

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point.

$\therefore$ In a Laurent series expansion a_{-1} is the residue of $f(z)$ at $z = a$.

$$\therefore \text{Residue of } f(z) \text{ at } (z = a) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$\therefore \oint_C f(z) dz = 2\pi i \text{ Res } f(a)$$

E] Cauchy's residue theorem

If $f(z)$ is analytic on and within a closed contour C enclosing a finite number of isolated singular points within 'C' then

$$\oint_C f(z) dz = 2\pi i \left[\text{Sum of residues at these singular points} \right]$$

$$= 2\pi i [r_1 + r_2 + r_3 + \dots + r_n]$$

where $r_1, r_2, \dots, r_n$ are the residues at the singular points within 'C'.

F] Working rules to find poles and residues :

1) If $f(z)$ has a simple pole at $z = a$ then

$$\text{Res } f(a) = \lim_{z \rightarrow a} (z-a) f(z)$$

2) If $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at $z = a$ then

$$\text{Res } f(a) = \frac{p(a)}{q'(a)} = \left[\frac{p(z)}{q'(z)} \right]_{z=a}$$

3) If $f(z)$ has a double pole at $z = a$ then

$$\text{Res } f(a) = \lim_{z \rightarrow a} \frac{d}{dz} (z-a)^2 f(z)$$

4) If $f(z)$ has a pole of order n at $z = a$ then

$$\text{Res } f(a) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \right]_{z=a}$$

Q.8 Evaluate $\oint_C \frac{z}{z^4 + 13z^2 + 36} dz$ where C is the circle $|z| = 2$.

Ans. : Let

$$I = \oint_C f(z) dz$$

[SPPU : May-07, Dec-07]

$$f(z) = \frac{z}{z^4 + 13z^2 + 36}$$

$$f(z) = \frac{z}{(z^2 + 9)(z^2 + 4)}$$

$$= \frac{z}{(z+3i)(z-3i)(z+2i)(z-2i)}$$

$f(z)$ has simple poles at $z = 3i, -3i, 2i, -2i$. Out of these 4 poles only two poles $z = 2i$ and $-2i$ lie inside $|z| = 2.5$.

By Residue theorem, we have

$$\oint_C f(z) dz = 2\pi i [r_1 + r_2]$$

where r_1, r_2 are residues of $f(z)$ at $z = 2i$ and $-2i$ respectively.

$$r_1 = \left[\frac{z}{(z+2i)(z-3i)(z+3i)} \right]_{z=2i} = \frac{1}{10}$$

$$r_2 = \left[\frac{z}{(z-2i)(z-3i)(z+3i)} \right]_{z=-2i} = \frac{1}{10}$$

$$I = \oint_C f(z) dz = 2\pi i \left[\frac{1}{10} + \frac{1}{10} \right] = \frac{2\pi i}{5}$$

Q.9 Evaluate the following integral using residue theorem

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} dz \text{ where } C \text{ is circle } |z| = \frac{3}{2}$$

[SPPU : Dec.-15]

$$\text{Ans. : Let } I = \oint \frac{4-3z}{z(z-1)(z-2)} dz$$

$$\text{Let } f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

$f(z)$ has a simple pole at $z = 0, 1, 2$ out of these poles $z = 0, 1$ lie inside and $z = 2$ lies outside of $|z| = \frac{3}{2}$.

$$I = \oint_C f(z) dz = 2\pi i [r_1 + r_2] \quad \dots (Q.9.1)$$

where

$$r_1 = \text{Residue of } f(z) \text{ at } z = 0$$

$$= \left[z \frac{(4-3z)}{z(z-1)(z-2)} \right]_{z=0} = \frac{4}{(-1)(-2)} = 2$$

$$r_2 = \text{Residue of } f(z) \text{ at } z = 1$$

$$r_2 = \left[(z-1) \frac{(4-3z)}{z(z-1)(z-2)} \right]_{z=1}$$

$$r_2 = \left[\frac{4-3}{1(1-2)} \right] = \frac{1}{-1} = -1$$

$\therefore$ Equation (Q.9.1) becomes

$$I = 2\pi i [2 + (-1)] = 2\pi i$$

Q.10 By using Cauchy's residue theorem evaluate $\oint \frac{e^{2z}}{z(z-1)^2} dz$ over

$$|z| = 3.$$

[SPPU : Dec.-12, 15, May-13]

$$\text{Ans. : Let } f(z) = \frac{e^{2z}}{z(z-1)^2}$$

$\therefore f(z)$ has a simple pole at $z = 0$ and double Pole at $z = 1$. All these poles lie within $|z| = 3$.

