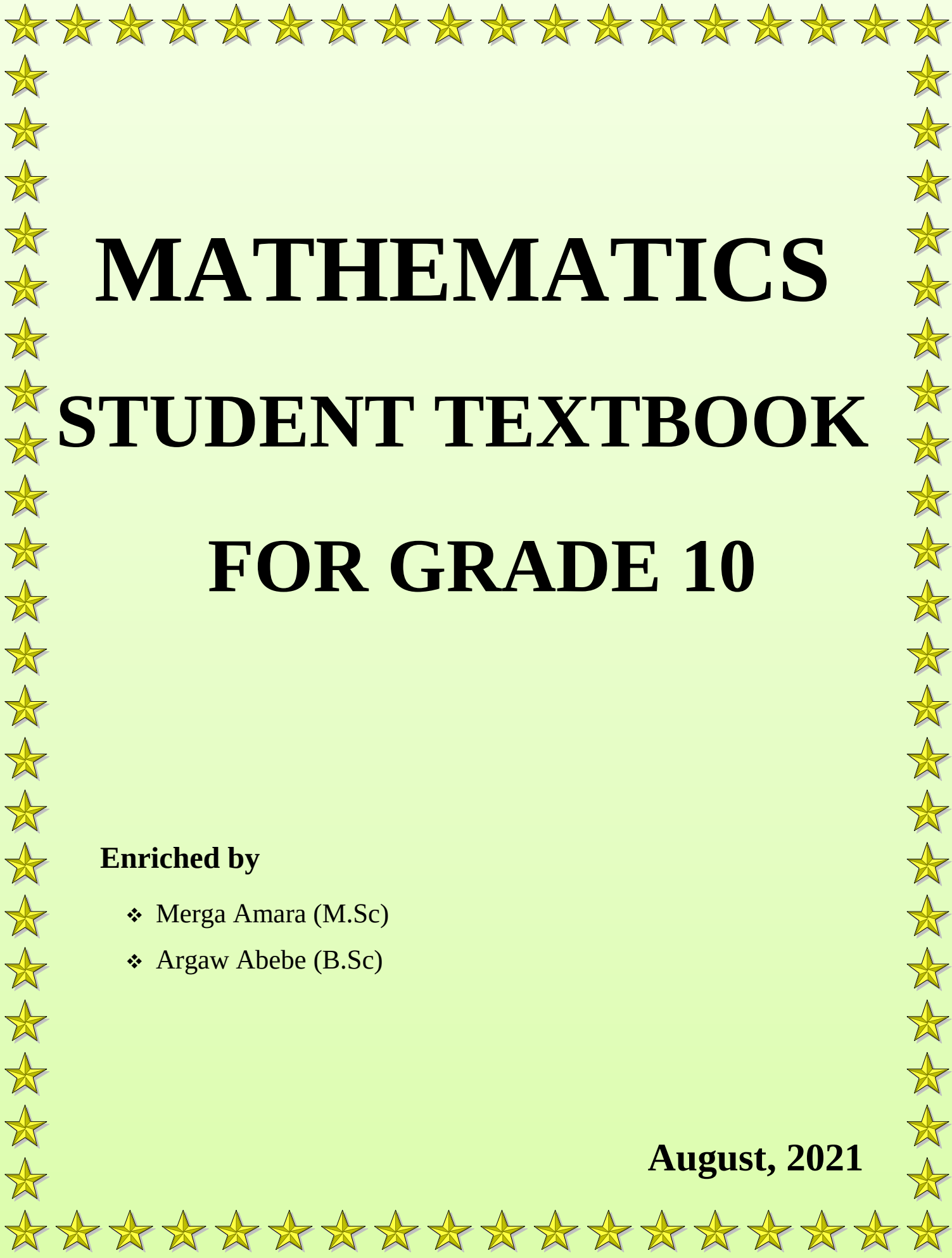


A decorative border of yellow stars with black outlines and a slight 3D effect, arranged in a rectangular frame around the central text.

MATHEMATICS

STUDENT TEXTBOOK

FOR GRADE 10

Enriched by

- ❖ Merga Amara (M.Sc)
- ❖ Argaw Abebe (B.Sc)

August, 2021



POLYNOMIAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *define polynomial functions.*
- *perform the four fundamental operations on polynomials.*
- *apply theorems on polynomials to solve related problems.*
- *determine the number of rational and irrational zeros of a polynomial.*
- *sketch and analyses the graphs of polynomial functions.*

Main Contents:

1.1. Introduction to polynomial functions

1.2. Theorems on polynomial

1.3. Zeros of polynomial functions

1.4. Graphs of polynomial functions

Key terms

Summary

Review Exercises

1.1 Introduction to Polynomial Functions

- In your previous grade mathematics course, you have learnt the concept of function and special types of function such as constant function, linear function and quadratic function which are special cases of the general class of polynomial function.
- Recall that a **function** f is a relation in which no two ordered pairs have the same first element which means that for any given x in the domain of f , there is a unique pair (x, y) belonging to the function f .
- A function f is a **linear function**, if it can be written in the form

$$f(x) = ax + b, a \neq 0 \text{ and } a, b \in \mathbb{R}$$

- Domain of $f = \mathbb{R}$ and Range of $f = \mathbb{R}$

For example; $f(x) = 5 - 3x$, $g(x) = \frac{1}{2} - \frac{2}{3}x$, $y = 8x$ are linear functions.

- The graphs of linear functions are **straight lines**.

- A function f is said to be **constant function**, if it can be written as $f(x) = b$, for any $b \in \mathbb{R}$.

- Domain of $f = \mathbb{R}$ and Range of $f = \{b\}$

For example; $f(x) = -3$, $y = \frac{5}{7}$, $h(x) = \pi - 2\sqrt{2}$, $g(x) = 0$ are constant functions.

- The graphs of constant functions are **horizontal straight lines**.

- ✓ A function f is said to be a **quadratic function**, if can be written in the form

$$f(x) = ax^2 + bx + c; \text{ where } a, b, \text{ and } c \in \mathbb{R}, a \neq 0,$$

- Domain of $f = \mathbb{R}$; Range of $f = \begin{cases} \left[\frac{4ac - b^2}{4a}, \infty \right), & \text{if } a > 0 \\ \left(-\infty, \frac{4ac - b^2}{4a} \right], & \text{if } a < 0 \end{cases}$

- The graph of quadratic function $f(x) = ax^2 + bx + c$ is a **parabola which opens upward** (if $a > 0$), or **opens downward** (if $a < 0$).

Therefore if $a > 0$, the graph of f has a minimum point and if $a < 0$, the graph of f has a maximum point. Both minima and maxima are the vertices of the parabolas.

EXERCISE 1.1

Classify each of the following functions as constant, linear, quadratic or none.

a. $f(x) = 1 - x^2$

b. $h(x) = \sqrt{2x - 1}$

c. $g(x) = 3 + 2^x$

d. $f(x) = 5 - \frac{4}{5}x$

e. $f(x) = 2\sqrt{3}$

f. $f(x) = \left(\frac{2}{3}\right)^{-1}$

g. $h(x) = 1 - |x|$

h. $f(x) = 12x^{-1}$

i. $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$

j. $k(x) = \frac{3}{4}(12 + 8x)$

k. $L(x) = \frac{(x+1)(x-2)}{x-2}$

l. $f(x) = x^4 - x + 1$

1.1.1. Definition of a Polynomial Function

Constant, linear, and quadratic functions are all special cases of a wide class of functions called **Polynomial Functions**.

Definition 1.1

Let n be a non – negative integer and let $a_n, a_{n-1}, \dots, a_1, a_0$ be real numbers with $a_n \neq 0$.

The function P defined by $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is called a **polynomial function** in variable x .

The expression $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$ is called **polynomial** in x .

Note: In the definition of a polynomial functions

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 ;$$

- i) $a_n, a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are called the **coefficients** of $P(x)$.
- ii) The number a_n , where $a_n \neq 0$, is called the **leading coefficient** and the term $a_n x^n$ is called the **leading term** of $P(x)$.
- iii) The number a_0 is called the **constant term** of $P(x)$.
- iv) If $a_n \neq 0$, then the number n (the highest exponent of power of x) is called the **degree** of $P(x)$.

Note: The domain of any polynomial function is the set of real number.

Example 1:

- a) $p(x) = 5x^4 + x^3 - \frac{3}{7}x^2 + \pi x - \sqrt{2}$ is a polynomial function with degree 4 and constant term $-\sqrt{2}$.
 - ✓ The leading term of P is $5x^4$ and the leading coefficient is 5.
 - ✓ The coefficients of x^3, x^2 and x are, respectively, 1, $-\frac{3}{7}$ and π .
 - ✓ The domain of P is $\mathbb{R}$.
- b) $f(x) = \frac{5}{2}x - 2x^3$ is a polynomial function with degree 3 and constant term 0.
 - ✓ The leading term of f is $-2x^3$ and the leading coefficient is -2 .
 - ✓ The coefficient of x^2 is 0 and the coefficient of x is $\frac{5}{2}$.
 - ✓ The domain of f is $\mathbb{R}$.
- c) $f(x) = 2x^3 + x^{-2} - 1$ is **NOT** polynomial function, since x has negative exponent, x^{-2} .
- d) $P(x) = 5$ is polynomial function known as **constant polynomial function**, since P can be written in the form cx^m uniquely as $P(x) = 5 = 5x^0$.
 - ✓ Hence, the degree of P is 0, its leading coefficient is 5 and the constant term is also 5.
 - ✓ The domain of P is $\mathbb{R}$.
- e) $f(x) = 0$ is polynomial function known as **zero polynomial**.
 - ✓ But there is no unique way of writing f in the form of cx^m , since

$$f(x) = 0 = 0x^0 = 0x^1 = 0x^2 = 0x^{10} = \dots\dots\dots$$
- f) $f(x) = \sqrt{x^4 + 1}$ has domain $\mathbb{R}$, but $\sqrt{x^4 + 1}$ cannot be expressed in the standard form of polynomial $a_n x^n + \dots + a_1 x + a_0$.
Hence, $f(x) = \sqrt{x^4 + 1}$ is **NOT** polynomial.

Remark: From the above examples, we conclude that for a function to be polynomial, if it satisfies the following two conditions:

Condition 1: Its domain is the set of real numbers.

Condition 2: It can be expressed in the standard form $a_n x^n + \dots + a_1 x + a_0$; where n is a non – negative integer and $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$ with $a_n \neq 0$.

Example 2: Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term.

a) $f(x) = \frac{2}{3}x^4 - 12x^2 + x + \frac{7}{8}$

✓ The domain of f is $\mathbb{R}$.

✓ It is a polynomial function with degree 4, leading coefficient $\frac{2}{3}$, constant term $\frac{7}{8}$.

b) $g(x) = \frac{x+1}{x+1}$

✓ Although, $g(x)$ can be expressed as $g(x) = \frac{x+1}{x+1} = 1$ which is in the standard form of polynomial, its domain $= \mathbb{R} \setminus \{-1\} \neq \mathbb{R}$.

✓ Therefore, g is **not** polynomial function.

c) $y = \frac{(x^2-3)(x-2)}{x-2}$

✓ y can be expressed as the standard form of polynomial as

$$\frac{(x^2-3)(x-2)}{x-2} = x^2 - 3, \text{ but its domain } = \mathbb{R} \setminus \{2\} \neq \mathbb{R}.$$

✓ Therefore, y is **not** polynomial function.

d) $P(x) = \frac{5x^3(x^2+1)}{x^2+1}$

✓ Here, the domain of $P = \mathbb{R}$, and $P(x)$ can be expressed as the standard form of polynomial as $\frac{5x^3(x^2+1)}{x^2+1} = 5x^3$.

✓ Its degree is 3, its leading coefficient is 5 and its constant term is 0.

✓ Therefore, P is polynomial function.

e) $g(x) = \sqrt{(x+1)^2}$

✓ Clearly, the domain of $g = \mathbb{R}$.

✓ g can be reduced to the form $g(x) = \sqrt{(x+1)^2} = |x+1|$. Which has no the standard form of polynomial.

✓ Thus, g is **not** polynomial function.

f) $k(x) = \sqrt{(x^2+3)^2}$

✓ Clearly, the domain of $k = \mathbb{R}$.

✓ k can be reduced to the form $k(x) = \sqrt{(x^2+3)^2} = |x^2+3| = x^2+3$. which is in the standard form of polynomial.

✓ Hence, k is polynomial function with degree = 2, leading coefficient = 1, and constant term = 3

g) $f(x) = 2x^{-4} + x^2 + 8x + 1$

✓ It is not polynomial function because one of its terms has negative power of x .

h) $g(x) = \frac{8}{5}x^{15}$

✓ It is a polynomial function with degree = 15, leading coefficient = $\frac{8}{5}$ and constant term = 0.

i) $h(x) = (2 - \sqrt{2}x)(1 + \sqrt{2}x^2) + 2x^3 - 5$

✓ Clearly, the domain of $h = \mathbb{R}$.

✓ $h(x)$ can be expressed and simplified as:

$$\begin{aligned} h(x) &= (2 - \sqrt{2}x)(1 + \sqrt{2}x^2) + 2x^3 - 5 \\ &= 2 + 2\sqrt{2}x^2 - \sqrt{2}x - 2x^3 + 2x^3 - 5 \\ &= 2\sqrt{2}x^2 - \sqrt{2}x - 3 \text{ which is in the standard form of polynomial.} \end{aligned}$$

✓ Hence, $h(x)$ is a polynomial function of degree = 2, leading coefficient = $2\sqrt{2}$ and constant term = -3

j) $g(x) = \frac{\pi^2+1}{\pi^2-1}$

✓ $g(x) = \frac{\pi^2+1}{\pi^2-1}$ it is constant polynomial function with degree = 0 ,

leading coefficient = $\frac{\pi^2+1}{\pi^2-1}$, and constant term = $\frac{\pi^2+1}{\pi^2-1}$

k) $f(x) = \sqrt{(x^2 - 1)^2}$

- $f(x) = \sqrt{(x^2 - 1)^2} = |x^2 - 1|$ which is **not** polynomial function, because $|x^2 - 1|$ has no the standard form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$

Note:

- Linear functions of the form $f(x) = ax + b$, $a \neq 0$, are polynomial functions of degree 1, leading coefficient a and constant term b .
- Non- zero constant function of the form $f(x) = b$, $b \neq 0$, are polynomial functions of degree 0, leading coefficient b and constant term b .
- Quadratic functions $f(x) = ax^2 + bx + c$, $a \neq 0$ are polynomials functions of degree 2, leading coefficient a and constant term c .

Example 3: For the polynomial expression $f(x) = \frac{x^2 + 3 - 6x^4}{4} + \frac{7}{8}x - x^3$

- What is the degree?
- What is the leading coefficient?
- What is the coefficient of x^3 ?
- What is the constant term?

Solution: Expanding $f(x)$, and collect like terms gives

$$f(x) = \frac{x^2 + 3 - 6x^4}{4} + \frac{7}{8}x - x^3 = -\frac{3}{2}x^4 - x^3 + \frac{1}{4}x^2 + \frac{7}{8}x + \frac{3}{4}. \text{ Hence,}$$

- degree of $f = 4$
- coefficient of $x^3 = -1$ and
- leading coefficient of $f = -\frac{3}{2}$
- the constant term is $= \frac{3}{4}$

Definition 1.2: A polynomial function $P(x)$ is said to be:

- a polynomial function over integers, if the coefficients of $P(x)$ are all integers.
- a polynomial function over rational number, if the coefficients of $P(x)$ are all rational numbers.
- a polynomial function over real number if the coefficients of $P(x)$ are all real numbers.

Note:

1. Since $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, every polynomial function over integer is a polynomial function over rational numbers; and every polynomial function over rational number is a polynomial function over real numbers.
2. Every polynomial function is a polynomial function over real numbers.

Example 4: $g(x) = \frac{2}{3}x^4 - 13x^2 + \frac{7}{8}$ is a polynomial function over rational number but **not** integers.

Example 5: Which of the following polynomial functions are polynomial functions over
 i. the integers? ii. the rational numbers? iii. the real numbers?

- a) $f(x) = 3x^5 - x^6 + 2x + 1$ d) $f(x) = \frac{2}{3}x^4 - \frac{3}{5}x^2 + 2x - 4$
 b) $f(x) = \sqrt[3]{4}x^2 - 3\sqrt{21}x^5 + 2x^2 - 8$ e) $f(x) = \frac{2x^3-1}{6} - \frac{x^3+3x}{3} + \frac{15}{2}$
 c) $f(x) = (\sqrt{3}x - 1)(\sqrt{3}x + 1) + 2$

Solution:

- a) f is polynomial function over integer, since all coefficients of f are integers.
 b) f is polynomial function over real numbers, since $\sqrt[3]{4}$ and $\sqrt{21}$ are **not** rational numbers.
 c) $f(x) = (\sqrt{3}x - 1)(\sqrt{3}x + 1) + 2 = 3x^2 + \sqrt{3}x - \sqrt{3}x - 1 + 2 = 3x^2 + 1$
 Hence, f is polynomial function over integer, since all coefficients of f are integers.
 d) f is polynomial function over rational numbers, since all coefficients of f are rational numbers.
 e) $f(x) = \frac{2x^3-1}{6} - \frac{x^3+3x}{3} + \frac{7}{2} = f(x) = \frac{x^3}{3} - \frac{5}{6} - \frac{x^3}{3} - x + \frac{7}{2} = -x + 8$
 Hence, f is polynomial function over integer.

Example 6: A popcorn producer is designing a new box for the popcorn. The marketing department has designed a box with the width 2cm less than the length and with the height 5cm greater than the length. The volume of each box must be 24cm^3 . What is the length of the box?



