DON BOSCO HIGH SCHOOL AND PREPARATORY, DILLA

MATHEMATICS *HANDOUT* FOR GRADE 11

UNIT SIX

MATRICES AND DETERMINANTS

Definition

Let $\mathbb{R}$ be the set of real numbers and *m* and *n* be positive integers.

A rectangular array of numbers in R of the form,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$ is called an *m* by *n* (*m*× *n*) matrix in $\mathbb{R}$.

In matrix A above:

- *m* is called the number of *rows* of *A*.
- *n* is called the number of *columns* of *A*.
- a_{ij} is called the ij^{th} element or entry of *A* which is an element in the i^{th} row and j^{th} column of *A*.
- *m*×*n* (read as *m* by *n*) is called the *size or order* of the matrix.

Example

Consider the matrix:

$$B = \begin{pmatrix} 1 & 4 \\ -2 & 6 \\ 5 & 0 \end{pmatrix}$$

matrix B is 3× 2 matrix with $a_{11} = 1, a_{21} = -2,\ a_{31} = 5,\ a_{12} = 4, a_{22} = 6\ and\ a_{32} = 0$

TYPES OF MATRICES

Read some types of matrices from textbook of grade 11on page 232.

Note:

In a square matrix, the diagonal containing the elements a_{11}, a_{22}, a_{33}, a_{44}, ……, a_{nn} is called the ***principal or main diagonal.***

The sum of all elements in the principal diagonal is called the ***trace of the matrix***.

Example

Identify the type(s) of each matrix given below.

$A = (2) \qquad B = (1 \quad 0 \quad 7) \quad C = \begin{pmatrix} 2 \\ -6 \\ 9 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

- **Matrix A** is column matrix, row matrix and also square matrix.
- **Matrix B** is a row matrix.
- **Matrix C** is column matrix.
- **Matrix D** is both square matrix and lower triangular matrix.
- **Matrix E** is square matrix, diagonal matrix, lower triangular matrix, upper triangular matrix, scalar matrix and unit (identity) matrix

Equality of matrices

Definition

Two matrices of the same order are said to be *equal*, if their corresponding elements are equal.

Example

Find the values of x , y , z and a which satisfy the matrix equation

$$\begin{pmatrix} \boldsymbol{x+3} & \boldsymbol{2y+x} \\ \boldsymbol{z-1} & \boldsymbol{4a-6} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{-7} \\ \boldsymbol{3} & \boldsymbol{2a} \end{pmatrix}$$

Solution : By the definition of equality of matrices, we have

x + 3 = 0 ……..(1)

2y + x = -7 ……(2)

z – 1 = 3 ……(3)

4a – 6 = 2a ……(4)

From (1) x = -3

Put the value of x in (2) , we get y = -2

From (3) z = 4

From (4) a = 3

OPERATIONS ON MATRICES

ADDITION AND SUBTRACTION

Definition: *Two matrices of the same order can be added (subtracted) by adding (subtracting) the corresponding elements of the two matrices.*

Example

Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 10 & 4 & 0 \\ 0 & 7 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 9 \\ 0 & 5 & 8 \\ 6 & 2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} -4 & 0 & 8 \\ 9 & 2 & 5 \\ 11 & 2 & 3 \end{pmatrix}$ then,

a) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1+1 & 3+0 & -2+9 \\ 10+0 & 4+5 & 0 \\ 0+6 & 7+2 & 3+(-3) \end{pmatrix} = \begin{pmatrix} 2 & 3 & 7 \\ 10 & 9 & 0 \\ 6 & 9 & 0 \end{pmatrix}$

b) $\mathbf{C} - \mathbf{B} = \begin{pmatrix} -4-1 & 0-0 & 8-9 \\ 9-0 & 2-5 & 5-8 \\ 11-6 & 2-2 & 3-(-3) \end{pmatrix} = \begin{pmatrix} -5 & 0 & -1 \\ 9 & -3 & -3 \\ 5 & 0 & 6 \end{pmatrix}$

c) $\mathrm{A} + \mathrm{C}$ exercise

d) $\mathrm{A} - \mathrm{C} + \mathrm{B}$ exercise

PROPERTIES OF MATRIX ADDITION

If A and B are *two matrices of the same order then,*

1. A + B = B + A (Commutative property
2. (A + B) + C = A + (B + C) (Associative property)
3. A + 0 = A = 0 + A (Existence of additive identity)
4. A + (-A) = 0 (Existence of additive invers

Multiplication of a matrix

Definition : If a matrix ***A*** is multiplied by a scalar ***r,*** then ***rA*** is the matrix obtained from ***A*** by multiplying each element of ***A*** by ***r***.

