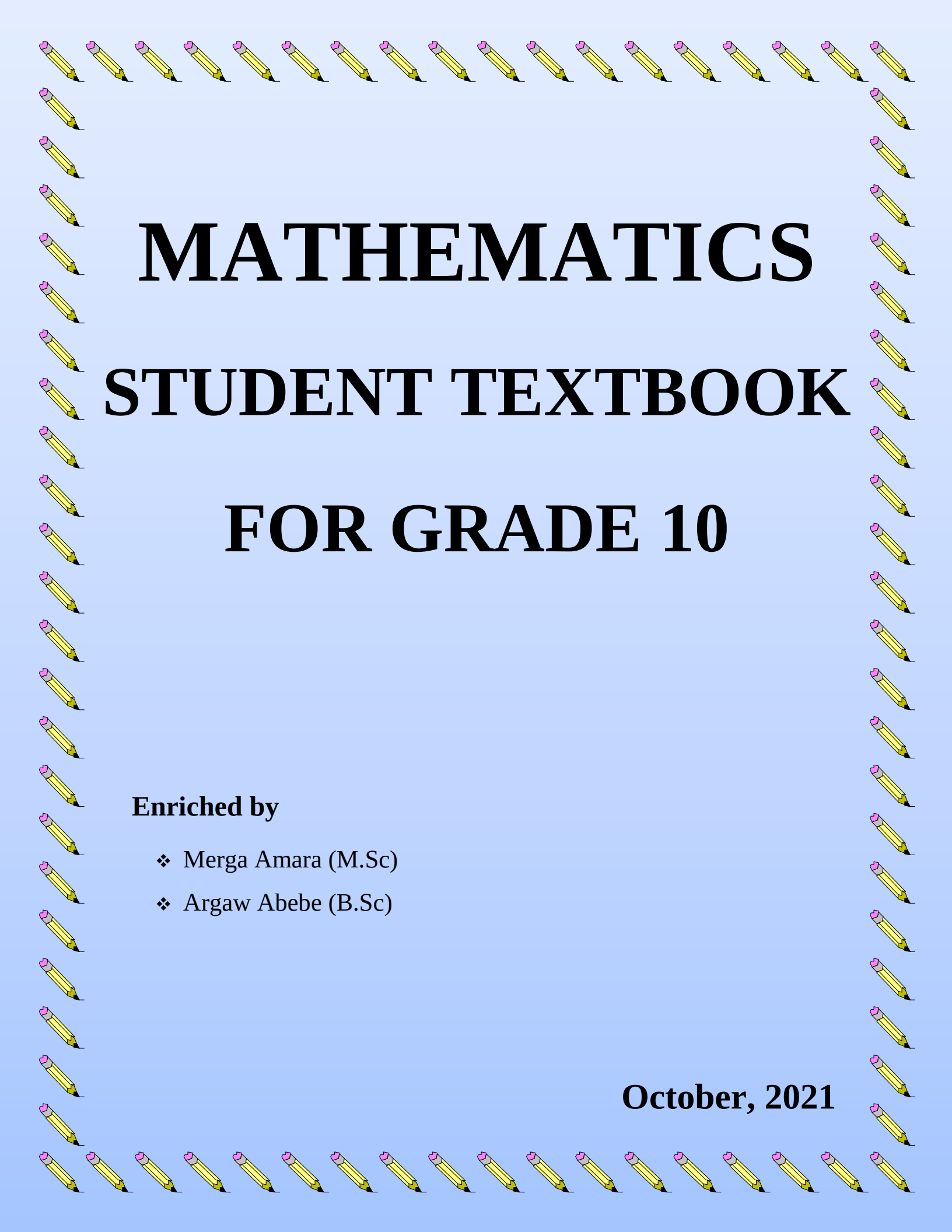




MATHEMATICS

STUDENT TEXTBOOK

FOR GRADE 10

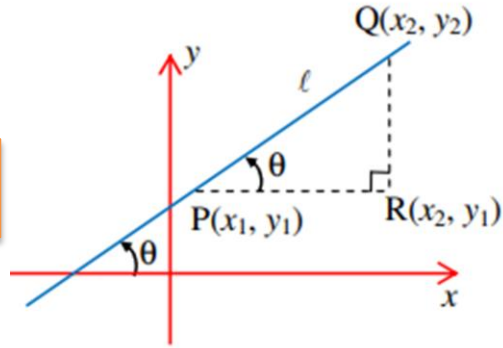
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October, 2021

Unit

4



COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *apply the distance formula to find the distance between any two given points in the coordinate plane.*
- *formulate and apply the section formula to find a point that divides a given line segment in a given ratio.*
- *write different forms of equations of a line and understand related terms.*
- *describe parallel or perpendicular lines in terms of their slopes.*

Main Contents:

4.1. Distance between two points

4.2. Division of a line segment

4.3. Equation of a line

4.4. Parallel and perpendicular lines

Key terms

Summary

Review Exercises

Introduction

Coordinate geometry studies about the use of algebraic (or analytic) methods in the solution of geometric problems with the help of coordinate plane (xy – coordinate plane).

The number plane, or Cartesian plane, is divided into four **quadrants** by two perpendicular lines called the **x -axis**, a horizontal line, and the **y -axis**, a vertical line. These axes intersect at a point called the **origin**. Once a unit distance has been chosen, the position of any point in the plane can be uniquely represented by an **ordered pair** of numbers (x, y) . For the point $(5, 3)$, for example, 5 is the x -coordinate and 3 is the y -coordinate, sometimes called the first and second coordinates. When developing trigonometry, the four quadrants are usually called the **first, second, third and fourth quadrants** as shown in the following diagram.

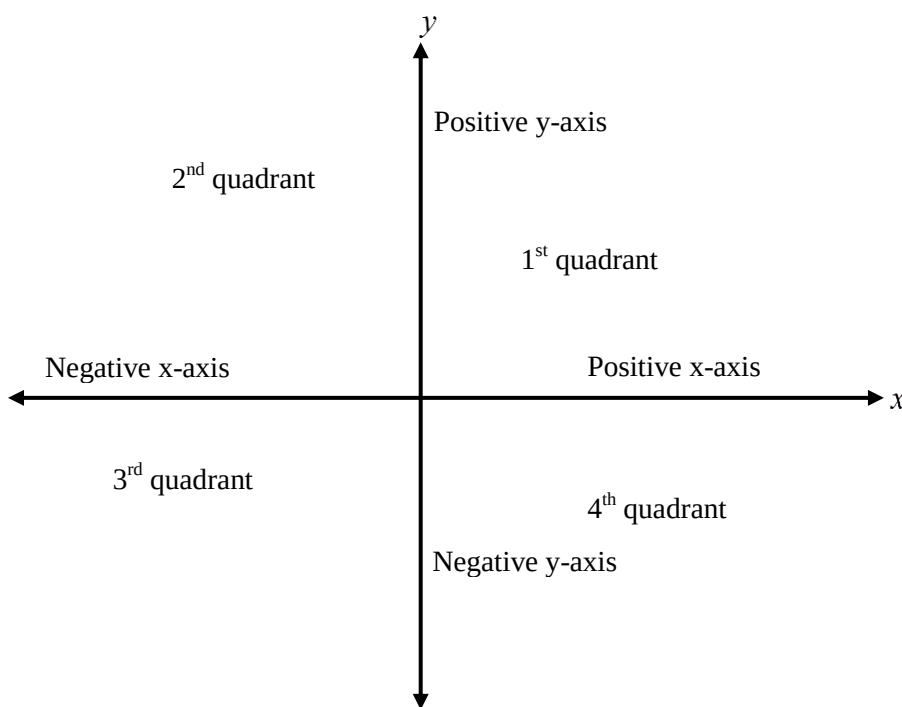


Figure 4.1

Suppose $p(a, b)$ be point on the xy -coordinate plane “ a ” is called the x -coordinate or **abscissa** and “ b ” is called the **y -coordinate** or **ordinate**.

$$1^{st} \text{ QUADRANT} = \mathbb{R}^+ \times \mathbb{R}^+ \Rightarrow \{(x, y) | x > 0, y > 0\}$$

$$2^{st} \text{ QUADRANT} = \mathbb{R}^- \times \mathbb{R}^+ \Rightarrow \{(x, y) | x < 0, y > 0\}$$

$$3^{rd} \text{ QUADRANT} = \mathbb{R}^- \times \mathbb{R}^- \Rightarrow \{(x, y) | x < 0, y < 0\}$$

$$4^{th} \text{ QUADRANT} = \mathbb{R}^+ \times \mathbb{R}^- \Rightarrow \{(x, y) | x > 0, y < 0\}$$

$$x - \text{axis} = \{(x, 0) | x \in \mathbb{R}\} \text{ and } y - \text{axis} = \{(0, y) | y \in \mathbb{R}\}$$

This means xy – coordinate = $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup x - \text{axis} \cup y - \text{axis}$

4.1. DISTANCE BETWEEN TWO POINTS

DISTANCE BETWEEN TWO POINTS ON A NUMBER LINE

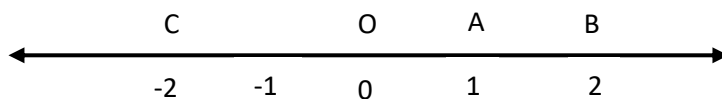


Figure 4.2

- $OA = 1 = |1|$
- $OB = 3 = |3|$
- $OC = 2 = |-2|$
- $AB = |2| = |3 - 1| = 2$
- $BC = |3 - (-2)| \text{ or } |-2 - 3| = 5$

In general on a number line the distance between points with coordinate x from the origin is given by $d = |x|$

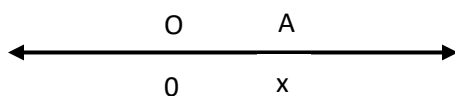


Figure 4.3

Note: The distance between two points A and B with coordinate x and y respectively is given by $d = |x - y|$ or $|y - x|$

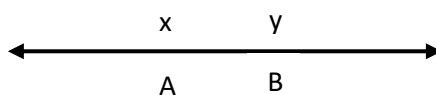


Figure 4.4

DISTANCE BETWEEN TWO POINTS IN A PLANE

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on the xy -coordinate plane. We can find the distance between the two points P and Q by considering three cases.

Case i: When P and Q are on a line parallel to the x -axis (that is, PQ is a horizontal segment) as in Figure 4.5.

Since the two points P and Q have the same y -coordinate (ordinate), the distance between P and Q is $PQ = |x_2 - x_1|$

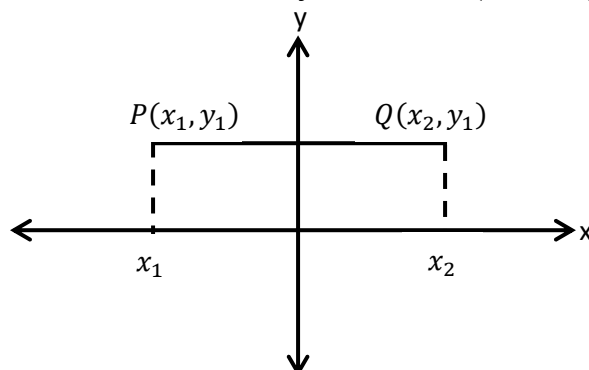


Figure 4.5

Case ii: When P and Q are on a line parallel to the y -axis (that is, PQ is a vertical segment) as in **Figure 4.6**.

Since the two points have the same x -coordinate (abscissa), the distance between P and Q is $PQ = |y_2 - y_1|$

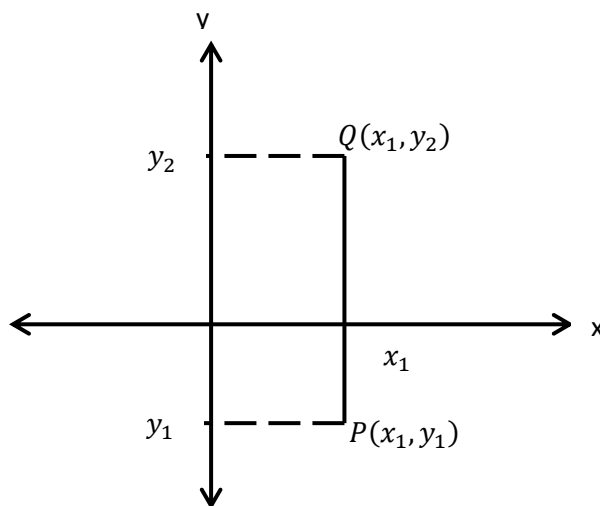


Figure 4.6

Case iii: When PQ is neither vertical nor horizontal (the general case). To find the distance between the points P and Q , draw a line passing through P parallel to the x -axis and draw a line passing through Q parallel to the y -axis. The horizontal line and the vertical line intersect at $R(x_2, y_1)$.

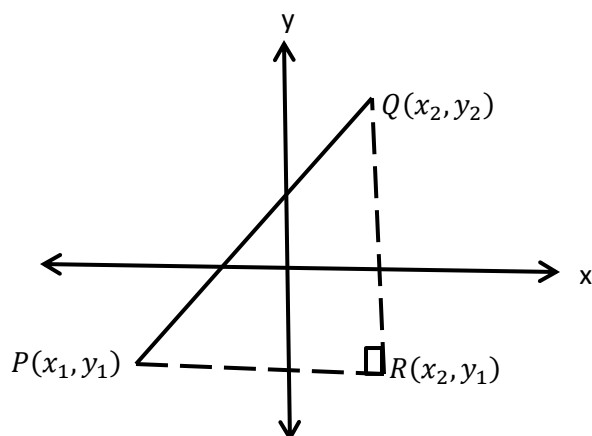


Figure 4.7

Using **case i** and **case ii**, we have $PR = |x_2 - x_1|$ and $RQ = |y_2 - y_1|$. Since $\triangle PRQ$ is a right-angled triangle at R , you can use **Pythagoras' Theorem** to find the distance between points P and Q as follows:

$$(PQ)^2 = (PR)^2 + (RQ)^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\text{Therefore, } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In general, the distance d between any two points $p(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

This is called the **distance formula**.