Solution:

- Write an equation to model of the volume of the box.
 Let x represents the length of the box in cm . Then the width is $x - 2$ and the height is $x + 5$.
 So the volume $V = lwh \Rightarrow V = x(x - 2)(x + 5) = 24$

$$\Rightarrow x^3 + 3x^2 - 10x = 24$$

$$\Rightarrow x^3 + 3x^2 - 10x - 24 = 0$$

$$\Rightarrow (x - 3)(x^2 + 6x + 8) = 0$$

$$\Rightarrow x = 3 \text{ or } x = -4 \text{ or } x = -2$$

But the length must be positive, so length of the box is 3cm

EXERCISE 1.2

1. Which of the following are polynomial functions and which are **not**? If **not** give the reason.

a) $f(x) = x^9 - 3\sqrt{2}x^5 + x^2 + 7$

h) $f(y) = \frac{5\sqrt{5}}{3 - \sqrt{3}}$

b) $f(t) = 2t^{-3} - t^{-2} + t^{-1} - t$

i) $f(x) = 312x^6$

c) $y = \frac{3}{2x^{-1}} - \frac{1}{x^{-2}} - x - 5$

j) $y = \sqrt{9x^4 - 6x^2 + 1}$

d) $f(t) = \frac{3}{t} + \frac{2}{t^2}$

k) $f(x) = \sqrt{x^2} + x + 3$

e) $h(y) = \frac{4y^2 - 5y^3 + 6}{8}$

l) $f(x) = \frac{3}{6 + x}$

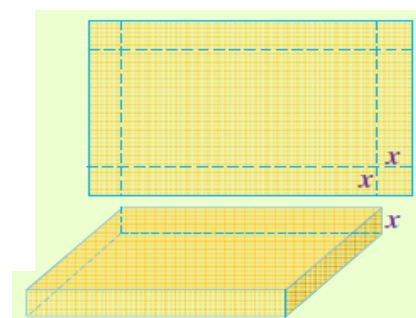
f) $k(a) = a^{\frac{1}{2}} + 3a + a^2$

m) $f(t) = \frac{4}{7} - 2\pi$

g) $g(x) = \left(x - \frac{2}{3}\right)\left(x + \frac{3}{4}\right)$

2. An open box is to be made from a 20cm. long square piece of material by cutting equal squares of length x cm. from the corners and turning up the sides, as shown in the figure.

- a. Verify that the volume of the box is given by the function $v(x) = 4x^3 - 80x^2 + 400x$
 b. Determine the domain of v .



OPERATIONS ON POLYNOMIAL FUNCTIONS

- Recall that the fundamental operations of algebra are addition, subtraction, multiplication and division.
- The first step in performing operations on polynomial functions is to use the commutative, associative and distributive laws in order to combine like terms.

A. Addition and Subtraction of Polynomial Functions

- Addition and subtraction of polynomials is based on addition and subtraction of like powers.
- Recall that** two powers having the same variables are said to be like if the powers have the same exponents possibly different coefficients.
- The sum and difference of like powers can be done, respectively, by adding and subtracting the coefficients of the powers.

For example; $7x^3$ and $-2x^3$ are like powers and their sum and differences are

i) $(7x^3) + (-2x^3) = [7 + (-2)]x^3 = 5x^3$, and

ii) $(7x^3) - (-2x^3) = [7 - (-2)]x^3 = 9x^3$

Note:

- 1) The sum of two polynomial functions f and g is written as $f + g$, and is defined as:

$$(f + g)(x) = f(x) + g(x), \text{ for all } x \in \mathbb{R}.$$
- 2) The difference of two polynomial functions f and g is written as $f - g$, and is defined as:

$$(f - g)(x) = f(x) - g(x), \text{ for all } x \in \mathbb{R}.$$

Remark: The sum and difference of two polynomials are obtained by adding and subtracting like terms or powers of the polynomials.

ACTIVITY

- a. What do you think about the degree of the sum and difference of polynomials?
- b. Is it possible to pre – determine these degrees from the degrees of the two polynomials?

Example 7:

- a) Let $f(x) = x^3 + 5x^2 - 3x - 1$ and $g(x) = 2x^3 - x$. Here, degree of f = degree of g . Then
 - i) $(f + g)(x) = (x^3 + 5x^2 - 3x - 1) + (2x^3 - x)$

$$= (x^3 + 2x^3) + (5x^2 + 0x^2) + (-3x + (-x)) + (-1 + 0)$$

$$= (1 + 2)x^3 + 5x^2 + (-3 - 1)x - 1$$

$$= 3x^3 + 5x^2 - 4x - 1; \text{ and}$$
 - ii) $(f - g)(x) = (x^3 + 5x^2 - 3x - 1) - (2x^3 - x)$

$$= (x^3 - 2x^3) + 5x^2 + (-3x - (-x)) - 1$$

$$= -x^3 + 5x^2 - 2x - 1$$

Thus, degree of $(f + g) = 3 = \text{degree of } f = \text{degree of } g$, and
 degree of $(f - g) = 3 = \text{degree of } f = \text{degree of } g$.

- b) Let $f(x) = 4x^3 - x + 2$ and $g(x) = 5 - 4x^3$. Then

$$(f + g)(x) = (4x^3 - x + 2) + (5 - 4x^3)$$

$$= (4x^3 + (-4x^3)) - x + (2 + 5)$$

$$= 7 - x, \text{ and}$$

$$(f - g)(x) = (4x^3 - x + 2) - (5 - 4x^3) = 8x^3 - x - 3$$

Thus, degree of $(f + g) = 1 < \text{degree of } f (= \text{degree of } g)$ and
 degree of $(f - g) = 3 = \text{degree of } f = \text{degree of } g$.

- c) Let $f(x) = x^4 + 2x^3 - 3x$ and $g(x) = x^2 - 5x + 3$. Here, degree of $f \neq \text{degree of } g$. Then

$$(f + g)(x) = (x^4 + 2x^3 - 3x) + (x^2 - 5x + 3) = x^4 + 2x^3 - 8x + 3; \text{ and}$$

$$(f - g)(x) = (x^4 + 2x^3 - 3x) - (x^2 - 5x + 3) = x^4 + 2x^3 + 2x - 3$$

Therefore, degree of $(f + g) = \text{degree of } (f - g) = 4 = \max\{\text{degree of } f, \text{degree of } g\}$

- d) Let $f(x) = 1 - x^4$ and $g(x) = x^4 - 1$. Here, degree of $f = \text{degree of } g = 4$. Then

$$(f + g)(x) = (1 - x^4) + (x^4 - 1) = 0$$

Therefore, $(f + g)$ is a zero polynomial, which has **no** degree.

- Thus, from above examples, we can conclude that:
 - If $f(x)$ and $g(x)$ have same degree, then the degree of $(f + g)(x)$ and the degree of $(f - g)(x)$ might be equal to or lower than the degree of $f(x)$ or the degree of $g(x)$, or $(f + g)(x)$ and $(f - g)(x)$ might be zero polynomial at all.

- If $f(x)$ and $g(x)$ have unequal degree, then the degree of $(f + g)(x)$ and the degree of $(f - g)(x)$ might be equal to the degree of $f(x)$ or the degree of $g(x)$, whichever has the highest degree.

In short: If f and g are non-zero polynomials, then:

1. $\deg(f + g) \leq \text{maximum}\{\deg(f), \deg(g)\}$ and
2. $\deg(f - g) \leq \text{maximum}\{\deg(f), \deg(g)\}$

B. Multiplication of polynomial functions

- ✓ Multiplication of polynomial function by a polynomial function is based on multiplication of powers.
- ✓ Multiplication of polynomial function by polynomial function is done by multiplying each term of one polynomial by every term of the second using distribution property of multiplication over addition and subtraction

Remark:

- i. For two powers ax^n and bx^m , their product is obtained as; $ax^n \times bx^m = (ab)x^{n+m}$
- ii. For three real numbers a, b and c ;
 $a \times (b + c) = (a \times b) + (a \times c)$ and $a \times (b - c) = (a \times b) - (a \times c)$
- Let $f(x)$ and $g(x)$ be two polynomial functions with degrees n and m , leading coefficients a_n and b_m , and constant terms a_0 and b_0 , respectively given by;
 $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0$
 where a_n and $b_m \neq 0$. then the product of f and g is given by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$= (a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0)(b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0)$$

$$= a_nx^n(b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0) + a_{n-1}x^{n-1}(b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0) + \dots + a_1x(b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0) + a_0(b_mx^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0)$$
- From the product obtained above, we can see that the product $f(x) \cdot g(x)$ is also polynomial function with
 - Leading term $= a_nx^n \times b_mx^m = a_nb_mx^{n+m}$
 - Degree $= n + m$
 - Leading coefficient $= a_nb_m$ and
 - Constant term $= a_0b_0$
- If one of the polynomial $f(x)$ and $g(x)$ is a zero polynomial, their product $f(x) \cdot g(x)$ is also a zero polynomial function.

Example 8:

- a) Let $f(x) = 2x^3 - x^2 + 3x - 2$ and $g(x) = x^2 - 2x + 3$. Find the product $f \cdot g$ and determine
 - i) the degree
 - ii) leading coefficient, and
 - iii) constant term of $f \cdot g$.
- b) Let $f(x) = \frac{1}{4}x^3 - 2x - \frac{7}{3}$ and $g(x) = 1 - 4x$. Then without finding the actual product, find
 - i) the leading term
 - ii) the degree
 - iii) the leading coefficient, and
 - iv) the constant term of $f \cdot g$.

Solution:

$$\begin{aligned} \text{a) } (f \cdot g)(x) &= f(x) \cdot g(x) = (2x^3 - x^2 + 3x - 2)(x^2 - 2x + 3) \\ &= 2x^3(x^2 - 2x + 3) - x^2(x^2 - 2x + 3) + 3x(x^2 - 2x + 3) - 2(x^2 - 2x + 3) \\ &= 2x^5 - 4x^4 + 6x^3 - x^4 + 2x^3 - 3x^2 + 3x^3 - 6x^2 + 9x - 2x^2 + 4x - 6 \end{aligned}$$

Then collecting like terms and simplifying gives

$$f(x) \cdot g(x) = 2x^5 - 5x^4 + 11x^3 - 11x^2 + 13x - 6$$

Hence; $f \cdot g$ is a polynomial function with

- ✓ degree = 5 = 3 + 2 = deg. of f + deg. of g
- ✓ leading coefficient = 2 = 2×1 = (lead. coeff. of f) $\times$ (lead. coeff. of g); and
- ✓ constant term = -6 = $(-2) \cdot (3)$ = (cons. term of f) $\times$ (cons. term of g)

$$\text{b) Let } f(x) = \frac{1}{4}x^3 - 2x - \frac{7}{3} \text{ and } g(x) = 1 - 4x.$$

	$f(x)$	$g(x)$	$f(x) \cdot g(x)$
leading term of	$\frac{1}{4}x^3$	$-4x$	$\left(\frac{1}{4}x^3\right)(-4x) = -x^4$
leading coefficient of	$\frac{1}{4}$	-4	$\left(\frac{1}{4}\right)(-4) = -1$
degree of	3	1	$3 + 1 = 4$
constant term of	$-\frac{7}{3}$	1	$\left(-\frac{7}{3}\right)(1) = -\frac{7}{3}$

C. Division of Polynomial Functions

- As we can divide positive integers by an at most an equal positive integers and obtain integer quotient and integer remainder, it is possible to divide a polynomial by a polynomial and obtain a polynomial quotient and a polynomial remainder.
- For instance, let us divide 743 by 52 as follow.
 - ✓ In **division 1** bellow, we divide 743 by 52 and obtain quotient 14.2 which is not integer. The reason for this is we continue the division process after obtaining the remainder 15 which is less than the divisor.

Division 1

$$\begin{array}{r} 14.2 \\ 52 \overline{) 743} \\ \underline{52} \\ 223 \\ \underline{208} \\ 150 \\ \underline{104} \\ 46 \end{array}$$

Division 2

$$\begin{array}{r} 14 \\ 52 \overline{) 743} \\ \underline{52} \\ 223 \\ \underline{208} \\ 15 \end{array}$$

- ✓ In **division 2** above, we divide 743 by 52, we stop the division process whenever we obtain remainder 15 which is less than the divisor, and we obtain integer quotient which is 14 and integer remainder 15.
- ✓ In **division 2**, 743 is called **dividend**, 52 is called **divisor**, 14 is called **quotient** and 15 is called **remainder**. The division can be written in the form of equation containing division sign as

$$\frac{743}{52} = 14 + \frac{15}{52}$$

Multiplying both sides by 52, we obtain an equation without division sign as

$$743 = (52)(14) + 15$$

This has the form of

$$\text{Dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

Note: The division technique above is known as **long division**.

- All the above things help you to understand division of polynomial functions.
 - Division of a polynomial function by a polynomial function is similar to that of division of integer by integer.
 - To divide polynomials using long division, first divide the first term of the dividend by the first term of the divisor. This is the first term of the quotient. Multiplying the new term of the quotient by the divisor, subtract this product from the dividend.
- Repeat these steps using the difference as the new dividend until the first term of the divisor is of a greater degree than the new dividend. The last new dividend whose degree is less than the divisor is the remainder.
- For example, consider division of polynomial function $f(x) = 2x^3 - 5x^2 - 3x + 1$ by polynomial function $g(x) = x^2 + x - 2$ using long division, as follow.

The diagram illustrates the long division of $2x^3 - 5x^2 - 3x + 1$ by $x^2 + x - 2$. The dividend is written at the top, and the divisor is written on the left. The quotient $2x - 7$ is written above the dividend. The steps are as follows:

$$\begin{array}{r}
 \text{Dividend} \\
 \begin{array}{r}
 2x^3 - 5x^2 - 3x + 1 \\
 \underline{2x^3 + 2x^2 - 4x} \\
 -7x^2 + x + 1 \\
 \underline{-7x^2 - 7x + 14} \\
 8x - 13 \\
 \hline
 \end{array} \\
 \text{Divisor: } x^2 + x - 2 \\
 \text{Quotient: } 2x - 7 \\
 \text{Remainder: } 8x - 13
 \end{array}$$

Annotations in the diagram include:

- quotient** (pointing to $2x - 7$)
- divisor** (pointing to $x^2 + x - 2$)
- Dividend** (pointing to $2x^3 - 5x^2 - 3x + 1$)
- remainder** (pointing to $8x - 13$)
- think* $\frac{2x^3}{x^2} = 2x$
- think* $\frac{-7x^2}{x^2} = -7$

- ✓ Here, to obtain a polynomial quotient and a polynomial remainder, we stop the division when the remainder is $r(x) = 2x - 9$, since the degree of the remainder = 1 is less than the degree of the divisor = 2.
- ✓ The division can be written in the form

$$\frac{2x^3 - 5x^2 - 3x + 1}{x^2 + x - 2} = (2x - 7) + \frac{8x - 13}{x^2 + x - 2}$$

Multiplying both sides by $x^2 + x - 2$, we obtain an equation of the form

$$2x^3 - 5x^2 - 3x + 1 = (2x - 7)(x^2 + x - 2) + (8x - 13)$$

This has the form of

$$\text{Dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

In this division,

- the **dividend** is $f(x) = 2x^3 - 5x^2 - 3x + 1$,
- the **divisor** is $g(x) = x^2 + x - 2$,
- the **quotient** is $q(x) = 2x - 7$, and
- the **remainder** is $r(x) = 8x - 13$.

Example 9: Divide polynomial $x^3 - x + 1$ by $x + 1$, using long division.

Determine the quotient and the remainder.

Solution:

$$\begin{array}{r}
 x^2 - x \\
 x + 1 \overline{) x^3 - x + 1} \\
 \underline{x^3 + x^2} \\
 -x^2 - x + 1 \\
 \underline{-x^2 - x} \\
 1
 \end{array}$$

In this division we have $\frac{x^3 - x + 1}{x + 1} = (x^2 - x) + \frac{1}{x + 1}$

This implies that $x^3 - x + 1 = (x + 1)(x^2 - x) + 1$

1.2 THEOREMS ON POLYNOMIALS

1.2.1 Polynomial Division Theorem

- As seen in the above examples, in division of a polynomial function by a polynomial function, we stop the division process when we obtain the remainder whose degree is less than the degree of the divisor or when the remainder is zero polynomial.
- This concept is generalized by the following theorem.

Theorem 1.1: Polynomial Division Theorem (Division Algorithm)

If $f(x)$ and $d(x)$ are two polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.

If the remainder $r(x) = 0$, $f(x)$ divides exactly into $d(x)$ or we say that division of $f(x)$ by $d(x)$ is exact.

- In the above Division Algorithm, $f(x)$ is **dividend**, $d(x)$ is **divisor**, $q(x)$ is **quotient**, and $r(x)$ is **remainder**.

Example 10: Divide polynomial $x^3 + 1$ by $x^2 - x + 1$ using long division. Determine the quotient and the remainder. Is the division exact?

Solution: The long division to divide $x^3 + 1$ by $x^2 - x + 1$ is given below.

From the long division we can see that

- the **quotient** is $q(x) = x + 1$, and
- the **remainder** is $r(x) = 0$.

Hence, **the division is exact**.

As written in Division Algorithm,

$$f(x) = q(x)d(x) + r(x)$$

$$x^3 + 1 = (x + 1)(x^2 - x + 1) + 0$$

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

- Thus, the **quotient** $x + 1$ is the **factor** of the dividend $x^3 + 1$.
- The **divisor** $x^2 - x + 1$ is also the **factor** of the dividend $x^3 + 1$.
(That is $x + 1$ and $x^2 - x + 1$ are the factors of $x^3 + 1$.)
- The product $(x + 1)(x^2 - x + 1)$ a **factorized form** of $x^3 + 1$.

Remark: In division of a polynomial (dividend) with degree n by a polynomial (divisor) of degree m , where $n \geq m$,

- The remainder is a zero polynomial or a polynomial of degree less than the degree of the divisor.
- The degree of the quotient $= n - m = \text{degree of } f - \text{degree of } g$.

Example 11: Divide polynomial $f(x)$ by $d(x)$ using long division. Determine the quotient and the remainder. Is the division exact?