Example:

Let $A = \begin{pmatrix} -4 & 0 & 8 \\ 9 & 2 & 5 \\ 11 & 7 & 3 \end{pmatrix}$**, t**hen find 3A, $\frac{2}{3}A$ and $-\frac{1}{3}A$

Solution: $3A = 3\begin{pmatrix} -4 & 0 & 8 \\ 9 & 2 & 5 \\ 11 & 7 & 3 \end{pmatrix} = \begin{pmatrix} -4\times 3 & 0\times 3 & 8\times 3 \\ 9\times 3 & 2\times 3 & 5\times 3 \\ 11\times 3 & 7\times 3 & 3\times 3 \end{pmatrix} = \begin{pmatrix} -12 & 0 & 24 \\ 27 & 6 & 15 \\ 33 & 21 & 9 \end{pmatrix}$

$\frac{2}{3}A = \frac{2}{3}\begin{pmatrix} -4 & 0 & 8 \\ 9 & 2 & 5 \\ 11 & 7 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{8}{3} & 0 & \frac{16}{3} \\ 6 & \frac{4}{3} & \frac{10}{3} \\ \frac{22}{3} & \frac{14}{3} & 2 \end{pmatrix}$

$-\frac{1}{3}A =$ _____ **(exercise)**

Note:

- *To multiply any two matrices the number of columns of the first matrix must equal to the number of rows of the second matrix.*
- *The operation is done row by column in such a way that each element of the row is multiplied by the corresponding element of the column and then the products are added.*

Example:

1. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ then,

$$AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{21}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

2. Let $A = (1 \quad -3 \quad 2)$, $B = \begin{pmatrix} 7 \\ 2 \\ 11 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 & 0 \\ -6 & 2 & -5 \\ 0 & 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & -7 \\ 2 & 4 & 0 \\ 7 & 0 & 8 \end{pmatrix}$, then

i) $AB = (1 \quad -3 \quad 2)\begin{pmatrix} 7 \\ 2 \\ 11 \end{pmatrix} = (1x7 + (-3)x2 + 2x11) = (7 + (-6) + 22) = (23)$

ii) $CB = \begin{pmatrix} 1 & -1 & 0 \\ -6 & 2 & -5 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 1x7 - 1x2 + 0x11 \\ -6x7 + 2x2 - 5x11 \\ 0x7 + 2x2 + 3x11 \end{pmatrix} = \begin{pmatrix} 7 - 2 + 0 \\ -42 + 4 - 55 \\ 0 + 4 + 33 \end{pmatrix} = \begin{pmatrix} 5 \\ -93 \\ 37 \end{pmatrix}$

iii) AD **(exercise)** **V) CD** **(exercise)**

iv) DA **(exercise)** **Vi) BD** **(exercise)**

Definition:

1. The ***Transpose*** of a matrix ***A*** denoted by A^T, is the matrix found by interchanging the rows and c olumns of *A*.
2. A square matrix *A* is called a ***symmetric*** matrix if $A^T = A$.
3. A square matrix *A* is called a ***skew-symmetric*** matrix if $A^T + A = 0$

Exercise:

Let $\boldsymbol{A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ -1 & 4 \end{pmatrix}}$, $\boldsymbol{B = \begin{pmatrix} 1 & 2 & 0 \\ -5 & -4 & 3 \end{pmatrix}}$ **and** $\boldsymbol{C = \begin{pmatrix} 4 & 3 & 7 \\ 0 & -6 & 5 \\ 1 & -2 & 3 \end{pmatrix}}$**, then**

$A^T = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 7 & 4 \end{pmatrix}$ $B^T = \begin{pmatrix} 1 & -5 \\ 2 & -4 \\ 0 & 43 \end{pmatrix}$ **and** $C^T = \begin{pmatrix} 4 & 0 & 1 \\ 3 & -6 & -2 \\ 7 & 5 & 3 \end{pmatrix}$

Activity: Dear students try to solve questions on page 234 exercise 6.2 of number 5, 6, 7, 8, 9 and 10.

DETERMINANTS OF MATRICES

The determinant of a matrix A, denoted by $det(A) = |A|$, is a real number.

Example:

i) **Let A** $= \begin{pmatrix} a & d \\ b & c \end{pmatrix}$ be a matrix of order 2 × 2, then the determinant of A is defined as:

det(A) $= |A| = \begin{vmatrix} a & d \\ b & c \end{vmatrix} = ac - bd$.

Let A $= \begin{pmatrix} 2 & -1 \\ 6 & 5 \end{pmatrix}$**, then**

$|A| = \begin{vmatrix} 2 & -1 \\ 6 & 5 \end{vmatrix} = 2x5 - 6x(-1) = 10 + 6 = 16$ **and**

$|A^T| = \begin{vmatrix} 2 & 6 \\ -1 & 5 \end{vmatrix} = 10 + 6 = 16$

ii) Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column).