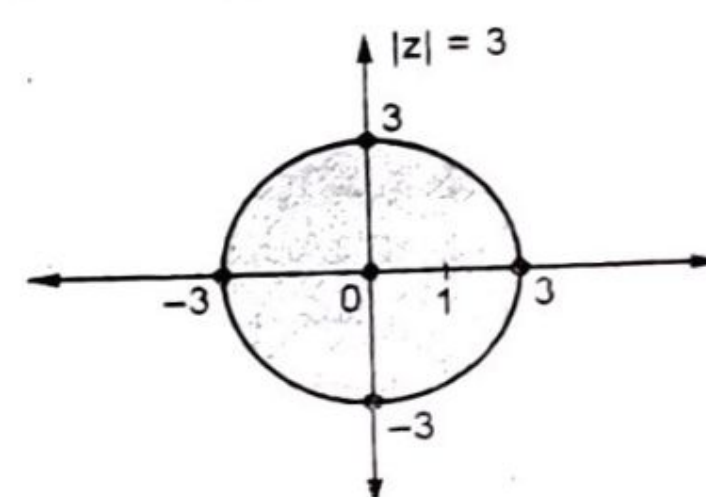


Fig. Q.10.1

$\therefore$ By Residue theorem, we have

$$\oint \frac{e^{2z}}{z(z-1)^2} dz = 2\pi i [r_1 + r_2] \quad \dots (Q.10.1)$$

where r_1 and r_2 are residue of $f(z)$ at $z = 0$ and $z = 1$ respectively.

Now $r_1 = \text{Residue of } f(z) \text{ at } z = 0$

$$= [(z-0)f(z)]_{z=0}$$

$$= \left[z \times \frac{e^{2z}}{z(z-1)^2} \right]_{z=0} = \left[\frac{e^{2z}}{(z-1)^2} \right]_{z=0}$$

$$r_1 = \frac{1}{1} = 1$$

and $r_2 = \text{Residue of } f(z) \text{ at } z = 1$

$$= \left[\frac{d}{dz} (z-1)^2 f(z) \right]_{z=1} = \left[\frac{d}{dz} (z-1)^2 \frac{e^{2z}}{z(z-1)^2} \right]_{z=1}$$

$$= \left[\frac{d}{dz} \frac{e^{2z}}{z} \right]_{z=1} = \left[\frac{ze^{2z} - e^{2z}}{z^2} \right]_{z=1}$$

$$r_2 = \left[\frac{2e^2 - e^2}{1} \right] = e^2$$

Equation (Q.10.1) becomes

$$\oint \frac{e^{2z}}{z(z-1)^2} dz = 2\pi i [1 + e^2]$$

11.5 : Evaluation of Real Definite Integrals

[SPPU : Dec.-05, 06, 07, 08, 09,
May-05, 06, 07, 08, 09, 10, 11, 12]

Consider the integral of the type

$$I = \int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta \quad (11.5)$$

where the integrand $f(\cos \theta, \sin \theta)$ is rational function of $\sin \theta, \cos \theta$.

Let $z = e^{i\theta} \therefore dz = i e^{i\theta} d\theta = i z d\theta$

$\therefore d\theta = \frac{dz}{iz}$

Now $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2} \left(z + \frac{1}{z} \right)$

Decode

A Guide for Engineering Students

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

Here as θ varies from 0 to 2π , z describes the unit circle the anticlockwise sense. Substituting in equation (11.5), we get,

$$I = \oint_{|z|=1} f \left[\frac{1}{2} \left(z + \frac{1}{z} \right), \frac{1}{2i} \left(z - \frac{1}{z} \right) \right] \frac{dz}{iz}$$

$$\frac{dz}{iz} = \oint_{|z|=1} \phi(z) dz$$

where $\phi(z)$ is a rational function of z . Now, use Cauchy theorem to evaluate I .

Note If $f(2a-x) = f(x) \forall x$ then

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$$

Q.11 Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

[SPPU : Dec.-11]

Ans. : Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$

Also as θ varies from 0 to 2π , z describes the unit circle the anticlockwise sense.

$$I = \oint_{|z|=1} \frac{1}{5 + 3 \left(z + \frac{1}{z} \right)} \frac{dz}{iz}$$

$$= \frac{2}{i} \int_{|z|=1} \frac{dz}{3z^2 + 10z + 3} = \frac{2}{i} \int_{|z|=1} \frac{dz}{(3z+1)(z+3)}$$

Let $f(z) = \frac{1}{(3z+1)(z+3)}$

Decode

A Guide for Engineering Students

$f(z)$ has simple poles at $z = -\frac{1}{3}$ and $z = -3$.

Out of these two poles only $z = -\frac{1}{3}$ lies inside $|z| = 1$.