- $f(x) = 2x^3 + 1$ by $d(x) = 5x^3 - x + 2$
- $f(x) = x^5 + 4x^3 - 6x^2 - 8$ by $d(x) = x^2 + 3x + 2$
- $f(x) = x^3 - x - 1$ by $d(x) = 2x^2 + x + 1$
- $f(x) = x^4 - x^3 + x^2 - x + 1$ by $d(x) = x + 1$

Solution:

- From the long division, we have,

$$\text{Quotient } q(x) = \frac{2}{5} \text{ and}$$

$$\text{Remainder } r(x) = \frac{2}{5}x + \frac{1}{5} \neq 0$$

Hence, the division is **not** exact.

$$\begin{array}{r} \frac{2}{5} \\ 5x^3 - x + 2 \overline{) 2x^3 + 1} \\ \underline{2x^3 - \frac{2}{5}x + \frac{4}{5}} \\ \frac{2}{5}x + \frac{1}{5} \end{array}$$

b) From the long division,

- quotient;
 $q(x) = x^3 - 3x^2 + 11x - 33$
- remainder;
 $r(x) = 77x + 58 \neq 0$
Hence, the division is **not** exact.

$$\begin{array}{r}
 x^3 - 3x^2 + 11x - 33 \\
 \underline{x^2 + 3x + 2 } x^5 + 4x^3 - 6x^2 - 8 \\
 x^5 + 3x^4 + 2x^3 \\
 \hline
 -3x^4 + 2x^3 - 6x^2 - 8 \\
 -3x^4 + 9x^3 + 6x^2 \\
 \hline
 11x^3 - 8 \\
 11x^3 + 33x^2 + 22x \\
 \hline
 -33x^2 - 22x - 8 \\
 -33x^2 - 99x + 66 \\
 \hline
 77x + 58
 \end{array}$$

c) From the long division,

- ✓ quotient;
 $q(x) = \frac{1}{2}x - \frac{1}{4},$
- ✓ remainder;
 $r(x) = -\frac{5}{4}x - \frac{3}{4} \neq 0$
Hence, the division is **not** exact.

$$\begin{array}{r}
 \frac{1}{2}x - \frac{1}{4} \\
 \underline{2x^2 + x + 1 } x^3 - x - 1 \\
 x^3 + \frac{1}{2}x^2 + \frac{1}{2}x \\
 \hline
 -\frac{1}{2}x^2 - \frac{3}{2}x - 1 \\
 -\frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{4} \\
 \hline
 -\frac{5}{4}x - \frac{3}{4}
 \end{array}$$

- d) From the long division, when $x^5 + 1$ divided by $x + 1$, we obtain, quotient $q(x) = x^4 - x^3 + x^2 - x + 1$ and remainder is $r(x) = 0$
Hence, the division is **not** exact.
Thus, $x^4 - x^3 + x^2 - x + 1$ and $x + 1$ are the factors of $x^5 + 1$.
This can be written as;
 $x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$

$$\begin{array}{r}
 x^4 - x^3 + x^2 - x + 1 \\
 \underline{x + 1 } x^5 + 1 \\
 x^5 + x^4 \\
 \hline
 -x^4 + 1 \\
 -x^4 - x^3 \\
 \hline
 x^3 + 1 \\
 x^3 + x^2 \\
 \hline
 -x^2 + 1 \\
 -x^2 - x \\
 \hline
 x + 1 \\
 x + 1 \\
 \hline
 0
 \end{array}$$

EXERCISE 1.3

1. In the following ,find
 - c. $f + g$ b. $f - g$ c. $f \times g$ d. the degree of $f + g, f - g$ and $f \times g$
 - i. $f(x) = 1 - x^3 + 2x^2, g(x) = x^2 + 2x^3 - x^2 + 8x + 1$
 - ii. $f(x) = x^3 - x^2 + 2x + 1, g(x) = 1 - x^2$
 - iii. $f(x) = x^3 - x^2 + 1, g(x) = \frac{1}{2}x - 1$
 - iv. $f(x) = x^2 - x + 2, g(x) = x^3 - 2x^2 + 8$
2. In each of the following divide $f(x)$ by $g(x)$ such that, find the quotient $q(x)$ and the remainder
 - i. $f(x) = x^3 - x^2 + 2x + 1, g(x) = 1 - x^2$
 - ii. $f(x) = 4x^6 - 2x^5 + x^3 - 6x + 8, g(x) = 2x^3 - x^2 + x - 1$
 - iii. $f(x) = x^3 - x^2 + 1, g(x) = \frac{1}{2}x - 1$
 - iv. $f(x) = x^4 + 1, g(x) = x + 1$
 - v. $f(x) = \frac{2}{3}x^3 - 3x^2 - 5x + 3, g(x) = 2x + 3$
 - vi. $f(x) = x^2 + 2x + 4, g(x) = x^3 + 4$

1.2.2 The Remainder Theorem

- Above, in Division Algorithm, we have seen that when a polynomial $f(x)$ divide a polynomial $d(x)$ such that $d(x) \neq 0$, and degree of $d(x) \leq$ degree of $f(x)$, we obtain quotient polynomial $q(x)$ and remainder polynomial $r(x)$ satisfying the equation

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $d(x)$.

- Now, in this subtopic you will answer the question that what happens to the remainder $r(x)$, if the divisor is linear polynomial of the form $x - c$ or $ax + b, a \neq 0$.

Consider division of a non-constant polynomial $f(x)$ by a linear polynomial $d(x) = x - c, c \in \mathbb{R}$.

Since the divisor is linear, by Division Algorithm, the remainder $r(x)$ must be constant. Denote this constant by $r(x) = R$

Then, by Division Algorithm, we have

$$f(x) = d(x)q(x) + r(x)$$

$$f(x) = (x - c)q(x) + R$$

Now, compute the value of $f(c)$ as follow.

$$f(c) = (c - c)q(x) + R = 0 \cdot q(x) + R$$

$$f(c) = R$$

This shows that the remainder is $R = f(c)$.

This proves the following theorem known as **remainder theorem**.

Theorem 1.2: (Remainder Theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let $c \in \mathbb{R}$. If $f(x)$ is divided by the linear polynomial $x - c$, then the remainder is $f(c)$.

Example 12: Using Remainder Theorem, find the remainder if:

- $f(x) = x^3 + 2x^2 - 6$ divided by $d(x) = x - 3$
- $g(x) = 1 + 5x + x^2 - 2x^5$ divided by $d(x) = x + 2$
- $f(x) = 2 + 7x^{25}$ divided by $d(x) = x - 1$

Solution

- $f(x) = x^3 + 2x^2 - 6$. Here, $x - c = x - 3$ implies $c = 3$.

Then by Division Algorithm, the remainder is

$$R = f(3) = (3)^3 + 2(3)^2 - 6 = 27 + 18 - 6 = 39$$

- $g(x) = 1 + 5x + x^2 - 2x^5$. Here, $x - c = x + 2$ implies $c = -2$.

Then by Division Algorithm, the remainder is

$$R = g(-2) = 1 + 5(-2) + (-2)^2 - 2(-2)^5 = 1 - 10 + 4 + 64 = 59$$

- $f(x) = 2 + 7x^{25}$. Here, $x - c = x - 1$ implies $c = 1$.

Then by Division Algorithm, the remainder is

$$R = f(1) = 2 + 7(-1)^{25} = 2 + 7(-1) = 2 - 7 = -5$$

Corollary: When a polynomial $f(x)$ of degree greater than or equal to 1 is divided by the linear polynomial of the form $ax + b$, the remainder is $f\left(-\frac{b}{a}\right)$.

Example 13: Using Remainder Theorem, find the remainder if

- $f(x) = 5 - 4x^2 + x^3 + 12x^5$ divided $d(x) = 2x - 1$
- $f(x) = 6x^3 + x^2 - x + 2$ divided $d(x) = 3x + 2$

Solution:

- Here, $ax + b = 2x - 1$ implies $a = 2$ and $b = -1$. This implies $-\frac{b}{a} = -\frac{(-1)}{2} = \frac{1}{2}$

Then, the remainder is

$$R = f\left(-\frac{b}{a}\right) = f\left(\frac{1}{2}\right) = 5 - 4\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + 12\left(\frac{1}{2}\right)^5 = 5 - 1 + \frac{1}{8} + \frac{3}{8} = 4 + \frac{1}{2} = \frac{9}{2}$$

- Here, $ax + b = 3x + 2$ implies $a = 3$ and $b = 2$. This implies $-\frac{b}{a} = -\frac{2}{3}$

Then, the remainder is

$$\begin{aligned} R &= f\left(-\frac{b}{a}\right) = f\left(-\frac{2}{3}\right) = 6\left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right) + 2 \\ &= -\frac{16}{9} + \frac{4}{9} + \frac{2}{3} + 2 = \frac{-16 + 4 + 6 + 18}{9} = \frac{12}{9} = \frac{4}{3} \end{aligned}$$

Example 14: When the polynomial $f(x) = x^7 - kx^6 + 5x^3 - x + 11$ is divided by $x + 1$, the remainder is 15, what is the values of k .

Solution: $x - c = x + 1$ implies $c = -1$

By Remainder Theorem, the remainder is

$$R = f(c) \Rightarrow f(-1) = 15$$

$$\Rightarrow (-1)^7 - k(-1)^6 + 5(-1)^3 - (-1) + 11 = 15$$

$$\Rightarrow -1 - k - 5 + 1 + 11 = 15 \Rightarrow k = -11$$

Example 15: When a polynomial $f(x) = ax^3 + bx^2 - 2x + 8$ is divided by $x - 1$ and $x + 1$, the remainder are 3 and 5, respectively. Find the values of a and b .

Solution:

When $f(x) = ax^3 + bx^2 - 2x + 8$ is divided by $x - 1$ the remainder is $f(1) = 3$.

$$\Rightarrow a(1)^3 + b(1)^2 - 2(1) + 8 = 3 \Rightarrow a + b = -3 \text{ -----} (*)$$

When $f(x) = ax^3 + bx^2 - 2x + 8$ is divided by $x + 1$ the remainder is $f(-1) = 5$.

$$\Rightarrow a(-1)^3 + b(-1)^2 - 2(-1) + 8 = 5 \Rightarrow -a + b = 5 \text{ -----} (**)$$

Then solve equations (*) and (**) simultaneously.

$$\begin{array}{r} \downarrow \begin{cases} a + b = -3 \\ -a + b = 5 \end{cases} \\ \hline \end{array}$$

$$\Rightarrow 2a = -3 - (-5) = 2 \Rightarrow a = 1$$

$$\Rightarrow a + b = -3 \Rightarrow b = -3 - a = -3 - 1 = -4$$

Therefore, the values of a and b are $a = 1$ and $b = -4$

EXERCISE 1.4

- Find the remainder when:
 - $f(x) = x^3 - x^2 + 7x + 11$ divided $d(x) = x + 2$.
 - $g(x) = 1 + 5x + 2x^2 - \frac{8}{3}x^4$ divided $d(x) = 2x - 3$.
 - $h(x) = 1 + x^{17}$ divided $d(x) = x + 1$.
- When the polynomial $P(x) = ax^4 - 2bx^3 + cx + 5$ is divided by $x - 1$ and $x + 1$, the remainders are 2 and 3, respectively. If $P\left(\frac{1}{2}\right) = -1$, then find the values of a , b and c .
- When a polynomial $P(x)$ is divided by $x + 2$, the remainder is 3 and when $P(x)$ is divided by $x - 1$ the remainder is 2. Find the remainder when $P(x)$ is divided by $(x + 2)(x - 1)$.

1.2.3 The Factor Theorem

- Another important objective of this chapter is factorization of a polynomial and determining its factors, if any.
- Factorizing a polynomial means writing the polynomial as a product of two or more polynomials.
- For instance, consider the polynomials $x^2 + 2x - 3$ and $2x^3 + 7x^2 + 2x + 7$.
 - As you have learnt in your previous grade mathematics $x^2 + 2x - 3$ can be factorized as

$$x^2 + 2x - 3 = x^2 + 3x - x - 3; \text{ since } 3 + (-1) = 2 \text{ and } 3 \times (-1) = -3$$

$$= x(x + 3) - 1(x + 3) = (x + 3)(x - 1)$$
 Hence, $(x + 3)(x - 1)$ is the factorized form of $x^2 + 2x - 3$.
 Moreover, $x + 3$ and $x - 1$ are the factors of $x^2 + 2x - 3$.
 Letting $f(x) = x^2 + 2x - 3$, you can see that $f(-3) = f(1) = 0$

- b) Some polynomials can be factorized by grouping, which can be applied for $2x^3 + 7x^2 + 2x + 7$ as follow:

$$2x^3 + 7x^2 + 2x + 7 = x^2(2x + 7) + 1(2x + 7) = (x^2 + 1)(2x + 7)$$

Hence, $x^2 + 1$ and $2x + 7$ are the factors of $2x^3 + 7x^2 + 2x + 7$.

Moreover, $(x^2 + 1)(2x + 7)$ is its factorized form.

Since $2x + 7$ is linear and $x^2 + 1$ cannot be factorized more over $\mathbb{R}$, we say that $(x^2 + 1)(2x + 7)$ is the **complete factorized form** of $2x^3 + 7x^2 + 2x + 7$.

- In addition, let us do the following activity.

Consider the polynomial, $f(x) = x^3 - 5x^2 + 2x + 8$.

$$f(2) = (2)^3 - 5(2)^2 + 2(2) + 8 = 8 - 20 + 4 + 8 = 0$$

This shows that, by Remainder Theorem, when $f(x)$ divided by $x - 2$, the remainder is 0.

By Division Algorithm, $f(x) = d(x)q(x) + r(x)$, for some polynomial

- $x^3 - 5x^2 + 2x + 8 = (x - 2)q(x) + 0$; for some polynomial $q(x)$.

- $x^3 - 5x^2 + 2x + 8 = (x - 2)q(x)$

Hence, $x - 2$ is a factor of $f(x)$.

$q(x)$ is also another factor of $f(x)$, which is the quotient of division of $f(x)$ by $x - 2$.

From the long division, we have

$$q(x) = x^2 - 3x - 4$$

$(x - 2)(x^2 - 3x - 4)$ is the factorized form of $f(x)$, but it is incomplete.

$$\begin{array}{r} x^2 - 3x - 4 \\ x - 2 \overline{) x^3 - 5x^2 + 2x + 8} \\ \underline{x^3 - 2x^2} \\ -3x^2 + 2x + 8 \\ \underline{-3x^2 + 6x} \\ -4x + 8 \\ \underline{-4x + 8} \\ 0 \end{array}$$

To factorize $f(x)$ completely, factorize $x^2 - 3x - 4$ into its linear factors, as follow.

$$x^2 - 3x - 4 = x^2 - 4x + x - 4 = x(x - 4) + 1(x - 4) = (x + 1)(x - 4)$$

Therefore, the **complete factorized** form $f(x)$ is

$$\begin{aligned} x^3 - 5x^2 + 2x + 8 &= (x - 2)(x^2 - 3x - 4) \\ &= (x - 2)(x + 1)(x - 4) \end{aligned}$$

$x - 2$, $x + 1$ and $x - 4$ are the linear factors of $x^3 - 5x^2 + 2x + 8$.

All the above discussions can be summarized using a theorem known as the **Factor Theorem**, stated below, which is helpful in checking whether a linear polynomial is a factor of a given polynomial or not.

Theorem 1.3: Factor Theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number. Then $x - c$ is a factor of $f(x)$, if and only if $f(c) = 0$.

For example, consider the polynomial, $P(x) = x^3 - 3x^2 - x + 3$. Then

$$P(-1) = (-1)^3 - 3(-1)^2 - (-1) + 3 = -1 - 3 + 1 + 3 = 0$$

Hence, by Factor theorem, $x - (-1) = x + 1$ is a factor of $P(x) = x^3 - 3x^2 - x + 3$.

Remark: The Factor theorem has two parts.

i) If $x - c$ is a factor of $f(x)$, then $f(c) = 0$; and

ii) If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

Corollary: Let $f(x)$ be a polynomial of degree greater than or equal to one, and let $a, b \in \mathbb{R}$, where $a \neq 0$. Then $ax + b$ is a factor of $f(x)$, if and only if $f(-\frac{b}{a}) = 0$.

Example 16:

1) In each of the following, use the factor theorem to determine whether or not $g(x)$ is a factor of $f(x)$.

a. $f(x) = x^{15} + 1$; $g(x) = x + 1$,

b. $f(x) = 3x^4 + 7x^2 - x - 2$; $g(x) = x - 1$

c. $f(x) = 4x^4 - 2x^3 - 2x^2 + 7x - 3$; $g(x) = 2x - 1$

d. $f(x) = x^3 - 3x^2 - 4x + 12$; $g(x) = x + 2$

Solution

d. $x + 1 = x - c$ implies $c = -1$. Then $f(-1) = (-1)^{15} + 1 = -1 + 1 = 0$.

Thus, by Factor Theorem, $x + 1$ is a factor of $x^{15} + 1$.

e. $x - 1 = x - c$ implies $c = 1$. Then

$$f(1) = 3(1)^4 + 7(1)^2 - (1) - 2 = 3 + 7 - 1 - 2 = 7 \neq 0.$$

Therefore, $x - 1$ is **NOT** a factor of $f(x) = 3x^4 + 7x^2 - x - 2$

c. $2x - 1 = ax + b$ implies $a = 2$ and $b = -1 \Rightarrow -\frac{b}{a} = -\frac{-1}{2} = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right) - 3 = \frac{1}{4} - \frac{1}{4} - \frac{1}{2} + \frac{7}{2} - 3 = 0$$

Thus, by Factor Theorem, $2x - 1$ is a factor of $4x^4 - 2x^3 - 2x^2 + 7x - 3$.

e. Here, $c = -2$ and

$$\begin{aligned} f(-2) &= (-2)^3 - 3(-2)^2 - 4(-2) + 12 \\ &= -8 - 12 + 8 + 12 = 0 \end{aligned}$$

Therefore, $x + 2$ is a factor of $f(x)$.

Let us extend this question up to factorization of $f(x)$.