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:

Let $\boldsymbol{A}=\begin{pmatrix}\mathbf{4} & \mathbf{3} & \mathbf{7}\\ \mathbf{0} & \mathbf{-6} & \mathbf{5}\\ \mathbf{1} & \mathbf{-2} & \mathbf{3}\end{pmatrix}$, **then**

$$|\boldsymbol{A}|=\begin{vmatrix}4 & 3 & 7\\ 0 & -6 & 5\\ 1 & -2 & 3\end{vmatrix}=4\begin{vmatrix}-6 & 5\\ -2 & 3\end{vmatrix}-3\begin{vmatrix}0 & 5\\ 1 & 3\end{vmatrix}+7\begin{vmatrix}0 & -6\\ 1 & -2\end{vmatrix}$$

$$=4(-18+10)-3(0-5)+7(0+6)$$

$$=-32+15+42=\mathbf{25}$$

Minor and co-factors

(i) **Minor** of an element $\boldsymbol{a_{ij}}$ of the determinant of matrix ***A*** is the determinant obtained by deleting $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column, and it is denoted by M_{ij}.

(ii) **Co-factor** of an element $\boldsymbol{a_{ij}}$ is given by $\boldsymbol{A_{ij}} = (-\mathbf{1})^{\boldsymbol{i+j}}\,\boldsymbol{M_{ij}}$.

(iii) Value of determinant of a matrix A is obtained by the sum of products of elements of a row (or a column) with corresponding co-factors. For example: $|\boldsymbol{A}| = \boldsymbol{a_{11}A_{11}} + \boldsymbol{a_{12}A_{12}} + \boldsymbol{a_{13}A_{13}}$.

(iv) If elements of a row (or column) are multiplied with co-factors of elements of any other row (or column), then their sum is zero. For example: $\boldsymbol{a_{11}A_{21}} + \boldsymbol{a_{12}A_{22}} + \boldsymbol{a_{13}A_{23}} = \mathbf{0}$.

Example: Find the minor and co-factors of the entries $a_{11},\ a_{13},\ a_{23}\ and\ a_{22}$ of the matrix

$$\boldsymbol{A}=\begin{pmatrix}\mathbf{4} & \mathbf{3} & \mathbf{7}\\ \mathbf{0} & \mathbf{-6} & \mathbf{5}\\ \mathbf{1} & \mathbf{-2} & \mathbf{3}\end{pmatrix}$$

The minor of $a_{11} = M_{11} = \begin{vmatrix}-6 & 5\\ -2 & 3\end{vmatrix} = -18+10 = -8.$

The minor of $a_{13} = M_{13} = \begin{vmatrix}0 & -6\\ 1 & -2\end{vmatrix} = 0+6 = 6.$

The minor of $a_{23} = M_{23} = \begin{vmatrix}4 & 3\\ 1 & -2\end{vmatrix} = -8-3 = -11.$

The minor of $a_{22} = M_{22} = \begin{vmatrix}4 & 7\\ 1 & 3\end{vmatrix} = 12-7 = 5.$

The co-factor of $a_{11} = C_{11} = (-1)^{1+1}M_{11} = (-1)^2\begin{vmatrix}-6 & 5\\ -2 & 3\end{vmatrix} = \begin{vmatrix}-6 & 5\\ -2 & 3\end{vmatrix} = -8$

The co-factor of $a_{13} = C_{13} = (-1)^{1+3}M_{13} = (-1)^4\begin{vmatrix}0 & -6\\ 1 & -2\end{vmatrix} = 6.$

The co-factor of $a_{23} = C_{23} = (-1)^{2+3}M_{23} = (-1)^5 x(-11) = 11.$

The co-factor of $a_{22} = C_{22} = (-1)^{2+2}M_{22} = (-1)^4 x5 = 5.$

THE ADJOINT OF A MATRIX

The ***adjoint*** of a square matrix is defined as the transpose of the co-factors of each elements.

Adjoint of ***A*** is denoted by ***adj A***, *i.e.*, $\boldsymbol{adj\ A} = (\boldsymbol{C_{ij}})^{\boldsymbol{T}}$.

Example: Find the adjoint of a matrix $\boldsymbol{A} = \begin{pmatrix} \mathbf{4} & \mathbf{3} & \mathbf{7} \\ \mathbf{0} & \mathbf{-6} & \mathbf{5} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \end{pmatrix}$.

