Kotebe Metropolitan University

Department of Mathematics



Solution Manual for Ethiopian University Entrance Examination (EUEE) Mathematics for Natural Science GINBOT 2010 & 2011/May 2018 & 2019

Prepared By:

Dr. Tsegaye Simon

Dr. Margie Balcha

Mr. Temesgen Debas

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MATHEMATICS FOR NATURAL SCIENCE

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SOLUTION MANUAL

1. If $f(x) = x^3 - 2x + 1$, then which one of the following is NOT true?

   (A) $f(c) = \frac{1}{4}$ for some $c \in [-1,0]$

   (B) $f(c) = \frac{1}{2}$ for some $c \in [0,1]$

   (C) $f(c) = 0$ for some $c \in [-2,0]$

   (D) $f(c) = 3$ for some $c \in [1,2]$

   Solution:

   Define $g(x) = f(x) - \frac{1}{4}$. Then $g(-1) = f(-1) - \frac{1}{4} = \frac{7}{4}$ and $g(0) = f(0) - \frac{1}{4} = \frac{3}{4}$. As both values are positive, we can't use an intermediate value theorem to conclude that $g(c) = 0$ for some $c \in [-1,0]$. That is, $g(c) = f(c) - \frac{1}{4} = 0$ or $f(c) = \frac{1}{4}$ is not true for some $c \in [-1,0]$.

   Answer: A

2. If $f(x) = \ln(2^{\tan x})$, then what is the value of $f'(0)$?

   (A) 1

   (B) $-2\ln 2$

   (C) $\frac{\ln 2}{2}$

   (D) $\ln 2$

   Solution:

   Using chain rule, $f'(x) = \left(\frac{1}{2^{\tan x}}\right)(2^{\tan x}\ln 2)(sec^2 x)$

   $$\Rightarrow f'(0) = \left(\frac{1}{2^{\tan 0}}\right)(2^{\tan 0}\ln 2)(sec^2 0)$$

   $$\Rightarrow f'(0) = \ln 2$$

   Answer: D

3. If $f(x) = ax - b$ and $f^{-1}(x+1) = \frac{1}{2}x + 2$, for some $x \in R$, then what must be the values of a and b?

(A) $a = \frac{1}{2}$, $b = -2$ (C) $a = 1$, $b = 1$

(B) $a = 2$, $b = 3$ (D) $a = 2$, $b = 2$

Solution:

To determine the inverse function, set $x = ay - b$.

$$\Rightarrow y = \frac{1}{a}x + \frac{b}{a}$$

$$\Rightarrow f^{-1}(x) = \frac{1}{a}x + \frac{b}{a}$$

$$\Rightarrow f^{-1}(x+1) = \frac{1}{a}(x+1) + \frac{b}{a}$$

$$\Rightarrow \frac{1}{a}(x+1) + \frac{b}{a} = \frac{1}{2}x + 2$$

$$\Rightarrow \frac{1}{a} = \frac{1}{2} \text{ and } \frac{1+b}{a} = 2, \text{ then } a = 2 \text{ and } b = 3$$

Answer: B

4. If $2\begin{pmatrix} 2x & x \\ -5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$, then what is the value of x?

(A) 2 (B) -2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Solution:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} \frac{d}{\det A} & \frac{-b}{\det A} \\ \frac{-c}{\det A} & \frac{a}{\det A} \end{pmatrix}$.

The inverse of the matrix $\begin{pmatrix} 2x & x \\ -5 & -3 \end{pmatrix}$ is then $\begin{pmatrix} 2x & x \\ -5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{x} & 1 \\ \frac{-5}{x} & -2 \end{pmatrix}$

Since $2\begin{pmatrix} 2x & x \\ -5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$, $\begin{pmatrix} \frac{6}{x} & 2 \\ \frac{-10}{x} & -4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$.

From equality of two matrices, we have $\frac{6}{x} = 3$ and $\frac{-10}{x} = -5$. Then $x = 2$.

Answer: A

5. What is the area of the region bounded by the lines $x = 0, x = 2$, and $y = 1$ and the curve $y = e^{2x}$?

(A) $\frac{e^2}{2} - \frac{1}{2}$ (C) $\frac{e^3}{3} - \frac{1}{3}$

(B) $\frac{e^4}{2} - \frac{5}{2}$ (D) $-\frac{e^4}{2} + \frac{5}{2}$

Solution:

Since $y = e^{2x}$ is greater than $y = 1$ on the interval $[0\ , 2]$, the area of the bounded region is given by $A = \int_0^2 (e^{2x} - 1)dx = \left(\frac{e^{2x}}{2} - x\right) = \left(\frac{e^4}{2} - 2\right) - \left(\frac{1}{2} - 0\right) = \frac{e^4}{2} - \frac{5}{2}$

Answer: B

6. What is the solution of $\int 4x\left(\ln x + \frac{1}{x^2}\right)dx$?

(A) $4x^2(\ln x + 1) - 2x^2 + c$

(B) $x^2(2\ln x - 1) + 4\ln x + c$

(C) $x^2(2\ln x + 1) + 4\ln x + c$

(D) $x^2(4\ln x - 1) + 2\ln x + c$

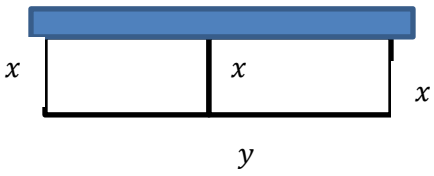
Solution: Applying sum rule for integration and integration by parts, we have

$$\int 4x\left(\ln x + \frac{1}{x^2}\right)dx = 4\int xln\ x\ dx + 4\int \frac{1}{x}dx = 4\left(\frac{x^2\ln x}{2} - \int \frac{x}{2}dx\right) + 4\ln x + c$$

$$= x^2(2\ln x - 1) + 4\ln x + c$$

Answer: B

7. A man wants to fence two identical rectangular enclosures in a field alongside a straight river as shown in the figure.



What is the maximum area of each enclosure that he can make with 192meter fencing material if the side along the river does not need a fence?

(A) $1530m^2$ (B) $1564m^2$ (C) $1536m^2$ (D) $1664m^2$

Solution:

The area of one of the enclosure is given by $A = \frac{1}{2}xy$ and the fence is

$3x + y = 192$. i.e. $y = 192 - 3x$ Substituting this in the area function, we have,

$A = \frac{x(192-3x)}{2} = \frac{192x-x^2}{2}$, $0 \le x \le 64$ otherwise $A < 0$.

Differentiating both sides, we get $A'(x) = \frac{192-6x}{2}$, to obtain maximum,

$$A'(x) = 0 \implies \frac{192-6x}{2} = 0 \implies x = 32 \text{ is a critical number.}$$

The maximum value of A occur either at this critical number or at the end of the interval $[0,64]$.Since $A(0) = 0, A(32) = 1536$ *and* $A(64) = 0$ the maximum value is 1536. Then the maximum area of each enclosure is $1536m^2$.

Answer: C

8. If f is the greatest integer function and g is the absolute value function, then what is the value of $(f \circ g)\left(\frac{3}{2}\right) + (g \circ f)(-\frac{4}{3})$?

(A) 1 (B) 2 (C) -1 (D) 3

Solution:

By definition of composition of functions:

$$(f \circ g)\left(\frac{3}{2}\right) + (g \circ f)\left(-\frac{4}{3}\right) = f\left(g\left(\left(\frac{3}{2}\right)\right) + g\left(f\left(-\frac{4}{3}\right)\right)\right).$$

By definition of absolute value $g\left(\frac{3}{2}\right) = \left|\frac{3}{2}\right| = \frac{3}{2}$ and by definition of greatest integer function $f\left(-\frac{4}{3}\right) = \left[\!\left[-\frac{4}{3}\right]\!\right] = -2$.

Hence $f\left(g\left(\left(\frac{3}{2}\right)\right) + g\left(f\left(-\frac{4}{3}\right)\right) = f\left(\frac{3}{2}\right) + g(-2) = 1 + |-2| = 1 + 2 = 3\right.$

Since $f\left(\frac{3}{2}\right) = \left[\!\left[\frac{3}{2}\right]\!\right] = 1.$

Answer: D

9. Which one of the following is true about the graph of the function $f(x) = \frac{x^2+5x+6}{x^2-4} + 3$?

(A) A horizontal asymptote of the graph is $y = 4$.

(B) The vertical asymptote of the graph are $x = -2$ and $x = 2$.

(C) The graph has a hole at $x = 2$.