Dividing $f(x)$ by its factor $x + 2$, gives quotient

$$q(x) = x^2 - 5x + 6 \text{ and remainder}$$

$$r(x) = 0$$

Then by Division Algorithm,

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 4x + 12 \\ &= (x + 2)(x^2 - 5x + 6) \end{aligned}$$

which is the factorized form of $f(x)$, **but incomplete**.

$$\begin{aligned} x^2 - 5x + 6 &= x^2 - 2x - 3x + 6 \\ &= x(x - 2) - 3(x - 2) = (x - 2)(x - 3) \end{aligned}$$

Therefore, the **complete factorized** form of $f(x)$ is

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 4x + 12 \\ &= (x + 2)(x^2 - 5x + 6) \\ &= (x + 2)(x - 2)(x - 3) \end{aligned}$$

$$\begin{array}{r} x^2 - 5x + 6 \\ x + 2 \overline{) x^3 - 3x^2 - 4x + 12} \\ \underline{x^3 + 2x^2} \\ -5x^2 - 4x + 12 \\ \underline{-5x^2 - 10x} \\ 6x + 12 \\ \underline{6x + 12} \\ 0 \end{array}$$

Example 17: In each of the following find a number k such that

- $x + 2$ is a factor of $3x^4 - 8x^2 - kx + 6$.
- $3x - 2$ is a factor of $6x^3 - 4x^2 + 2kx - k - 3$.

Solution:

- Let $f(x) = 3x^4 - 8x^2 - kx + 6$ then $f(2) = 0$.

$$f(2) = 3(2)^4 - 8(2)^2 - k(2) + 6 = 3(16) - 8(4) - 2k + 6 = 0$$

$$\Rightarrow 48 - 32 - 2k + 6 = 22 - 2k = 0$$

$$\Rightarrow k = -11$$
- $3x - 2 = ax + b \Rightarrow -\frac{b}{a} = \frac{2}{3}$.
 Let $f(x) = 6x^3 - 4x^2 + 2kx - k - 3$. Then

$$f\left(\frac{2}{3}\right) = 0 \Rightarrow 6\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 2k\left(\frac{2}{3}\right) - k - 3 = 0$$

$$\Rightarrow \frac{16}{9} - \frac{16}{9} + \frac{4}{3}k - k - 3 = 0 \Rightarrow \frac{1}{3}k = 3.$$

 Therefore, $k = 9$

Example 18:

- Let $f(x) = x^4 - 2hx^3 + hx^2 - x + k$. If $x - 2$ is a factor of $f(x)$ and $f(-1) = 3$, then find the values of h and k .
- If $x - 2$ is a common factor of the expressions: $x^2(m+n)x - n$ and $2x^2 + (m-1)x + (m+2n)$, then find the values of m and n .

Solution:

- Using the factor theorem $x - 2$ is a factor of $f(x)$ implies $f(2) = 0$

$$f(2) = (2)^4 - 2h(2)^3 + h(2)^2 - 2 + k = 0$$

$$16 - 16h + 4h - 2 + k = 0$$

$$-12h + k = -14 \dots\dots\dots(1)$$

$$f(-1) = 3$$

$$f(-1) = (-1)^4 - 2h(-1)^3 + h(-1)^2 - (-1) + k = 3$$

$$1 - 2h(-1) + h + 1 + k = 3$$

$$1 + 2h + h + 1 + k = 3$$

$$3h + k = 1 \dots\dots\dots(2)$$

Then solve equations (1) and (2) simultaneously,

$$\begin{cases} -12h + k = -14 \\ 3h + k = 1 \end{cases}$$

we obtain $h = 1$ and $k = -2$.

b. $x - 2$ is a common factor of the expression $x^2(m+n)x - n$ implies:

$$2^2(m+n)2 - n = 0 \Rightarrow 4(m+n)2 - n = 0$$

$$\Rightarrow 8(m+n) - n = 0$$

$$\Rightarrow 8m + 7n = 0 \dots\dots\dots(1) \text{ and}$$

c. $x - 2$ is a common factor of the expression $2x^2 + (m-1)x + (m+2n)$ implies:

$$2(2)^2 + (m-1)2 + (m+2n) = 0 \Rightarrow 8 + (m-1)2 + (m+2n) = 0$$

$$\Rightarrow 8 + 2m - 2 + m + 2n = 0$$

$$\Rightarrow 3m + 2n = -6 \dots\dots\dots(2)$$

Now solving equations (1) and (2)

$$\begin{cases} 8m + 7n = 0 \\ 3m + 2n = -6 \end{cases} \Rightarrow m = -\frac{42}{5} \text{ and } n = \frac{48}{5}$$

Example 19: If $x + 3$ and $x - 2$ are factors of $f(x) = x^4 + x^3 - 7x^2 - x + 6$, then

a) Find the complete factorized form of $f(x)$.

b) Find the other factors of $f(x)$.

Solution:

First divide $f(x)$ by $x + 3$ to obtain a cubic polynomial $q_1(x)$ and again divide the resulting quotient $q_1(x)$ by $x - 2$ to obtain a quadratic polynomial $q_2(x)$. Lastly, factorize $q_2(x)$.

$$\begin{array}{r} x^3 - 2x^2 - x + 2 = q_1(x) \\ x + 3 \overline{) x^4 + x^3 - 7x^2 - x + 6} \\ \underline{x^4 + 3x^3} \\ -2x^3 - 7x^2 - x + 6 \\ \underline{-2x^3 - 6x^2} \\ -x^2 - x + 6 \\ \underline{-x^2 - 3x} \\ 2x + 6 \\ \underline{2x + 6} \\ 0 \end{array}$$

$$\begin{array}{r} x^2 - 1 = q_2(x). \\ x - 2 \overline{) x^3 - 2x^2 - x + 2 = q_1(x)} \\ \underline{x^3 - 2x^2} \\ -x + 2 \\ \underline{-x + 2} \\ 0 \end{array}$$

And we have
 $x^2 - 1 = (x - 1)(x + 1)$

To summarize the above divisions:

In the first division, we have

$$\begin{aligned} f(x) &= x^4 + x^3 - 7x^2 - x + 6 \\ &= (x + 3)(x^4 + x^3 - 7x^2 - x + 6) \\ &= (x + 3)(x - 2)(x^2 - 1) \\ &= (x + 3)(x - 2)(x - 1)(x + 1) \end{aligned}$$

Thus, the complete factorized form of $f(x)$ is

$$f(x) = (x + 3)(x - 2)(x - 1)(x + 1)$$

Therefore, $x + 3$, $x - 2$, $x - 1$ and $x + 1$ are the factors of $f(x)$.

EXERCISE 1.5

- In each of the following find a number k such that
 - $x + 3$ is a factor of $x^5 - kx^4 - 6x^3 - x^2 + 4x + 29$
 - when $f(x) = x^2 + (k - 2)x - k^2 - 3k + 5$ is divided by $x + k$ the remainder is -1 .
- When the polynomial $P(x) = x^3 + 4x^2 - nx - 3$ divided by $x + 1$ and $x - 2$, the remainders are $5m + 4$ and $-n - 10$. Find the values of m and n .
- When the polynomial $f(x) = x^3 - (2c - 1)x^2 + cx - 3$ divided by $x - 1$, it gives remainder 5. What will be the remainder when $f(x)$ is divided by $x + 1$.
- What number must be added to $f(x) = x^3 - 5x^2 + 2x - 7$ so that $x - 2$ is the factor of $f(x)$?
- Find a polynomial function of degree 3 such that $f(2) = 24$ and $x - 1$, x and $x + 2$ are factors of the polynomial.

1.3. ZEROS OF A POLYNOMIAL FUNCTION

For instance, consider the function $f(x) = \frac{x^2 - 4}{x + 1}$.

The domain of $f = \mathbb{R} \setminus \{-1\}$.

For $x = 2 \in \text{domain}$, $f(2) = \frac{(2)^2 - 4}{2 + 1} = \frac{4 - 4}{3} = \frac{0}{3} = 0$. In this case we say that 2 is a **zero** of $f(x)$.

In general, given a function f , a number in the domain of f that makes f zero is called a **zero** of f .

Definition 1.2: For a polynomial function P and a real number c , if $P(c) = 0$, then c is a **zero** of P .

For instance, for a polynomial function $f(x) = 2x^3 - x^2 + x - 2$,

$$f(1) = 2(1)^3 - 1^2 + 1 - 2 = 2 - 1 + 1 - 2 = 0$$

Therefore, $x = 1$ is a zero of f .

Note : If f is a polynomial function and $c \in \mathbb{R}$, then the following statements are equivalent.

- $f(c) = 0$
- $x - c$ is a factor of $f(x)$.
- $f(x)$ is divisible by $x - c$.
- When $f(x)$ is divided by $x - c$, the remainder is 0.

Note. A polynomial function of degree n has at most n real zeros. But it may have no real zeros.

Example 20: Find the zeros of each of the following polynomial functions.

- | | |
|--|--------------------------------|
| a) $f(x) = 5$ | f) $f(y) = y^2 + 4$ |
| b) $y = 3 - \frac{2}{5}x$ | g) $h(x) = 6x^3 + 5x^2 - x$ |
| c) $f(x) = \frac{2x-1}{3} - \frac{x+2}{2} - 2$ | h) $f(t) = t^3 - t^2 + 2t - 2$ |
| d) $f(x) = 2x^5(x-3)(\pi x+1)(3-2x)$ | i) $y = x^4 - 3x^2 - 4$ |
| e) $g(t) = 3 - 2t - 2t^2$ | |

Solution:

- Since $f(x) = 5 \neq 0$ for all real number x , $f(x)$ has no any real zero.
- $y = 0 \Rightarrow 3 - \frac{2}{5}x = 0 \Rightarrow 3 = \frac{2}{5}x \Rightarrow x = 3\left(\frac{5}{2}\right) = \frac{15}{2}$ is the zero of the function.
- $f(x) = 0$
 $\frac{2x-1}{3} - \frac{x+2}{2} - 2 = 0$. Multiply both sides by LCM(2, 3) = 6
 $\Rightarrow 6 \times \left(\frac{2x-1}{3} - \frac{x+2}{2} - 2\right) = 0 \times 6 \Rightarrow 2(2x-1) - 3(x+2) - 12 = 0$
 $\Rightarrow 4x - 2 - 3x - 6 - 12 = 0 \Rightarrow x - 20 = 0$
 $\Rightarrow x = 20$
 Therefore, $x = 20$ is the zero of f .
- By Product Property, $f(x) = 2x^5(x-3)(\pi x+1)(3-2x) = 0$
 $\Rightarrow 2x^5 = 0$ or $x - 3 = 0$ or $\pi x + 1 = 0$ or $3 - 2x = 0$
 $\Rightarrow x^5 = 0$ or $x = 3$ or $\pi x = -1$ or $3 = 2x$
 $\Rightarrow x = 0$ or $x = 3$ or $x = -\frac{1}{\pi}$ or $x = \frac{3}{2}$
 Therefore; $0, 3, -\frac{1}{\pi}$ and $\frac{3}{2}$ are the zeros of $f(x)$.
- $g(t) = 0 \Rightarrow 3 - 2t - 2t^2 = 0$, which is a quadratic equation with $a = -2, b = -2, c = 3$.
 Then by quadratic formula, we have

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-2)(3)}}{2(-2)} = \frac{2 \pm \sqrt{4 + 24}}{-4} = \frac{2 \pm \sqrt{28}}{-4} = \frac{2 \pm 2\sqrt{7}}{-4} = \frac{-1 \pm \sqrt{7}}{2}$$

 Therefore; $\frac{-1 - \sqrt{7}}{2}$ and $\frac{-1 + \sqrt{7}}{2}$ are the zeros of $f(x)$.
- $f(y) = y^2 + 4 = 0 \Rightarrow y^2 = -4 \Rightarrow y = \sqrt{-4} \notin \mathbb{R}$
 Therefore; $f(y) = y^2 + 4$ has **NO** any real zero.
- $h(x) = 6x^3 + 5x^2 - x = x(6x^2 + 5x - 1) = 0$
 $\Rightarrow x = 0$ or $6x^2 + 5x - 1 = 0$ Then factorize the quadratic part

$$\Rightarrow x = 0 \text{ or } 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow x = 0 \text{ or } 6x(x + 1) - (x + 1) = 0$$

$$\Rightarrow x = 0 \text{ or } (6x - 1)(x + 1) = 0 \Rightarrow x = 0 \text{ or } 6x - 1 = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{6} \text{ or } x = -1$$

Hence; $x = 0$, $x = \frac{1}{6}$ and $x = -1$ are the zeros of $f(x)$.

h) $f(t) = t^3 - t^2 + 2t - 2 = 0$

This equation can be solved by using Factorizing by Grouping.

$$t^3 - t^2 + 2t - 2 = (t^3 - t^2) + (2t - 2) = t^2(t - 1) + 2(t - 1) = (t - 1)(t^2 + 2) = 0$$

$$\Rightarrow t - 1 = 0 \text{ or } t^2 + 2 = 0$$

$$\Rightarrow t = 1 \text{ or } t = \sqrt{-2} \notin \mathbb{R}$$

Hence; $t = 1$ is the only zero of $f(t)$.

i) $y = x^4 - 3x^2 - 4$. This can be solved by using substitution as follow.

Let $u = x^2$. Then the equation can be written as

$$x^4 - 3x^2 - 4 = u^2 - 3u - 4 = u^2 - 4u + u - 4 = u(u - 4) + 1(u - 4) \\ = (u + 1)(u - 4)$$

Since, $u = x^2$, we have,

$$x^4 - 3x^2 - 4 = (u + 1)(u - 4) = (x^2 + 1)(x^2 - 4) = (x + 2)(x - 2)(x^2 + 1)$$

Here we can see that $(x + 2)(x - 2)(x^2 + 1)$ is the complete factorized form of $x^4 - 3x^2 - 4$.

$$\Rightarrow x + 2, x - 2 \text{ and } x^2 + 1 \text{ are the factors of } x^4 - 3x^2 - 4$$

$$\text{Hence, } x^4 - 3x^2 - 4 = 0 \text{ implies } (x + 2)(x - 2)(x^2 + 1) = 0$$

$$\Rightarrow x + 2 = 0 \text{ or } x - 2 = 0, \text{ but } x^2 + 1 \neq 0 \text{ for all } x \in \mathbb{R}.$$

Hence; $x = -2$ and $x = 2$ are the zero of $y = x^4 - 3x^2 - 4$.

● ***In your discussion above, you have seen that how the zeros of a polynomial function obtained.***

What do you think about the converse process?

For instance, what will be a polynomial function of degree 2 whose zeros are -3 and 1 ? How many such polynomials exist? Is it unique?

- By Factor Theorem, -3 and 1 are zeros of a polynomial function f of degree 2, then $x + 3$ and $x - 1$ are its factors.

Thus, $f(x) = (x + 3)(x - 1)$ is one possible such quadratic polynomial.

The polynomials

$y = 2(x + 3)(x - 1)$, $y = -5(x + 3)(x - 1)$, $y = \frac{3}{2}(x + 3)(x - 1)$ are also polynomial functions of degree 2 with zeros $x = 1$ and $x = -3$.

In general; $f(x) = k(x + 3)(x - 1)$, where k is any non-zero constant, is a polynomial function of degree 2 with zeros $x = 1$ and $x = -3$.

- If the functional value of f at any point x other than the zeros of f is given, the value of k can be determined. In this case the polynomial f is uniquely determined.
- For example, if $f(2) = 7$, then

$$f(2) = k(2 + 3)(2 - 1) = 7 \Rightarrow k(5)(1) = 7 \Rightarrow k = \frac{7}{5}$$

Hence, we obtain

$$f(x) = \frac{7}{5}(x + 3)(x - 1) = \frac{7}{5}x^2 + \frac{14}{5}x - \frac{21}{5}.$$

Thus, we have the following statement.

If $x_1, x_2, x_3, \dots, x_n$ are n real zeros of a polynomial function f of degree n , then f is given by

$$f(x) = a(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n)$$

where k is any non-zero constant.

Example 21: Find a polynomial function f of degree three such that $f(0) = f(-1) = f(1)$ and $f(3) = -48$.

Solution: $f(0) = f(-1) = f(1)$ implies that $-1, 0$ and 1 are zeros of f .

A cubic polynomial f with zeros $-1, 0$ and 1 is given by

$$\begin{aligned} f(x) &= a(x - x_1)(x - x_2)(x - x_3) \\ &= a(x + 1)(x - 0)(x - 1) = ax(x + 1)(x - 1) \\ f(3) &= -48 \text{ implies } a(3)(3 + 1)(3 - 1) = -48 \\ \Rightarrow a(3)(4)(2) &= -48 \quad \Rightarrow a = -\frac{48}{24} = -2 \end{aligned}$$

Hence, the required cubic polynomial is

$$f(x) = -2x(x + 1)(x - 1) = 2(x - x^3)$$

EXERCISE 1.6

- Find the zeros of each of the following polynomial functions.
 - $f(x) = 2x^3 - 5x^2 + x + 2$
 - $f(x) = x^3 + 3x^2 - x - 3$
 - $f(x) = 2x^3 + 5x^2 - 11x + 4$
- Find a polynomial function of degree 3 such that $f(2) = 24$ and $x - 1, x$ and $x + 2$ are factors of the polynomial
- $p(x)$ is a polynomial function of degree 3 such that $p(-2) = p(0) = p(2) = 0$ and $p(1) = -5$. then find $p(x)$?