Example 1: Find the distance between the given points:

a. $A(2,5)$ and $B(-3,5)$

c. $R\left(-2, \frac{1}{2}\right)$ and $S\left(\frac{1}{2}, 2\right)$

b. $P(4,9)$ and $Q(4,-1)$

d. $R\left(\frac{3}{2}, \frac{5}{2}\right)$ and $S\left(\frac{1}{2}, \frac{7}{2}\right)$

Solution:

a. $A(2,5)$ and $B(-3,5)$

$$\begin{aligned} AB = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{or more simply } AB = |x_2 - x_1| = |-3 - 2| \\ &= \sqrt{(-3 - 2)^2 + (5 - 5)^2} \\ &= \sqrt{(-5)^2 + (0)^2} = 5 \text{ units} \end{aligned}$$

b. $P(4,9)$ and $Q(4,-1)$

$$\begin{aligned} PQ = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{or, more simply } PQ = |y_2 - y_1| = |-1 - 9| \\ &= \sqrt{(4 - 4)^2 + (-1 - 9)^2} \\ &= \sqrt{(0)^2 + (-10)^2} = 10 \text{ units} \end{aligned}$$

c. $R\left(-2, \frac{1}{2}\right)$ and $S\left(\frac{1}{2}, 2\right)$

$$\begin{aligned} RS = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\left(\frac{1}{2} - (-2)\right)^2 + \left(2 - \frac{1}{2}\right)^2} \\ &= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{25}{4} + \frac{9}{4}} = \frac{\sqrt{34}}{2} \text{ units} \end{aligned}$$

d. $R\left(\frac{3}{2}, \frac{5}{2}\right)$ and $S\left(\frac{1}{2}, \frac{7}{2}\right)$

$$\begin{aligned} RS = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{2} - \frac{3}{2}\right)^2 + \left(\frac{7}{2} - \frac{5}{2}\right)^2} \\ &= \sqrt{\left(\frac{-2}{2}\right)^2 + \left(\frac{2}{2}\right)^2} = \sqrt{1 + 1} = \sqrt{2} \text{ units} \end{aligned}$$

Example 2: KM is a straight line of 13units. If K has the co-ordinates(2,5) and M has the co-ordinates(x, -7), find the possible value of x.

Solution:

$$\begin{array}{ccc} & 13 \text{ units} & \\ \hline K(2, 5) & & M(x, -7) \end{array}$$

We have, $d = KM = \sqrt{(x - 2)^2 + (-7 - 5)^2}$

$$(KM)^2 = \left(\sqrt{(x - 2)^2 + (-7 - 5)^2}\right)^2 \dots \dots \text{squaring both sides}$$

$$(KM)^2 = (x - 2)^2 + (-12)^2.$$

$$(KM)^2 = x^2 - 4x + 4 + 144$$

$$(KM)^2 = x^2 - 4x + 148$$

But $KM = 13$ (given)

$$\Rightarrow (13)^2 = x^2 - 4x + 148$$

$$\Rightarrow 169 = x^2 - 4x + 148$$

$$\Rightarrow x^2 - 4x + 148 - 169 = 0$$

$$\begin{aligned}\Rightarrow x^2 - 4x - 21 &= 0 \\ \Rightarrow (x + 3)(x - 7) &= 0 \\ \Rightarrow (x + 3) = 0 \text{ or } (x - 7) &= 0 \\ \Rightarrow x = -3 \text{ or } x = 7\end{aligned}$$

Hence, the possible values of x are 7 and -3

From the value of x above, we can conclude that the coordinate of point M is either in **quadrant III** or **quadrant IV**.

Example 3: $A(-3,2), B(-5,-5), C(2,-3)$ and $D(4,4)$ are the four points in a plane. Show that $ABCD$ is rhombus but not a square.

Solution: The given points are $A(-3,2), B(-5,-5), C(2,-3)$ and $D(4,4)$

$$\therefore d = AB = \sqrt{(-5 - (-3))^2 + (-5 - 2)^2} = \sqrt{(-2)^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53} \text{ units}$$

$$d = BC = \sqrt{(2 - (-5))^2 + (-3 - (-5))^2} = \sqrt{(7)^2 + (2)^2} = \sqrt{49 + 4} = \sqrt{53} \text{ units}$$

$$d = CD = \sqrt{(4 - 2)^2 + (4 - (-3))^2} = \sqrt{(2)^2 + (7)^2} = \sqrt{4 + 49} = \sqrt{53} \text{ units}$$

$$d = DA = \sqrt{(-3 - 4)^2 + (2 - 4)^2} = \sqrt{(-7)^2 + (-2)^2} = \sqrt{49 + 4} = \sqrt{53} \text{ units}$$

$$\therefore AB = BC = CD = DA$$

$\therefore ABCD$ is either a rhombus or a square

$$\text{Diagonal } AC = \sqrt{(2 - (-3))^2 + (-3 - 2)^2} = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} \text{ units}$$

$$\text{Diagonal } BD = \sqrt{(4 - (-5))^2 + (4 - (-5))^2} = \sqrt{(9)^2 + (9)^2} = \sqrt{162} \text{ units}$$

$$\therefore \text{Diagonal } AC \neq \text{diagonal } BD$$

$\therefore ABCD$ is a rhombus but not a square

Example 4: Name the type of triangle PQR formed by the points $P(\sqrt{2}, \sqrt{2})$, $Q(-\sqrt{2}, -\sqrt{2})$ and $R(-\sqrt{6}, \sqrt{6})$

Solution:

$$d = PQ = \sqrt{(-\sqrt{2} - \sqrt{2})^2 + (-\sqrt{2} - \sqrt{2})^2} = \sqrt{(-2\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{16} = 4 \text{ units}$$

$$d = PR = \sqrt{(-\sqrt{6} - \sqrt{2})^2 + (\sqrt{6} - \sqrt{2})^2} = \sqrt{6 + 2\sqrt{12} + 2 + 6 - 2\sqrt{12} + 2} = \sqrt{16} = 4 \text{ unit}$$

$$d = RQ = \sqrt{(-\sqrt{2} + \sqrt{6})^2 + (-\sqrt{2} - \sqrt{6})^2} = \sqrt{6 - 2\sqrt{12} + 2 + 6 + 2\sqrt{12} + 2} = \sqrt{16} = 4 \text{ units}$$

Since $PQ = PR = QR = 4 \text{ units}$

$\therefore$ Points P, Q and R form an equilateral triangle

4.2. DIVISION OF A LINE SEGMENT

Consider the horizontal line segment with end-points A (x_1, y) and B (x_2, y) as shown in **figure 4.8**. In terms of the coordinates of A and B, determine the coordinates of the point P (x_0, y_0) that divides AB internally in the ratio $m : n$.

Clearly, the ratio of the line segment AP to the line segment PB is given by $\frac{AP}{PB}$.

The distance between A and P is $AP = x_0 - x_1$

The distance between P and B is $PB = x_2 - x_0$.

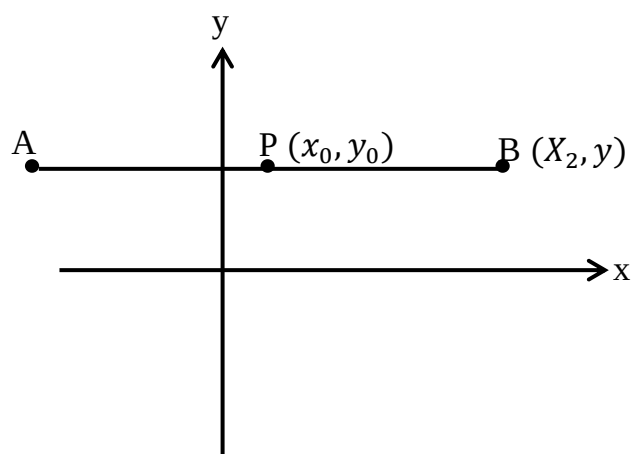


Figure 4.8

Therefore, $\frac{AP}{PB} = \frac{m}{n}$ i.e. $\frac{x_0 - x_1}{x_2 - x_0} = \frac{m}{n}$

Solving this equation for x_0

$$\Rightarrow n(x_0 - x_1) = m(x_2 - x_0)$$

$$\Rightarrow nx_0 - nx_1 = mx_2 - mx_0$$

$$\Rightarrow nx_0 + mx_0 = nx_1 + mx_2$$

$$\Rightarrow x_0(n + m) = nx_1 + mx_2$$

$$\Rightarrow x_0 = \frac{nx_1 + mx_2}{n + m}$$

Since AB is parallel to the x-axis (AB is a horizontal line segment) and obviously, $y_0 = y$, therefore, the

point $P(x_0, y_0)$ is $\left(\frac{nx_1 + mx_2}{n + m}, y\right)$.

Given a line segment PQ with end point coordinates P (x_1, y_1) and Q (x_2, y_2) , let us find the coordinates of the point R dividing the line segment PQ internally in the ratio $m: n$. i.e. $\frac{PR}{RQ} = \frac{m}{n}$ where m and n are given positive real numbers.

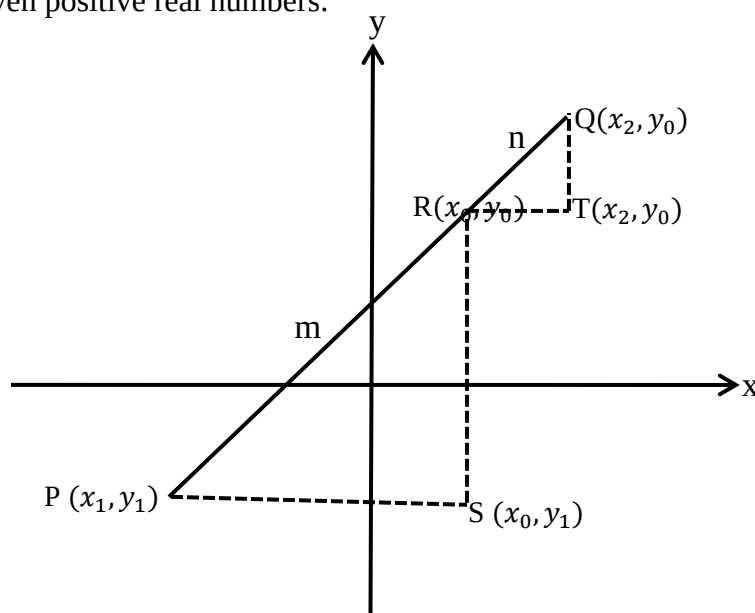


Figure 4.9

Let the coordinates of R be (x_0, y_0) . Consider the points (x_1, x_2) and (y_1, y_2) . If you draw lines through the points P, Q and R parallel to the axes as shown in **figure 4.9**, the points S and T have the coordinates (x_0, y_1) and (x_2, y_0) , respectively.

$$PS = x_0 - x_1, RT = x_2 - x_0, SR = y_0 - y_1 \text{ and } TQ = y_2 - y_0$$

Since triangles PSR and RTQ are similar (**Why?**)

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ and } \frac{SR}{TQ} = \frac{PR}{RQ}$$

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{m}{n} \text{ and } \frac{y_0 - y_1}{y_2 - y_0} = \frac{m}{n}$$

$$\Rightarrow n(x_0 - x_1) = m(x_2 - x_0) \text{ and } n(y_0 - y_1) = m(y_2 - y_0)$$

$$\Rightarrow nx_0 - nx_1 = mx_2 - mx_0 \text{ and } ny_0 - ny_1 = my_2 - my_0$$

$$\Rightarrow nx_0 + mx_0 = nx_1 + mx_2 \text{ and } ny_0 + my_0 = ny_1 + my_2$$

$$\Rightarrow x_0(n + m) = nx_1 + mx_2 \text{ and } y_0(n + m) = ny_1 + my_2$$

$$\Rightarrow x_0 = \frac{nx_1 + mx_2}{n + m} \text{ and } y_0 = \frac{ny_1 + my_2}{n + m}$$

The point $R(x_0, y_0)$ dividing the line segment PQ internally in the ratio $m:n$ is given by

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n + m}, \frac{ny_1 + my_2}{n + m} \right).$$

This is called the **section formula**.

Example 1: Find the coordinate of a point R that divides the line segments with end-points $A(-3, 3)$ and $B(12, -7)$ in the ratio $2:3$.

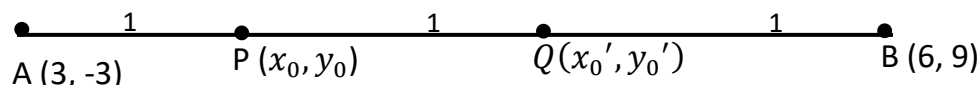
Solution: Put $(x_1, y_1) = (-3, 3)$, $(x_2, y_2) = (12, -7)$, $m = 2$ and $n = 3$, using the section formula, you have:

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n + m}, \frac{ny_1 + my_2}{n + m} \right) = \left(\frac{3 \times -3 + 2 \times 12}{3 + 2}, \frac{3 \times 3 + 2 \times -7}{3 + 2} \right) = \left(\frac{-9 + 24}{5}, \frac{9 - 14}{5} \right) = (3, -1)$$

Therefore, R is $(3, -1)$

Example 2: A line segment has end-points $(3, -3)$ and $(6, 9)$ and it is divided into three equal parts. Find the coordinate of the points that trisect the segment.

Solution: Let $P(x_0, y_0)$ and $Q(x'_0, y'_0)$ be points which trisect the line segment joining the points $(3, -3)$ and $(6, 9)$



The first point $P(x_0, y_0)$ divides the line segment in the ratio $1:2$ and hence

$$(x_1, y_1) = (3, -3), (x_2, y_2) = (6, 9), m = 1 \text{ and } n = 2$$

$$P(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n + m}, \frac{ny_1 + my_2}{n + m} \right) = \left(\frac{2 \times 3 + 1 \times 6}{2 + 1}, \frac{2 \times -3 + 1 \times 9}{2 + 1} \right) = \left(\frac{12}{3}, \frac{3}{3} \right) = (4, 1)$$

Therefore the first point $P(x_0, y_0) = (4, 1)$

The second point $Q(x'_0, y'_0)$ divides the line segment in the ratio $2:1$.

Thus

$$(x_1, y_1) = (3, -3), (x_2, y_2) = (6, 9), m = 2 \text{ and } n = 1$$

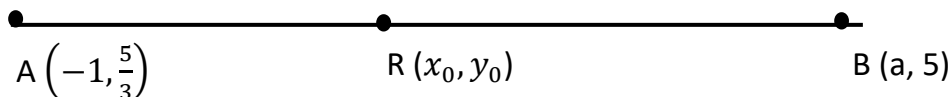
$$Q(x_0', y_0') = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right) = \left(\frac{1 \times 3 + 2 \times 6}{2+1}, \frac{1 \times -3 + 2 \times 9}{2+1} \right) = \left(\frac{15}{3}, \frac{15}{3} \right) = (5, 5)$$

Therefore the second point $Q(x_0', y_0') = (5, 5)$

Example 3: The line segment joining $A\left(-1, \frac{5}{3}\right)$ and the points $B(a, 5)$ is divided in the ratio 1:3 at P the point where the line segment AB intersects y-axis. Calculate

- the values of a
- the co-ordinate of P

Solution:



$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right) = \left(\frac{3 \times -1 + 1 \times a}{1+3}, \frac{3 \times \frac{5}{3} + 1 \times 5}{1+3} \right) = \left(\frac{-3+a}{4}, \frac{10}{4} \right)$$

$$\therefore x = \frac{-3+a}{4} \text{ and } y = \frac{10}{4}$$

- AB intersect y-axis at p

$$\begin{aligned} \therefore x = 0 &\Rightarrow \frac{-3+a}{4} = 0 \\ &\Rightarrow a = 3 \end{aligned}$$

- The co-ordinate of P are $\left(0, \frac{5}{2}\right)$

Example 4: In what ratio does the point (5,4) divide the line segment joining the points(2,1) and (7,6)?