(D) The graph has a y-intercept at $(0, -\frac{3}{2})$.

Solution:

$f(x) = \frac{x^2+5x+6}{x^2-4} + 3 = \frac{4x^2+5x-6}{x^2-4} = \frac{(x+2)(4x-3)}{(x+2)(x-2)}$.

Since, $\lim_{x\to\infty} \frac{(x+2)(4x-3)}{(x+2)(x-2)} = 4$. The graph has a horizontal asymptote at $y = 4$. Hence A is true. B is false because the graph has a vertical asymptote only at $x = 2$. C is false because the graph has a hole at $x = -2$. And D is false the graph has y-intercept at$x = 0$, i.e.$f(0) = \frac{-6}{-4} = \frac{3}{2}$ or at$(0, \frac{3}{2})$.

Answer: A

10. The earth's orbit has a semi-major axis $a \approx 149.6Gm$ $(gigameters)$ and an eccentricity of $e \approx 0.017$. what is the approximate value of the semi-minor axis?

(A) 149.58Gm (C) 149.06Gm

(B) 145.32Gm (D) 152.14Gm

Solution:

The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b) is\ e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$ where a is the semi-major axis and b is the semi-minor axis to be determined.

We know that $e \approx 0.017 = \frac{\sqrt{(149.6)^2-b^2}}{149.6} \implies \sqrt{(149.6)^2 - b^2} = 2.5432 \implies (149.6)^2 - b^2 = 6.47 \implies b^2 = 22380.16 - 6.47 \implies$b=149.58.

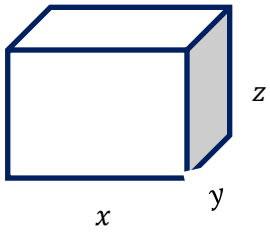
Answer: A

11. A water tank is rectangular parallelepiped with base length 3m, width 2m, and height 2.5m. If water is flowing in to the tank at the rate of $0.12m^3$/sec, then how fast does the level of the water rise up in the tank?

(A) 0.04m/sec (C) 0.02m/sec

(B) 0.03m/sec (D) 0.06m/sec

Solution:



Let x, y and z be the length, width and height of the rectangular parallelepiped respectively. The volume (V) of the parallelepiped is given by $= xyz$. If water is flowing in to the tank as time goes the level of the water rise up, that is the value of z varies while the values of x and y remain constant, and this relation is given as $\frac{dV}{dt} = xy\frac{dz}{dt}$ where t is measured in sec. hence we have

$0.12m^3 = (3m)(2m)\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{0.12m^3}{6m^2} = 0.02m.$

Answer: **C**

12. What is the sum of the infinite series $\sum_{k=0}^{\infty} 5(\frac{2^k+5^k}{10^k})$?

(A) 15 (B) $\frac{25}{4}$ (C) $\frac{65}{4}$ (D) 5

Solution:

$\sum_{k=0}^{\infty} 5\left(\frac{2^k+5^k}{10^k}\right) = 5\sum_{k=0}^{\infty}\left(\frac{2^k}{10^k}+\frac{5^k}{10^k}\right) = 5\sum_{k=0}^{\infty}\left(\left(\frac{2}{10}\right)^k+\left(\frac{5}{10}\right)^k\right) = 5\sum_{k=0}^{\infty}\left(\left(\frac{1}{5}\right)^k+\left(\frac{1}{2}\right)^k\right) =$

$5\left[\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k+\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\right].$

But $\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k$ is a geometric series with common ratio r=$\frac{1}{5}$ and first term $G_1 = 1$,

since $r = \frac{1}{5} < 1$ the geometric progression converges to $= \frac{G_1}{1-r} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$, and also$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k$ is

a geometric series with common ratio $r = \frac{1}{2} < 1$ which converges to $s = \frac{1}{1-\frac{1}{2}} = 2.$

Hence $\sum_{k=0}^{\infty} 5\left(\frac{2^k+5^k}{10^k}\right) = 5\left[\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k+\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\right] = 5\left(\frac{5}{4}+2\right) = \frac{65}{4}.$

Answer: **C**

13. The marks of 50 students is given below:

Marks	$0-10$	$10-20$	$20-30$	$30-40$	$40-50$
No. of students	5	8	f_1	10	f_2

If the median of the data is 26, what are the values of f_1 and f_2?

(A) $f_1 = 20, f_2 = 7$ (C) $f_1 = 15, f_2 = 12$

(B) $f_1 = 12, f_2 = 15$ (D) $f_1 = 7, f_2 = 20$

Solution:

The formula to find the median in a group data which is located in the median class is $= l + \left(\frac{\frac{n}{2}-f_b}{f_1}\right)w$, where $l = 20$ (lower boundary of the median class).

$n = 50$ (sum of frequencies).

f_b =13 (cumulative frequency before the median class).

w: the side of the median class (class width).

f_1 (frequency of the median class) is to be determined.

Hence we have $26 = 20 + \left(\frac{\frac{50}{2}-13}{f_1}\right)10 \Rightarrow 26 = 20 + \frac{120}{f_1} \Rightarrow \frac{120}{f_1} = 6 \Rightarrow f_1 = 20$

and $f_2 = 50 - 43 = 7$.

Answer: A

14. There are three children in a room with ages four, five, and six. If a five year-old child enters the room, and then which of the following statement is correct?

(A) Mean age will stay the same but the standard deviation will increase.

(B) Mean age will stay the same but the standard deviation will decrease.

(C) Mean age and standard deviation will increase.

(D) Mean age and standard deviation will stay the same.

Solution:

The mean age of the three children is $m = \frac{4+5+6}{3} = 5$. If a five year age inters the room the mean age is $m' = \frac{4+5+5+6}{4} = 5$. That is, the mean age will stay the same. The standard deviation $= \sqrt{\frac{(4-5)^2+(5-5)^2+(6-5)^2}{3}} = 0.667$. If a five year child inters the room the standard deviation is $s' = \sqrt{\frac{(4-5)^2+(5-5)^2+(5-5)^2+(6-5)^2}{4}} = 0.5$. hence the standard deviation will decrease.

Answer: B

15. For real numbers x and y which of the following is NOT true?

(A) $(\forall x)(\forall y)(x^2 + y^2 \geq -1)$ (C) $(\forall x)(\exists y)(y \geq x^2 + 1)$

(B) $(\exists x)(\forall y)(y \geq x^2 + 1)$ (D) $(\exists x)(\exists y)\ (y \geq x^2 + 1)$

Solution:

A) since $y^2 \geq 0$ *and* $x^2 \geq 0$ for all real numbers x and y, it follows that $y^2 + x^2 \geq -1$ is true for all real numbers x and y. Thus the quantified statement $(\forall x)(\forall y)(x^2 + y^2 \geq -1)$ is true.

B) $(\exists x)(\forall y)(y \geq x^2 + 1)$ is false since no fixed value of x will make this true for all y in the set of real numbers. E.g. If $x = 2$, then $y \geq 2^2 + 1 = 5$ is false for some y, say $y = 3$.

C) $(\forall x)(\exists y)(y \geq x^2 + 1)$ means that for every integer x, there is an integer y such that $y \geq x^2 + 1$. Let $x = a$ then $y \geq a^2 + 1$ is true statement.

D) $(\exists x)(\exists y)\ (y \geq x^2 + 1)$ means that there is an integer x and there is an integer y such that $y \geq x^2 + 1$. This statement is true, if x=2 and y=7,then $7 \geq 4 + 1$.

Answer: B

16. The second and fifth terms of a geometric progression are $-\frac{1}{2}$ and $\frac{1}{16}$, respectively, what is the sum of the first eight terms of the sequence?

(A) $\frac{128}{85}$ (C) $\frac{85}{128}$

(B) $\frac{255}{256}$ (D) $\frac{256}{255}$

Solution:

The n[th] term of a geometric sequence is given by $G_n = r^{n-1}G_1$, given that $G_2 = -\frac{1}{2} = rG_1$ *and* $G_5 = \frac{1}{16} = r^4G_1$.

Then the ratio, $\frac{G_5}{G_2} = \frac{r^4G_1}{rG_1} = \frac{\frac{1}{16}}{-\frac{1}{2}} \quad \Rightarrow r^3 = -\frac{1}{8} \quad \Rightarrow r = -\frac{1}{2}$ and $G_1 = 1$.

The sum of n terms of a geometric progression is given by $s_n = \frac{G_1(1-r^n)}{1-r}, r \neq 1$.