Solution:

The co-factor of $a_{11} = \mathbf{C_{11}} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} -6 & 5 \\ -2 & 3 \end{vmatrix} = \begin{vmatrix} -6 & 5 \\ -2 & 3 \end{vmatrix} = -8.$

The co-factor of $a_{12} = \mathbf{C_{12}} = (-1)^{1+2}M_{12} = (-1)^3 \begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} = -(0-5) = 5.$

The co-factor of $a_{13} = \mathbf{C_{13}} = (-1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} 0 & -6 \\ 1 & -2 \end{vmatrix} = 6.$

The co-factor of $a_{21} = C_{21} = (-1)^{1+2}M_{21} = (-1)^3 \begin{vmatrix} 3 & 7 \\ -2 & 3 \end{vmatrix} = -(9+14) = -23.$

The co-factor of $a_{22} = \mathbf{C_{22}} = (-1)^{2+2}M_{22} = (-1)^4 \begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix} = 5$

The co-factor of $a_{23} = \mathbf{C_{23}} = (-1)^{2+3}M_{23} = (-1)^5 \begin{vmatrix} 4 & 3 \\ 1 & -2 \end{vmatrix} = -(-8-3) = \mathbf{11}.$

The co-factor of $a_{31} = \mathbf{C_{31}} = (-1)^{3+1}M_{31} = (-1)^4 \begin{vmatrix} 3 & 7 \\ -6 & 5 \end{vmatrix} = 15+42 = \mathbf{57}.$

The co-factor of $a_{32} = \mathbf{C_{32}} = (-1)^{3+2}M_{32} = (-1)^5 \begin{vmatrix} 4 & 7 \\ 0 & 5 \end{vmatrix} = \mathbf{-20}$

The co-factor of $a_{33} = \mathbf{C_{33}} = (-1)^{3+3}M_{33} = (-1)^6 \begin{vmatrix} 4 & 3 \\ 0 & -6 \end{vmatrix} = \mathbf{-4}$

matrix $\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{-8} & \mathbf{5} & \mathbf{6} \\ \mathbf{-23} & \mathbf{5} & \mathbf{11} \\ \mathbf{57} & \mathbf{-20} & \mathbf{-4} \end{pmatrix}$

$$\mathbf{adjA} = \begin{pmatrix} -8 & 5 & 6 \\ -23 & 5 & 11 \\ 57 & -20 & -4 \end{pmatrix}^{T} = \begin{pmatrix} \mathbf{-8} & \mathbf{-23} & \mathbf{57} \\ \mathbf{5} & \mathbf{5} & \mathbf{-20} \\ \mathbf{6} & \mathbf{11} & \mathbf{-4} \end{pmatrix}$$

PROPERTIES OF DETERMINANTS

For any square matrix A, |A| satisfies the following properties.

i) The determinant of a matrix A and its transpose are equal.

ii) If we interchange any two rows (or columns), then sign of the determinant changes.

iii) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

iv) Multiplying a determinant by *k* means multiplying the elements of only one row (or one column) by *k*.

v) If we multiply each element of a row (or a column) of a determinant by constant *k*, then value of the determinant is multiplied by *k*.

vi) If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

EXERCISE

1. Using the properties of determinants, evaluate the following.

a) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

b) $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

c) $\begin{vmatrix} x + 4 & x & x \\ x & x + 4 & x \\ x & x & x + 4 \end{vmatrix}$

d) $\begin{vmatrix} x + a & y & z \\ x & y + a & z \\ x & y & z + a \end{vmatrix}$

e) $\begin{vmatrix} 3x & -x + y & -x + z \\ x - y & 3y & z - y \\ x - z & y - z & 3z \end{vmatrix}$

f) $\begin{vmatrix} x + 4 & 0 & x - 9 \\ 3x & 0 & x + 2 \\ x & 0 & x + 4 \end{vmatrix}$

2. Solve for ***x*** in each of the following equations:

a) $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$

b) $\begin{vmatrix} 2 & -2 & 1 \\ x & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 1$

INVERSE OF A SQUARE MATRIX

Definition

A square matrix A is said to be ***invertible or non-singular***, if and only if there is a square matrix B such that AB = BA = I, where I is the identity matrix that has the same order as A.

Note:

- *The inverse of a square matrix, if it exists, is unique.*
- *Only a square matrix can have an inverse.*
- *The inverse of matrix A, whenever it exists, is denoted by A^{-1}.*
- *A and A^{-1} have the same order.*
- *A matrix that does not have an inverse is called **singular.***

Theorem:

1. A square matrix *A* is invertible, if and only if $|A| \neq 0$. If *A* is invertible, then $A^{-1} = \frac{1}{|A|} adjA$.
2. If *A* and *B* are two invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Example:

Find the inverse, if it exists, for each of the following matrices

a) $A = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$ b) $B = \begin{pmatrix} \mathbf{4} & \mathbf{3} & \mathbf{7} \\ \mathbf{0} & \mathbf{-6} & \mathbf{5} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \end{pmatrix}$

Solution:

a) $detA = \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 15 - 8 = 7 \neq 0$. Therefore, A is invertible.