Solution: Let the ratio be $m:n$ that the point (5,4) divides the line segment joining the points(2,1), (7,6).

$$(x_1, y_1) = (2, 1), (x_2, y_2) = (7, 6)$$

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right)$$

$$(5, 4) = \left(\frac{n \times 2 + m \times 7}{n+m}, \frac{n \times 1 + m \times 6}{n+m} \right)$$

$$5 = \frac{2n+7m}{n+m} \Rightarrow 5(n+m) = 2n+7m$$

$$5n + 5m = 2n + 7m$$

$$5n - 2n = 7m - 5m$$

$$3n = 2m$$

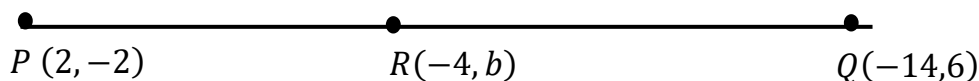
$$\frac{3n}{2n} = \frac{2m}{2n} \dots \text{dividing each term by } 2n$$

$$\frac{3}{2} = \frac{m}{n} \Rightarrow m:n = 3:2$$

Therefore $m:n = 3:2$

Example 5: In what ratio does the point $(-4, b)$ divide the line segment joining the points $P(2, -2)$, $Q(-14, 6)$? Find the value of b.

Solution:



The point $(-4, b)$ divides the line segment joining the points $P(2, -2)$ and $Q(-14, 6)$ in the ratio $m: n$

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right)$$

$$(-4, b) = \left(\frac{n \times 2 + m \times -14}{n+m}, \frac{n \times -2 + m \times 6}{n+m} \right)$$

$$-4 = \frac{2n - 14m}{n+m} \Rightarrow -4(n+m) = 2n - 14m$$

$$-4n - 4m = 2n - 14m$$

$$-4n - 2n = -14m + 4m$$

$$-6n = -10m$$

$$\frac{-6n}{-10n} = \frac{-10m}{-10n} \dots \dots \text{dividing each term by } -10n$$

$$\frac{6}{10} = \frac{m}{n}$$

$$\therefore m:n = 3:5$$

$$b = \frac{-2n + 6m}{n+m} = \frac{-2 \times 5 + 6 \times 3}{5+3} = \frac{-10+18}{8} = 1$$

$$\therefore b = 1$$

THE MID- POINT FORMULA

A point that divides a line segment into two equal parts is the mid-point of the segment.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the end-points of $\overline{PQ}$.

If $PR = RQ$ (the case where $m = n$), then R is the mid-point of the line segment PQ .

Now let us derive the mid-point formula.

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right)$$

$$= \left(\frac{nx_1 + nx_2}{n+n}, \frac{ny_1 + ny_2}{n+n} \right) = \left(\frac{n(x_1 + x_2)}{2n}, \frac{n(y_1 + y_2)}{2n} \right) \text{ (as } m = n \text{)}$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

This is the formula used to find the **mid-point** of the line segment PQ whose end points are $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The **mid-point** of the line segment joining the points (x_1, y_1) and $Q(x_2, y_2)$ is given by

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 1: Find the co-ordinates of the mid-point of the line segments with end-points:

- $A(-3, 4)$ and $B(1, 5)$
- $G(\sqrt{5}, 1 - \sqrt{3})$ and $H(3\sqrt{5}, 1 + \sqrt{3})$
- $P(a + 3, 5b)$ and $Q(2a - 1, 3b + 4)$

Solution:

$$a. \quad M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$x_0 = \frac{x_1 + x_2}{2} \text{ and } y_0 = \frac{y_1 + y_2}{2}$$

$$x_0 = \frac{-3+1}{2} = -1 \text{ and } y_0 = \frac{4+5}{2} = \frac{9}{2}$$

$$\text{Therefore } M(x_0, y_0) = \left(-1, \frac{9}{2} \right)$$

$$b. \quad M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$x_0 = \frac{x_1 + x_2}{2} \text{ and } y_0 = \frac{y_1 + y_2}{2}$$

$$x_0 = \frac{\sqrt{5}+3\sqrt{5}}{2} = 2\sqrt{5} \text{ and } y_0 = \frac{1-\sqrt{3}+1+\sqrt{3}}{2} = 1$$

$$\text{Therefore } M(x_0, y_0) = (2\sqrt{5}, 1)$$

$$c. \quad M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$x_0 = \frac{x_1 + x_2}{2} \text{ and } y_0 = \frac{y_1 + y_2}{2}$$

$$x_0 = \frac{a+3+2a-1}{2} = \frac{3a+2}{2} \text{ and } y_0 = \frac{5b+3b+4}{2} = \frac{8b+4}{2} = 4b + 2$$

$$\text{Therefore } M(x_0, y_0) = \left(\frac{3a+2}{2}, 4b + 2 \right)$$

Example 2: The mid-point of the line segment joining the points $(3m, 6)$ and $(-4, 3n)$ is $(1, 2m - 1)$ find the values of m and n .

Solution: The mid-point of the line segment joining two points $(3m, 6)$ and $(-4, 3n)$ is $(1, 2m - 1)$

$$(x_1, y_1) = (3m, 6), (x_2, y_2) = (-4, 3n) \text{ and } (x_0, y_0) (1, 2m - 1)$$

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$\text{If } (1, 2m - 1) = \left(\frac{3m-4}{2}, \frac{6+3n}{2} \right) \text{ then;}$$

$$1 = \frac{3m-4}{2} \text{ and } 2m - 1 = \frac{6+3n}{2}$$

$$2 = 3m-4 \text{ and } 4m-2 = 6+3n$$

$$m = 2 \text{ and } 4(2)-2 = 6+3n$$

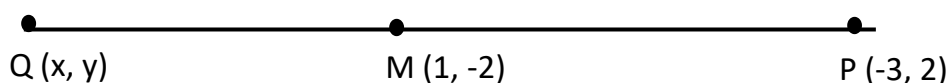
$$m = 2 \text{ and } 8-2-6 = 3n \Rightarrow n = 0$$

$$\text{Hence } m = 2 \text{ and } n = 0$$

Example 3: If co-ordinate of the mid-point of the line segment PQ is $(1, -2)$ and the co-ordinates of P is $(-3, 2)$, then find the co-ordinate of Q.

Solution: Let the co-ordinate of Q be (x, y) , co-ordinate of P is $(-3, 2)$ (given)

The mid-point of PQ is $(1, -2)$ (given), then



$$(x_1, y_1) = (x, y), (x_2, y_2) = (-3, 2) \text{ and } (x_0, y_0) (1, -2)$$

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$\text{If } (1, -2) = \left(\frac{x-3}{2}, \frac{y+2}{2} \right) \text{ then;}$$

$$1 = \frac{x-3}{2} \text{ and } -2 = \frac{y+2}{2}$$

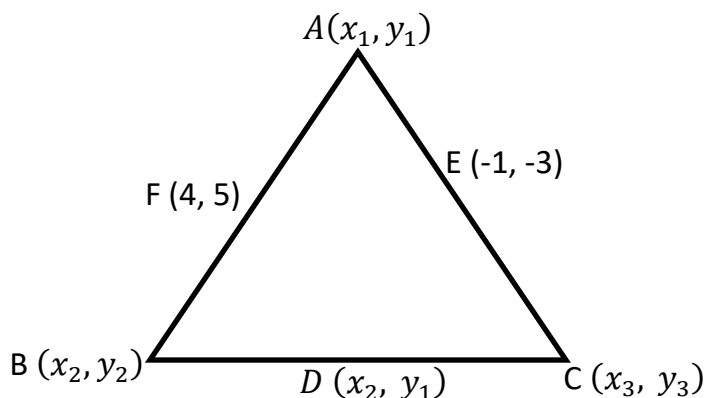
$$2 = x - 3 \text{ and } -4 = y + 2$$

$$x = 5 \text{ and } y = -6$$

Hence co-ordinate of Q is $(5, -6)$.

Example 4: The mid-point of the sides BC , CA and AB of $\triangle ABC$ are $D(2, 1)$, $E(-1, -3)$ and $F(4, 5)$, respectively. Find the co-ordinates of A, B and C

Solution: Let the co-ordinates of A, B and C be (x_1, y_1) , (x_2, y_2) and (x_3, y_3)



- D is the mid-point of BC
 $\therefore \frac{x_2 + x_3}{2} = 2 \text{ and } \frac{y_2 + y_3}{2} = 1$
 $x_2 + x_3 = 4 \text{ and } y_2 + y_3 = 2$
 $x_2 + x_3 = 4 \dots (1)$
 $y_2 + y_3 = 2 \dots (2)$
- E is the mid-point of AC
 $\therefore \frac{x_1 + x_3}{2} = -1 \text{ and } \frac{y_1 + y_3}{2} = -3$
 $x_1 + x_3 = -2 \text{ and } y_1 + y_3 = -6$
 $x_1 + x_3 = -2 \dots (3)$
 $y_1 + y_3 = -6 \dots (4)$
- F is the mid-point of AB
 $\therefore \frac{x_1 + x_2}{2} = 4 \text{ and } \frac{y_1 + y_2}{2} = 5$
 $x_1 + x_2 = -8 \text{ and } y_1 + y_2 = 10$
 $x_1 + x_2 = -8 \dots (5)$
 $y_1 + y_2 = 10 \dots (6)$

$$\Rightarrow \text{from 1 and 3, we get } x_1 - x_2 = -6 \dots (7)$$

$$\Rightarrow \text{from 2 and 4, we get } y_1 - y_2 = -8 \dots (8)$$

$$\Rightarrow \text{from 7 and 5, we get } 2x_1 = 2 \Rightarrow x_1 = 1$$

$$\Rightarrow \text{from 8 and 6, we get } 2y_1 = 2 \Rightarrow y_1 = 1$$

$$\Rightarrow \text{from 7 and 8, we get } x_2 = 7 \text{ and } y_2 = 9$$

$$\Rightarrow \text{from 1 and 2, we get } x_3 = -3 \text{ and } y_3 = -7$$

Hence, co-ordinates of A, B and C are $(1, 1)$, $(7, 9)$ and $(-3, -7)$ respectively.

Exercise 4.1

- In each of the following, find the distance between the two given points:
 - $A(\sqrt{2}, 1)$ and $B(1, \sqrt{2})$
 - $C(\sqrt{2}a, c)$ and $D(\sqrt{2}b, c)$
 - $P(a, -b)$ and $Q(-a, b)$
 - the origin and $E\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- Let $A(3, -7)$ and $B(-1, 4)$ be two adjacent vertices of square. Calculate the area of the square.
- Let $P(3, 5)$ and $B(1, -3)$ are two opposite vertices of a square. Find its area.
- Show that the plane figure with vertices:
 - $A(5, -1), B(2, 3)$ and $C(1, 1)$ is a right angled triangle.
 - $A(-4, 3), B(4, -3)$ and $C(3\sqrt{3}, 4\sqrt{3})$ is an equilateral triangle.
 - $A(2, 3), B(6, 8)$ and $C(7, -1)$ is an isosceles triangle.
 - $A(0, 2), B(\sqrt{3}, -1)$ and $C(0, -2)$ is a right angle triangle.
 - $A(9, 0), B(9, 6), C(-9, 6)$ and $(-9, 0)$ are the vertices of rectangle.
 - $A(-1, 0), B(3, 1), C(2, 2)$ and $(-2, 1)$ are the vertices of parallelogram.
- Find h such that the points $A(h, 1), B(2, 7)$ and $C(-6, -7)$ are vertices of a right triangle with right angle at the vertex A.
- The triangle PQR has vertices $P(8, 6), Q(0, 2)$ and $R(2, r)$, find the values of r when the triangle PQR:
 - has a right angle at P
 - has a right angle at Q
 - has a right angle at R
 - Is isosceles with $RQ = RP$
- Three points A, B and C have coordinates $(1, 3), (3, 5)$ and $(-1, x)$. Find the values of x in each of the following cases:
 - $AB = AC$
 - AB is perpendicular to BC
 - A, B and C are collinear
- If the distance between the points $(2, -2)$ and $(-1, x)$ is 5, then find the possible value of x .
- If the mid-point of the line segment joining the points $A(3, 4)$ and $B(k, 6)$ is $P(x, y)$ and $x + y - 10 = 0$, find the value of k .
- Find the area of the triangle ABC with $A(1, -4)$ and the mid-points of the sides through A being $(2, -1)$ and $(0, -1)$.
- The line segment joining the points $A(3, 2)$ and $B(5, 1)$ is divided at the point P in the ratio 1: 2 and it lies on the line $3x - 18y + k = 0$, find the values of k .
- If $D\left(-\frac{1}{2}, \frac{5}{2}\right), E(7, 3)$ and $F\left(\frac{7}{2}, \frac{7}{2}\right)$ are the mid-points of the sides of $\triangle ABC$, then find the area of $\triangle ABC$.
- The points $A(2, 9), E(y, 5)$ and $F(5, 5)$ are the vertices of a triangle ABC right angled at B. Find the values of y and hence the area of $\triangle ABC$.
- Find the ratio in which the point $P\left(\frac{3}{4}, \frac{5}{12}\right)$ divides the line segment joining the points $A\left(\frac{1}{2}, \frac{3}{2}\right)$ and $B(2, -5)$.
- If $P(9a - 2, -b)$ divides line segment joining $A(3a + 1, -3)$ and $B(8a, 5)$ in the ratio 3: 1 find the values of a and b .

16. Find the coordinates of the point R on the line segment joining the points $P(-1,3)$ and $Q(2,5)$ such that $PR = \frac{3}{5}PQ$.
17. Show that the line segment joining the points $A(-5,8)$ and $B(10,-4)$ is trisected by the co-ordinates axis and also find the points of trisection of AB.
18. The co-ordinates of two points A and B are $(-3,3)$ and $(12,-7)$ respectively P is a point on the line segment AB such that $AP:PB = 2:3$ find the co-ordinates of P.
19. If $(2,1)$, $(-4,-1)$ and $(-2,-3)$ are the vertices of a rectangle taken in order, then find the fourth vertex of the rectangle.