1.3.1 Zeros and Their Multiplicities

➤ if $f(x)$ is a polynomial function of degree $n, n \geq 1$, then a root of the equation $f(x) = 0$ is called a zero of f

Definition 1.3: if $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then c is said to be a zero of multiplicity k off

Example 22: given that -1 and 2 are zeros of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$, determine their multiplicity

Solution: by the factor theorem $(x+1)$ and $(x-2)$ are factors of $f(x)$ hence $f(x)$ can be divided by $(x + 1)(x - 2) = x^2 - x - 2$ giving you

$$\begin{aligned} f(x) &= (x^2 - x - 2)(x^2 + 2x + 1) \\ &= (x + 1)(x - 2)(x + 1)^2 \\ &= (x + 1)^3(x - 2) \end{aligned}$$

$\therefore -1$ is a zero of multiplicity 3 and 2 is a zero of multiplicity 1

Example 23: for each of the following list the zeros of the given polynomial and state the multiplicity of each zero

a. $f(x) = x^{12} \left(x - \frac{2}{3}\right)$

0 is a zero of multiplicity 12 and $\frac{2}{3}$ is a zero of multiplicity 1

b. $g(x) = 3(x - \sqrt{2})^2(x + 1)^3$

$\sqrt{2}$ is a zero of multiplicity 2 and -1 is a zero of multiplicity 3

c. $h(x) = 3x^6(\pi - x)^5(x - (\pi + 1))^3$

0 is a zero of multiplicity 6, π is a zero of multiplicity 5 and $\pi + 1$ is a zero of multiplicity 3

d. $f(x) = 2(x - \sqrt{3})^5(x + 5)^9(1 - 3x)$

$\sqrt{3}$ is a zero of multiplicity 5, -5 is a zero of multiplicity 9 and $\frac{1}{3}$ is a zero of multiplicity 1

e. $f(x) = x^3 - 3x^2 + 3x - 1$

First factorize f(x) the factorized form of f(x) are $(x-1)(x-1)(x-1) = (x - 1)^3$ 1 is a zero of multiplicity 3

f. $f(x) = x^3 + x^2 - 5x + 3$

The factorized form of f(x) are $(x-1)(x-1)(x+3) = (x - 1)^2(x + 3)$ 1 is a zero of multiplicity 2 and -3 is a zero of multiplicity 1

g. $f(x) = x^4 + 3x^3 + 3x^2 + x$ (**exercise**)

1.3.2 Location Theorem

Polynomial function with rational coefficients may have no rational zeros

Example 24: the zeros of $x^2 - 4x - 2$ are $\frac{4+2\sqrt{6}}{2}$ and $\frac{4-2\sqrt{6}}{2}$

The polynomial function has rational and irrational zeros

Example: the zeros of $p(x) = x^3 - x^2 - 2x + 2$ are $-\sqrt{2}, 1$ and $\sqrt{2}$

ACTIVITY

1. in each of the following determine whether the zeros of the corresponding function are rational, irrational or neither

a. $f(x) = x^2 + 2x + 2$ has no real root

b. $f(x) = x^3 + x^2 - 2x - 2$

The zeros of f(x) are -1, $\sqrt{2}$ and $-\sqrt{2}$ rational and irrational zeros

c. $f(x) = (x + 1)(2x^2 + x - 3)$ The zeros of f(x) are -1, 1 and $\frac{-3}{2}$ rational zeros

d. $f(x) = x^4 - 5x^2 + 6$ The zeros of f(x) are $-\sqrt{3}, \sqrt{3}, \sqrt{2}$ and $-\sqrt{2}$ irrational zeros

Theorem 1.4 Location theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b .

Example 25: Let $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$. Construct a table of values and use the location theorem to calculate the zeros of f between successive integers.

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	210	38	-10	-6	2	-10	-42	-70	-44	102

Since $f(-2) = 38 > 0$ and $f(-1) = -10 < 0$ we see that the value of $f(x)$ changes from +ve to -ve between -2 and -1. Hence by the location theorem there is a zero of $f(x)$ between $x = -2$ and $x = -1$.

Since $f(0) = -6 < 0$ and $f(1) = 2 > 0$ there is also one zero between $x = 0$ and $x = 1$.

$f(1) = 2 > 0$ and $f(2) = -10 < 0$ there is also a zero between $x = 1$ and $x = 2$.

$f(5) = -44 < 0$ and $f(6) = 102 > 0$ there is also a zero between $x = 5$ and $x = 6$.

Example 26: Using the location theorem show that the polynomial $f(x) = x^5 - 2x^2 - 1$ has a zero between $x = 1$ and $x = 2$.

Solution: $f(1) = 1^5 - 2(1)^2 - 1 = -2$ and $f(2) = (2)^5 - 2(2)^2 - 1 = 23 > 0$. Here $f(1)$ is negative and $f(2)$ is positive. Therefore there is a zero between $x = 1$ and $x = 2$.

Example 27: Using the location theorem show that the polynomial $f(x) = x^5 - 2x^2 - 1$ has a zero between $x = 1$ and $x = 2$.

Solution: $f(1) = 1^5 - 2(1)^2 - 1 = -2$ and $f(2) = (2)^5 - 2(2)^2 - 1 = 23 > 0$. Here $f(1)$ is negative and $f(2)$ is positive. Therefore there is a zero between $x = 1$ and $x = 2$.

Example 28: In each of the following use the table of values for the polynomial function $f(x)$ to locate zeros of $y = f(x)$.

a.

x	-5	-3	-1	0	2	5
$f(x)$	7	4	2	-1	3	-6

$f(-1) = 2 > 0$ and $f(0) = -1 < 0 \therefore$ there is a zero between $x = -1$ and $x = 0$.

$f(0) = -1 < 0$ and $f(2) = 3 > 0 \therefore$ there is a zero between $x = 0$ and $x = 2$.

$f(2) = 3 > 0$ and $f(5) = -6 < 0 \therefore$ there is a zero between $x = 2$ and $x = 5$.

b.

x	-6	-5	-4	-3	-2	-1	0	1	2
$f(x)$	-21	-10	8	-1	-5	6	4	-3	18

1.3.3 Rational Root Test

The rational root test relates the possible rational zeros of a polynomial with integer coefficients to the leading coefficients and to the constant term of the polynomial

Theorem 1.5 Rational root test

If the rational number $\frac{p}{q}$ in its lowest terms is a zero of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ with integer coefficients, then p must be a factor of a_0 and q must be a factor of a_n

Example 29: in each of the following find all the rational zeros of the polynomial

a. $f(x) = x^3 - x + 1$ b. $g(x) = 2x^3 + 9x^2 + 7x - 6$ c. $g(x) = \frac{1}{2}x^4 - 2x^2 - \frac{1}{2}x^2 + 2x$

a. $f(x) = x^3 - x + 1$

the leading coefficient is 1 and the constant term is 1 .hence ,as these are factors of the constant term ,the possible rational zeros are ± 1

using the remainder theorem, test these possible zeros

$f(1) = (1)^3 - (1) + 1 = 1$, $f(-1) = (-1)^3 - (-1) + 1 = 1$ so we can conclude that the given polynomial has no rational zeros

b. $g(x) = 2x^3 + 9x^2 + 7x - 6$

$a_n = a_3 = 2$ and $a_0 = -6$

Possible values of p are factors of -6.these are $\pm 1, \pm 2, \pm 3$ and ± 6

Possible values of q are factors of 2 .these are ± 1 and ± 2

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$ of these 12 possible rational zeros,at most 3 can be the zeroes of g

Check that $f(-3) = 0, f(-2) = 0$ and $f(\frac{1}{2}) = 0$

Using the factor theorem, we can factorize $g(x)$ as $2x^3 + 9x^2 + 7x - 6 = (x+3)(x+2)(2x-1)$,so $g(x) = 0$ at $x = -3, x = -2$ and $x = \frac{1}{2}$

Therefore -3,-2 and $\frac{1}{2}$ are the only (rational) zeros of $g(x)$

c. $g(x) = \frac{1}{2}x^4 - 2x^2 - \frac{1}{2}x^2 + 2x$

let $h(x) = 2g(x)$.thus $h(x)$ will have the same zeros, but has integer coefficients

$h(x) = x^4 - 4x^3 - x^2 + 4x$

x is a factor ,so $h(x) = x(x^3 - 4x^2 - x + 4) = xk(x)$

$k(x)$ has constant term of 4 and leading coefficient of 1.the possible rational zeros are $\pm 1, \pm 2, \pm 4$

Using the remainder theorem , $k(1) = 0, k(-1) = 0$ and $k(4) = 0$.so by the factor theorem

$k(x) = (x-1)(x+1)(x-4)$.hence $h(x) = x(x-1)(x+1)(x-4)$ and $g(x) = \frac{1}{2}h(x) = \frac{1}{2}(x-1)(x+1)(x-4)$

Therefore, the zero of $g(x)$ are $0, \pm 1$ and ± 4

d. $f(x) = 12x^3 - 16x^2 - 5x + 3$

Possible values of p are factors of 3.these are $\pm 1, \pm 3$

Possible values of q are factors of 12 .these are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}$, of these 18 possible rational zeros at most 3 can be the zeros of $f(x)$

Check that $f\left(\frac{3}{2}\right) = 0, f\left(\frac{1}{3}\right) = 0$ and $f\left(-\frac{1}{2}\right) = 0$

Using the factor theorem, we can factorize $f(x) = 12x^3 - 16x^2 - 5x + 3 = \left(x - \frac{3}{2}\right)\left(x - \frac{1}{3}\right)\left(x + \frac{1}{2}\right)$

$= (2x - 3)(3x - 1)(2x + 1)$, so $f(x) = 0$ at $x = \frac{3}{2}, x = \frac{1}{3}$ and $x = -\frac{1}{2}$

Therefore $\frac{3}{2}, \frac{1}{3}$, and $-\frac{1}{2}$ are the only (rational) zeros of $f(x)$

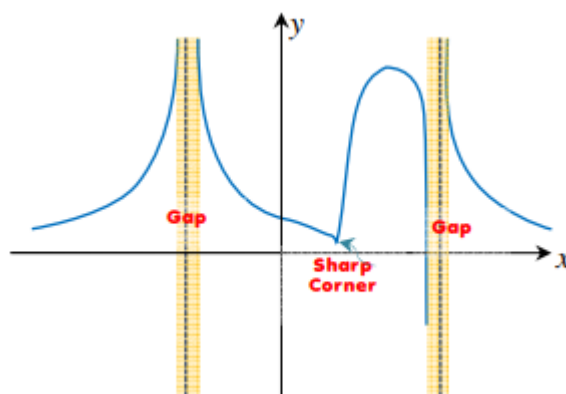
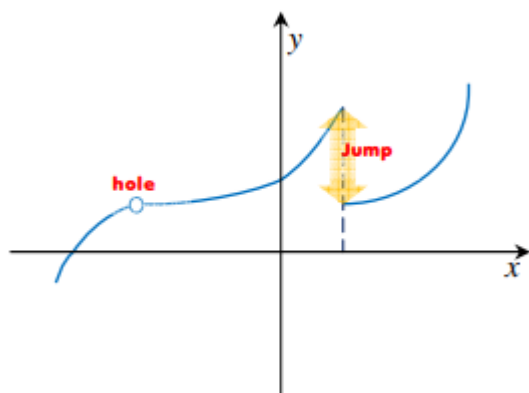
1.4 GRAPHS OF POLYNOMIAL FUNCTIONS

ACTIVITY

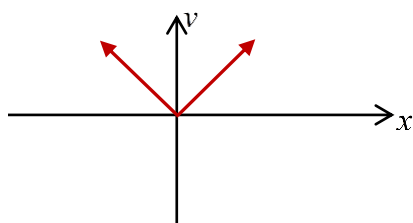
- sketch the graph of each of the following polynomial functions:
 - $f(x) = 3$
 - $f(x) = -2.5$
 - $g(x) = x - 2$
 - $g(x) = -3x + 1$
- Let $f(x) = x^2 - 4x + 5$
 - Copy and complete the table of values given below.

X	-2	-1	0	1	2	3	4
$f(x) = x^2 - 4x + 5$							

- Plot the points with coordinates (x, y) , where $y = f(x)$ on the xy -coordinate plane.
 - Join the points in b above by a smooth curve to get the graph of f . what do you call the graph of f ? Given the domain and range of f .
- Construct a table of values for each of the following polynomial functions and sketch the graph:
 - $f(x) = x^2 - 3$
 - $g(x) = -x^2 - 2x + 1$
 - $h(x) = x^3$
 - $p(x) = 1 - x^4$
 - ✓ The graph of polynomial function has no jumps, gaps and holes
 - ✓ It has no sharp corners
 - ✓ The graph of a polynomial function is a smooth and continuous curve which means there is no break anywhere on the graph
 - ✓ The following are not graphs of polynomial



- ✓ Functions with graphs that are not continuous are not polynomial functions
- ✓ The graph of the function $f(x) = |x|$: it has a sharp corner at the point $(0, 0)$ and hence $f(x) = |x|$ is not a polynomial function



If is not a polynomial function

- ✓ The point of which the graph of a function crosses (meets) the coordinates axes are important to note.
- ✓ If the graph of a function crosses the x-axis at $(x, 0)$, then x , is the x-intercept of the graph
- ✓ If the graph of f crosses the y-axis at the point $(0, y)$, then y , is the y-intercept of the graph of f .
 - $f(x) = ax + b, a \neq 0$
 - x - intercept of $f(x) = 0$
 - $ax + b = 0, x = \frac{-b}{a}$
 - x - int, $x = \frac{-b}{a}$
 - y - intercept, $x = 0$
 - $y = b$ is the y-intercept

Example 30:

Find x and y intercept of the following and draw the graph

a. $f(x) = -3x + 5$

Solution:

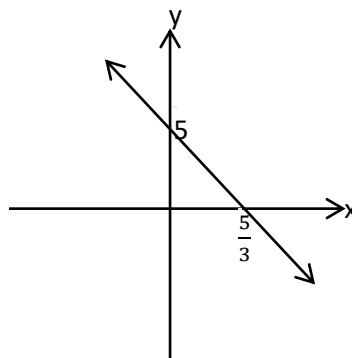
x - intercept: $y = f(x) = 0 = -3x + 5$

$$\Rightarrow -3x = -5 \Rightarrow x = \frac{5}{3}$$

y - intercept: $x = 0 \Rightarrow f(0) = -3(0) + 5 = 5$

Domain is the set of real numbers.

Range is the set of real numbers.



b. $f(x) = x^2 - 4x + 3$

Solution:

x - intercept: $y = f(x) = 0 = x^2 - 4x + 3$

$$\Rightarrow x = 1 \text{ or } x = 3$$

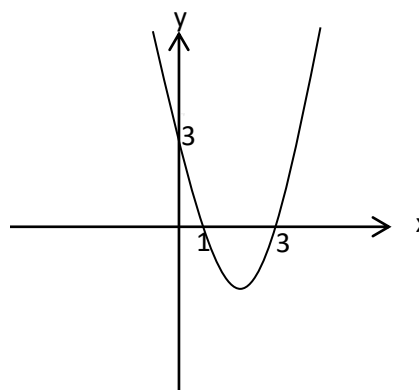
y - intercept: $x = 0 \Rightarrow f(0) = 3$

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(\frac{4}{2}, f\left(\frac{4}{2}\right)\right)$

$$\Rightarrow (2, f(2)) = (2, -1)$$

Domain is the set of real numbers.

Range $= [-1, \infty)$.



Note: the graph of any quadratic function $f(x) = ax^2 + bx + c$ has at most two x-intercept and one y-intercept

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note: the graph of a constant polynomial (polynomial of degree zero) is always horizontal line below or above x-axis depending on the sign of the constant term

Graph

Properties of $f(x) = c, c \in R$ graph

- ❖ $f(x)$ is a constant function
- ❖ its graph is a straight line parallel to the x axis passing through the point $(0, c)$
- ❖ the domain of f is real number
- ❖ the range is $\{c\}$

Example 31: draw the graph of the following

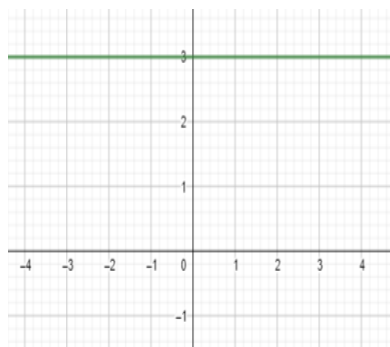
a. $f(x) = 3$

b. $f(x) = -2$

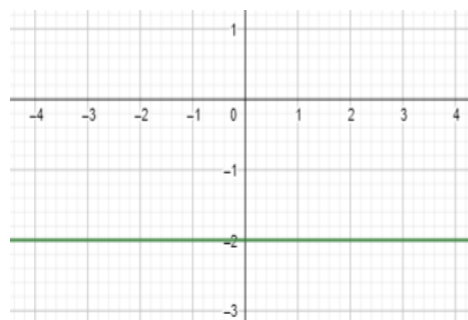
c. $f(x) = 0$

Solution:

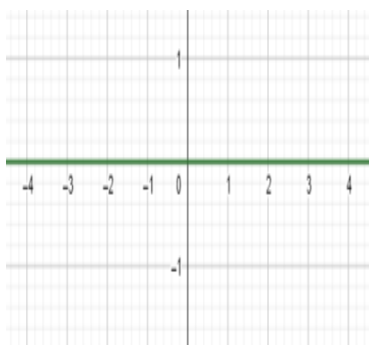
a.



b.



c.



Properties of $f(x) = ax + b$, $a \neq 0$ graph

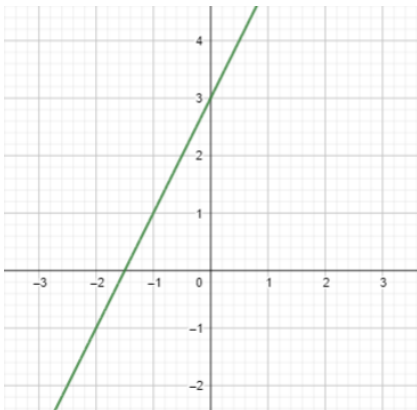
- its graph is a straight line
- the domain of f is real number
- the range of f is real number
- x- intercept of f is $\left(\frac{-b}{a}, 0\right)$
- Y-intercept of f is $(0, b)$
- slope of the graph is a
- is $a > 0$ the function is increasing
- is $a < 0$ the function is decreasing

Example 32: draw the graph of the following

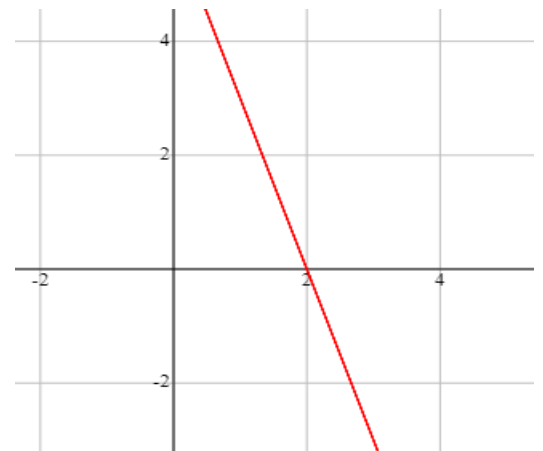
- a. $f(x) = 2x + 3$ b. $f(x) = -3x + 6$ c. $f(x) = 2x$ d. $f(x) = -3x$

Solution:

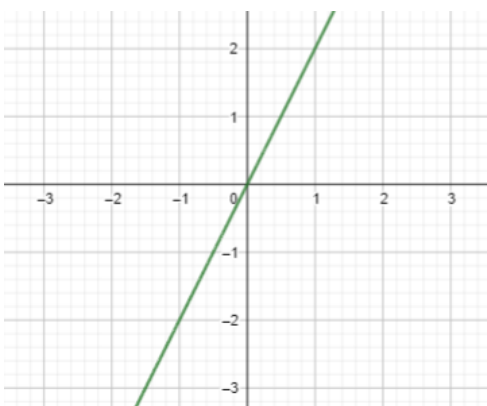
a.