$C_{11} = |3| = 3$ $C_{21} = -|4| = -4$

$C_{12} = -|2| = -2$ $C_{22} = |5| = 5$

$\boldsymbol{adjA} = \begin{pmatrix} 3 & -2 \\ -4 & 5 \end{pmatrix}^T = \begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{pmatrix}$

b) $\boldsymbol{detB} = \begin{vmatrix} \mathbf{4} & \mathbf{3} & \mathbf{7} \\ \mathbf{0} & \mathbf{-6} & \mathbf{5} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \end{vmatrix} = \mathbf{4}\begin{vmatrix} \mathbf{-6} & \mathbf{5} \\ \mathbf{-2} & \mathbf{3} \end{vmatrix} - \mathbf{3}\begin{vmatrix} \mathbf{0} & \mathbf{5} \\ \mathbf{1} & \mathbf{3} \end{vmatrix} + \mathbf{7}\begin{vmatrix} \mathbf{0} & \mathbf{-6} \\ \mathbf{1} & \mathbf{-2} \end{vmatrix}$

$= \mathbf{4}(\mathbf{-18} + \mathbf{10}) - \mathbf{3}(\mathbf{0} - \mathbf{5}) + \mathbf{7}(\mathbf{0} + \mathbf{6})$

$= \mathbf{-32} + \mathbf{15} + \mathbf{42} = \mathbf{25} \neq \mathbf{0}$. Therefore, B is invertible.

as we have done in previous example $\boldsymbol{adjB} = \begin{pmatrix} \mathbf{-8} & \mathbf{-23} & \mathbf{57} \\ \mathbf{5} & \mathbf{5} & \mathbf{-20} \\ \mathbf{6} & \mathbf{11} & \mathbf{-4} \end{pmatrix}$

$B^{-1} = \frac{1}{|B|} adjB = \frac{1}{25}\begin{pmatrix} -8 & -23 & 57 \\ 5 & 5 & -20 \\ 6 & 11 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{8}{25} & -23 & \frac{57}{25} \\ \frac{1}{5} & \frac{1}{5} & -\frac{4}{5} \\ \frac{6}{25} & \frac{11}{25} & -\frac{4}{25} \end{pmatrix}$.

Activity: Dear students try to solve all questions on page 245-246 under exercise 6.4

SYSTEMS OF EQUATIONS WITH TWO OR THREE VARIABLES

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $\mathrm{A} = \begin{pmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{pmatrix}, \mathrm{X} = \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} \mathrm{d}_1 \\ \mathrm{d}_2 \\ \mathrm{d}_3 \end{pmatrix}$.

Matrix A is called *the coefficient matrix* of the system and the matrix

$\begin{pmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 & \mathrm{d}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 & \mathrm{d}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 & \mathrm{d}_3 \end{pmatrix}$ is called *the augmented matrix* of the system.

Then, the system of equations can be written as, $AX = B$, i.e,

$$\begin{pmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{pmatrix}\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_1 \\ \mathrm{d}_2 \\ \mathrm{d}_3 \end{pmatrix}$$

To solve systems of linear equations, you may recall, we use either the substitution method or the elimination method. The method of elimination is more systematic than the method of substitution. It can be expressed in matrix form and matrix operations can be done by computers. The method of elimination is based on equivalent systems of equations.

To change a system of equations into an equivalent system, we use any of the following three ***elementary*** (also called ***Gaussian***) ***operations.***

Swapping	Interchange two equations of the system.
Rescaling	Multiply an equation of the system by a non-zero constant.
Pivoting	Add a constant multiple of one equation to another equation of the system.

Elementary row operations

The objective of using elementary row operation is to get as many zeros as possible in the coefficient of a given system of equation in writing the augmented matrix.