By Cauchy's residue theorem

$$I = \frac{2}{i} \cdot 2\pi i \left[\text{Residue of } f(z) \text{ at } z = -\frac{1}{3} \right]$$

$$\begin{aligned} \text{Res } f\left(-\frac{1}{3}\right) &= \left[\left(z + \frac{1}{3}\right) \cdot \frac{1}{(3z+1)(z+3)} \right]_{z=-\frac{1}{3}} \\ &= \frac{1}{3} \cdot \frac{1}{\left(-\frac{1}{3} + 3\right)} = \frac{1}{8} \end{aligned}$$

$$I = 4\pi \cdot \frac{1}{8} = \frac{\pi}{2}$$

12 Using Contour integration, evaluate $\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4 \sin \theta} d\theta$.

[SPPU : Dec.-05, May-11, 12]

Ans. : Let $I = \int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4 \sin \theta} d\theta$

Substituting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$

$$I = \oint_C \frac{-\frac{1}{4} \left(z - \frac{1}{z} \right)^2}{5 + 4 \cdot \frac{1}{2i} \left(z - \frac{1}{z} \right)} \frac{dz}{iz}$$

where C is $|z| = 1$

$$= \frac{1}{i} \oint_C \frac{\left(-\frac{1}{4} \right) (z^2 - 1)^2}{5z^3 + 2i \left(\frac{z^2 - 1}{z} \right)} dz$$

$$= -\frac{1}{4i} \oint_C \frac{(z^2 - 1)^2}{5z^3 - 2i z^4 + 2i z^2} dz$$

$$= -\frac{1}{4i} \oint_C \frac{(z^2 - 1)^2}{z^2 (-2i z^2 + 5z + 2i)} dz$$

Let $f(z) = \frac{(z^2 - 1)^2}{z^2 (-2i z^2 + 5z + 2i)}$

$$= \frac{(z^2 - 1)^2}{-z^2 (2i) \left[z^2 - \frac{5}{2i} z - 1 \right]}$$

$$= \frac{(z^2 - 1)^2}{(-2i) z^2 \left(z - \frac{1}{2i} \right) (z + 2i)}$$

$\therefore f(z)$ has simple poles at $z = \frac{1}{2i}$ and $z = -2i$ and double pole at $z = 0$.

Out of these poles $z = 0$ and $z = \frac{1}{2i}$ lie within a circle $|z| = 1$.

$$\begin{aligned} \text{Res } f(0) &= \left\{ \frac{d}{dz} \left[\frac{(z^2 - 1)^2}{(-2i) \left(z - \frac{1}{2i} \right) (z + 2i)} \right] \right\}_{z=0} \\ &= \left\{ \frac{\left(z - \frac{1}{2i} \right) (z + 2i) 2(z^2 - 1)(2z) - (z^2 - 1)^2 \left(2z - \frac{5}{2i} \right)}{(-2i) \left(z - \frac{1}{2i} \right)^2 (z + 2i)^2} \right\}_{z=0} \\ &= \left\{ \frac{0 - 1 \left(-\frac{5}{2i} \right)}{(-2i) \left(-\frac{1}{2i} \right)^2 (2i)^2} \right\} = \frac{5}{2i} \cdot \frac{1}{(-2i) \left(-\frac{1}{4} \right) (-4)} = \frac{5}{4} \end{aligned}$$

$$\begin{aligned}
 \text{Res } f\left(\frac{1}{2i}\right) &= \left[\left(z - \frac{1}{2i} \right) \frac{(z^2 - 1)^2}{(-2i) z^2 \left(z - \frac{1}{2i} \right) (z + 2i)} \right]_{z = \frac{1}{2i}} \\
 &= \frac{\left(-\frac{1}{4} - 1 \right)^2}{(-2i) \left(-\frac{1}{4} \right) \left(\frac{1}{2i} + 2i \right)} = \frac{25}{16} \times \frac{1}{(-2i) \left(-\frac{1}{4} \right) \left(-\frac{3}{2i} \right)} \\
 &= \frac{25}{16} \cdot \frac{1}{\left(-\frac{3}{4} \right)} = -\frac{25}{12}
 \end{aligned}$$

$$\begin{aligned}
 \therefore I &= \oint_C f(z) dz = \frac{2\pi i}{-4i} \left[\text{Res } f(0) + \text{Res } f\left(\frac{1}{2i}\right) \right] \\
 &= \frac{2\pi i}{-4i} \left[\frac{5}{4} - \frac{25}{12} \right] = \frac{2\pi i}{-4i} \left(\frac{-10}{12} \right) \\
 I &= \frac{5\pi}{12}
 \end{aligned}$$

□□□