4.3. EQUATION OF A LINE

4.3.1. Gradient (Slope) of a Line

From your everyday experience, you might be familiar with the idea of gradient (slope). A **hill** may be **steep** or may rise very slowly. The number that describes the **steepness** of a hill is called the **gradient (slope)** of the hill. We measure the gradient of a hill by the ratio of the **vertical rise** to the **horizontal run**.

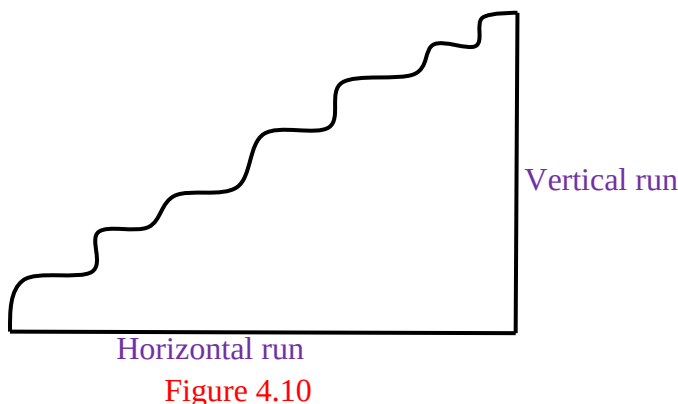


Figure 4.10

In coordinate geometry, the gradient of a non- vertical straight line is the ratio of "change in **y-coordinates**" to the corresponding "change in **x-coordinates**". That is, the slope of a line through P and Q is the ratio of the vertical distance from P to Q to the horizontal distance from P to Q.

If we denote the gradient of a line by the letter m , then $m = \frac{\text{change in y-coordinates}}{\text{change in x-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}; x_1 \neq x_2$.

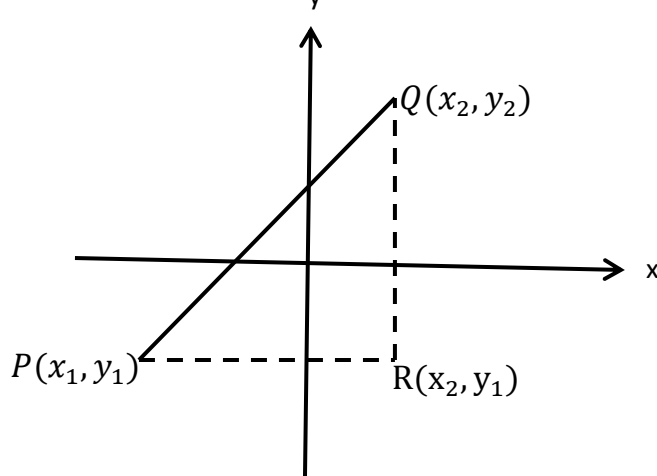


Figure 4.11

Definition 4.1

If (x_1, y_1) and (x_2, y_2) are points on a line with $x_1 \neq x_2$, then the **gradient** of the line, denoted by m , is given by

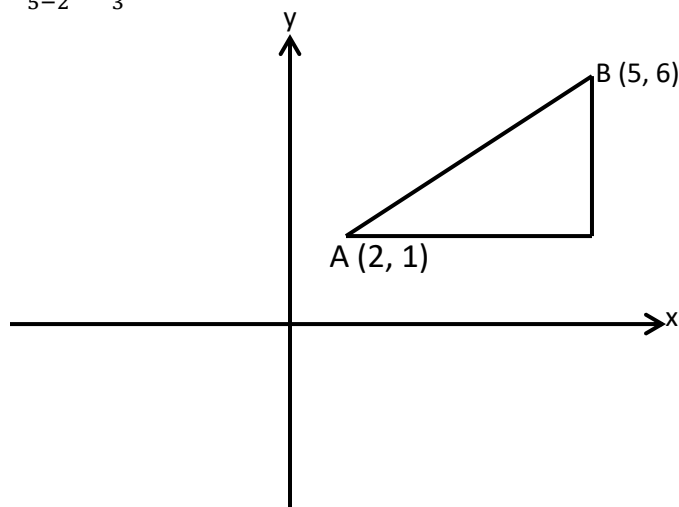
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note:

- As you move from A to B along the interval the y -value increases as the x -value increases. The gradient is **positive**.

For Example: Given the points $A(2,1)$ and $B(5,6)$ gradient of AB

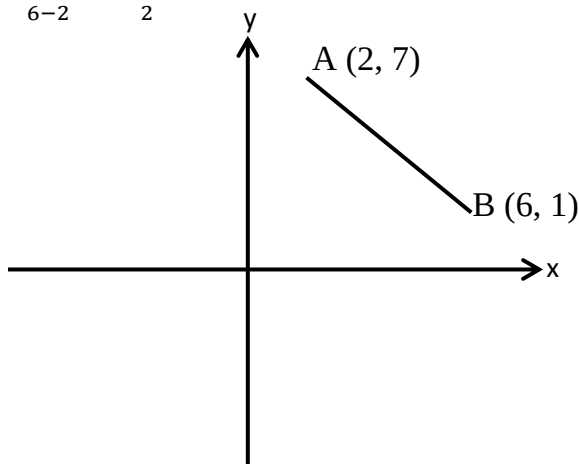
$$\text{Gradient of } AB = \frac{6-1}{5-2} = \frac{5}{3}$$



- As we move from A to B the y value decreases as the x value increases. The gradient is **negative**.

For Example: Given the points $A(2,7)$ and $B(6,1)$ gradient of AB

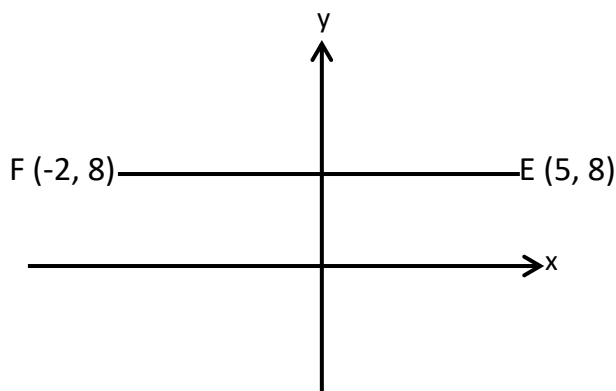
$$\text{Gradient of } AB = \frac{1-7}{6-2} = -\frac{3}{2}$$



3. The gradient of any horizontal line is **zero** i.e. $m = 0$

For Example: Given the points $E(5,8)$ and $F(-2,8)$ gradient of EF

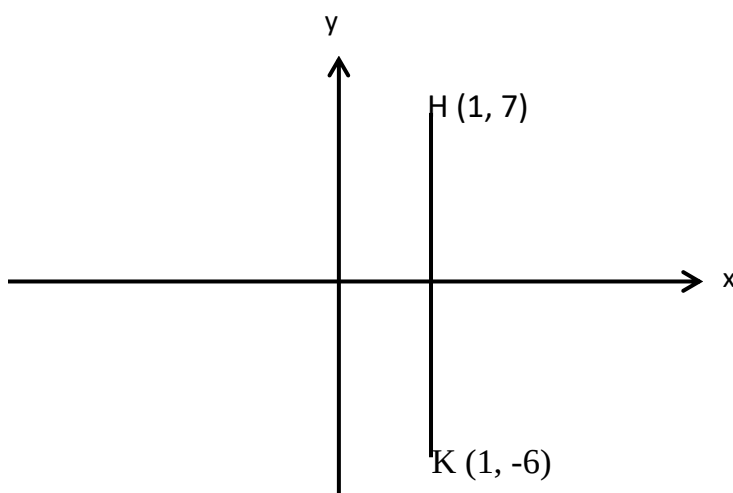
$$\text{Gradient of } EF = \frac{8-8}{-2-5} = -\frac{0}{7} = 0$$



4. The gradient of any vertical line is **not defined**

For Example: Given the points $H(1,7)$ and $K(1,-6)$ gradient of HK

$$\text{Gradient of } HK = \frac{-6-7}{1-1} = -\frac{13}{0} = \text{undefined (no gradient)}$$



Example 1: Find the gradient of the line passing through each of the following pairs of points:

a. $P(-3,1)$ and $Q(4,-5)$

d. $A\left(\frac{5}{6}, -3\right)$ and $B\left(\frac{5}{6}, 1\right)$

b. $P(5,-10)$ and $Q(-6,-7)$

e. $H(\sqrt{2}, -9)$ and $K(2\sqrt{2}, -7)$

c. $A\left(-\frac{3}{4}, -8\right)$ and $B\left(-\frac{1}{6}, \frac{1}{2}\right)$

f. $F(-2,9)$ and $G\left(-\frac{4}{5}, 9\right)$

Solution:

a. $P(-3,1)$ and $Q(4,-5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-1}{4-(-3)} = -\frac{6}{7}$$

b. $P(5, -10)$ and $Q(-6, -7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - (-10)}{-6 - 5} = -\frac{3}{11}$$

c. $A\left(-\frac{3}{4}, -8\right)$ and $B\left(-\frac{1}{6}, \frac{1}{2}\right)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{2} - (-8)}{-\frac{1}{6} - \left(-\frac{3}{4}\right)} = \frac{\frac{1}{2} + 8}{-\frac{1}{6} + \frac{3}{4}} = \frac{102}{7}$$

d. $A\left(\frac{5}{6}, -3\right)$ and $B\left(\frac{5}{6}, 1\right)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-3)}{\frac{5}{6} - \frac{5}{6}} = \frac{4}{0} \text{ which is undefined}$$

e. $H(\sqrt{2}, -9)$ and $K(2\sqrt{2}, -7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - (-9)}{2\sqrt{2} - \sqrt{2}} = \frac{-7 + 9}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

f. $F(-2, 9)$ and $G\left(-\frac{4}{5}, 9\right)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 9}{-\frac{4}{5} - (-2)} = \frac{0}{-\frac{4}{5} + 2} = 0$$

4.3.2 Slope of a Line in terms of Angle of Inclination

Definition: The angle of inclination of a line ℓ is the angle between the line ℓ and the positive x-axis in anticlockwise direction

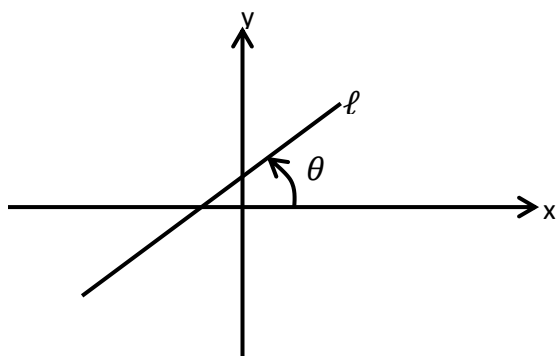


Figure 4.12

Note:

1. The angle of inclination of a line is such that $0 \leq \theta \leq 180^\circ$
2. The angle of inclination of horizontal line is zero i.e. $\theta = 0$

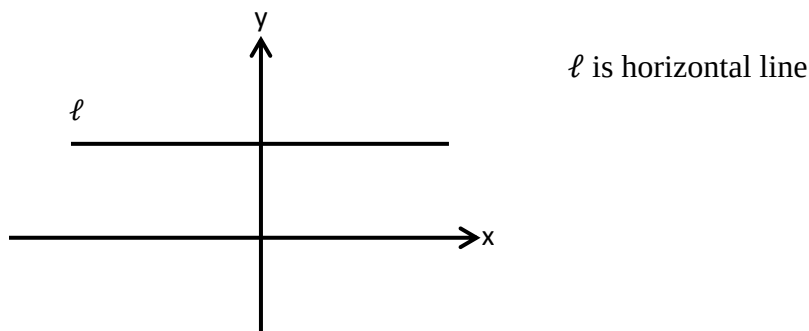


Figure 4.13

3. ℓ is a vertical line i.e. $\theta = 90^\circ$

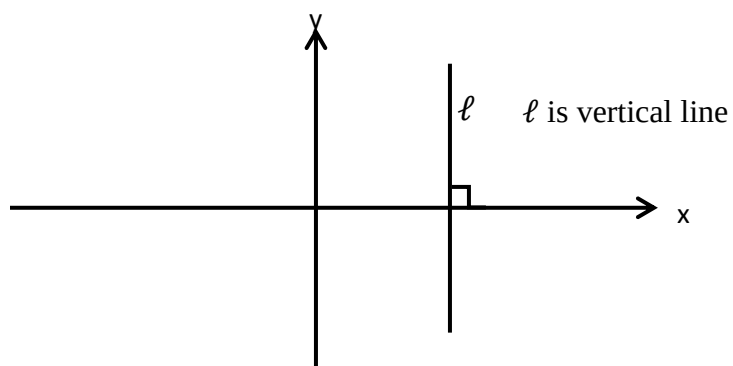


Figure 4.14

4. ℓ is inclined to the left i.e. $90^\circ < \theta < 180^\circ$

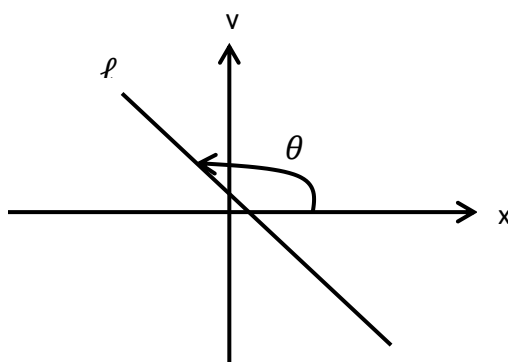
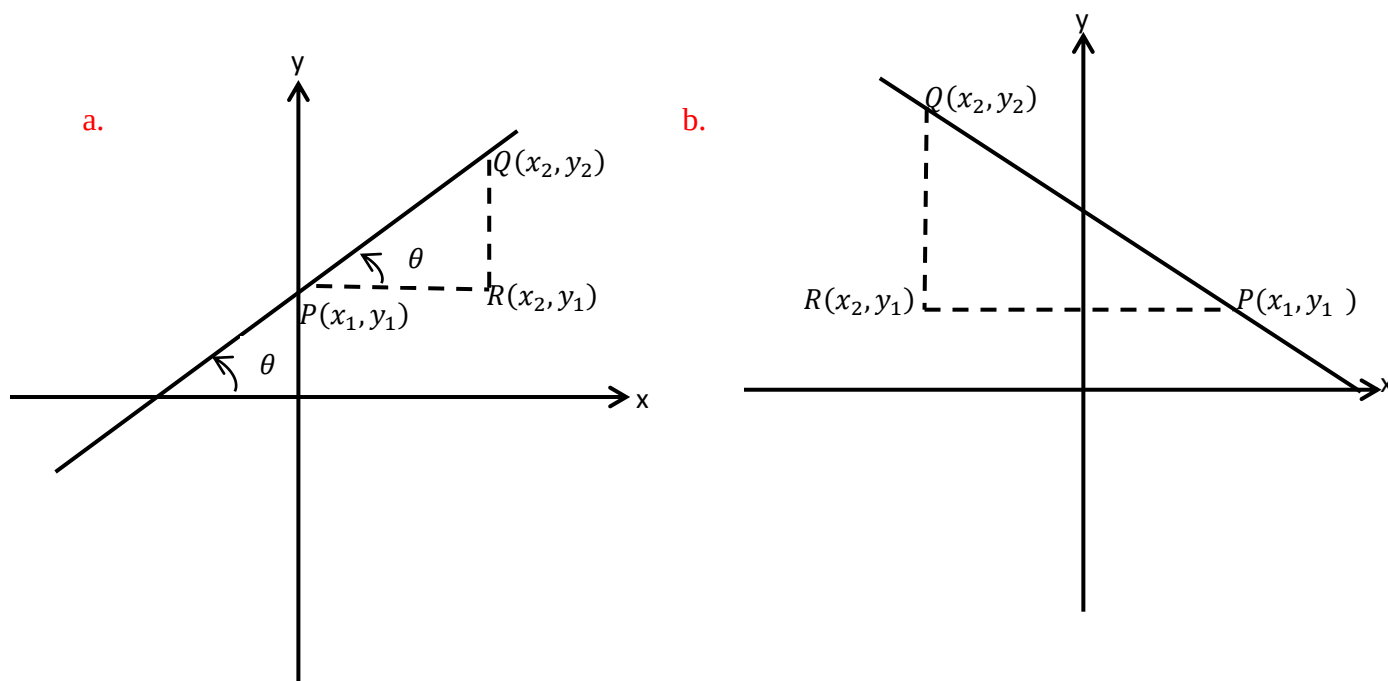