Swapping	Interchanging two rows of a matrix: $\boldsymbol{R_i \leftrightarrow R_j}$
Rescaling	Multiplying a row of a matrix by a non-zero constant r: $\boldsymbol{R_i \leftrightarrow rR_j}$
Pivoting	Adding a constant multiple of one row of the matrix onto another row: $\boldsymbol{R_i \leftrightarrow R_i + rR_j}$

Example: Solve the following system of equations using elementary row operation.

a) $\begin{cases} \boldsymbol{2x + 5y = 1} \\ \boldsymbol{3x + 2y = 7} \end{cases}$ **b)** $\begin{cases} x - 3y + z = -1 \\ 2x + y - 4z = -1 \\ 6x - 7y + 8z = 7 \end{cases}$

Solution: (a)

The augmented matrix	$\begin{pmatrix} 2 & 5 & 1 \\ 3 & 2 & 7 \end{pmatrix}$	The objective is to get as many zeros as possible in the coefficients.
$\mathrm{R}_2 \leftrightarrow \mathrm{R}_2 - \frac{3}{2}\mathrm{R}_1$	$\begin{pmatrix} 2 & 5 & 1 \\ 3 - \frac{3}{2}\times 2 & 2 - \frac{3}{2}\times 5 & 7 - \frac{3}{2}\times 1 \end{pmatrix}$ $= \begin{pmatrix} 2 & 5 & 1 \\ 0 & -\frac{11}{2} & \frac{11}{2} \end{pmatrix}$	A zero is obtained in the a_{21} position. Note that the other elements of row 2 are also changed.

The last matrix corresponds to the system of equation:

$$\begin{cases} \mathbf{2x + 5y = 1} \\ \mathbf{-\frac{11}{2}y = \frac{11}{2}} \end{cases}$$

Now, easily we can calculate from equation 2: $\boldsymbol{y = -1}$, and substituting $y = -1$ in equation 1: $\boldsymbol{x = 3}$.

The solution set is $\{(\mathbf{3}, \mathbf{-1})\}$

(b)

The augmented matrix	$\begin{pmatrix} 1 & -3 & 1 & -1 \\ 2 & 1 & -4 & -1 \\ 6 & -7 & 8 & 7 \end{pmatrix}$	The objective is to get as many zeros as possible in the coefficients.
$\mathrm{R}_2 \leftrightarrow \mathrm{R}_2 - 2\mathrm{R}_1$	$\begin{pmatrix} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 6 & -7 & 8 & 7 \end{pmatrix}$	A zero is obtained in the a_{21} position. Note that the other elements of row 2 are also changed.
$\mathrm{R}_3 \leftrightarrow \mathrm{R}_3 - 6\mathrm{R}_1$	$\begin{pmatrix} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 11 & 2 & 13 \end{pmatrix}$	A zero is obtained in the a_{31} position. Note that the other elements of row 3 are also changed.
$\mathrm{R}_3 \leftrightarrow \mathrm{R}_3 - \frac{11}{7}\mathrm{R}_2$	$\begin{pmatrix} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 0 & \frac{80}{7} & \frac{80}{7} \end{pmatrix}$	A zero is obtained in the a_{32} position. Note that the other elements of row 3 are also changed

The last matrix corresponds to the system of equation: $\begin{cases} x - 3y + z = -1 \\ 7y - 6z = 1 \\ \frac{80}{7}z = \frac{80}{7} \end{cases}$

Since this equation and the given equation are equivalent, they have the same solutions. Thus by back- substituting $\mathbf{z = 1}$ from the 3rd equation into the 2nd, we get, $\boldsymbol{y = 1}$ and back-substituting $\boldsymbol{z = 1}$ and $y = 1$ in the 1st equation, we get $x = 1$.

The solution set is $\{(1, 1, 1)\}$.

Echelon Form

A matrix is said to be in Row Echelon Form if,

1. A zero row (if there is) comes at the bottom.
2. The first non-zero element in each non-zero row is 1.

3. The number of zeros preceding the first non-zero element in each non-zero row except the first row is greater than the number of such zeros in the preceding row.

Example: Which of the following matrices are in echelon form?

a) $\begin{pmatrix} 1 & -4 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -3 & 4 & -1 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 1 & \frac{80}{7} \end{pmatrix}$ c) $\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 8 \\ 0 & 1 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

Solution:

A** and **B are in echelon form.

C is not in echelon form because the number of zeros in the second and third row are equal.

D is not in echelon form because in the third row the first non-zero element is not equals to 1.

Note:

1. A matrix is in ***Row Reduced Echelon*** form, if and only if, it is in echelon form and also the first non-zero element in each nonzero row is the only non-zero element in its column.
2. Both echelon form and *row reduced echelon* form have a great role in solving a system of equation with a number of variables.