Therefore $s_8 = (1)\frac{\left(1-\left(-\frac{1}{2}\right)^8\right)}{1-\left(-\frac{1}{2}\right)} = \frac{1-\frac{1}{256}}{\frac{3}{2}} = \frac{\frac{255}{256}}{\frac{3}{2}} = \frac{85}{128}$.

Answer: **C**

17. Which one of the following is equals to $f(x) = \sqrt{(x+4)^2}$, for every $x \in \mathbb{R}$?

(A) $g(x) = |x+4|$ (C) $g(x) = x+4$

(B) $g(x) = x+2$ (D) $g(x) = |x|+4$

Solution:

If $a \in \mathbb{R}$, then $\sqrt{a^2} = |a|$.

Therefore, $\sqrt{(x+4)^2} = |x+4|$.

Answer: **A**

18. Which one of the following is a valid argument?

(A) If I don't change my oil regularly, my engine will die. My engine died, thus I didn't change my oil regularly

(B) If you do every problem in the book, then you will learn the subject. You learned the subject. Thus you did every problem in the book.

(C) If I am literate, then I can read and write. I can read but I can't write. thus I am not literate

(D) If it rains or snows, then my roof leak. My roof is leaking. Thus it is raining and snowing

Solution:

i. The arguments which are given in A in terms of symbol are given as shown below.

p: I don't change my oil regularly

q: my engine will die

Then the argument is $p \Rightarrow q, q \vdash p$

Then the truth value of the compound statement $(p \Rightarrow q) \wedge q \Rightarrow p$ is obtained in the table below

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge q$	$(p \Rightarrow q) \wedge q \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Hence $(p \Rightarrow q) \wedge p \Rightarrow p$ is not tautology means the given argument is invalid

ii. The arguments which are given in B in terms of symbol are given as shown below.

p: you do every problem in the book

q: you will learn the subject

Then the argument is $p \Rightarrow q, q \vdash p$

Then the truth value of the compound statement $(p \Rightarrow q) \wedge p \Rightarrow p$ is obtained in the table below

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge q$	$(p \Rightarrow q) \wedge q \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Hence the compound statement $(p \Rightarrow q) \wedge q \Rightarrow p$ is not tautology

Therefore $(p \Rightarrow q) \wedge q \Rightarrow p$ is invalid

iii. The arguments which are given in C in terms of symbol are given as shown below.

p: I am literate

q: I can read

r: I can write

Then the argument is $p \Rightarrow (q \wedge r), q \wedge \neg r \vdash \neg p$

Then the truth value of the compound statement $p \Rightarrow (q \wedge r), q \wedge \neg r \vdash \neg p$ is obtained in the table below

p	q	r	$\neg p$	$\neg r$	$q \wedge r$	$q \wedge \neg r$	$p \Rightarrow (q \wedge r)$	$(p \Rightarrow (q \wedge r)) \wedge (q \wedge \neg r)$	$(p \Rightarrow (q \wedge r)) \wedge (q \wedge \neg r) \Rightarrow \neg p$
T	T	T	F	F	T	F	T	F	T
T	T	F	F	T	F	T	F	F	T
T	F	T	F	F	F	F	F	F	T
T	F	F	F	T	F	F	F	F	T
F	T	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	T	T	T
F	F	T	T	F	F	F	T	F	T
F	F	F	T	T	F	F	T	F	T

Hence the compound statement $p \Rightarrow (q \wedge r), q \wedge \neg r \vdash \neg p$is tautology

Therefore $p \Rightarrow (q \wedge r), q \wedge \neg r \vdash \neg p$ is valid

iv. The arguments which are given in D in terms of symbol are given as shown below.

P: It rains

q: it snows

r: my roof leak

Then the argument is $(p \vee q) \Rightarrow r, r \vdash p \wedge q$

Then the truth value of the compound statement $(p \vee q) \Rightarrow r, r \vdash p \wedge q$ is obtained in the table below

p	q	r	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow r$	$((p \vee q) \Rightarrow r) \wedge r$	$p \vee q) \Rightarrow r) \wedge r \Rightarrow p \wedge q$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	T
T	F	T	T	F	T	T	F
T	F	F	T	F	F	F	T
F	T	T	T	F	T	T	F
F	T	F	T	F	F	F	T
F	F	T	F	F	T	T	F
F	F	F	F	F	T	F	T

Hence it is not tautology.

Therefore the argument is invalid

Answer: **C**

19. Suppose 2500 items are produced by a machine and 2% of the product are randomly selected and tested. If 5 of the tested items have a defect, then what is the probability that an item produced by the machine has no defect?

(A) 0.80 (B) 0.90 (C) 0.85 (D) 0.95

Solution:

Suppose that A is the machine and out of the produce 2% are randomly selected and tested .from the randomly selected 5 is tested item have a defective means that

$P(A) = 5 * 2\% = \frac{10}{100} = 0.1$

Then $P(A) + P(D) = 1$

$P(D) = 1 - P(A)$

$P(D) = 1 - 0.1 = 0.9$

Answer: B

20. If the region enclosed by the graphs $f(x) = x^2$ and $g(x) = x^3$ from $x = 0$ to $x = 1$ rotates about the x−axis. What is the volume of the solid of revolution ?

(A) $\frac{2\pi}{35}$ cubic units (C) $\frac{2\pi}{5}$ cubic units

(B) $\frac{2\pi}{25}$ cubic units (D) $\frac{2\pi}{27}$ cubic units

Solution:

According to the given function $f(x) = x^2$ is greater than from $g(x) = x^3$ on the given interval $[0, 1]$ and by using washer method we have

$V(s) = \pi \int_0^1 ((f(x))^2 - (g(x))^2)dx$

$V(s) = \pi \int_0^1 ((x^2)^2 - (x^3)^2)dx$

$V(s) = \pi \int_0^1 (x^4 - x^6)dx$

$V(s) = \pi(\frac{x^5}{5} - \frac{x^7}{7})$

$V(s) = \pi(\frac{1}{5} - \frac{1}{7})$

$V(s) = \frac{2\pi}{35}$ Cubic units

Answer: A

21. What is the limit of the sequence $1,\frac{2}{2^2-1^2},\frac{3}{3^2-2^2},\frac{4}{4^2-3^2},\ldots..?$

(A) 0 (C) 1

(B) $\frac{1}{2}$ (D) $\frac{3}{2}$

Solution:

According to the given term of the sequence we can represent the general term in terms of n as $a_n=\frac{n}{n^2-(n-1)^2}$, with, $n=1,2,3,4,5$ ...., then after the limit of the sequence is

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{n}{n^2-(n-1)^2}$

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{n}{n^2-(n^2-2n+1)}$

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{n}{n^2-n^2+2n-1}$

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{n}{2n-1}$

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{2n-1}{n}}$

$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2-0}=\frac{1}{2}$

Answer: B

22. At what value of x does the function $f(x)=\frac{4x^3}{3}-x^4$ attains its maximum value?

(A)-1 (B)0 (C)$\frac{4}{3}$ (D)1

Solution:

First let as determine the critical point of the given function which makes the first derivative of the function is zero

$f(x)=\frac{4x^3}{3}-x^4$

$f'(x)=4x^2-4x^3=0$

$4x^2-4x^3=0$

$4x^2(1-x)=0$, which implies $x=0$ with multiplicity 2 and $x=1$ are the critical point of the given function.

Then using second derivative test to obtain the maximum value of the given function.

$f'(x)=4x^2-4x^3$

$f''(x)=8x-12x^2$, then for $x=1$, $f''(1)=8-12=-4<0$

Therefore the given function has maximum value at $x=1$

Answer: D

23. Fatuma can solve 90% of the problems given in a book and Mesfin can solve 70%.What is the probability that at least one of them will solve the problem, selected at random from the book?