Figure 4.16

For a non-vertical line, the tangent of this angle is the slope of the line. Observe the following figure 4.17



In **figure 4.17a** above, as $y_2 - y_1$ represents the distance RQ and $x_2 - x_1$ represents the distance PR , the slope of the straight line PQ is actually represented by the ratio:

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\angle RPQ)$$

$$\therefore m = \tan \theta$$

A line making an acute angle of inclination θ with the positive direction of the x-axis has positive slope. Similarly, a line with obtuse angle of inclination θ (**see figure 4.17b**), has negative slope.

$$\text{slope of } l (m) = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_1 - x_2} = -\frac{y_2 - y_1}{x_2 - x_1} = -\tan(180^\circ - \theta) = -(-\tan \theta) = \tan \theta$$

In general, the slope of a line may be expressed in terms of the coordinates of two points (x_1, y_1) and (x_2, y_2) on the line as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta, \quad x_2 \neq x_1$$

Where θ is the anticlockwise angle between the positive x-axis and the line.

Note:

- i. if ℓ is horizontal, $\theta = 0^\circ$
 $m = \tan \theta = \tan 0^\circ = 0$
- ii. if ℓ is vertical, $\theta = 90^\circ$
 $m = \tan \theta = \tan 90^\circ = \nexists$ (does not exist)
- iii. if ℓ is inclined to the right,
 $0 < \theta < 90^\circ$ (θ is acute or θ is first quadrantal angle)
 $m = \tan \theta > 0$
- iii. if ℓ is inclined to the left,
 $90^\circ < \theta < 180^\circ$ (θ is obtuse or θ is second quadrantal angle)
 $m = \tan \theta < 0$

Example 2: Find the slope m of the lines whose angle of inclinations are given below

- a. $\theta = 0^\circ$
- b. $\theta = 60^\circ$
- c. $\theta = 150^\circ$
- d. $\theta = 75^\circ$
- e. $\theta = \frac{\pi}{2} \text{ rad}$

Solution:

- a. $\theta = 0^\circ \Rightarrow m = \tan \theta = \tan 0^\circ = 0$
- b. $\theta = 60^\circ \Rightarrow m = \tan \theta = \tan 60^\circ = \sqrt{3}$
- c. $\theta = 150^\circ \Rightarrow m = \tan \theta = \tan 150^\circ = -\tan(180^\circ - 150^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$
- d. $\theta = 75^\circ \Rightarrow m = \tan \theta = \tan 75^\circ = 2 + \sqrt{3}$
- e. $\theta = \frac{\pi}{2} \text{ rad} \Rightarrow m = \tan \frac{\pi}{2} \text{ rad} = \text{does not exist}$

Example 3: Find the angle of inclination of the following pairs of points

- a. $A(1, 4)$ and $B\left(\frac{1}{2}, \frac{7}{2}\right)$
- b. $A(0, -2)$ and $B(3, -5)$
- c. $A(3, 7)$ and $B(\sqrt{3}, 7)$
- d. $A(5, -1)$ and $B(5, 3)$
- e. $A(2, \sqrt{6})$ and $B(\sqrt{2}, 2\sqrt{3})$

Solution:

a. $A(1,4)$ and $B\left(\frac{1}{2}, \frac{7}{2}\right)$

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{7}{2} - 4}{\frac{1}{2} - 1} = \frac{\frac{7-8}{2}}{\frac{1-2}{2}} = \frac{\frac{-1}{2}}{\frac{-1}{2}} = 1$$

$$\tan \theta = 1 \Rightarrow \theta = 45^\circ$$

b. $A(0, -2)$ and $B(3, -5)$

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - (-2)}{3 - 0} = -\frac{3}{3} = -1$$

$$\tan \theta = -1 \Rightarrow \theta = 135^\circ$$

c. $A(3,7)$ and $B(\sqrt{3}, 7)$

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 7}{\sqrt{3} - 3} = \frac{0}{\sqrt{3} - 3} = 0$$

$$\tan \theta = 0 \Rightarrow \theta = 0$$

The line is horizontal

d. $A(5, -1)$ and $B(5, 3)$

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{5 - 5} = \frac{3 + 1}{0} = \frac{4}{0} \text{ does not exist}$$

The line is vertical

e. $A(2, \sqrt{6})$ and $B(\sqrt{2}, 2\sqrt{3})$

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2\sqrt{3} - \sqrt{6}}{\sqrt{2} - 2} = \frac{\sqrt{3}(2 - \sqrt{2})}{\sqrt{2} - 2} = \frac{\sqrt{3}(2 - \sqrt{2})}{-(2 - \sqrt{2})} = -\sqrt{3}$$

$$\tan \theta = -\sqrt{3} \Rightarrow \theta = 120^\circ$$

Collinear points

Definition: Three or more points are said to be collinear if they lie on the same line.

Note: In coordinate geometry three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if

$$m_{\overline{AB}} = m_{\overline{BC}} = m_{\overline{AC}} \text{ that means } \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$$

Example 4: Check whether the following triplets of points are collinear or not:

a. $A(1,1)$, $B(-1,5)$ and $C(-3,9)$

b. $A(0,3)$, $B(-2,1)$ and $C(1,2)$

c. $A(\sqrt{2}, 1)$, $B(1, \sqrt{2} - 1)$ and $C(\sqrt{2} + 1, \sqrt{2} + 1)$

Solution:

a. $A(1,1)$, $B(-1,5)$ and $C(-3,9)$

Check $m_{\overline{AB}} = m_{\overline{BC}} = m_{\overline{AC}}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$$

$$\frac{5 - 1}{-1 - 1} = \frac{9 - 5}{-3 - 1} = \frac{9 - 1}{-3 - 1} \Rightarrow -2 = -2 = -2$$

$\therefore A, B$ and C are collinear

b. $A(0,3)$, $B(-2,1)$ and $C(1,2)$

Check $m_{\overline{AB}} = m_{\overline{BC}} = m_{\overline{AC}}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$$

$$\frac{1-3}{-2-0} = \frac{2-1}{1-(-2)} = \frac{2-3}{1-0} \Rightarrow 1 \neq \frac{1}{3} \neq -1$$

$\therefore A, B$ and C are not collinear

- c. $A(\sqrt{2}, 1), B(1, \sqrt{2} - 1)$ and $C(\sqrt{2} + 1, \sqrt{2} + 1)$

$$\text{Check } m_{\overline{AB}} = m_{\overline{BC}} = m_{\overline{AC}}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$$

$$\frac{\sqrt{2} - 1 - 1}{1 - \sqrt{2}} = \frac{\sqrt{2} + 1 - (\sqrt{2} - 1)}{\sqrt{2} + 1 - 1} = \frac{\sqrt{2} + 1 - 1}{\sqrt{2} + 1 - \sqrt{2}} \Rightarrow \sqrt{2} = \sqrt{2} = \sqrt{2}$$

$\therefore A, B$ and C are collinear

Example 5: Find the possible values of k so that the following triplets of points are collinear:

- $A(k, -1), B(3 + k, 2)$ and $C(0, 5)$
- $A(1, k), B(3, 2)$ and $C(4k, -1)$
- $A(-1, k - 1), B(k, k - 3)$ and $C(k - 6, 3)$

Solution:

- a. $A(k, -1), B(3 + k, 2)$ and $C(0, 5)$

$$m_{\overline{AB}} = m_{\overline{BC}}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\frac{2 - (-1)}{3 + k - k} = \frac{5 - 2}{0 - (3 + k)} \Rightarrow \frac{3}{3} = \frac{3}{-3 - k}$$

$$1 = \frac{3}{-3 - k} \Rightarrow -3 - k = 3$$

$$\Rightarrow k = -6$$

- b. $A(1, k), B(3, 2)$ and $C(4k, -1)$

$$m_{\overline{AB}} = m_{\overline{BC}}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \Rightarrow \frac{2 - k}{3 - 1} = \frac{-1 - 2}{4k - 3}$$

$$\Rightarrow \frac{2 - k}{2} = \frac{-3}{4k - 3}$$

$$\Rightarrow (2 - k)(4k - 3) = (2)(-3)$$

$$\Rightarrow -4k^2 + 11k - 6 = -6$$

$$\Rightarrow -4k^2 + 11k = 0$$

$$\Rightarrow k(-4k + 11) = 0$$

$$\Rightarrow k = 0 \text{ or } -4k + 11 = 0$$

$$\Rightarrow k = 0 \text{ or } k = \frac{11}{4}$$

- c. $A(-1, k - 1), B(k, k - 3)$ and $C(k - 6, 3)$

$$m_{\overline{AB}} = m_{\overline{BC}}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \Rightarrow \frac{k - 3 - (k - 1)}{k - (-1)} = \frac{3 - (k - 3)}{k - 6 - k}$$

$$\Rightarrow \frac{-2}{k + 1} = \frac{6 - k}{-6}$$

$$\Rightarrow (-2)(-6) = (k + 1)(6 - k)$$

$$\Rightarrow 12 = 6k + 6 - k^2 - k$$

$$\Rightarrow 12 = -k^2 + 5k + 6$$

$$\Rightarrow -k^2 + 5k - 6 = 0$$

$$\Rightarrow (-k + 3)(k - 2) = 0 \Rightarrow k = 3 \text{ or } k = 2$$

Exercise 4.2

- Find the gradient of the line passing through the following points:
 - $A(4,3)$ and $B(8,11)$
 - $P(3,7)$ and $Q(1,9)$
 - $C(\sqrt{2}, -9)$ and $D(2\sqrt{2}, -7)$
 - $R(-5, -2)$ and $S(7,8)$
 - $E(5,8)$ and $F(-2,8)$
 - $H(1,7)$ and $K(1, -6)$
 - $H(\sqrt{2}, 1)$ and $K(1, \sqrt{2} - 1)$
 - $H(1, \sqrt{2} - 1)$ and $K(\sqrt{2} + 1, \sqrt{2} + 1)$
- $A(2, -3)$, $B(7,5)$ and $C(-2,9)$ are vertices of triangle ABC. Find the gradient of each of the sides of the triangle.
- Show that the points $A(a, 0)$, $B(0, b)$ and $C(3a, -2b)$ are collinear.
- Find the possible values of k so that the following triplets of points are collinear:
 - $A(1, k)$, $B(3,2)$ and $C(4k, -1)$
 - $A(k + 1, 2k)$, $B(3k, 2k + 3)$ and $C(5k - 1, 5k)$
 - $A(-\sqrt{2}, 0)$, $B(-1, k - \sqrt{2})$ and $C(2\sqrt{2}, 3k)$
- Which of the following points are collinear?
 - $A(2,5)$, $B(-1, -1)$ and $C(-4, -7)$
 - $A(1, -1)$, $B(-2,4)$ and $C(0,1)$
 - $A(7, -3)$, $B(-2,1)$ and $C(6\frac{1}{2}, 3)$
 - $A(-4,6)$, $B(-1,12)$ and $C(-7,0)$
 - $A(4,5)$, $B(7,6)$ and $C(6,3)$
 - $A(5, -5)$, $B(-3,5)$ and $C(7,10)$
 - $A(3\frac{1}{2}, -1)$, $B(0, 2\frac{3}{5})$ and $C(1,2)$
 - $A(1, \pi - \pi^2)$, $B(\pi, 0)$ and $C(1 + \pi, \pi)$
 - $A(0, -4)$, $B(1, -\frac{7}{2})$ and $C(-1, \frac{9}{2})$
- Find the slope of the line whose angle of inclination is:
 - 30°
 - 120°
 - $\frac{5}{6}\pi \text{ rad}$
 - $\frac{2}{3}\pi \text{ rad}$
- Find the angle of inclination of the following pairs of points:
 - $P(3,7)$ and $Q(1,9)$
 - $E(5,8)$ and $F(-2,8)$
 - $C(\sqrt{2}, -9)$ and $D(2\sqrt{2}, -7)$
- Find the angle of inclination of the line if its slope is:
 - $-\sqrt{3}$
 - $-\frac{\sqrt{3}}{3}$
 - 1
 - $\frac{1}{\sqrt{3}}$
 - 0
 - 1

4.3.3 Different forms of Equations of a Line

The equation of a straight line can be expressed in different forms. Some of these are:

1. Equation of lines parallel to the co-ordinate axes,
2. Origin-slope form,
3. Two-intercept form,
4. The point-slope form,
5. The slope-intercept form and
6. The two-point form.

1. Equation of a lines parallel to the co-ordinate axes

Let $P(x, y)$ be any point on a line parallel to x -axis at a distance b from it. Hence the equation of this line is $y = b$. Similarly $x = a$ is the equation of the line parallel to the y -axis and at a distance a from it.