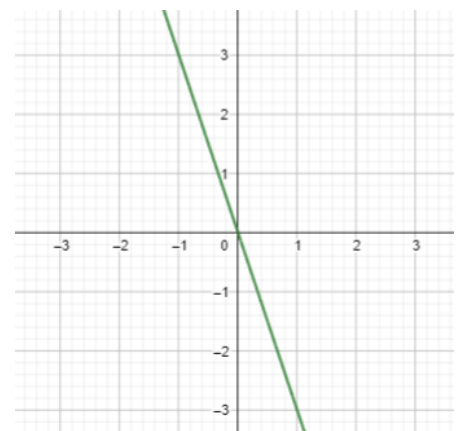
b.



c.



d.



Graph of quadratic function $f(x) = ax^2 + bx + c$, a, b , and $c \in R$, $a \neq 0$

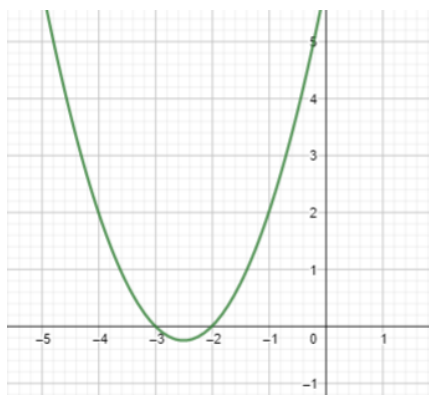
- ✓ Graph of quadratic function is a curve known as parabola
- ✓ If $a > 0$ the parabola opens up ward
- ✓ If $a < 0$ the parabola opens down ward
- ✓ Vertex of the parabola (turning point) $v = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a} \right)$
- ✓ The domain of the function is real number
- ✓ The range of the function is $y \geq \frac{4ac-b^2}{4a}$, $a > 0$, $y \leq \frac{4ac-b^2}{4a}$, $a < 0$

Example 33: draw the graph of the following

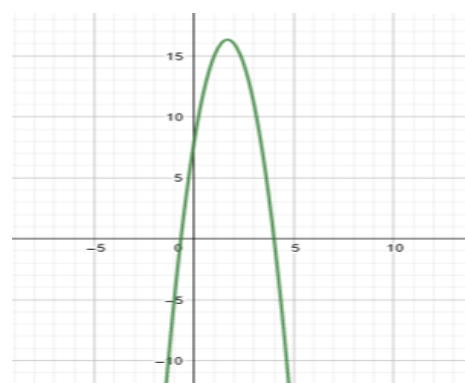
- a. $f(x) = x^2 + 5x + 6$ b. $f(x) = -3x^2 + 10x + 8$ c. $f(x) = x^2 - 2x$ d. $f(x) = -2x^2 + 2$ e. $f(x) = 3x^2$

Solution:

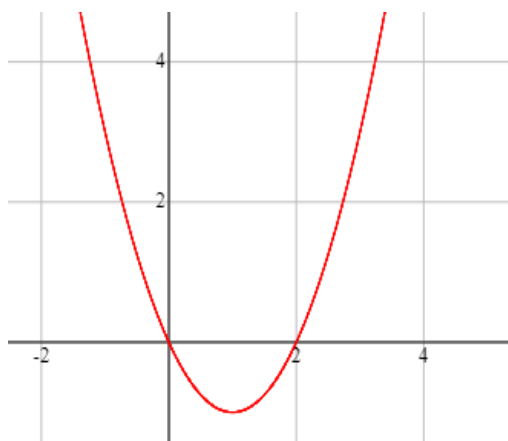
a.



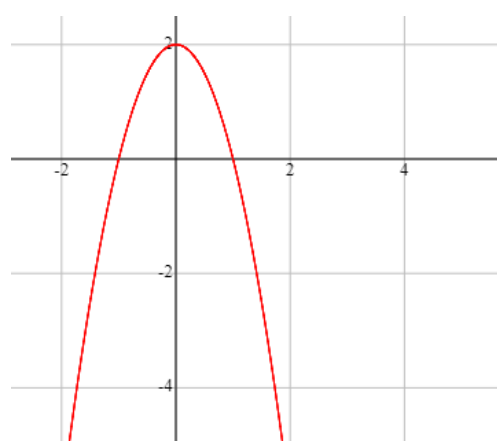
b.



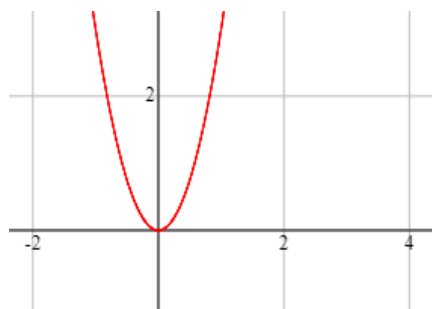
c.



d.

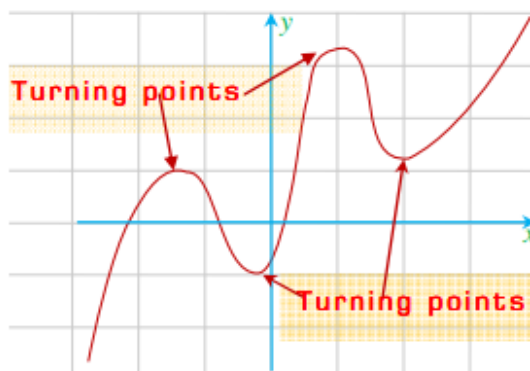


e.



Note: the graph of polynomial function of degree n meets the x – axis at most n times

- Every polynomial function of degree n has at most n zeros
- The graph of polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ eventually rises or falls



The above graph shows an example of a polynomial function whose graphs has peaks and valleys. The term peak refers to a local maximum and valley refers to a local minimum. Such a point is often called a turning point of the graph.

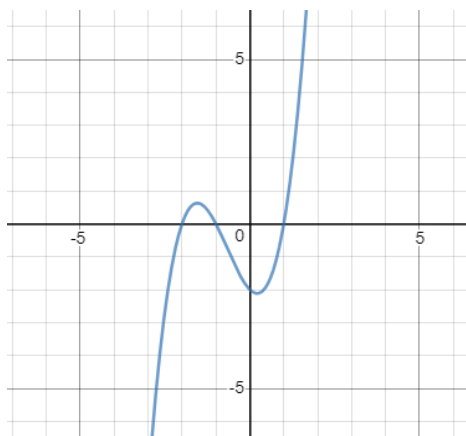
- a point of f that is either a maximum point or minimum point on its domain is called a local extremum point of f .
- the graph of a polynomial function of degree n has at most $n - 1$ turning points. The above observation can be generalized as follows.
 1. if c is a zero of odd multiplicity of a function f , then the graph of the function crosses the x -axis at $x = c$ and does not have a relative extremum at $x = c$
 2. if c is a zero of even multiplicity, then the graph of the function touches (but does not cross) the x – axis at $x = c$ and has a local extremum at $x = c$

Example 34: Draw the graph of the following

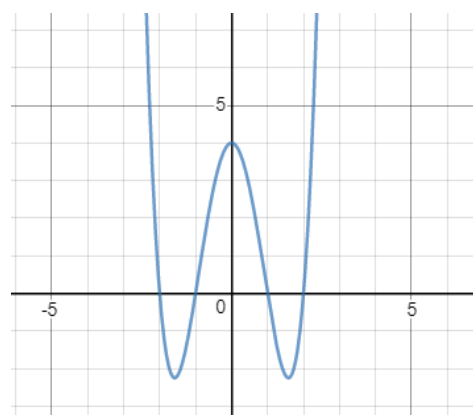
- | | | |
|---------------------------------|---------------------------------|------------------------------|
| a. $f(x) = x^3 + 2x^2 - x - 2$ | b. $f(x) = x^4 - 5x^2 + 4$ | c. $f(x) = -x^4 - 3x^3 + 4x$ |
| d. $g(x) = x(x - 2)^2(x + 2)^4$ | e. $f(x) = x^3 + x^2 - 6x - 10$ | f. $f(x) = 2x^2 - 2x^4$ |

Solution:

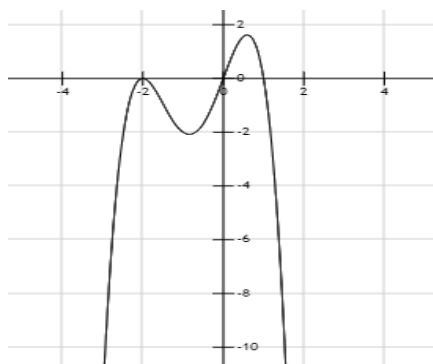
a.



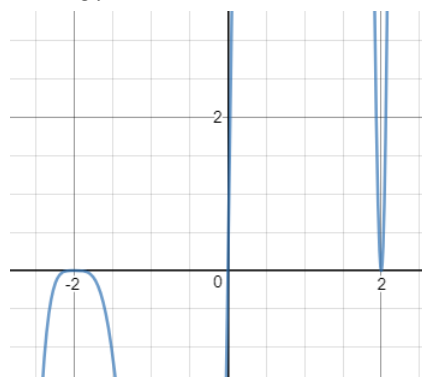
b.



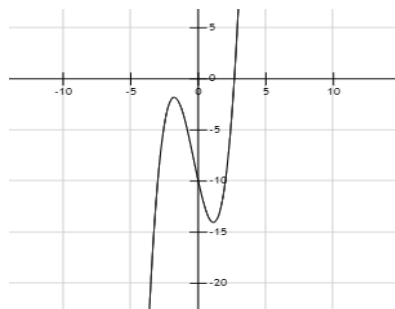
c.



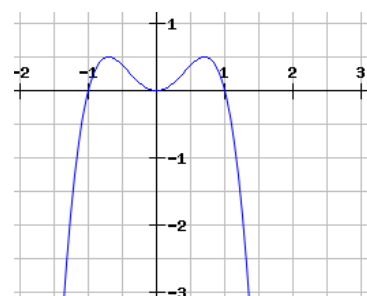
d.



e.



f.



Note:

1.

The graph of a polynomial function with leading term $a_n x^n$ has the following right and left behaviour:		
	n – even	n – odd
$a_n > 0$	Up to left and up to right	Down to left and up to right
$a_n < 0$	Down to left and down to right	Up to left and down to right

- If the multiplicity of the root c is an **odd** number, then the graph of the function **crosses** the x – axis at $x=c$.
- If the multiplicity of the root c is an **even** number, then the graph of the function **touches**(**but does not cross**) the x – axis at $x=c$.

End Behavior

If the value of a is positive for the quadratic function $p(x) = ax^2 + bx + c$, the graph opens upward. If a is negative, the graph opens downward. The sign of a determines the **end behavior** of the graph.

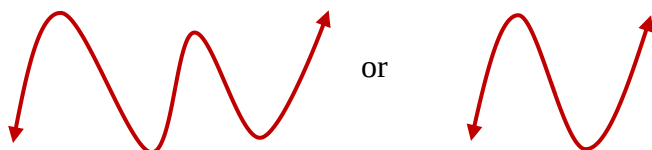
In general, the end behavior of the graph of a polynomial function is determined by the sign of the leading coefficient and the parity (odd or even) of the degree.

End Behavior of Graphs of Polynomial Functions

Suppose that ax^n is the dominating term of a polynomial function P of **odd degree**. (The **dominating term** is the term of greatest degree.)

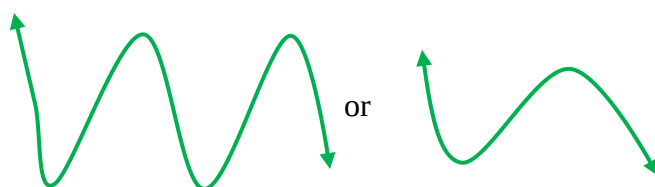
- If $a > 0$, then as $x \rightarrow \infty$, $p(x) \rightarrow \infty$ and as $x \rightarrow -\infty$, $p(x) \rightarrow -\infty$.

Therefore, the end behavior of the graph is of the type shown in **figure** below.



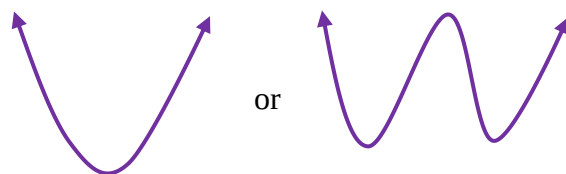
- If $a < 0$, then as $x \rightarrow \infty$, $P(x) \rightarrow -\infty$ and as $x \rightarrow -\infty$, $P(x) \rightarrow \infty$

Therefore, the end behavior of the graph is of the type shown in **figure** below.

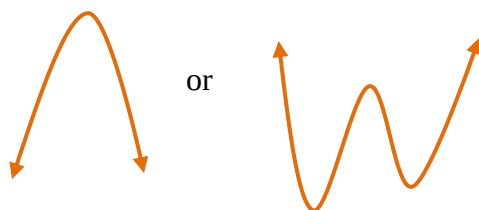


Suppose that ax^n is the dominating term of a polynomial function P of **even degree**.

1. If $a > 0$, then as $|x| \rightarrow \infty, p(x) \rightarrow \infty$. Therefore, the end behavior of the graph is of the type shown in **figure** below.



2. If $a < 0$, then as $|x| \rightarrow \infty, p(x) \rightarrow -\infty$. Therefore, the end behavior of the graph is of the type shown below.



Example 35: sketch the graph of the following:

a. $f(x) = \frac{1}{2}(x+3)^2(x^2-1)$

c. $h(x) = -x(x-3)^2(2x+3)$

b. $g(x) = \frac{1}{4}(1-x)(x^2+1)(x-2)$

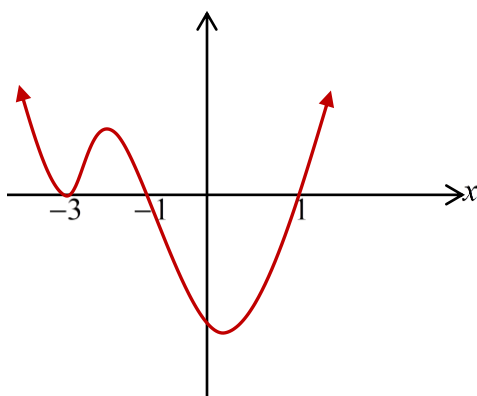
Solution:

- To draw the graph of $f(x) = \frac{1}{2}(x+3)^2(x^2-1) = \frac{1}{2}(x+3)^2(x-1)(x+1)$ let us use the sign chart:

The zeros of f are -3 , -1 and 1 . We are now ready to construct the sign chart of the function and write the zeros in ascending order in horizontal line as follows:

	-3	-1	1
$x+3$	-	0	+
$(x+3)^2$	+	0	+
$x-1$	-	-	0
$x+1$	-	0	+
$(x-1)(x+1) = (x^2-1)$	+	0	+
$f(x) = \frac{1}{2}(x+3)^2(x^2-1)$	+	0	+

Since the algebraic sign of the function f is positive on the interval $(-\infty, -3) \cup (-3, -1) \cup (1, \infty)$ the graph of the function lies above the x -axis and the sign of f is negative on $(-1, 1)$ the graph of the function lies below the x -axis.

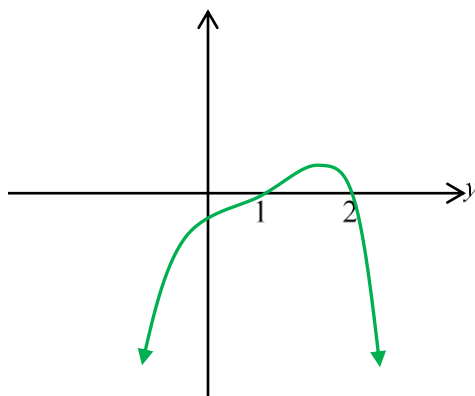


- b. To draw the graph of $g(x) = \frac{1}{4}(1-x)(x^2+1)(x-2)$ let us use the sign chart:

The zeros of g are 1 and 2, but $x^2 + 1$ has no real solution. We are now ready to construct the sign chart of the function and write the zeros in ascending order in horizontal line as follows:

		1		2	
$1-x$	+	0	-		-
(x^2+1)	+		+		+
$x-2$	-		-	0	+
$g(x) = \frac{1}{4}(1-x)(x^2+1)(x-2)$	-	0	+	0	-

Since the algebraic sign of the function g is positive on the interval $(1, 2)$ the graph of the function lies above the x -axis and the sign of g is negative on $(-\infty, 1) \cup (2, \infty)$, so the graph of the function lies below the x -axis.

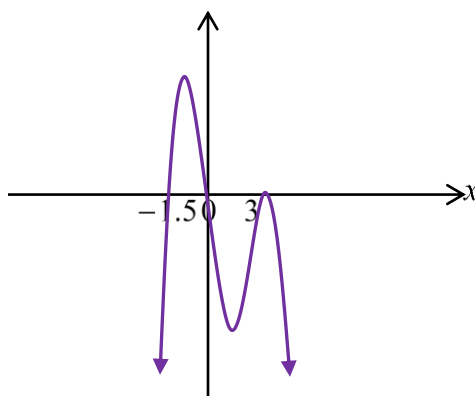


c. To draw the graph of $h(x) = -x(x-3)^2(2x+3)$ let us use the sign chart:

The zeros of h are $-\frac{3}{2}$, 0 and 3. Now write the zeros in ascending order in horizontal line as follows:

	$-\frac{3}{2}$	0	3
$-x$	+	+	-
$(x-3)^2$	+	+	+
$(2x+3)$	-	+	+
$h(x) = -x(x-3)^2(2x+3)$	-	+	-

Since the algebraic sign of the function h is positive on the interval $\left(-\frac{3}{2}, 0\right)$ the graph of the function lies above the x -axis and the sign of h is negative on $\left(-\infty, -\frac{3}{2}\right) \cup (0, 3) \cup (3, \infty)$ the graph of the function lies below the x -axis.