(A) 0.77 (B) 0.97 (C) 0.87 (D) 0.67

Solution:

According to the given from the problem we have given that

Probability that A will solve a problem is 0.9

Probability that B will solve a problem is 0.7

Probability that both A and B will be $0.9 * 0.7 = 0.63$

$P(A \cup B)$ solved from the problem is equal to $p(A) + p(B) - P(A \cap B)$ which implies that

$P(A \cup B) = p(A) + p(B) - P(A \cap B)$

$P(A \cup B) = 0.9 + 0.7 - 0.63 = 0.97$

Answer: B

24. What is the polar form of $\frac{7-i}{3-4i}$?

(A) $\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ (C) $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(B) $2\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$ (D) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Solution:

We have given the complex number $z = \frac{7-i}{3-4i}$ and let as multiply by the conjugate of the denominator as: $z = \frac{7-i}{3-4i} = \left(\frac{7-i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{21+28i-3i+4}{9+16} = \frac{25+25i}{25} = 1 + i$

$z = 1 + i$, and also from this we have

$r = \sqrt{1+1} = \sqrt{2}$,

The argument $\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$, then the polar form is

$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Answer: D

25. If $\neg p \Rightarrow r$ is false and $p \Leftrightarrow p$ is True, then which of the following is True?

(A) $p \vee (\neg q \wedge r)$ (C) $\neg p \wedge (q \Rightarrow r)$

(B) $\neg p \Rightarrow (q \vee r)$ (D) $p \Leftrightarrow (\neg q \vee r)$

Solution:

According the given compound statement we have

i $p \Leftrightarrow p$ is True implies that both p and q have either both are True or Both are False

ii $\neg p \Rightarrow r$, is false implies that r is False and $\neg p$ is True in all case.

iii From case (i) and (ii) we have the truth value of P is False implies that q is also False and r is False.

Hence the truth value of the compound statement is

$p \vee (\neg q \wedge r) = F \vee (\neg F \wedge F) = F.$

$\neg p \Rightarrow (q \vee r) = \neg F \Rightarrow (F \vee F) = F$

$\neg p \wedge (q \Rightarrow r) = \neg F \wedge (F \Rightarrow F) = T$

$p \Leftrightarrow (\neg q \vee r) = F \Leftrightarrow (\neg F \vee F) = F$

Answer: **C**

26. Which one of the following is the set of critical numbers of $f(x) = \frac{3}{8}x^{8/3} - 6x^{2/3}$?

(A) $\{-2,2\}$ (C) $\{-2,0,2\}$

(B) $\{-1,1\}$ (D) $\{-1,0,1\}$

Solution:

Critical number is belongs in the domain of the given function which makes the first derivative of the function is zero or undefined which are obtained as follow

$f(x) = \frac{3}{8}x^{8/3} - 6x^{2/3}$

$f'(x) = x^{5/3} - 4x^{-\frac{1}{3}} = \frac{x^2-4}{x^{\frac{1}{3}}} = 0$, which gives the critical point is $x = 0$ and the value of x which makes $x^2 - 4 = 0$ implies that $x = -2$,or $x = 2$

Hence the set of critical number is $\{-2,0,2\}$

Answer: **C**

27. The center of a circle on the line $y = 2x$ and the line $x = 1$ is tangent to the circle at $(1,6)$.How long is the radius of the circle?

(A) 5 (B) 4 (C) 2 (D) 3

Solution:

Let the coordinate of the center of the circle lying on the line $y = 2x$ be $(a, 2a)$ and its radius is r, then the equation of the circle will be

$(x-a)^2 + (y-2a)^2 = r^2$ (1)

The point of tangency $(1, 6)$ is lie on the circle.so

$(1-a)^2 + (6-2a)^2 = r^2$

$1 - 2a + a^2 + 36 - 24a + 4a^2 = r^2$

$5a^2 - 26a + 37 = r^2$ (2)

Now we have given the equation of the tangent line $x = 1$ which is vertical line and the equation of the normal line to the given circle which pass through the point of tangency is horizontal line with its slope is 0 implies that

$\frac{a-1}{b-6} = 0$, which gives $a - 1$ (3)

By substituting equation (3) in equation (1) we have

$5a^2 - 26a + 37 = 5 * 1^2 - 26 * 1 + 37 = 5 - 26 + 37 = r^2$

$r^2 = 16$, $r = \pm 4$

But r is the norm between the center of the circle and the point of tangency which is positive

Answer: B

28. If there are two children in a family, what is the probability that there is at least one girl in the family?

(A) $\frac{1}{2}$ (C) $\frac{2}{3}$

(B) $\frac{1}{4}$ (D) $\frac{3}{4}$

Solution:

Let G=girl and B=Boy, then the sample space $S = \{GG, GB, BG, BB\}$.

Thus, the probability of at least one girl $= 1 - P(BB) = 1 - \frac{1}{4} = \frac{3}{4}$.

Answer: D

29. Everyday a person saves 5cents more than the amount he saved on the previous day. His target is to save the total amount of 3225 cents by the end of 30 days. How much must be the starting saving to meet the target?

(A) 50 cents (B) 25 cents (C) 35 cents (D) 60 cents

Solution:

Let a_1 be the starting save, a_2 the second save, a_3 the third save… and a_{30} the last save. And $S_{30} = 3225$cents total save.

$a_2 = a_1 + 5, a_3 = a_2 + 5, a_4 = a_3 + 5$ etc, thus, it forms an arithmetic progression with common difference $d = 5$cents.

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

$$3225 = \frac{30}{2}(2a_1 + (30-1)5) \Leftrightarrow \frac{3225}{15} = 2a_1 + 29 * 5$$

$$\Leftrightarrow 215 = 2a_1 + 145$$

$$\Leftrightarrow 2a_1 = 215 - 145 = 70 \Leftrightarrow a_1 = 35\text{cents}$$

Answer: C

30. What is the value of $\int_1^2 \frac{lnx}{x^2} dx$?

(A) $\frac{-ln2}{2} + \frac{1}{2}$ (C) $\frac{-ln2}{2} - \frac{1}{2}$

(B) $\frac{ln2}{2} - \frac{1}{2}$ (D) $\frac{ln2}{2} + \frac{1}{2}$

Solution:

$\int_1^2 \frac{lnx}{x^2} dx$, using integration by part, let $u = lnx \Leftrightarrow du = \frac{1}{x}dx, dv = \frac{1}{x^2}dx \Leftrightarrow v = \frac{-1}{x}$

$\int_1^2 \frac{lnx}{x^2} dx = -\frac{1}{x}lnx - \int \left(-\frac{1}{x}\right)\frac{1}{x}dx = -\frac{1}{x}lnx + \int_1^2 \frac{1}{x^2}dx = -\frac{1}{x}lnx - \frac{1}{x}$ on [1,2]

$$= -\left(\frac{1}{2}ln2 - \frac{1}{1}ln1\right) - \left(\frac{1}{2} - 1\right) = -\frac{1}{2}ln2 + \frac{1}{2} = -\frac{ln2}{2} + \frac{1}{2}$$

Answer: A

31. If the circle passing through the point $(-1,0)$ touches the y-axis at $(2,0)$, then what is the equation of the circle?

(A) $x^2 + y^2 + 5x + 4y + 4 = 0$ (C) $x^2 + y^2 + 5x - 4y + 4 = 0$

(B) $x^2 + y^2 - 5x + 4y + 4 = 0$ (D) $x^2 + y^2 - 5x - 4y + 4 = 0$

Solution:

Let (h,k) is center of the circle and radius r, and the equation of the circle is:

$$(x-h)^2+(y-k)^2=r^2 \Leftrightarrow (-1-h)^2+(0-k)^2=r^2=(0-h)^2+(2-k)^2$$

$$=h^2+2h+1+k^2=h^2+k^2-4k+4 \Leftrightarrow 2h+1=-4k+4$$

As the circle touches the y-axis at(2,0), its radius is perpendicular to the y-axis

Thus, the value of $k=2$,since the radius is horizontal.

$2h+1=-4k+4 \Leftrightarrow 2h=-4(2)+4-1=-5$

$h=-\frac{5}{2}$ and its radius is r from the point$\left(-\frac{5}{2},2\right)$to (0,2) which is $r=\frac{5}{2}$.

The equation of the circle is $\left(x+\frac{5}{2}\right)^2+(y-2)^2=\left(\frac{5}{2}\right)^2 \Leftrightarrow x^2+5x+\frac{25}{4}+y^2-4y+4=\frac{25}{4}$

Therefore the equation of the circle is: $-x^2+y^2+5x-4y+4=0$

Answer: C

32. Let z be a complex number and $w=3+4i$. If $\frac{z^2+1}{z+i}=|w|z-\frac{1}{i}\overline{w}$, then what is the value of z?