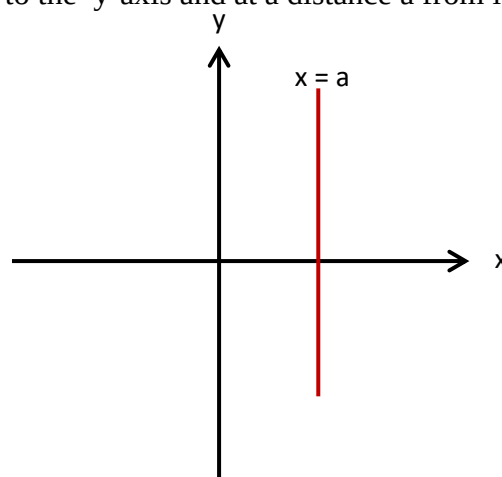
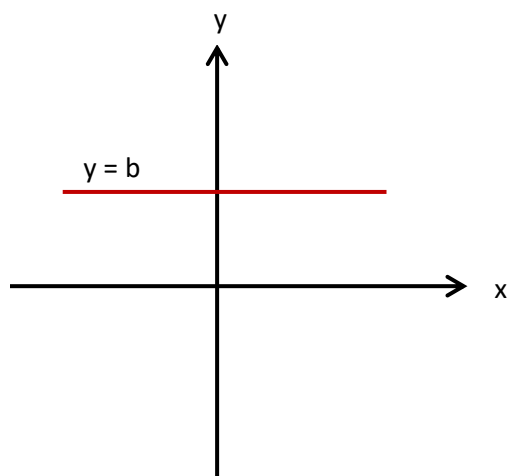


Figure 4.18

2. Origin-slope form

If the equation of a line passing through the origin and having the slop m , then the required equation of this line is, $y = mx$

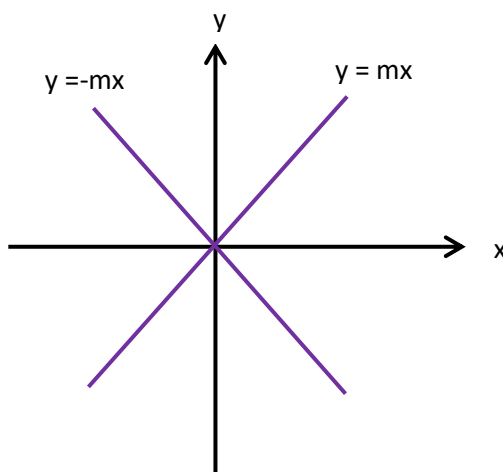


Figure 4.19

3. Two-intercept form

Let a straight line intersect the coordinate axes making intercepts of a and b on x -axis and y -axis respectively, then the required equation of the line is;

$$y = \frac{a}{b}x + b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

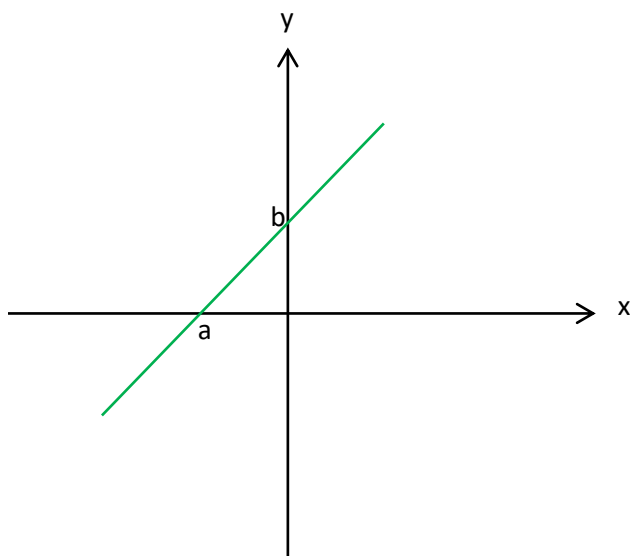


Figure 4.20

Example 6: Find equation of the line having x -intercept 3 and y -intercept -5.

Solution: $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{3} + \frac{y}{-5} = 1$ or $5x - 3y - 15 = 0$

4. The point-slope form of equation of a line

Suppose you want to find the equation of the straight line which passes through the point with coordinates (x_1, y_1) and which has slope m . Again, let the point with given coordinates be $A(x_1, y_1)$. Take any other point on the line, say B , with coordinate (x, y) as shown in figure 4.21.

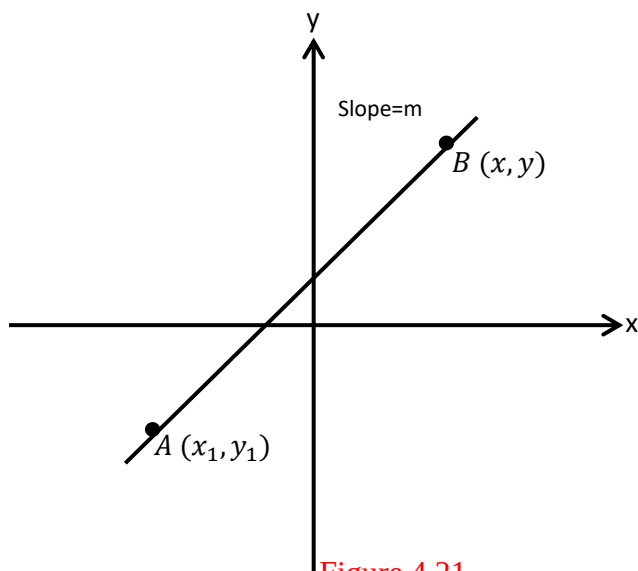


Figure 4.21

Then the slope of $\overline{AB}$ is $\frac{y-y_1}{x-x_1}$

$$\Rightarrow y - y_1 = m(x - x_1) \text{ which is the same as } y = y_1 + m(x - x_1).$$

This equation is called the **point-slope form** of equation of a line

Example 7: Find the equation of the straight line with slope $-\frac{2}{3}$ and which passes through the point

$$\left(3, \frac{1}{3}\right).$$

Solution: Given $m = -\frac{2}{3}$ and $point(x_1, y_1) = \left(3, \frac{1}{3}\right)$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - \frac{1}{3} = -\frac{2}{3}(x - 3)$$

$$\Rightarrow y = -\frac{2}{3}x + 2 + \frac{1}{3} = -\frac{2}{3}x + \frac{7}{3}$$

$$\Rightarrow y = -\frac{2}{3}x + \frac{7}{3} \text{ or } 2x + 3y - 7 = 0$$

Example 8: Find the equation of the straight line with slope 3 and which passes through the point $(2, -1)$.

Solution: Given $m = 3$ and $point(x_1, y_1) = (2, -1)$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - (-1) = 3(x - 2)$$

$$\Rightarrow y + 1 = 3x - 6$$

$$\Rightarrow y = 3x - 6 - 1 = 3x - 7$$

$$\Rightarrow y = 3x - 7$$

5. The slope-intercept form of equation of a line

Consider the equation $y = mx + b$. When $x = 0$, $y = b$. Also, when $x = 1$, $y = m + b$ as shown in figure 4.22. You can see that $P(0, b)$ is the point where the line with equation $y = mx + b$ crosses the y -axis. (b is called the y -intercept of the line).

Let Q be $(1, m + b)$. Using the coordinates of points P and Q , show that the slope of the straight line passing through P and Q is m . Writing the equation of this line through the point $(0, b)$ with slope m , using the point-slope form, gives;

$$y - b = m(x - 0) \Rightarrow y = mx + b$$

Where m is slope of the line and b is y -intercept of the line. This equation is called the **slope-intercept form** of the equation of a line.

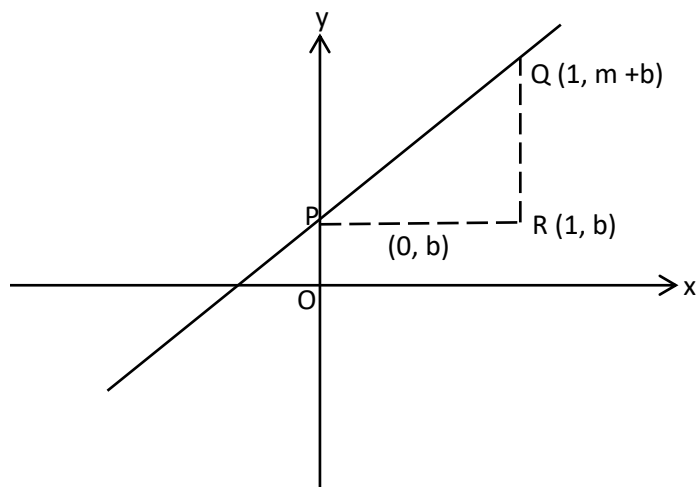


Figure 4.22

Example 9: Find the equation of the line with slope -1 and y-intercept 2.

Solution: Here, $m = -1$ and $b = 2$

Therefore, the equation of the line is $y = -x + 2$ or $x + y - 2 = 0$.

Example 10: Find the equation of the line with slope $\frac{3}{2}$ and y-intercept -3 .

Solution: Here, $m = \frac{3}{2}$ and $b = -3$

Therefore, the equation of the line is $y = \frac{3}{2}x - 3$ or $3x - 2y - 6 = 0$

6. The two-point form of equation of a line

Finally, let us look at the situation where the slope of a non-vertical line is not given but two points on the line are given.

Consider a straight line which passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. If $R(x, y)$ is any point on the line other than $P(x_1, y_1)$ or $Q(x_2, y_2)$, then the slope of $\overline{PR}$ is $m = \frac{y - y_1}{x - x_1}$, $x \neq x_1$ and the slope of $\overline{PQ}$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$ but the slope of $\overline{PR} = \text{the slope of } \overline{PQ}$.

$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ This equation is called the **two-point form of equation of a line**.

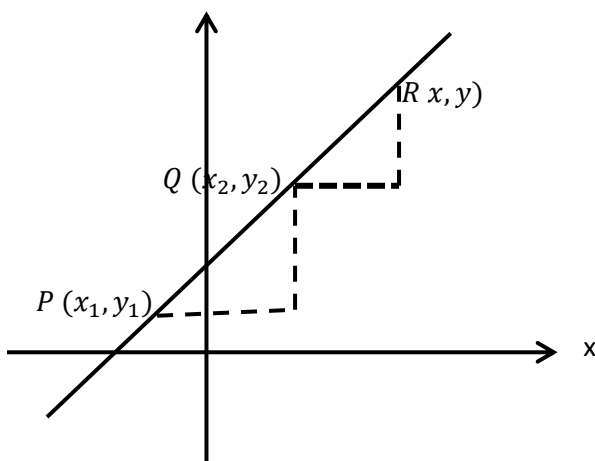


Figure 4.23

Example 11: Find the equation of the line passing through the point $P(3, -7)$ and $Q(-2, -5)$.

Solution: Taking $(3, -7)$ as (x_1, y_1) and $(-2, -5)$ as (x_2, y_2) , use the two-point form to get the equation of the line:

$$\begin{aligned}\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-(-7)}{x-3} = \frac{-5-(-7)}{-2-3} \\ &\Rightarrow \frac{y+7}{x-3} = \frac{-5+7}{-5} = \frac{2}{-5} \\ &\Rightarrow (-5)(y+7) = (2)(x-3) \\ &\Rightarrow -5y-35 = 2x-6 \\ &\Rightarrow -2x-5y-35+6 = 0 \\ &\Rightarrow -2x-5y-29 = 0\end{aligned}$$

Example 12: Find the equation of the line passing through the point $P(-5, 1)$ and $Q(1, -3)$.

Solution: Taking $(-5, 1)$ as (x_1, y_1) and $(1, -3)$ as (x_2, y_2) , use the two-point form to get the equation of the line:

$$\begin{aligned}\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-1}{x-(-5)} = \frac{-3-1}{1-(-5)} \\ &\Rightarrow \frac{y-1}{x+5} = \frac{-4}{6} \\ &\Rightarrow \frac{y-1}{x+5} = \frac{-2}{3} \\ &\Rightarrow (3)(y-1) = (-2)(x+5) \\ &\Rightarrow 3y-3 = -2x-10 \\ &\Rightarrow 2x+3y-3+10 = 0 \\ &\Rightarrow 2x+3y+7 = 0\end{aligned}$$

7. The general equation of a line

A first degree (linear) equation in x and y is an equation of the form; $Ax + By + C = 0$ where A , B and C are fixed real numbers such that $A \neq 0$ or $B \neq 0$.

All the different forms of equations of lines discussed above can be expressed in the form $Ax + By + C = 0$.

Conversely, one can show that any linear equation in x and y is the equation of a line.

Suppose a linear equation in x and y is given as $Ax + By + C = 0$.

If $B \neq 0$, then the equation: $Ax + By + C = 0$ may be solved for y as follows:

$$By = -Ax - C \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$$

This equation is of the form $y = mx + b$, and therefore represents a straight line with slope $m = -\frac{A}{B}$ and y -intercept $b = -\frac{C}{B}$.

What will be the equation $Ax + By + C = 0$, if $B = 0$ and $A \neq 0$?

Example 13: Find the slope and y -intercept of the line whose general equation is:

- $3x + 2y + 4 = 0$
- $2x + 3y - 7 = 0$

Solution: a. Solving for y the equation $3x + 2y + 4 = 0$ gives $2y = -3x - 4$
 $y = -\frac{3}{2}x - 2$ so, slope is $m = -\frac{3}{2}$ and y -intercept is $b = -2$.
 b. Solving for y the equation $2x + 3y - 7 = 0$ gives $3y = -2x + 7$
 $y = -\frac{2}{3}x + \frac{7}{3}$ so, slope is $m = -\frac{2}{3}$ and y -intercept is $b = \frac{7}{3}$.