Example: Reduce each of the following matrices into reduced - echelon form.

a) $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 5 & -1 & -4 \\ 2 & 5 & 4 & -9 \\ -1 & 1 & -2 & 11 \end{pmatrix}$ c) $\begin{pmatrix} 3 & -2 & 3 & 8 \\ 2 & 1 & -1 & 1 \\ 5 & -3 & 2 & 4 \end{pmatrix}$

Solution:

a) $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix} \underline{\mathrm{R}_1 \leftrightarrow \frac{1}{5}\mathrm{R}_1} \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

$$\underline{\mathrm{R}_2 \leftrightarrow \mathrm{R}_1 + \mathrm{R}_2} \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 1 & 4 \end{pmatrix} \underline{\mathrm{R}_3 \leftrightarrow \mathrm{R}_3 - \mathrm{R}_2} \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & \frac{21}{5} \end{pmatrix}$$

$\underline{R_3 \leftrightarrow \frac{5}{21}R_3}\begin{pmatrix} \mathbf{1} & \mathbf{0} & -\frac{\mathbf{1}}{\mathbf{5}} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{5}} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$ *is a* reduced - echelon form of $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$.

$\Rightarrow$ **Dear students try question b and c as an exercise**

Note:

1. When the augmented matrix is changed into either echelon form or reduced-echelon form and if the last non-zero row has numerical coefficients which are all zero while having non-zero constant part, then the system has ***no solution.***
2. When the augmented matrix is changed into either echelon form or reduced-echelon form and if the number of non-zero rows is less than the number of variables, then the system has an ***infinite solution.***

Exercise

1. Find the values of ***c*** for which this system has an infinite number of solutions. $\begin{cases} 2x - 4y = 6 \\ -3x + 6y = c \end{cases}$
2. For what values of ***c*** does $\begin{cases} x + 2y - 3z = 5 \\ 2x - y - z = 8 \\ cx + y + 2z = 14 \end{cases}$ have a unique solution?

Solution:

1. Construct the augmented matrix $\begin{pmatrix} 2 & -4 & 6 \\ -3 & 6 & c \end{pmatrix} \underset{R_2 \leftrightarrow R_2 + \frac{3}{2}R_1}{R_1 \leftrightarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 9 + c \end{pmatrix}$

   Now, for the system to have infinite solution

   $9 + c = 0 \Rightarrow c = -9$

   There will be only one equation $2x - 4y = 6$

   Solving for $x = 3+2y$, the solution set will be

   $\{(3 + 2y, y): y \in \mathbb{R}\}$

   This is an infinite set.

2. Construct the augmented matrix $\begin{pmatrix} 1 & 2 & -3 & 5 \\ 2 & -1 & -1 & 8 \\ c & 1 & 2 & 14 \end{pmatrix}$

   $\underset{R_3 \rightarrow cR_1 + R_3}{R_2 \rightarrow -2R_1 + R_2} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -5 & 5 & -2 \\ 0 & 1 - 2c & 2 + 3c & 14 - 5c \end{pmatrix}$

$$\underline{\mathrm{R}_2 \leftrightarrow -\frac{1}{5}\mathrm{R}_2} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 1-2c & 2+3c & 14-5c \end{pmatrix}$$

$$\underline{\mathrm{R}_3 \leftrightarrow \mathrm{R}_3 + (2\mathrm{c}-1)\mathrm{R}_2} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 0 & 3+c & \frac{68-21c}{5} \end{pmatrix}$$

Hence, the system has a unique solution if $c + 3 \neq 0$.

An alternative way is to use the fact that the system has a unique solution, if $|A| \neq 0$

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ c & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ c & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ c & 1 \end{vmatrix}$$

$$= (-2+1) - 2(4+c) - 3(2+c)$$
$$= -5c - 15$$

Now $|A| \neq 0 = -5c - 15 \neq 0$

$$\Rightarrow \boldsymbol{c \neq -3}$$

CRAMER'S RULE

Determinants can be used to solve systems of linear equations with equal number of equations and unknowns.

The method is practicable, when the number of variables is either 2 or 3.

Consider the system $\begin{cases} a_1x + b_1y = c \\ a_2x + b_2y = d \end{cases}$

Let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ and constant column vector $\begin{pmatrix} c \\ d \end{pmatrix}$.

Obtaining D_x and D_y by replacing the first and second columns by the constant column vector, respectively.

$D_x = \begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c \\ a_2 & d \end{vmatrix}$, then $\boldsymbol{x = \frac{D_x}{D}}$ $\boldsymbol{and}$ $\boldsymbol{y = \frac{D_y}{D}}$

The method is called ***Cramer's rule*** for a system with two equations and two unknowns.