(A) $-4-2i$ (C) $-1+i$

(B) $-1-i$ (D) $4-2i$

Solution:

$\overline{w}=\overline{3+4i}=3-4i$ and $|w|=\sqrt{3^2+4^2}=5$

$\frac{z^2+1}{z+i}=|w|z-\frac{1}{i}\overline{w} \Leftrightarrow \frac{z^2+1}{z+i}=5z-\frac{1}{i}(3-4i)$

$=\frac{z^2+1}{z+i}=5z+3i+4 \Leftrightarrow z^2+1=(5z+3i+4)(z+i), z\neq i$

$z^2+1=5z^2+3iz+4z+5iz-3+4i \Leftrightarrow 4z^2+8iz+4z+4i-4=0$

$\Leftrightarrow z^2+(2i+1)z+i-1=0$(Which is quadratic equation in z)

Using the quadratic formula:- $z=\frac{-(2i+1)\pm\sqrt{(2i+1)^2-4(1)(-i-1)}}{2(1)}=\frac{(-2i-1)\pm\sqrt{-4+4i+1-4i+4}}{2}$

$$z=\frac{(-2i+1)\pm\sqrt{1}}{2}=\frac{(-2i-1)\pm 1}{2}=-i-1 \; or \; -i$$

Answer: B

33. What is the value of the left- hand side limit, $\lim_{x\to 0^-}\frac{sinx}{x^2}\sqrt{4x^3+9x^2}$?

(A) Does not exist (C) 2

(B) 3 (D) -3

Solution:

$$\lim_{x\to 0^-}\frac{sinx}{x^2}\sqrt{4x^3+9x^2}=\lim_{x\to 0^-}\frac{sinx}{x}\lim_{x\to 0^-}\frac{|x|}{x}\sqrt{4x+9}$$

$$=\lim_{x\to 0^-}\frac{sinx}{x}\lim_{x\to 0^-}(-1)\sqrt{4x+9}=1(-1)\sqrt{9}=-3$$

Answer: D

34. If $\begin{vmatrix}-1 & 1 & 2\\ 3 & 2 & x\\ 2 & 4 & 1\end{vmatrix}=\begin{vmatrix}-x & 3 & 2\\ 2 & 2 & 3\\ 1 & -1 & -2\end{vmatrix}$, then what is the value of x?

(A) $-\frac{2}{5}$ (C) $\frac{2}{3}$

(B) $\frac{2}{5}$ (D) $-\frac{2}{3}$

Solution:

Evaluating the determinant of the two matrices,

$$-1\begin{vmatrix}2 & x\\ 4 & 1\end{vmatrix}-1\begin{vmatrix}3 & x\\ 2 & 1\end{vmatrix}+2\begin{vmatrix}3 & 2\\ 2 & 4\end{vmatrix}=-x\begin{vmatrix}2 & 3\\ -1 & -2\end{vmatrix}-3\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}+2\begin{vmatrix}2 & 2\\ 1 & -1\end{vmatrix}$$

$$\Leftrightarrow -1(2-4x)-(3-2x)+2(12-4)=-x(-6+3)-3(-4-3)+2(-2-2)$$

$$\Leftrightarrow -2+4x-3+2x+24-8=6x-3x+21-8$$

$$\Leftrightarrow 6x-3x=13-11\Leftrightarrow x=\frac{2}{3}$$

Answer: C

35. Consider the following system of equations: $\begin{cases}x-2y+z=1\\ -x+y+z=3\\ 3x-5y+z=k\end{cases}$

How much must be the value of k so that the system has a solution?

(A) 7 (B) 1 (C) -1 (D) 0

Solution:

Let $A=\begin{bmatrix}1 & -2 & 1\\ -1 & 1 & 1\\ 3 & -5 & 1\end{bmatrix}$ be the coefficient matrix, and $|A|=\begin{vmatrix}1 & -2 & 1\\ -1 & 1 & 1\\ 3 & -5 & 1\end{vmatrix}$

$$|A|=1\begin{vmatrix}1 & 1\\ -5 & 1\end{vmatrix}+2\begin{vmatrix}-1 & 1\\ 3 & 1\end{vmatrix}+1\begin{vmatrix}-1 & 1\\ 3 & -5\end{vmatrix}=(1+5)+2(-1-3)+(5-3)=0$$

$|A|=0$, thus to have a solution, replacing the constant matrix in one of the column, must have determinant zero.

$$\begin{vmatrix}1 & -2 & 1\\ -1 & 1 & 3\\ 3 & -5 & k\end{vmatrix}=0\Leftrightarrow 1\begin{vmatrix}1 & 3\\ -5 & k\end{vmatrix}+2\begin{vmatrix}-1 & 3\\ 3 & k\end{vmatrix}+1\begin{vmatrix}-1 & 1\\ 3 & -5\end{vmatrix}=0$$

$$\Leftrightarrow (k+15)+2(-k-9)+1(5-3)=0$$

$$\Leftrightarrow k+15-2k-18+2=0\Leftrightarrow k=-1$$

Answer: **C**

36. What is the simplified form of $\frac{a^{-1}b^{-1}}{a^{-3}-b^{-3}}$?

(A) $\frac{a^2b^2}{b^2-a^2}$ (C) $\frac{a^3-b^3}{ab}$

(B) $\frac{a^3-b^3}{a-b}$ (D) $\frac{a^2b^2}{b^3-a^3}$

Solution:

$$\frac{a^{-1}b^{-1}}{a^{-3}-b^{-3}}=\frac{(ab)^{-1}}{\frac{1}{a^3}-\frac{1}{b^3}}=\frac{(ab)^{-1}}{\frac{b^3-a^3}{a^3b^3}}=\frac{(ab)^{-1}}{b^3-a^3}(a^3b^3)=\frac{a^2b^2}{b^3-a^3}$$

Answer: **D**

37. If $f(x)=ax^3+\frac{b}{x}+5$ has local minimum at (2,-3), what are the values of a and b?

(A) $a=\frac{1}{4},b=12$ (C) $a=-\frac{1}{4},b=-12$

(B) a $=\frac{1}{4}$, b $=-12$ (D) $a=\frac{1}{4},b=-12$

Solution:

$f(x)=ax^3+\frac{b}{x}+5\Leftrightarrow f(2)=8a+\frac{b}{2}+5=-3\Leftrightarrow 16a+b=-16$ and

$f'(x)=3ax^2-\frac{b}{x^2}\Leftrightarrow f'(2)=12a-\frac{b}{4}=0\Leftrightarrow b=48a$, thus,

$16a+b=-16\Leftrightarrow 16a+48a=-16\Leftrightarrow a=-\frac{1}{4}$ and $b=48\left(-\frac{1}{4}\right)=-12$

Answer: **C**

38. What is the value of $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx$?

(A) $\frac{1}{e}ln(x^e + e^{x+1}) + c$

(B) $\frac{1}{e}ln(x^e + e^x) + c$

(C) $\frac{1}{e}ln((x+1)^e + e^{x+1}) + c$

(D) $\frac{1}{e}ln(x^{e+1} + e^{x+1}) + c$

Solution:

Let $u = x^e + e^x \Longrightarrow \frac{du}{dx} = ex^{e-1} + e^x = \frac{ex^e + xe^x}{x} \Longrightarrow$

$dx = \frac{x}{ex^e + xe^x}du$.Thus,$\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx = \int \frac{x^{e-1}+e^{x-1}}{u}.\frac{x}{ex^e+xe^x}du = \int \frac{ex^e+xe^x}{xeu}.\frac{x}{ex^e+xe^x}du =$

$\frac{1}{e}\int \frac{1}{u}du = \frac{1}{e}\,n(u) + c = \frac{1}{e}ln(x^e + e^x) + c$

Answer: B

39. If $h(x) = \sqrt{1+\sqrt{x}}$, then which of the following is equal to $h'(x)$?

(A) $\frac{1}{2\sqrt{1+\sqrt{x}}}$

(B) $\frac{1}{4\sqrt{x+x\sqrt{x}}}$

(C) $\frac{x}{2\sqrt{1+\sqrt{x}}}$

(D) $\frac{x}{4\sqrt{x+x\sqrt{x}}}$

Solution:

$$h'(x) = \frac{(1+\sqrt{x})'}{2\sqrt{1+\sqrt{x}}} = \frac{1}{2\sqrt{x}.2\sqrt{1+\sqrt{x}}} = \frac{1}{4\sqrt{x+x\sqrt{x}}}.$$

Answer: B

40. If $(x) = \frac{1}{x}$, then what is the value of $f^{(n)}(x)$?

(A) $f^{(n)}(x) = \frac{(-1)^n n!}{x^n}$

(B) $f^{(n)}(x) = \frac{(-1)^{n+1}(n+1)!}{x^{n+1}}$

(C) $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n-1}}$

(D) $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

Solution:

$f'(x) = \frac{-1}{x^2} = \frac{(-1)^1 1!}{x^{1+1}}$, $f''(x) = \frac{2}{x^3} = \frac{(-1)^2 2!}{x^{2+1}}$,

$f'''(x) = \frac{-6}{x^4} = \frac{(-1)^3 3!}{x^{3+1}}$, . . ., $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$.