Exercise 4.3

- Find equation of the line passing through the given points:
 - $A(-3,1)$ and $B(4,-5)$
 - $P(-4,9)$ and $Q(-1,1)$
 - $C(1,1)$ and $D(1 + \sqrt{2}, 1 - \sqrt{2})$
 - $P(-2,0)$ and the origin
 - $M(\pi, \pi)$ and $N(3\pi, -5\pi)$
 - $R(1\frac{1}{2}, -\frac{5}{2})$ and $S(-\frac{3}{2}, 1)$
 - $P(4, -1)$ and $Q(4, -4)$
- Find the equation of the line with slope m , and passing through the given point p :
 - $m = 5$ and $P(0,7)$
 - $m = -\frac{1}{2}$ and $P(10, -8)$
 - $m = \frac{1}{4}$ and $P(-2, -3)$
 - $m = 1\frac{2}{3}$ and $P(1,1)$
 - $m = \sqrt{2}$ and $P(\sqrt{2}, -\sqrt{2})$
 - $m = -\pi$ and $P(0,0)$
 - $m = -1$ and $P(\frac{1}{3}, \frac{3}{2})$
- Find the equation of the line with slope m and y-intercept b :
 - $m = \frac{3}{2}$; $b = -\frac{1}{2}$
 - $m = 2$; $b = 3$
 - $m = \frac{2}{3}$; $b = -2$
 - $m = -\sqrt{2}$; $b = -1$
 - $m = 1\frac{1}{2}$; $b = -\frac{3}{5}$
 - $m = 2\pi$; $b = 2$
- For each of the following equations, find the slope and y-intercept:
 - $\frac{3}{5}x - \frac{4}{5}y + 8 = 0$
 - $2x - 3y + 5 = 0$
 - $-y + 3 = 0$
 - $y + 2 = 2(x - 3y + 1)$
 - $x + \frac{1}{2}y - 2 = 0$
 - $3(y - 2x) = y + \frac{1}{2}(1 - 2x)$
 - $\frac{1}{2}x - \frac{2}{3}y + 1 = y + x$
- If the line passes through points $A(1, -2)$ and $B(-3,0)$, then find:
 - The point-slope form of equation of the line.
 - The slope-intercept form of equation of the line.
 - The two-point form of equation of the line.
 - The general form of equation of the line.
- Find equation of a line:
 - passing through the point $p(\sqrt{2}, -2)$ and having y-intercept $\frac{1}{2}$.
 - passing through the point $p(-1,5)$ and having x-intercept 3.
 - passing through the point $p(1\frac{1}{2}, -\frac{1}{2})$ and having angle of inclination $\frac{2}{3}\pi$ rad.
- Find the angle of inclination of the following lines:
 - $2x + 2y - 1 = 0$
 - $\sqrt{6}x - \sqrt{2}y = 0$
 - $-x + y + 2 = 0$
- A triangle has vertices at $A(-1,1)$, $B(1,3)$ and $C(3,1)$, then:
 - find equation of the lines containing the sides of the triangle.
 - is the triangle a right-angle triangle?
 - what are the intercept of the line passing through B and C?

4.4. PARALLEL AND PERPENDICULAR LINES

I. PARALLEL LINES

Theorem 4.1

If two non-vertical lines ℓ_1 and ℓ_2 are parallel to each other, then they have the same slope.

Suppose ℓ_1 and ℓ_2 be two parallel lines with slopes m_1 and m_2 and angle of inclination θ and β respectively as shown in the figure 4.24 $\ell_1 \parallel \ell_2$ are the x-axis is transversal line

$\Rightarrow \theta$ and β Corresponding angle

$\Rightarrow \tan \theta = \tan \beta$

$\therefore m_1 = m_2$

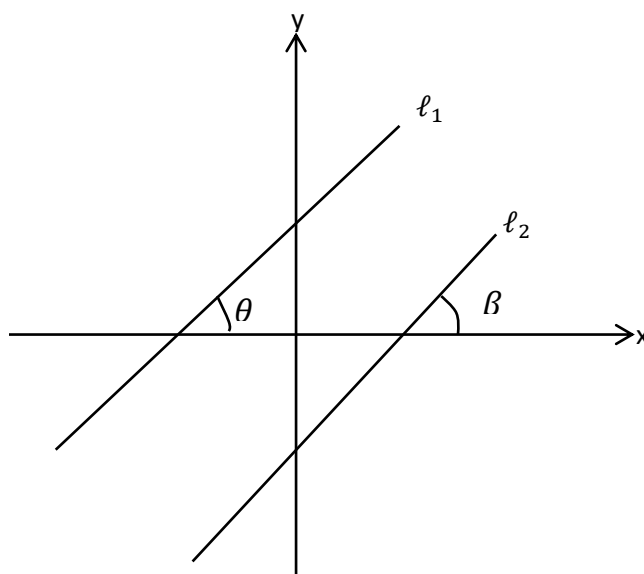


Figure 4.24

The converse is also true that is two non-vertical lines ℓ_1 and ℓ_2 are parallel if and only if they have the same slope i.e. $\ell_1 \parallel \ell_2$ if and only if $m_1 = m_2$.

Any two vertical lines are always parallel.

Example 1: Check whether the following pairs of lines are parallel:

- ℓ_1 : $A(-1,2)$ and $B(2,6)$
 ℓ_2 : $P(1,3)$ and $Q(-2,-1)$

Solution:

$$\text{Slope of } \overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{2 - (-1)} = \frac{4}{3}$$

$$\text{Slope of } \overline{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 3}{-2 - 1} = \frac{4}{3}$$

Since $\overline{AB}$ and $\overline{PQ}$ have the same slope, $\overline{AB}$ is parallel to $\overline{PQ}$ i.e. $\overline{AB} \parallel \overline{PQ}$

- b. $\ell_1: A(2, -5) \text{ and } B(2, 1)$
 $\ell_2: P(1, 3) \text{ and } Q(1, \sqrt{3})$

Solution:

ℓ_1 is vertical line and ℓ_2 also vertical line

$$\therefore \overline{AB} \parallel \overline{PQ}$$

- c. $\ell_1: A(0, -3) \text{ and } B(2, 6)$
 $\ell_2: P(2, -5) \text{ and } Q(3, 0)$

Solution:

$$\text{Slope of } \overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-3)}{2 - 0} = \frac{9}{2}$$

$$\text{Slope of } \overline{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-5)}{3 - 2} = \frac{5}{1} = 5$$

Since $\overline{AB}$ and $\overline{PQ}$ have the different slope, $\overline{AB}$ is not parallel to $\overline{PQ}$ i.e. $\overline{AB} \nparallel \overline{PQ}$

Example 2: Check whether the following pairs of lines are parallel:

- a. $\ell_1: \sqrt{2}x - y = 0$
 $\ell_2: \sqrt{3}y - \sqrt{6}x + 1 = 0$
 b. $\ell_1: y = 1 - 3x$
 $\ell_2: 3y - x + 1 = 0$

Solution:

- a. $\ell_1: \sqrt{2}x - y = 0$
 $\ell_2: \sqrt{3}y - \sqrt{6}x + 1 = 0$

From ℓ_1 and ℓ_2 express y in terms of x i.e. $\sqrt{2}x - y = 0 \Rightarrow y = \sqrt{2}x$ slope of ℓ_1 is $\sqrt{2}$

$$\sqrt{3}y - \sqrt{6}x + 1 = 0 \Rightarrow y = \sqrt{2}x - \frac{1}{\sqrt{3}} \text{ Slope of } \ell_2 \text{ is } \sqrt{2}$$

slope of $\ell_1 = \text{slope of } \ell_2$

$$\therefore \ell_1 \parallel \ell_2$$

- b. $\ell_1: y = 1 - 3x$
 $\ell_2: 3y - x + 1 = 0$

Solution:

From ℓ_1 and ℓ_2 express y in terms of x i.e. $y = 1 - 3x \Rightarrow y = -3x + 1$ slope of ℓ_1 is -3

$$3y - x + 1 = 0 \Rightarrow y = \frac{1}{3}x - \frac{1}{3} \text{ Slope of } \ell_2 \text{ is } \frac{1}{3}$$

slope of $\ell_1 \neq \text{slope of } \ell_2$

$$\therefore \ell_1 \nparallel \ell_2$$

II. PERPENDICULAR LINES

Theorem 4.2

Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 \times m_2 = -1$

Proof:

Suppose ℓ_1 is perpendicular to ℓ_2

Let m_1 and m_2 be the slopes of ℓ_1 and ℓ_2 , respectively.

Let $R(x_0, y_0)$ be the point of intersection and choose $P(x_1, y_1)$ and $Q(x_2, y_2)$ on ℓ_1 and ℓ_2 respectively draw triangles QSR and RTP as shown in figure 4.25

$\triangle QSR$ and $\triangle RTP$ are similar

$$\frac{PT}{RT} = \frac{RS}{QS}$$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{x_0 - x_2}{y_2 - y_0} = -\left(\frac{x_2 - x_0}{y_2 - y_0}\right)$$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{-1}{\frac{y_2 - y_0}{x_2 - x_0}}$$

$$m_1 = \frac{1}{m_2} \text{ or } m_1 \times m_2 = -1$$

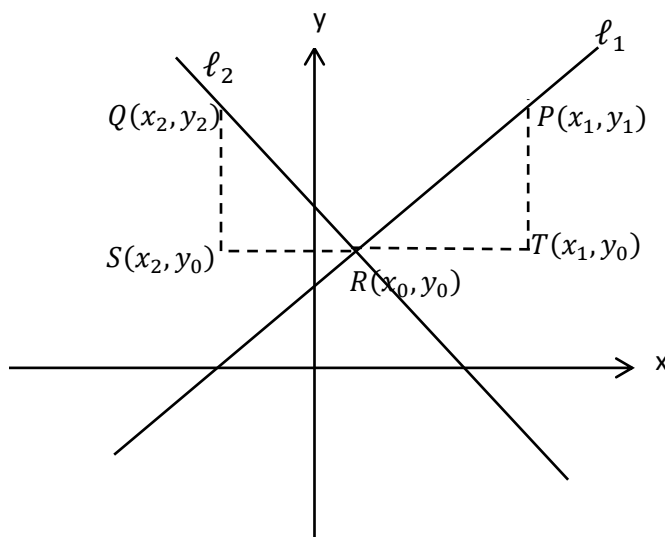


Figure 4.25

Note:

If one of the lines is a vertical line, then the other line must be a horizontal line which has slope zero.

Example 3: Check whether or not the following pair of lines passing through the given pairs of points are perpendicular:

- $\ell_1: A(1, -2)$ and $B\left(\frac{1}{3}, 0\right)$
 $\ell_2: P\left(3, \frac{2}{3}\right)$ and $Q(-2, -1)$
- $\ell_1: A(-3, 5)$ and $B(2, -5)$
 $\ell_2: P(1, 4)$ and $Q(1, 5)$

Solution:

- $\ell_1: A(1, -2)$ and $B\left(\frac{1}{3}, 0\right)$

$$\ell_2: P\left(3, \frac{2}{3}\right) \text{ and } Q(-2, -1)$$

$$\text{Slope of } \ell_1 (m_1) \text{ is } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-2)}{\frac{1}{3} - 1} = \frac{2}{-\frac{2}{3}} = -3$$

$$\text{Slope of } \ell_2 (m_2) \text{ is } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - \frac{2}{3}}{-2 - 3} = \frac{-\frac{5}{3}}{-5} = \frac{1}{3}$$

$$m_1 \times m_2 = -3 \times \frac{1}{3} = -1$$

$\therefore \ell_1$ is perpendicular to ℓ_2

- $\ell_1: A(-3, 5)$ and $B(2, -5)$

$$\ell_2: P(1, 4) \text{ and } Q(1, 5)$$

$$\text{Slope of } \ell_1 (m_1) \text{ is } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 5}{2 - (-3)} = \frac{-10}{5} = -2$$

$$\text{Slope of } \ell_2 (m_2) \text{ is } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 4}{1 - 1} = \frac{1}{0} \text{ does not exist}$$

$$m_1 \times m_2 \neq -1$$

$\therefore \ell_1$ is not perpendicular to ℓ_2

Example 4: Suppose ℓ_1 passes through $p(-2, -5)$ and $Q(3, 4)$ find the slope m_2 of any line ℓ_2 that is

- Parallel to ℓ_1
- perpendicular to ℓ_1

Solution: The slope of ℓ_1 is;

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-5)}{3 - (-2)} = \frac{9}{5}. \text{ So,}$$

- The slope of line ℓ_2 parallel to ℓ_1 is $m_1 = \frac{9}{5}$
- The slope of line ℓ_2 perpendicular to ℓ_1 is $m_2 = -\frac{1}{m_1} = -\frac{5}{9}$

Example 5: Find the equation of the line passing through $p(-1, 2)$ and perpendicular to the line $\ell: 2x - 5y - 4 = 0$.

Solution: From $2x - 5y - 4 = 0 \Rightarrow y = \frac{2}{5}x - \frac{4}{5}$ so, $m_1 = \frac{2}{5}$

Let the slope of the required line be m_2 , then, $m_1 \times m_2 = -1$ gives $m_2 = -\frac{1}{m_1} = -\frac{5}{2}$

Therefore, the required equation of the line is $y - y_1 = m(x - x_1)$

$$y - 2 = -\frac{5}{2}(x - (-1)) \text{ i.e. } y = -\frac{5}{2}x - \frac{1}{2}$$

Exercise 4.4

- Check whether the following pairs of lines are parallel
 - $\ell_1: A(2, 4)$ and $B(-3, -1)$
 $\ell_2: P(-2, -3)$ and $Q(5, 4)$
 - $\ell_1: A(1, 1)$ and $B(2, 2)$
 $\ell_2: P(1, 5)$ and $Q(-2, -1)$
 - $\ell_1: A(3, -5)$ and $B(-1, 3)$
 $\ell_2: P(2, -9)$ and $Q(0, -5)$
- Check whether the following pairs of lines are parallel
 - $\ell_1: 2x + 3y + 4 = 0$
 $\ell_2: 2x + 3y - 4 = 0$
 - $\ell_1: -3x - 2y - 4 = 0$
 $\ell_2: 3x - 2y + 4 = 0$
 - $\ell_1: 2x + 5y + 1 = 0$
 $\ell_2: 4x + 10y + 2 = 0$
- If the lines passing through points $A(-4, 5)$ and $B(3, t)$ is parallel to the line passing through $C(1, 3)$ and $D(-4, 1 - 2t)$ then find the possible value of t
- Check when the following pair of lines passing through the given pair of points are perpendicular
 - $\ell_1: A(2, 5)$ and $B(2, -3)$
 $\ell_2: P(3, 1)$ and $Q(-5, 1)$
 - $\ell_1: A(1, \sqrt{2} - 1)$ and $B(\sqrt{2}, 1)$
 $\ell_2: P(-\sqrt{2}, 6)$ and $Q(1 + \sqrt{2}, \sqrt{2})$
- Find the slope of the line that is perpendicular to the line joining $P(2, -3)$ and $Q(-3, -2)$
- Use slope to show that the quadrilateral ABCD with vertices $A(-5, -2)$, $B(-3, 1)$, $C(3, 0)$ and $D(1, -3)$ is parallelogram.
- Let ℓ be the line with equation $2x - 3y = 6$. Find the slope-intercept form of the equation of the line that passes through the point $P(2, -1)$ and is
 - Parallel to ℓ
 - perpendicular to ℓ