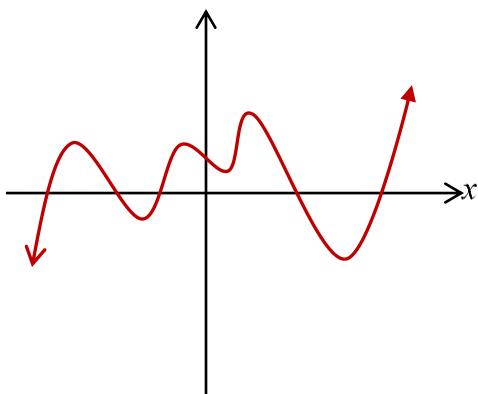
Note:

1. A function f is said to be even if and only if the domain of f is symmetric set i.e. for $x \in S$, $-x \in S$ and $f(x) = f(-x)$ for each x in its domain. The graph of even function is symmetric with respect to y -axis.
2. A function f is said to be odd if and only if the domain of f is symmetric set i.e. for $x \in S$, $-x \in S$ and $f(-x) = -f(x)$ for each x in its domain. The graph of even function is symmetric with respect to the origin.

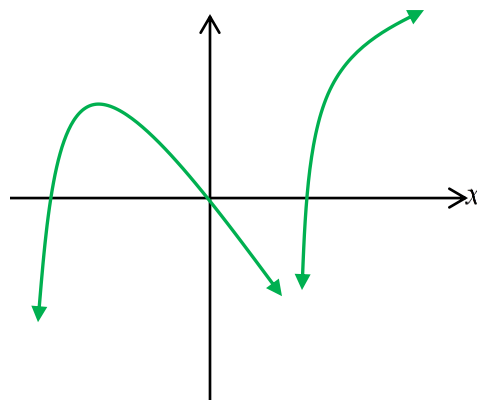
EXERCISE 1.7

1. In each of the following, decide whether the given graph could possibly be the graph of a polynomial.

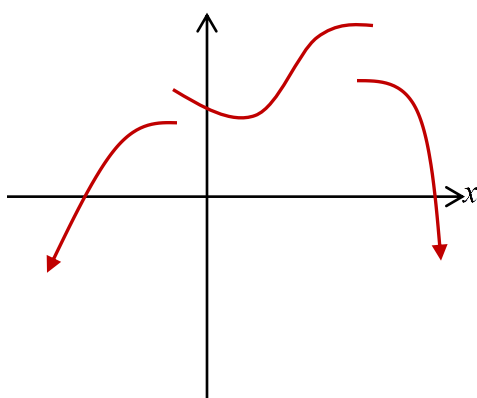
a.



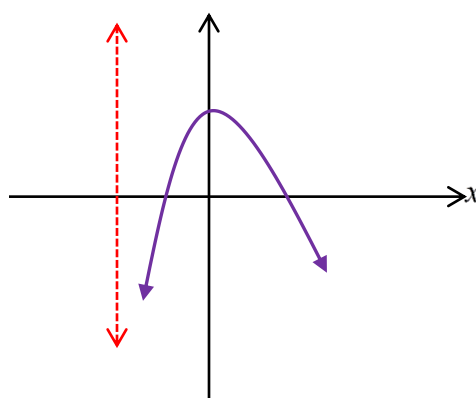
b.



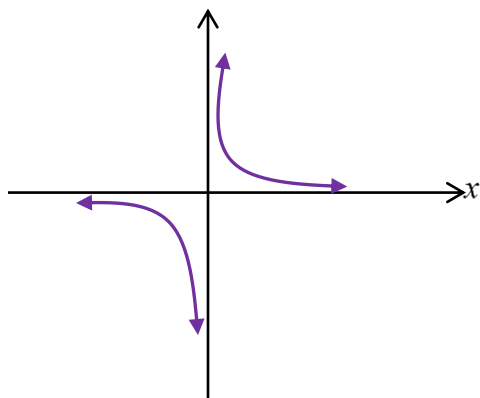
c.



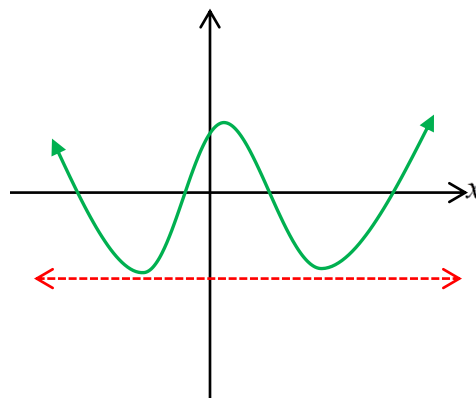
d.



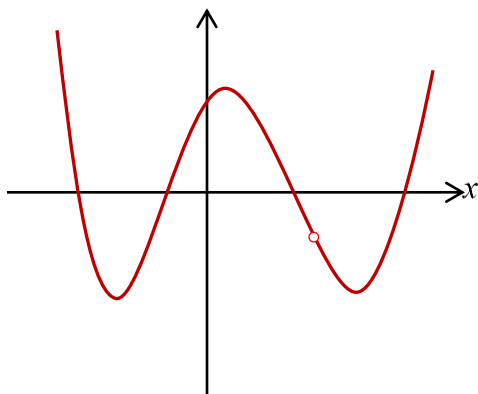
e.



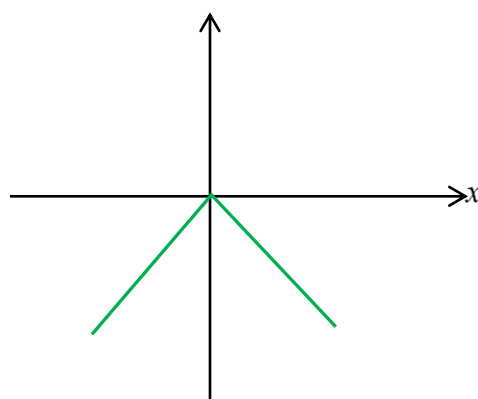
f.



g.

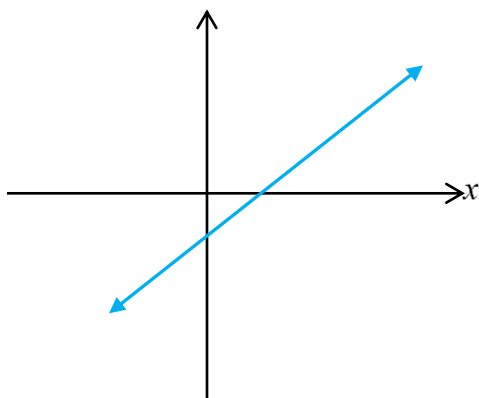


h.

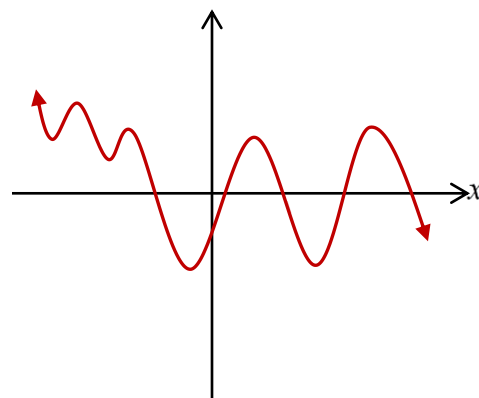


2. Find the possible degree for each of the following graphs and indicate the sign of the leading coefficient.

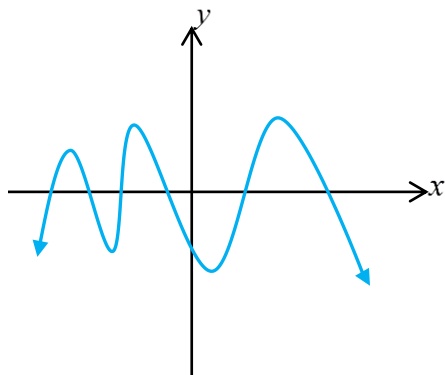
a.



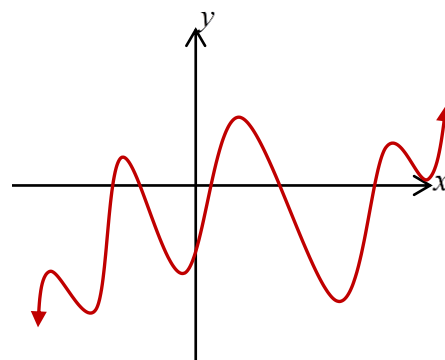
b.



c.



d.



3. In each of the following, factor the given function (if necessary) and sketch its graph.

a. $f(x) = (x-1)(x+5)(x+2)$

b. $g(x) = -x(x-1)(x+2)^2$

c. $h(x) = x^4 - 5x^2 + 4$

4. Write down a polynomial function with roots:

a. 1, 2, 3, 4

b. 2, -4

c. 12, -1, -6

5. Make a table of values and draw the graph of each of the following polynomial functions:

a. $f(x) = 4x^2 - 11x + 3$

c. $f(x) = \frac{1}{4}(x-2)^2(x+2)^2$

b. $f(x) = -1 - x^2$

d. $f(x) = x^3 + x^2 - 6x + 3 - 10$



Key Terms

constant function	linear function	rational root
constant term	local extremum	remainder theorem
degree	location theorem	turning points
domain	multiplicity	x – intercept
factor theorem	polynomial division theorem	y – intercept
leading coefficient	polynomial function	zero(s) of a polynomial
leading term	quadratic function	



Summary

1. A polynomial is an expression that can be written in the form:
 $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$, where n is a non-negative integer and x is a variable.
2. A polynomial function is a function whose rule is given by a polynomial.
3. The domain of any polynomial function is the set of real numbers.
4. If polynomial function $f(x)$ is divided by a non-zero polynomial $d(x)$, then there is a quotient $q(x)$ and a remainder $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where either $r(x) = 0$ or has a degree less than the degree of the divisor $d(x)$.
5. The remainder in polynomial division is 0 when the divisor is a factor of the dividend. In this case the other factor is the quotient.
6. If a polynomial function $f(x)$ is divided by a first degree polynomial of the form $x - c$, then the remainder is the number $f(c)$.
7. The number c is the zero of the polynomial $f(x)$ if and only if $x - c$ is factor of $f(x)$.
8. If a rational number $\frac{p}{q}$ (in lowest form) is a zero of the polynomial
 $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$, where coefficients $a_n, a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are integers, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n .
9. The graph of a polynomial function is far from the x –axis when $|x|$ is large.
10. The graph of a polynomial function of degree n , meets x –axis, at most n times.
11. If the multiplicity of the root c of a polynomial function is an odd number, then the graph of the function crosses the x –axis at $x = c$.
12. If the multiplicity of the root c of a polynomial function is an even number, then the graph of the function touches, but does not cross the x –axis at $x = c$.
13. The graph of a polynomial function of degree n has at most $n - 1$ turning points.
14. The graph of every polynomial function has no sharp corners; it is a smooth and continuous curve.
15. Let a and b be real numbers such that $a < b$. If $f(a)$ and $f(b)$ have opposite signs, then the polynomial function $f(x)$ has a root between a and b . Consequently, if $f(x)$ has no roots in an interval, then either $f(x)$ is positive for every x in an interval or $f(x)$ is negative for every x in an interval.

REVIEW EXERCISES ON UNIT 1

1. In each of the following state the quotient and remainder when the first polynomial is divided by the second.

a. $5x^3 + 7x - 4; x - 3$

e. $3x^3 - 2x^2 - 27x - 18; x + 1$

b. $4x^3 - 7x^2 + 2x - 3; x^2 + 2x - 1$

f. $3x^4 + 2x^2 - 6x + 1; x + 1$

c. $x^3 - 1; x + 1$

g. $x^5 + 2x^4 - 6x^3 + x^2 - 5x + 1; x^3 + 1$

d. $2x^3 + 5x^2 + 11x - 1; x^2 - x + 3$

i. $1 + x + 5x^5 - 4x^4 + 3x^3 - 2x^2; 2 + x + x^2 + x^3$

e. $5x^5 + x^2 + 3x^3 - 4x^4 - 10; 2 + x^2 - x$

2. Prove the following:

When a polynomial $f(x)$ is divided by a first degree polynomial $ax + b$, the remainder is $f\left(-\frac{b}{a}\right)$.

3. Which of the following expressions are polynomials

b. $\frac{(x-1)(x+1)}{x+1}$

h. $2^x + 3x^2$

o. $\sqrt{x^2 + 2x + 1}$

c. $\frac{2}{3}\pi - \sqrt{3}$

i. $\frac{x^3 - 5x^2 + x - 5}{x^2 + 1}$

p. $\sqrt[3]{x^3}$

d. $\sqrt{(x+3)^2}$

j. $|x - 2| + 4$

q. $\sqrt{x^2}$

e. $x - 2x^{-3} + x^2 + 6$

k. $\frac{1}{2\sqrt{2}-3}x$

r. $\sqrt{(x^2 + 1)^2}$

f. $x^3 + \frac{1}{x} - 12x + 6$

l. $\frac{3}{6+x}$

s. $\frac{4x^2(x^4 + x^2)}{(x^2 + 1)}$

g. $\sqrt{2x} + 9x^2 - 7x$

m. $\frac{3}{t} + \frac{5}{t^2}$

t. $\sqrt{x^2} + 2x + 2$

h. $\sqrt{3}x - \frac{4}{3}x^2 + \frac{7}{2}$

n. $\sqrt{9x^4 + 6x^2 + 1}$

4. For each of the following polynomials state the **degree**, the **leading coefficient**, **leading term** and **constant term**

a. $f(x) = x(1 - x)(3x^2 + 2)$

f. $f(x) = (2x^2 - 3)^4$

b. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - 1$

g. $f(x) = x^2(x^2 - 4x + 4)(2x^2 - x^3)$

c. $g(x) = \frac{(x-1)(3-x^2)}{3} + \frac{x^2(2x-3)}{6} - 5x + 1$

d. $g(x) = (x + 2)(x - 4)(x - 2)^2$

e. $k(x) = \frac{x-3x^5}{3} + \frac{3\sqrt{3}x^4-12x}{6} - x^2 + x^3 + 8x - 4$

5. For any polynomial function f , which of the following are defined polynomial and which are not?

a. $f \times f$

e. $\frac{f}{f}$

b. $\sqrt{f^2}$

f. $f^2 - f^3 + 3f$

c. $\sqrt{2f}$

g. $3f - 2$

d. $\sqrt[3]{f^3}$

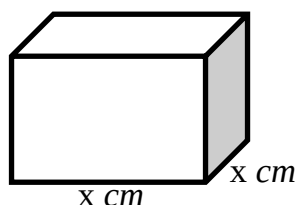
6. Find the sum $p(x) + q(x)$, the difference $p(x) - q(x)$ and the product $p(x) \times q(x)$
 - a. $p(x) = 3x^2 - 5x + 2, q(x) = x^2 + 7x - 3$
 - b. $p(x) = 2x^2 + 7, q(x) = 2x^3 - 5x + 2$
 - c. $p(x) = x + 5, q(x) = x^3 - 3x + 7$
 - d. $p(x) = 1 - 5x, q(x) = x^3 + 7$
 - e. $p(x) = 4x + 3, q(x) = 2x^3 + 5x^2 - 7x + 1$
7. Factorize of the following polynomials

a. $x^3 + 3x^2 - 6x - 8$	f. $4x^3 + 3x^2 - x$
b. $x^3 - 7x^2 + 36$	g. $x^4 - 3x^2 - 4$
c. $x^4 + 2x^3 - 3x^2 - 4x + 4$	h. $x^4 - 4$
d. $x^4 - x^3 - 7x^2 + x + 6$	i. $25 - 4x^2$
e. $15x^3 - 6x^2 - 5x + 2$	j. $x^8 - 1$
8. When the polynomial $f(x) = 4x^5 - 2hx^3 + x^2 - 5$ divided by $x + 1$, it leaves remainder 2. find the remainder when $f(x)$ divided by $2x - 1$
9. If $p(x) = x^3 + ax^2 + bx - 6$ is divided by $x+1$ and $x-2$. find the values of a and b
10. $x + 1$ is a factor of $f(x) = x^3 + mx^2 + nx - 3$, and when $f(x)$ is divided by $x+3$ the remainder is - 24. find m and n
11. If $x^2 - 4$ is a factor of the polynomial $f(x) = x^3 - 3nx^2 + (1 - k)x - 4k$, then find $h^2 + k^2$
12. What number should be added to the polynomial $2x^3 - x^2 + 5x - 1$, so that the resulting polynomial is divisible by $2x - 1$
13. When the polynomial $p(x) = ax^4 - bx^3 + cx - 8$ is divided by $x-1$, the remainder is 2, divided by $x-2$ the remainder is 14 and divided by $x+1$ the remainder is 0, find the values of a, b and c
14. When a polynomial function $p(x) = mx^3 + 4x^2 - nx - 3$ divided by $x + 1$, it gives remainder $5m + 4$ and $p(x)$ is divided by $x - 2$. then find the values of m and n
15. Solve the following polynomial equation
 - a. $x^3 - 3x^2 - 10x + 24 = 0$
 - b. $x^4 - 2x^2 + 1 = 0$
 - c. $(x^2 - 9)^4 - 1 = 0$
 - d. $\left(x - \frac{1}{x}\right)^2 + 3\left(x - \frac{1}{x}\right) + 2 = 0$
 - e. $(x^2 + 2x - 3)^2 + (x^2 + 2x - 3) - 2 = 0$
 - f. $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$
16. Find a polynomial function f of degree 3 such that $f(10) = 17$ and the zeros of f are 0, 5 and 8
17. If $p(x)$ is a polynomial of degree 3 with $p(0) = p(1) = p(-1) = 0$ and $p(2) = 6$ then
 - a. Show that $p(-x) = -p(x)$
 - b. Find the interval in which $p(x)$ is less than zero
18. In each of the following, find the zeros and indicate the multiplicity of each zero
 - a. $f(x) = (x + 6)(x - 3)^2$
 - b. $f(x) = x^4 - 4x^3 + 7x^2 - 12x + 12$
 - c. $f(x) = \frac{1}{2}(x - 2)^4(x + 3)^3(1 - x)$
 - d. $g(x) = -5(x^2 - 1)^3(x + 1)^2(x^2 + 2x + 1)$