Answer: D

41. Among 2000 students who took a regional exam, the percentile of certain students score is 90. Which of the following is correct about the student's score?

(A) The student have answered 90% of the questions correctly.

(B) The score of the student is as good as that of 90% of the students.

(C) The student's score is the same as the top 10% of the scores.

(D) The student's score is greater than or equal to that of 1800 students.

Solution:

Percentage =2000*$\frac{90}{100} = 1800$ that means the students score is greater than 1800 students score

Answer: D

42. Let $f(x) = 2e^x - k\sin x + 1$. If the equation of the tangent line to the graph of f at $(0,3)$ is $= 5x + 3$, then the value of k?

(A) -3 (B) 3 (C) 5 (D) -2

Solution:

The slope of the tangent line is $m = 5$ and from the derivative of f,

$f'(0)$ is the slope of the tangent line to the curve of $f(x)$ at $(0,3)$

$\Rightarrow f'(0) = 5 \Rightarrow 2e^0 - k\cos 0 = 5 \Rightarrow 2 - k = 5 \Rightarrow k = -3$

Answer: A

43. If $a_1 = 2, a_2 = 6, a_3 = 10, a_4 = 14, \ldots$ ...then $\sum_1^{100} a_n =$ _____

(A) 20,020 (C) 20,200

(B) 20,000 (D) 22,000

Solution:

$a_2 = a_1 + 4 = a_1 + 4(2-1), a_3 = a_2 + 4 = a_1 + 8 = a_1 + 4(3-1),$
$a_4 = a_3 + 4 = a_1 + 4(4-1), \ldots, a_{100} = a_1 + 4(100-1) = a_1 + 4(99).$

Then $\sum_1^{100} a_n = a_1 + a_2 + a_3 \ldots . + a_{100} = a_1 + a_1 + 4(1) + a_1 + 4(2) + a_1 + 4(3) + \cdots + a_1 + 4(99)$

$= a_1 + a_1 + a_1 + a_1 + \cdots + a_1 + 4(1 + 2 + 3 + \cdots + 99) = 100a_1 + 4\left(\frac{n(n+1)}{2}\right), \quad n = 99$

$= 200 + 2\big(n(n+1)\big)$

$= 200 + 2\big(99(100)\big) = 200 + 19{,}800 = 20{,}000$

Answer: B

44. Which one of the following is the multiplicative inverse of $= \frac{3+4i}{4-5i}$?

(A) $\frac{8}{25} - \frac{31i}{25}$ (C) $-\frac{8}{25} + \frac{31i}{25}$

(B) $-\frac{8}{25} - \frac{31i}{25}$ (D) $\frac{8}{25} + \frac{31i}{25}$

Solution:

Let $\boldsymbol{z = \frac{3+4i}{4-5i} \neq 0}$. Therefore the multiplicative inverse exists and which is $\frac{1}{z} = \frac{1}{\frac{3+4i}{4-5i}} = \frac{4-5i}{3+4i}$, then multiplying both numerator and denominator of $\frac{4-5i}{3+4i}$ by the conjugate of 3+4i, we obtain $\frac{(4-5i)(3-4i)}{(3+4i)(3-4i)} = \frac{12-16i-15i-20}{9+16} = \frac{-8-31i}{25} = -\frac{8}{25} - \frac{31i}{25}$.

Answer: B

45. If A is a 3×3 matrix and $det(A) = 5$, then how much is det$(2A^TA)$?

(A) 20 (B) 100 (C) 50 (D) 200

Solution:

det$(2A^TA)$=$2^3 det(A^t) det(A) = 8\big(det(A) det(A)\big) = 8\big(5(5)\big) = 200$

Answer: D

46. Which one of the following is true about the pair of lines:$3x + 9y - 24 = 0$ and $4x + 12y + 32 = 0$?

(A) Parallel and distinct lines (C) Intersecting lines

(B) Perpendicular lines (D) Representing the same lines

Solution:

Let $L_1: 3x + 9y - 24 = 0$ *and* $L_2: 4x + 12y + 32 = 0$ the slopes of L_1=$-\frac{1}{3}$ and L_2=$-\frac{1}{3}$

$9y = 24 - 3x$ and $12y = -32 - 4x$

$y = \frac{-1}{3}x + \frac{8}{3}$ and $y = \frac{-1}{3}x + \frac{-8}{3}$ implies the y intercept are different ,hence the lines are distinct.

Answer: A

47. If $F(x) = \int_0^x e^{-t}\,dt$ then what is the value of $F'(x)$?

(A) $-e^{-x} - 1$ (C) $\frac{e^{-x+1}}{-x+1}$

(B) e^{-x} (D) $-e^{-x}$

Solution: By the fundamental theorem of calculus, $F(x) = -e^{-t}|_0^x$

$F(x) = -e^{-x} + 1$, then $F'(x) = (-e^{-x} + 1)' = e^{-x}$.

Answer: B

48. What is the value of $\lim_{h\to 0} \frac{8\left(\frac{1}{2}+h\right)^2 - 8\left(\frac{1}{2}\right)^2}{h}$?

(A) 8 (C) 0

(B) 4 (D) The limit doesn't exist

Solution: $\lim_{h\to 0} \frac{8\left(\frac{1}{2}+h\right)^2 - 8\left(\frac{1}{2}\right)^2}{h} = \lim_{h\to 0} \frac{8[(\frac{1}{2})^2 + h + h^2 - \left(\frac{1}{2}\right)^2]}{h} =$

$\lim_{h\to 0} \frac{8(h+h^2)}{h} = \lim_{h\to 0} 8(1+h) = 8$

Answer: A

49. What is the value of $\lim_{n\to\infty} \left(\frac{3^n+2^n}{6^n}\right)$?

(A) 1 (C) $\frac{5}{6}$

(B) ∞ (D) 0

Solution: $\lim_{n\to\infty} \left(\frac{3^n+2^n}{6^n}\right) = \lim_{n\to\infty} (\frac{3^n}{6^n} + \frac{2^n}{6^n}) = \lim_{n\to\infty} \left(\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) = 0$

Answer: D

50. What is the value of $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$?

(A) 3 (C) -2

(B) 2 (D) the limit does not exist.

Solution: $\lim_{x\to 2} \left(\frac{x^3-8}{x^2-4}\right) = \lim_{x\to 2} \left(\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}\right) = \lim_{x\to 2} \left(\frac{x^2+2x+4}{x+2}\right) = \frac{2^2+2(2)+4}{2+2} = \frac{12}{4} = 3$

Answer: A

51. What is the greatest lower bound of $a_n = \{(-1)^n \frac{1}{n+1}\}_{n=0}^{\infty}$?

(A) -1 (B) 0 (C) -0.5 (D) 0.5

Solution: Let $a_n = \{(-1)^n \frac{1}{n+1}\}_{n=0}^{\infty}$,then $\{(-1)^n \frac{1}{n+1}\}_{n=0}^{\infty} = \left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots \frac{(-1)^n}{n+1} \ldots.\right\}$

Here the lower bounds of a_n are less than or equal to $\frac{-1}{2}$ then the greatest lower bound $a_n = \{(-1)^n \frac{1}{n+1}\}_{n=0}^{\infty}$ is $\frac{-1}{2}$.

Answer: C

52. Ship **A** and **B** depart from the same point at the same time on the course N60^{0}E and N40^{0}E, respectively. If the speed of ship **A** is 20 km per hour and the speed of ship **B** is 30 km per hour, what is the distance between the two ships just after 30 minutes of their departure?
[You may take: $cos(40^0) = 0.77$, $cos(20^0) = 0.94$, $sin(20^0) = 0.34$]
(A) $\sqrt{43}$ km (B) $\sqrt{40}$ km (C) $\sqrt{50}$ km (D) $\sqrt{53}$ km

Solution:

Given ship A and B departing from the same point (say O) at the same time on the course N60^{0}E and N40^{0}E.

Now since the speed of ship **A** is 20 km per hour and the speed of ship **B** is 30 km per hour, it is clear to see that after 30 minutes, ship A is $\frac{1}{2}(20)$km$= 10$km far from O on the course N60^{0}E and ship B is $\frac{1}{2}(30)$km$= 15$km far from O on the course N40^{0}E.