8. Find the equation of a line passing through the point p and parallel to the line ℓ for
 - a. $\ell: 2x - 5y - 4 = 0; P(-1, 2)$
 - b. $\ell: 3x + 6 = 0; P(4, -6)$
9. Determine which of the following pairs of lines whose equations are given are perpendicular or parallel or neither
 - a. $3x - y + 5 = 0$ and $x + 3y - 1 = 0$
 - b. $3x - 4y + 1 = 0$ and $4x - 3y + 1 = 0$
 - c. $2x + y = 1$ and $-2x - y = 1$
 - d. $3x - 2y = 0$ and $3x + 2y = 2$
 - e. $4x - 10y + 8 = 0$ and $10x + 6y - 3 = 0$
 - f. $2x + 2y = 4$ and $x + y = 10$
 - g. $x + y = 1$ and $y - 2x = 2$
 - h. $x + y = 0$ and $-x + y = 1$
10. If the line $4y - x - 3 = 0$ and $y - \frac{3}{a}x = 0$ are perpendicular lines then find the value a
11. Find the value of k for which the lines with equations $\frac{x-1}{2} = \frac{2y+4}{5}$ and $\frac{x+6}{4} = \frac{y-2}{k}$ are parallel
12. Show that the plane figures with vertices
 - a. $A(6,1), B(5,6), C(-4,3)$ and $D(-3, -2)$ is parallelogram
 - b. $A(2,4), B(1,5), C(-2,2)$ and $D(-1,1)$ is rectangles



Key Terms

analytic geometry	general equation of a line	point-slope form
angle of inclination	horizontal line	slope (gradient)
coordinate geometry	inclination of a line	slope-intercept form
coordinates	mid-point	steepness
equation of a line	non-vertical line	two-point form
collinear points		



Summary

1. If a point P has coordinates (a, b) , then the number a is called the x -coordinate or abscissa of P and b is called the y -coordinate or ordinate of P .
2. The distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
3. The point $R(x_0, y_0)$ dividing the line segment PQ , internally, in the ratio $m: n$ is given by

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right)$$
 where $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the end-points.
4. The mid-point of a line segment whose end-points are $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$
5. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points on a line with $x_1 \neq x_2$, then the slope (gradient) of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.
6. If θ is the angle between the positive x -axis and the line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, $x_1 \neq x_2$, then the slope of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$.
7. The graph of the equation $x = c$ is the vertical line through $P(c, 0)$ and has no slope.
8. The equation of the line with slope m and passing through the point $P(x_1, y_1)$ is given by $y - y_1 = m(x - x_1)$.
9. The equation of the line with slope m and y -intercept b is given by $y = mx + b$.
10. The equation of the line passing through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, $x_1 \neq x_2$.
11. The graph of every first degree (linear) equation $Ax + By + C = 0$, $A, B \neq 0$ is a straight line and every straight line is a graph of a first degree equation.
12. Two non-vertical lines are parallel, if and only if they have the same slope.
13. Let ℓ_1 be a line with slope m_1 and ℓ_2 be a line with slope m_2 . Then ℓ_1 and ℓ_2 are perpendicular lines if and only if $m_1 \times m_2 = -1$.
14. Three or more points are said to be **collinear** if they lie on the same line.

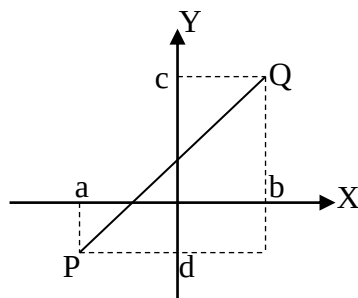
REVIEW EXERCISES ON UNIT 4

1. Show that the points $(6,6)$, $(2,3)$ and $(4,7)$ are the vertices of a right-angle triangle.
2. Prove that $(-2, -1)$, $(1,0)$, $(4,3)$ and $(1,2)$ be the vertices of a parallelogram.
3. Determine the coordinates of the vertices of the triangle PQR if the mid-points of its sides PQ , QR and RP have coordinates $(2,5)$, $(-4,3)$ and $(4, -1)$ respectively.
4. Find the coordinates of a point C dividing a line in the ratio $7:3$ connecting the points $P(-2,9)$ and $Q(8, -1)$.
5. Find the area of a triangle ABC whose vertices are $A(3,8)$, $B(-4,2)$ and $C(5, -1)$.
6. The points $A(2,9)$, $B(x, 5)$ and $C(5,5)$ are the vertices of a triangle ABC right-angled at B . Find the value of x and hence the area of $\triangle ABC$.
7. $ABCD$ is a rhombus. The co-ordinate of the points A and C are $(3,6)$ and $(-1,2)$ respectively. Write down the equation of the diagonal BD .
8. Find the equation of the line passing through the points $(1,2)$ and making an angle 30° with y -axis.
9. If the points A, B, C and D have coordinates $(1,2)$, $(7,5)$, $(9,8)$ and $(3,5)$ respectively, then:
 - i. find the gradients of the lines AB , BC , CD and DA .
 - ii. what do these gradients tell you about the quadrilateral $ABCD$.
 - iii. find the area of the quadrilateral.
10. Find the equation of the lines:
 - a. Parallel to $y = 3x - 2$ and passing through $(0,0)$.
 - b. Parallel to $y = 3x$ and passing through $(2,5)$.
 - c. Parallel to $2x + y - 3 = 0$ and passing through $(-2,5)$.
11. Find the equations of the lines :
 - a. Perpendicular to $y = 3x$ and passing through $(0,0)$.
 - b. Perpendicular to $y = 2x + 3$ and passing through $(4,3)$.
 - c. Perpendicular to $2x + y = 4$ and passing through $(4, -3)$.
12. Find an equation of the line through the intersection of the lines $x - y - 4 = 0$ and $7x + y + 20 = 0$:
 - a. Parallel
 - b. perpendicular
13. Find the area of the region bounded by the triangle whose sides are $7x - y - 10 = 0$, $10x + y - 41 = 0$ and $3x + 2y + 3 = 0$.
14. Find the equation of the line through $(-4, -6)$ and perpendicular to a line having slope $-\frac{3}{2}$.
15. Find the equation of the line:
 - a. Through $A(-6,5)$ having slope 7 .
 - b. Through $A(-8,5)$ having slope undefined.
 - c. y -intercept -7 and slope -5 .
16. If the lines with equations $2 - ky = y - 3x$ and $\frac{kx-1}{3} = \frac{2k-3y}{2}$ are perpendicular, then find the values of k .
17. If the line passing through points $(-3,4)$ and $(2, t + 3)$ is parallel to the line passing through points $(-2, t)$ and $(1,1)$, then find the values of t .

PRACTICE QUESTIONS ON UNIT 4

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

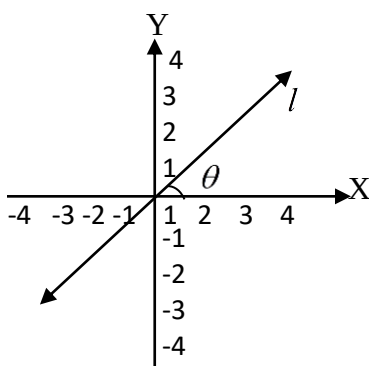
- If line l_1 passes through the points $(5, x)$ and $(-1, 3)$ and line l_2 contains the points $(x, 6)$ and $(2, 0)$, then the value of x for which the two lines are perpendicular is:
A. $\frac{2}{5}$ B. $\frac{5}{2}$ C. 5 D. $\frac{1}{2}$
- If a line passes through $(2, 8)$ and $(-5, 15)$, then what is the degree measure of the angle of inclination that this line makes with positive x -axis?
A. 30° B. 45° C. 135° D. 225°
- If the line passing through points $(2, 8)$ and $(-7, t + 4)$ is parallel to the line passing through points $(1, t)$ and $(4, -2)$, then what is the value of t ?
A. $-\frac{1}{2}$ B. $\frac{5}{2}$ C. -5 D. 1
- Which one of the following is the equation of a line that is perpendicular to the line with equation $2x + 3y + 4 = 0$?
A. $3y - 2x + 4 = 0$ C. $3x - 2y + 4 = 0$
B. $-3x - 2y - 4 = 0$ D. $2x + 3y - 4 = 0$
- P and Q are points given on the coordinate plane as shown in the figure below.



Thus, the distance between point P and Q is equal to:

- $\sqrt{(a+c)^2 + (b+d)^2}$ C. $\sqrt{(c-a)^2 + (b-d)^2}$
 - $\sqrt{(b-a)^2 + (c-d)^2}$ D. $\sqrt{(a-c)^2 + (b-d)^2}$
- What are the co-ordinates of a point that divides the line segment joining points A $(2, 3)$ and B $(5, -7)$ in the ratio 3:4?
A. $\left(\frac{23}{7}, \frac{9}{7}\right)$ B. $\left(\frac{2}{7}, -\frac{9}{7}\right)$ C. $\left(-\frac{23}{7}, -\frac{2}{7}\right)$ D. $\left(\frac{23}{7}, -\frac{9}{7}\right)$
 - If a line with x-intercept 4 and y-intercept -6 is given, then its slope is equal to _____.
A. $-\frac{2}{3}$ B. $-\frac{3}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{2}$
 - The distance between P $(2, 3)$ and Q $(1, -1)$ is:
A. 17 units B. 16 units C. $\sqrt{17}$ units D. 9 units

9. In the figure below the line l makes an angle θ with x -axis. What is $\cos \theta$?



- A. $\frac{1}{3}$
 B. $\frac{\sqrt{5}}{5}$
 C. $\frac{1}{2}$
 D. $\frac{2\sqrt{5}}{5}$

10. Which one of the following pairs of equations represents perpendicular lines?
 A. $x + y = 0$ and $-x + y = 1$
 B. $2x + y = 1$ and $-2x - y = 1$
 C. $x + y = 1$ and $y - 2x = 2$
 D. $3x - 2y = 0$ and $3x + 2y = 2$
11. Which one of the following lines is parallel to the line $5x - 2y = 0$?
 A. $y = -\frac{5}{2}$
 B. $2x + 5y = -4$
 C. $-5x - 2y = 1$
 D. $-5x + 2y = 6$
12. If P (-1, 2) and Q (5, -3) are points on the coordinates plane, so that, Q is the midpoint of $\overline{PR}$, the what are the coordinates of p?
 A. (11, -8)
 B. $(\frac{7}{2}, -2)$
 C. $(-2, 5\frac{1}{2})$
 D. (2, -6)
13. Which one of the following is **true** about a second quadrant angle θ in standard position whose terminal side lies on the line $2x + y = 0$?
 A. $\sin \theta = \frac{1}{\sqrt{5}}$
 B. $\cos \theta = \frac{2}{\sqrt{5}}$
 C. $\sin \theta = \frac{2}{\sqrt{5}}$
 D. $\cos \theta = \frac{1}{\sqrt{5}}$
14. The equation of the line that passes through the point (2, -1) and perpendicular to the line $2x + 4y + 3 = 0$ is given by:
 A. $x + 2y = 0$
 B. $-2x + y - 4 = 0$
 C. $x - 2y + 2 = 0$
 D. $-2x + y + 5 = 0$
15. Which one of the following is the equation of a line whose x-intercept and y-intercept are 1 and -2 respectively?
 A. $-2x + y - 1 = 0$
 B. $x - y + 1 = 0$
 C. $2x - y - 2 = 0$
 D. $x - 2y - 1 = 0$
16. What is the equation of the line passing through mid-point of the line segment with end points (-1, 3) and (3, 1), and perpendicular to the line whose angle of inclination is double the angle of inclination of the line $2x + y = 7$?
 A. $3y + 4x = 10$
 B. $4y - 5x = 3$
 C. $2y - x = 3$
 D. $4y + 3x = 11$
17. What is the coordinate of the point R on the line segment with end points P(-3, 0) and Q(0, -3) such that $\frac{PR}{RQ} = \frac{2}{3}$?
 A. (-1, -1)
 B. $(-1, \frac{7}{5})$
 C. $(-\frac{9}{5}, -\frac{6}{5})$
 D. $(-\frac{7}{5}, 0)$

18. Let (a, b) and (b, a) be points such that $a \neq b$. Which of the following is the equation of the line through (a, b) and perpendicular to the line containing the given points?
 - A. $y = x + b - a$
 - B. $y = x - a - b$
 - C. $y = -x + b + a$
 - D. $y = -x - a + b$
19. The equation straight line passing through P $(-2, 1)$ and perpendicular to the line with equation $6x + 5y = 10$ is:
 - A. $6y + 5x + 16 = 0$
 - B. $6y - 5x + 16 = 0$
 - C. $y = 5x - 10$
 - D. $6y - 5x - 16 = 0$
20. If P $(2, -1)$ and Q $(-3, 5)$ are points on the coordinates plane, so that, P is the midpoint of $\overline{QR}$, then what are the coordinates of R?
 - A. $(7, 7)$
 - B. $(7, -7)$
 - C. $(6, 7)$
 - D. $(10, 11)$
21. Line l passes through $(0, 5)$ and $(-5, 0)$. What is the angle between the y -axis and l in radian measure?
 - A. $\frac{\pi}{4}$
 - B. $\frac{\pi}{3}$
 - C. $\frac{\pi}{2}$
 - D. $\frac{3}{4}\pi$
22. Suppose l_1 and l_2 are perpendicular lines intersecting at $(2, 1)$. If the angle of inclination of l_2 is 45° , what is the equation of l_1 ?
 - A. $y = -x + 2$
 - B. $y = -x + 3$
 - C. $y = x - 1$
 - D. $y = -2x + 5$
23. If a line with angle of inclination of $\frac{3}{4}\pi$ passes through $(0, 1)$, which one of the following is the equation of the line?
 - A. $y = -x + 1$
 - B. $y = x + 1$
 - C. $y = -x - 1$
 - D. $y = x - 1$
24. What is the distance from the origin to the line that passes through $(1, 0)$ and $(0, 1)$?
 - A. $\frac{1}{2}\sqrt{2}$
 - B. 1
 - C. $\frac{1}{2}$
 - D. $\sqrt{2}$
25. The equation of the line that passes through $(2, -1)$ and perpendicular to $3x + 4y = 6$ is:
 - A. $-4x + 3y = 5$
 - B. $4x - 3y = 5$
 - C. $4x + 3y = 11$
 - D. $-4x + 3y = -11$
26. If the graphs of the equations $ax + by + c = 0$ and $Ax + By + C = 0$ are perpendicular, then which of the following is necessarily **true**?
 - A. $aA + bB = 0$
 - B. $aB - bA = 0$
 - C. $aA - bB = 0$
 - D. $aB + bA = 0$