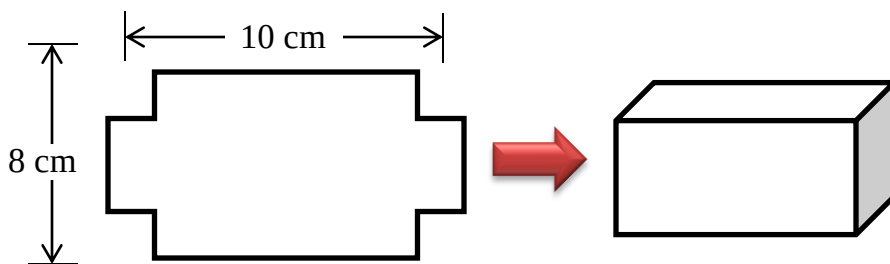
- e. $h(x) = (x^3 - 2x^2)^2(x^2 + 4)(x^2 - 4)$
- f. $f(x) = x^3 - 3x^2$
- g. $f(x) = (x^2 - 2x)(x^2 - 4)^2$

19. A box is to be built with a square base and height 2cm more than the length of the base

- a. Write an expression for the volume of the box
- b. If the volume is to be 45cm^3 , find a polynomial equation from which the dimension of the box can be found
- c. Find the dimensions that will give this volume



20. Four identical squares are cut from the corners of a rectangular sheet of metal 10cm by 8cm .the sides are folded along the dotted lines to form a rectangular box, as shown



- a. Find an expression for the volume of the box
 - b. Find the side length of the square that should be cut from each corner if the volume is to be 48cm^3
 - c. Find the dimensions of the rectangular box with this volume
21. Sketch the graph of the following polynomial functions
- a. $p(x) = (x + 2)(2x - 3)(x^2 - 1)$
 - b. $p(x) = (2x - 3)(x^3 + x^2 - 4x - 4)$
 - c. $p(x) = -x^3 - 3x^2 - 2x$
 - d. $p(x) = x^4 - 5x^2 + 4$
 - e. $p(x) = x^3 - 7x + 6$
 - f. $g(x) = (x + 1)^3(x - 2)^2(x + 3)$
 - g. $h(x) = 4x^4 - 15x^2 + 5x + 6$
 - h. $k(x) = 3x^4 - 3x^2 + 2$
 - i. $f(x) = 5x^6 - 7x^2$
 - j. $h(x) = 5x^7 - 7x^5 + 2x$

PRACTICE QUESTIONS ON UNIT 1

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

- Which of the following is **NOT** a polynomial?

A. $P(x) = |x - 2| + 4$

B. $P(x) = (-3x)\left(1 - \frac{2x}{5}\right)$

C. $P(x) = \frac{x^3 - 5x^2 + x - 5}{x^2 + 1}$

D. $P(x) = \frac{x^{50}}{3} + \sqrt{17}x^5 + \pi$
- Which of the following is true about the polynomial $P(x) = -x^2 + 6x - 8$?

A. It has no real zero.

B. Its maximum value is 1.

C. Its minimum value is -1 .

D. It has neither minimum nor maximum values.
- Let $P(x)$ be polynomial. If $P(x+t) = P(x) + P(t)$ for any two real numbers x and t , then which of the following is necessarily the form of $P(x)$?

A. $P(x) = c$ for some constant c .

B. $P(x) = x + c$ for some non-zero constant c .

C. $P(x) = ax$ for some constant a .

D. None of the above.
- If $P(x)$ is a polynomial and $Q(x) = P(x-1)$, then which of the following is **NOT** necessarily true?

A. If b is a zero of $Q(x)$, then $b-1$ is a zero of $P(x)$.

B. If b is a zero of $P(x)$, then $b+1$ is a zero of $Q(x)$.

C. If $x-1$ is a factor of $P(x)$, then $x-2$ is a factor of $Q(x)$.

D. $Q(x)$ and $P(x)$ have the same constant term.
- If $P(x)$ is a polynomial of degree three with $P(0) = P(1) = P(-1) = 0$ and $P(2) = 6$, then which of the following is **NOT** true?

A. $P(-x) = -P(x)$

B. $P\left(\frac{1}{x}\right) = \frac{1-x^2}{x^3}, x \neq 0$.

C. $P(3) = 48$.

D. The graph of P lies below x -axis for every x between 0 and 1.
- Which of the following is true about a polynomial function $f(x)$ of degree n ?

A. Its graph crosses the y -axis at exactly one point.

B. Its graph crosses the x -axis at least at n points.

C. $f(a)$ is the remainder if $f(x)$ is divided by $x+a$

D. None of the above.
- Let $f(x) = x^4 - 5x^2 + 4$. Which of the following is **NOT** true about the graph of $f(x)$?

A. 1 is an x -intercept

B. The graph crosses the x -axis four times.

C. The graph goes up without limit as x approaches positive infinity.

D. The range of the function is $(-2, \infty)$
- Which one of the following is **NOT** zero of the polynomial function $g(x) = 2x^3 - x^2 - 10x + 5$?

A. $\frac{1}{2}$

B. $\sqrt{5}$

C. $-\sqrt{5}$

D. 2

9. When divide the polynomial $f(x) = 3x^3 + 2x^2 - 19x + 6$ by $x - 1$, the remainder that you will get is _____
 A. 0 B. -8 C. 24 D. 8
10. For the polynomial functions $f(x) = x^2 - 2x + 2$ and $g(x) = x^2 + 2x + 2$, the product $(f \cdot g)(x) =$ _____.
 A. $x^4 + 4$ C. $x^4 - 4x^3 - 8x^2 - 8x + 4$
 B. $x^4 - 4x^3 - 6x^2 - 8x + 4$ D. $x^4 - 8x^2 + 4$
11. If $f(x)$ and $d(x)$ are polynomials, such that $d(x) \neq 0$ and degree of $d(x)$ is less than or equal to that of $f(x)$, then after long division of $f(x)$ by $d(x)$ there exist unique polynomials $q(x)$ and $r(x)$, such that $f(x) = d(x) \cdot q(x) + r(x)$. Which one of the following is correct about $r(x)$?
 A. degree of $r(x) < \text{degree of } q(x)$ C. $r(x) = 0$ or degree of $r(x) \leq \text{degree of } d(x)$
 B. degree of $r(x) \geq \text{degree of } d(x)$ D. $r(x) = 0$ or degree of $r(x) < \text{degree of } d(x)$
12. A polynomial function p is given by $p(x) = \frac{3x^3 - 4x^5 + x^2 - 6x + 1}{12} + \frac{x}{2} - x^4$. From the following statement which one is true about p ?
 A. In $p(x)$ the coefficient of x is 0 C. Its constant term is $\left(-\frac{1}{2}\right)$
 B. The degree of $p(x)$ is 3 D. Its leading coefficient is $\frac{1}{4}$
13. If $f(x)$ a polynomial of degree is n , where $n \geq 1$ and if c any real number, then which one of the following statements is true?
 A. If $f(c) = 0$ and $n = 1$, then $f(x) = k(x - c)$, for some non-zero real number k .
 B. If $f(c) = 0$ and $n = 2$, then $f(x) = (x - c)q(x) + r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree 1.
 C. If $f(c) = 0$ and $n > 1$, then $f(x) = (x - c)q(x) + r(x)$, where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is 1.
 D. If $f(c) = 0$ and $n > 2$, then $f(x) = (x - c)q(x)$, where $q(x)$ is a polynomial of degree 1.
14. What number must be added to $x^3 + 5x^2 + 6x + 4$ so that $x + 1$ is a factor?
 A. 2 B. -19 C. 19 D. -2
15. Let $f(x) = x^3 + 5x^2 - x + 8$ and $g(x) = x^2 + 2$. If $f(x)$ is divided by $g(x)$, then
 A. the constant term of the quotient is (-2) C. the remainder is $x + 5$
 B. the degree of the remainder is 1 D. the quotient is $3x + 2$
16. Let $f(x)$ and $g(x)$ be polynomials with $g(x) \neq 0$ and degree of $g(x) \leq \text{degree of } f(x)$. Therefore, according division theorem, there exist unique polynomials $q(x)$ and $r(x)$ such that,
 A. $f(x) = g(x) \cdot q(x) + r(x)$
 B. $f(x) = g(x) \cdot q(x) + r(x)$, and $r(x) = 0$ for any polynomials $f(x)$ and $g(x)$.
 C. $f(x) = g(x) \cdot q(x) + r(x)$, where degree of $r(x) < \text{degree of } g(x)$.
 D. $f(x) = g(x) \cdot q(x) + r(x)$, where degree of $r(x) > \text{degree of } g(x)$.

17. Which of the following function is neither even nor odd?

A. $k(x) = x^5$

C. $h(x) = 2x^2 - 3|x|$

B. $g(x) = x - 1$

D. $f(x) = x^4$

18. What are the zeros of the polynomial function $f(x) = x^3 + 3x^2 - x - 3$?

A. $-3, -1$ and 1

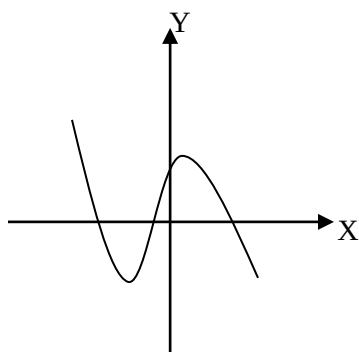
C. $-3, 2$ and 3

B. $-2, -1$ and 3

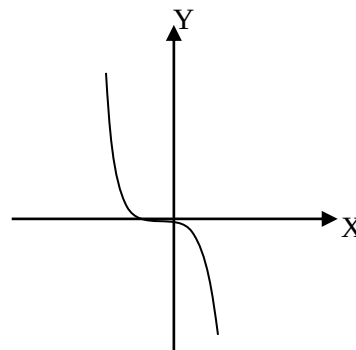
D. $-2, 1$ and 2

19. Which of the following can be the graph of a polynomial function of degree three with positive leading coefficient?

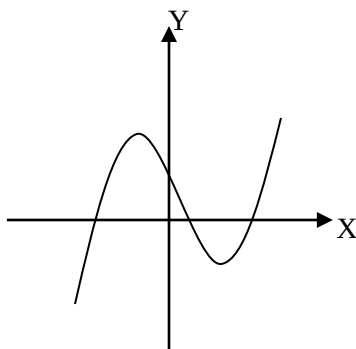
A.



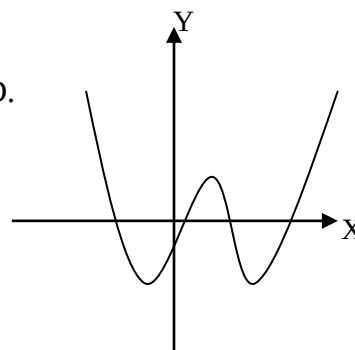
C.



B.



D.



20. The rational root of the polynomial equation $6x^4 + x^3 + 2x - 24 = 0$ is

A. $-\frac{3}{2}$

B. $\frac{7}{3}$

C. $-\frac{1}{3}$

D. $\frac{5}{2}$

21. Let $f(x) = (2x)(5x - 4) - 3x + 2$ and $g(x) = 5x^3 - x(x^2 - 2)$. Which of the following is **NOT** correct?

A. x is a factor of $(f \cdot g)(x)$

C. The degree of $(f \cdot g)(x)$ is 5

B. $\left(\frac{g}{f}\right)(1) = 6$

D. $(f \cdot g)(-1) = -6$

22. Which of the following is **NOT** true about the function $f(x) = x^{-5}$?

A. Its graph has no x-intercept.

C. It is a decreasing function.

B. It is an odd function.

D. $f(xy) = f(x)f(y)$

23. If f is a polynomial function given by $f(x) = 2x^3 + 4x^2 + 5x - 9$, then this function has root between_____.

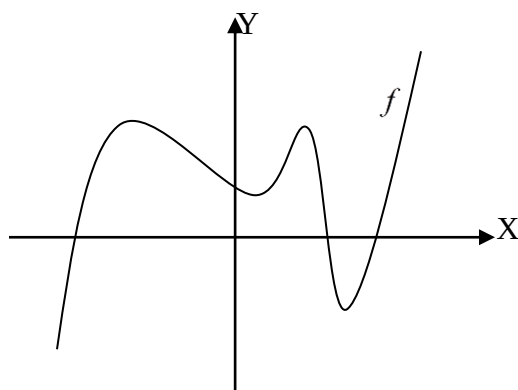
A. -1 and -2

C. 1 and 2

B. 0 and 1

D. -1 and 0

24. The graph of a certain polynomial function f is given below.



Which of the following is true about this function?

- A. Its constant term is negative.
 - B. The degree of f is even number.
 - C. Its leading coefficient is positive.
 - D. It has three local maximum.
25. Which one of the following is **NOT** true about the graph of the polynomial function $f(x) = 2x^5 - 7x^3 + 2x - 1$?
- A. It is a smooth and continuous curve.
 - B. It passes through the origin.
 - C. It has at most 4 turning points and intersects the x-axis at most 5 times.
 - D. As we move far from the origin, it rises upward to the right and falls down to the left.
26. Let p and q be polynomials given by $p(x) = 6x^5 + 4x^4 - 2x^3$ and $q(x) = 3x^2 + 2x - 1$. For $x \neq -1$ and $x \neq \frac{1}{3}$, which of the following is equal to $\frac{p(x)}{q(x)}$?
- A. $3x^3 + 2x + 1$
 - B. $2x^3 + 1$
 - C. $2x^3$
 - D. $3x^2$
27. Let f be a polynomial function in x . If the degree of $f(x)$ is greater than or equal to 1, then what will be the remainder when $f(x)$ is divided by $x + 3$?
- A. $x - 3$
 - B. 3
 - C. $f(-3)$
 - D. $f(3)$
28. If a polynomial function P is given by $P(x) = (3x + 2)(1 - 5x)(1 - x^2)$, then which one of the following statements is correct about $P(x)$?
- A. The degree of $P(x)$ is 3.
 - B. $P(x)$ is a polynomial over integer.
 - C. The leading coefficient of $P(x)$ is -15 .
 - D. The constant term of $P(x)$ is -2 .
29. Let $f(x) = ax^{10} + bx^5 - 1$. If $x + 1$ is a factor of $f(x)$ and when $f(x)$ is divided by $x - 1$ the remainder is 4, then a and b respectively equal to:
- A. $-1, 1$
 - B. $3, 2$
 - C. $1, 5$
 - D. $3, 1$
30. Which of the following is **true** about the polynomial function $p(x) = 3x^2 - (x + 1)^3 + 5$?
- A. The degree is 4
 - B. The leading coefficient is 1
 - C. The constant term is 4
 - D. The coefficient of x is 3

31. Which one of the following is a true statement?
- $x - 1$ is the factor of $x^n - 1$, for a natural number n .
 - If you divide $5x^3 + 4x^2 + x + 9$ by $x + 2$, then the remainder is -17 .
 - $x - 2$ is the factor of $x^4 + 2x^3 - 3x^2 - 4x + 4$.
 - If $x^4 + 4x^2 + 3$ is divided by $x - 1$; the constant term of the quotient is 3.
32. If $f(x)$ and $g(x)$ are logarithmic functions whose bases are reciprocals of each other, then which one of the following statement is **true**?
- They have reciprocal values for all x .
 - There are some equal values of x for which both $f(x)$ and $g(x)$ are positive.
 - They have opposite signs for all $x \neq 1$
 - They can never have equal values for any x .
33. Which of the following is **true** about $f(x) = x^5(x-1)^4(x^2+3)$?
- 0 is a root of $f(x)$ with the lowest multiplicity.
 - The graph of $f(x)$ crosses the x-axis only at 0.
 - The graph of $f(x)$ crosses the x-axis at $x = 1$.
 - $f(x)$ has three distinct real roots.
34. Which of the following functions touches but never crosses the x-axis?
- $f(x) = 1 - x^3$
 - B** $f(x) = x^4 - 1$
 - $f(x) = (x^2 - 1)^2$
 - $f(x) = x - x^5$
35. Which of the following is **true** about $f(x) = 10 + 3x - x^3$?
- f is increasing in $(-\infty, -1) \cup (1, \infty)$
 - f is decreasing for all $x \in \mathbb{R}$.
 - f is decreasing in $[-1, \infty)$
 - The maximum value of f in $[0, \infty)$ is 12.
36. Which of the following is a polynomial function?
- $f(x) = 5x^2 - 2x + \frac{1}{x} + 3$
 - $g(x) = \sqrt{(x^2 + 1)^2} + 8x^2 - (x - 2)^4$
 - $k(x) = \left(\frac{1}{2}\right)^x + 3$
 - $h(x) = \sqrt{(x - 2)^2} + \frac{1}{2}x + 4$
37. The solution set of the equation $x^3 + |x| = 0$ is:
- $\{0\}$
 - $\{0, 1\}$
 - $\{-1, 0\}$
 - $\{-1, 0, 1\}$
38. Which one of the following is the solution set of the equation, $2x^3 - 7x^2 + 2x + 3 = 0$?
- $\left\{-3, -\frac{1}{2}, 1\right\}$
 - $\left\{-3, \frac{1}{2}, 1\right\}$
 - $\left\{-\frac{1}{2}, 1, 3\right\}$
 - $\left\{-1, \frac{1}{2}, 3\right\}$
39. Which of the following is a polynomial of degree 3?
- $x^3 - 3x^{-2} + x$
 - $x^3 + x^2(1 - x) + 1$
 - $x^2 + x^2(x - 1) + \sin\left(\frac{\pi}{4}\right)$
 - $x^3 + 3x^2 + \sin(x) + 1$
40. Suppose that the point $(1, 0)$ is x-intercept of the graph of the polynomial $p(x) = x^3 - kx^2 + ax$ and when $p(x)$ is divided by $x - 2$ the remainder is 4. What is the value of k ?
- 1
 - 2
 - 1
 - 0