Let x be the distance between the two ships just after 30 minutes. Using …. We obtain $x^2 = (10)^2 + (15)^2 - 2(10)(15)\,cos(20^0)$.

Thus,

$$x = \sqrt{(10)^2 + (15)^2 - 2(10)(15)\,cos(20^0)} = \sqrt{325 - 300(0.94)} = \sqrt{325 - 282} = \sqrt{43}\,.$$

Answer: **A**

53. When the plane is rotated 45^0 about the point$(1,-2)$, then what would be the image of the point $(2,4)$?
(A) $\left(1 - \frac{\sqrt{2}}{2}, -2 + \frac{\sqrt{2}}{2}\right)$ (C) $\left(\frac{5\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$
(B) $\left(1 - \frac{5\sqrt{2}}{2}, -2 + \frac{7\sqrt{2}}{2}\right)$ (D) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Solution:

To find the image of the point A when the plane is rotated about the point B through an angle of θ we follow the following steps:

Step 1: Find $C = A - B$.

Step 2: Find the image of point D of the point C rotated about the origin through an angle of θ.

Step 3: Find $F = B + D$.

Then, F is the image of the point A when the plane is rotated about the point B through an angle of θ.

Simply, the image point $F = (a, b)$ of the point $A = (x_1, y_1)$ when the plane is rotated about the point $B = (x_0, y_0)$ through an angle of θ is given by the formula

$$a = x_0 + (x_1 - x_0)cos\theta - (y_1 - y_0)sin\theta$$

and

$$b = y_0 + (x_1 - x_0)sin\theta + (y_1 - y_0)cos\theta.$$

Given $A = (2,4), B = (1,-2)$ and $\theta = 45^0$. Thus,

$$a = 1 + (2-1)cos\theta - \big(4 - (-2)\big)sin\theta = 1 + 1\cos(45^0) - 6sin(45^0) = 1 - \frac{5\sqrt{2}}{2}$$

and

$$b = -2 + (2-1)sin\theta - \big(4 - (-2)\big)cos\theta = -2 + 6cos(45^0) + 1sin(45^0) = -2 + \frac{7\sqrt{2}}{2}.$$

Therefore, the required image point is $\left(1 - \frac{5\sqrt{2}}{2}, -2 + \frac{7\sqrt{2}}{2}\right)$.

Answer: **B**

54. If $u = 2\vec{j} - \vec{k}$ and $v = \vec{\imath} - 8\vec{j} + 3\vec{k}$, then what is the unit vector in the direction of $5u + v$?

(A) $\vec{\imath} + 2\vec{j} - 2\vec{k}$

(B) $\frac{2}{3}\vec{\imath} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$

(C) $\frac{1}{3}\vec{\imath} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

(D) $\frac{1}{3}\vec{\imath} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$

Solution:

If w is a vector in $\mathbb{R}^3$, then the unit vector in the direction of w is a vector $\overline{w}$ given by

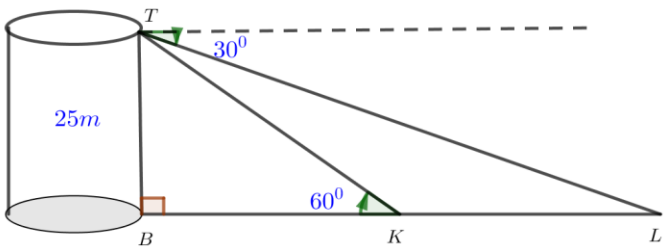
$$\overline{w} = \frac{w}{\|w\|}.$$

Now, $w = 5u + v = 5\big(2\vec{j} - \vec{k}\big) + \vec{\imath} - 8\vec{j} + 3\vec{k} = \vec{\imath} + 2\vec{j} - 2\vec{k}$ and $\|w\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

Thus, $\overline{w} = \frac{w}{\|w\|} = \frac{\vec{\imath} + 2\vec{j} - 2\vec{k}}{3} = \vec{\imath} + 2\vec{j} - 2\vec{k}$.

Answer: **D**

55. The diagram below is a representation of a $25\ m$ vertical observation tower TB and two cars **K** and **L** on a road. The angle of depression from **T** to car **L** is 30^0. The angle of elevation from car **K** to the top of the tower is 60^0. **B**, **K** and **L** lie in in a straight line and lie on the same horizontal plane as the base of the tower.



What is the distance between the two cars?

(A) $\frac{50\sqrt{3}}{2}$ m

(B) $50\sqrt{3}$ m

(C) $\frac{50\sqrt{3}}{3}$ m

(D) $50+\sqrt{3}$ m

Solution:

From the given, we can have

$$\tan(60^0)=\frac{25}{BK} \quad \text{and} \quad \tan(30^0)=\frac{25}{BK+KL}.$$

That is, $BK=\frac{25}{\sqrt{3}}$ and $BK+KL=25\sqrt{3}.$

Therefore, $KL=25\sqrt{3}-BK=25\sqrt{3}-\frac{25}{\sqrt{3}}=50\sqrt{3}$

Answer: **B**

56. Which of the following is the correct assertion that can be proved by the principle of Mathematical induction?

(A) $\frac{1}{n+1} \leq 0$, for each real number $n \geq 0$.

(B) $k! \geq 2^k$. for each integer $k \geq 4$.

(C) $2^p - 1$ is a prime for each prime integer p.

(D) $m! \leq 4^m$, for each positive integer m.

Solution:

A method of proving mathematical results based on the principle of mathematical induction: An assertion $A(x)$, depending on a natural number x, is regarded as proved if $A(1)$ has been proved and if for any natural number n the assumption that $A(n)$ is true implies that $A(n+1)$ is also true.

Therefore,

(A) Can't be proved by the principle of Mathematical induction as the assertion is defined on the set of real numbers.

(C) Can't be proved by the principle of Mathematical induction as if p is prime number $p+1$ might not be prime.

(D) We can't proceed using principle of Mathematical induction.

Answer: B

57. If the angle between the vectors $\vec{A} = (2, -1, 1)$ and $\vec{B} = (1, 1, \alpha)$ is $\frac{\pi}{3}$, then is what the value of α?

(A) 2 (B) -1 (C) 1 (D) -2

Solution:

If $\vec{A}$ and $\vec{B}$ are nonzero vectors in $\mathbb{R}^3$, then $\vec{A}.\vec{B} = \|\vec{A}\|\|\vec{B}\| cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

Now, for the given vectors $\vec{A} = (2, -1, 1)$ *and* $\vec{B} = (1, 1, \alpha)$ and the angle $\theta = \frac{\pi}{3}$, we obtain $\vec{A}.\vec{B} = 1 - \alpha$, $\|\vec{A}\| = \sqrt{6}$ and $\|\vec{B}\| = \sqrt{2 + \alpha^2}$.

Thus,

$$\vec{A}.\vec{B} = \|\vec{A}\|\|\vec{B}\| cos\theta \Rightarrow 1 - \alpha = \sqrt{6}\sqrt{2 + \alpha^2}\,\frac{1}{2} \Rightarrow (1 - \alpha)^2 = \left(\sqrt{6}\sqrt{2 + \alpha^2}\,\frac{1}{2}\right)^2$$

$$\Rightarrow \alpha^2 - 4\alpha + 4 = (x - 2)^2 = 0 \Leftrightarrow \alpha = 2.$$

Answer: A

58. Let ***L*** be the line given by the vector equation$(x, y) = (1,1) + t(\sqrt{3}, 1), t \in \mathbb{R}$. What is the equation of the image of the line ***L*** after being rotated 15^0 about $(1,1)$ and then translated by the vector$u = (-1,1)$.

(A) $x - y = 2$

(B) $-x + y = 2$

(C) $\sqrt{3}x - y = 2$

(D) $-x + \sqrt{3}y = 2$

Solution:

To find the image of the line L when the plane is rotated about the point B through an angle of θ and translated by the vector V we follow the following steps:

Take arbitrary two distinct points P and Q on the line L.

Step 1: Find the image of points R and S of points P and Q, respectively, when rotated about the point A through an angle of θ and translated by the vector V.

Step 2: Find the equation of the line L' through the two points R and S.

Then, L' is the image of the line L when the plane is rotated about the point B through an angle of θ and translated by the vector V.

Now given a line L given by $(x, y) = (1,1) + t(\sqrt{3}, 1), t \in \mathbb{R}$. Take points $P = (1,1)$ (when $t = 0$) and $Q = (1 + \sqrt{3}, 2)$ (when $t = 1$) on the line L.

Thus, $\cos(15^0) = cos(60^0 - 45^0) = cos(60^0)\cos(45^0) + sin(60^0)sin(45^0) = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

$\sin(15^0) = sin(60^0 - 45^0) = sin(60^0)\cos(45^0) - cos(60^0)sin(45^0) = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$.

The image of the point $P = (1,1)$ rotated about the point $A = (1,1)$ through an angle of $\theta = 15^0$ is $P = (1,1)$, because rotation fixes the center of rotation. Moreover, when P is translated by the vector $u = (-1,1)$ is $S = (1 - 1,1 + 1) = (0,2)$.

Now the image of the point $Q = (1 + \sqrt{3}, 2) = (x_1, y_1)$ rotated about the point $A = (1,1) = (x_0, y_0)$ through an angle of $\theta = 15^0$ is $P = (a, b)$ where

$$a = 1 + (1 + \sqrt{3} - 1)cos(15^0) - (2 - 1)sin(15^0) = 1 + \sqrt{2}$$

and

$$a = 11 + (1 + \sqrt{3} - 1)sin(15^0) + (2 - 1)cos(15^0) = 1 + \sqrt{2}.$$

Then when translated by $(1+\sqrt{2}, 1+\sqrt{2})$ by a vector $u = (-1,1)$ we have $S = (\sqrt{2}, 2+\sqrt{2})$.

Therefore, the line L' through the two points $R = (0,2)$ and $S = (\sqrt{2}, 2+\sqrt{2})$,

that is, $y = x + 2$.

Answer: B

59. What is the solution set of $sin^2x - sinx\ cosx = 0$ over $[0,2\pi]$?

(A) $\left\{0, \pi, \frac{5\pi}{4}, 2\pi\right\}$ (C) $\left\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}\right\}$

(B) $\left\{0, \frac{\pi}{4}, \pi, 2\pi\right\}$ (D) $\left\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi\right\}$

Solution:

For $\in \mathbb{R}$, $a.b = 0$ if and only if $a = 0$ or $b = 0$.

Note that $sin^2x - sinx\ cosx = sinx(sinx - cosx)$.

Thus, $sin^2x - sinx\ cosx = 0 \Leftrightarrow sinx(sinx - cosx) = 0 \Leftrightarrow sinx = 0$ or $sinx - cosx = 0$.

But, $sinx = 0 \Leftrightarrow x = 0 + 2n\pi$ or $x = \pi + 2n\pi$ for $n \in \mathbb{Z}$ and

$sinx = cosx \Leftrightarrow x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{5\pi}{4} + 2n\pi$ for $n \in \mathbb{Z}$.

Therefore, $x \in [0,2\pi]$ solving $sin^2x - sinx\ cosx = 0$ is $x = 0, x = 2\pi, x = \pi$ and $x = \frac{5\pi}{2}$.

Answer: A

60. Consider a rectangle $ABCD$ with base vertices $A = (0,3)$ and $B = (4,0)$ and the other vertices C and D, in the first-quadrant of the coordinate plane. If its height $\overline{BC}$ is half of the length of its base, then which of the following indicates the coordinates of vertex C?

(A) $(4,5/2)$ (C) $(5/2,-2)$

(B) $(11/2,2)$ (D) $(6,3/2)$

Solution:

Now, $AB = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$. Thus, $BC = \frac{1}{2}AB = \frac{5}{2}$.

The equation of the line through A and B is $y = \frac{-3}{4}x + 3$. Thus, $y = \frac{4}{3}x - \frac{16}{3}$ is the equation of the line through B and C (using the fact that the line through B and C is perpendicular to the line through A and B).

Let $C = (a,b)$. Thus, we have

$b = \frac{4}{3}a - \frac{16}{3}$ . ..... (1)

And $\frac{5}{2} = BC = \sqrt{(4-a)^2 + b^2} = \sqrt{a^2 - 8a + 16 + b^2}$.

$$\Rightarrow \frac{25}{4} = a^2 - 8a + 16 + b^2$$

$a^2 - 8a + \frac{39}{4} + b^2 = 0$ …….. (2)

Hence, using (1) and (2), we have

$$a^2 - 8a + 16 + (\frac{4}{3}a - \frac{16}{3})^2 = 0$$

$$\Rightarrow a^2 + \frac{16}{9}a^2 - 8a - \frac{128}{9}a + \frac{39}{4} + \frac{256}{9} = 0$$

$$\Rightarrow \frac{25}{9}a^2 - \frac{200}{9}a + \frac{1375}{18} = 0 \Rightarrow a^2 - 8a + \frac{55}{4} = 0$$

$$\left(a - \tfrac{5}{2}\right)\left(a - \tfrac{11}{2}\right) = 0 \Leftrightarrow a = \tfrac{5}{2} \text{ or } a = \tfrac{11}{2}$$

But b $= \frac{4}{3}$a $- \frac{16}{3} = -2 < 0$ if a $= \frac{5}{2}$, and$= \frac{4}{3}$a $- \frac{16}{3} = 2 < 0$ if a $= \frac{11}{2}$. However, since C is in the first quadrant it must be the case that $C = (11/2,2)$

Answer: B

61. $\forall n \in N$, $3^n - 2$ is prime that can be proved or disproved by which one of the following mathematical proofs?

(A) Disprove by counter example

(B) Proof by exhaustion

(C) Direct proof

(D) Proof by contradiction

Solution:

If n $= 1$, $3^1 - 2 = 3 - 2 = 1$ which is not a prime number. So, the claim is diproved by counter example.

Answer: A

62. If the point$(\alpha, 0,3)$ is on the sphere centered at$(1,2,3)$ with radius 2, what is the value of α?

(A) 0 (B) 2 (C) 1 (D) -3

Solution:

The equation of the sphere is given as $(x-1)^2+(y-2)^2+(z-3)^2=4$. Since $(\alpha,0,3)$ is a point on the sphere, $(\alpha-1)^2+(0-2)^2+(3-3)^2=4$.

$$\Rightarrow (\alpha-1)^2+4=4, \Rightarrow \alpha=1$$

Answer: C

63. What is the image of the circle $x^2+y^2-4x-6y+11=0$, when the origin is shifting to the point $(1,1)$ after translation of axes?

(A) $x^2+y^2-4x-2y+3=0$

(B) $x^2+y^2-4x-6y+3=0$

(C) $x^2+y^2+6x+8y-23=0$

(D) $x^2+y^2-6x-8y+23=0$

Solution:

The equation $x^2+y^2-4x-6y+11=0$ of the circle can be written as

$$\Rightarrow (x-2)^2+(x-3)^2=2$$

Then, when the center $(2,3)$ is translated by the vector $(1,1)$ it becomes $(3,4)$. As translation is a rigid motion, the image is also a circle with the same radius.

$$\Rightarrow (x-3)^2+(x-4)^2=2 \Rightarrow x^2+y^2-6x-8y+23=0$$

Answer: D

64. What is the value of $\cot 270^\circ+2\cos 90^\circ+4\,sec^2 180^\circ$?

(A) 4 (B) 8 (C) -2 (D) 7

Solution:

$$\cot 270^\circ+2\cos 90^\circ+4\,sec^2 180^\circ=\frac{\cos 270^\circ}{\sin 270^\circ}+2(0)+4\left(\frac{1}{(\cos 180^\circ)^2}\right)$$

$$=\frac{0}{-1}+0+4\left(\frac{1}{(-1)^2}\right)=4$$

Answer: A

65. Which one of the following is a valid assertion that can be proved by the principle of mathematical induction?

(A) $3n^2 + 25 < 3^n$, for every integer $n \geq 3$

(B) $2^n > n + 20$, for every integer $n \geq 4$

(C) $n^3 - n$ is divisible by 6, for every integer $n \geq 1$

(D) $n^2 \leq 2^n$, for every integer $n \geq 1$

Solution:

i. When $n = 1$:

$$\frac{1^3 - 1}{6} = \frac{0}{6} = 0 \qquad (true)$$

ii. When $n = k$ assume the statement is true

That is, $k^3 - k$ is divisible by 6

iii. When $n = k + 1$:

Show that $(k+1)^3 - (k+1)$ is divisible by 6

$$\Rightarrow (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + (3k^2 + 3k) = (k^3 - k) + 3k(k+1)$$

Since one of the consecutive integers $n \; and \; n+1$ is even, $3n(n+1)$ is divisible by 6 and $k^3 - k$ is divisible by 6 by assumption

Therefore, $n^3 - n$ is divisible by 6, for every integer $n \geq 1$

Answer: **C**

THE END!