Example: Use Cramer's rule to find the solution set of

a) $\begin{cases} 2x-4y=6 \\ -3x+6y=7 \end{cases}$ b) $\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ x+y+2z=14 \end{cases}$

Solution:

a) $D=\begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix}=12-(-12)=24\neq 0$

Thus, by Cramer's Rule, $\boldsymbol{x=\frac{D_x}{D}=\frac{\begin{vmatrix} 6 & 4 \\ 7 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix}}=\frac{36-28}{12+12}=\frac{8}{24}=\frac{1}{3}}$.

$$\boldsymbol{y=\frac{D_y}{D}=\frac{\begin{vmatrix} 2 & 6 \\ -3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix}}=\frac{14+18}{24}=\frac{32}{24}=\frac{4}{3}}.$$

b) $D=\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix}=1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix}-2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}-3\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$$=(-2+1)-2(4+1)-3(2+1)$$

$$=\boldsymbol{-18\neq 0}$$

Thus, by Cramer's Rule, $\boldsymbol{x=\frac{D_x}{D}=\frac{\begin{vmatrix} 5 & 2 & -3 \\ 8 & -1 & -1 \\ 14 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix}}=\frac{5(-2+1)-2(16+14)-3(8+14)}{-18}=\frac{131}{18}}$

$$\boldsymbol{y=\frac{D_y}{D}=\frac{\begin{vmatrix} 1 & 5 & -3 \\ 2 & 8 & -1 \\ 1 & 14 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix}}=\frac{1(16+14)-5(4+1)-3(28-8)}{-18}=\frac{65}{18}}$$

$$\boldsymbol{z=\frac{D_z}{D}=\frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 8 \\ 1 & 1 & 14 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix}}=\frac{1(-14-8)-2(28-8)+5(2+1)}{-18}=\frac{47}{18}}$$

The solution set is $x=\frac{131}{18}, y=\frac{65}{18}\ and\ z=\frac{47}{18}$.

UNIT SEVEN

THE SET OF COMPLEX NUMBERS

DEFINITION

A complex number z is an expression which is written in the form $\boldsymbol{z = x + iy}$, for some real numbers $\boldsymbol{x}$ and $\boldsymbol{y}$, where $\boldsymbol{i^2 = -1}$; the number x is called the real part of $\boldsymbol{z}$ and is denoted by $\boldsymbol{Re(z)}$ and the number y is called the imaginary part of $\boldsymbol{z}$ and is denoted by $\boldsymbol{Im(z)}$.

Example 1:

complex number	real part $Re(z)$	imaginary part $Im(z)$
$z = 3 + 4i$	3	4
$z = 3i$	0	3
$z = 4$	4	0
$z = \frac{1+2i}{3}$	$\frac{1}{3}$	$\frac{2}{3}$

2. Write the following without exponents.

a) $i^3 = i.i.i = i^2.i = (-1)i = -i$

b)) $i^4 = i.i.i.i = i^2.i^2 = (-1)(-1) = 1$

c) $i^{11} = (i^2)^5.i = (-1)^5.i = (-1)i = -i$

d) $i^{13} = (i^2)^6.i = (-1)^6.i = 1.i = i$

e) $i^{30} = (i^2)^{15} = (-1)^{15} = -1$

f) $i^{32} = (i^2)^{16} = (-1)^{16} = 1$

From the above example 2 we conclude that;

$$\boldsymbol{i^{2n} = \begin{cases} 1, if\ n\ is\ even \\ -1, if\ n\ is\ odd \end{cases}} \quad \textbf{and} \quad \boldsymbol{i^{2n+1} = \begin{cases} i, if\ n\ is\ even \\ -i, if\ n\ is\ odd \end{cases}}$$

EQUALITY OF COMPLEX NUMBERS

Suppose $z_1 = x + iy$ and $z_2 = a + ib$ are two complex numbers; then we define the equality of z_1 and z_2, written as $z_1 = z_2$, if and only if $x = a$ and $y = b$.

Example: If $x - 3i - = 2 + 12iy$, then $x = 2$ and $-3i = 12iy$.

Thus, $x = 2$ and $y = -\frac{1}{4}$

OPERATIONS ON COMPLEX NUMBERS

Given two complex numbers $z_1 = x + iy$ and $z_2 = a + ib$, we define the sum, difference, product and division of z_1 by z_2 of complex numbers as follow:

i) $z_1 + z_2 = (x + iy) + (a + ib) = (x + a) + i(y + b)$

ii) $z_1 - z_2 = (x + iy) - (a + ib) = (x - a) + i(y - b)$

iii) $z_1 . z_2 = (x + iy).(a + ib) = (ax - by) + i(bx + ay)$

iv) $\frac{z_1}{z_2} = \frac{(x+iy)}{(a+ib)} = \frac{(ax+by)}{a^2+b^2} + i\frac{(ay-bx)}{a^2+b^2}$, $z_2 \neq 0$

Example:1. Let $z_1 = 2 + 5i$ and $z_2 = 7 - 11i$, then

i) $z_1 + z_2 = (2 + 5i) + (7 - 11i) = \mathbf{9 - 6i}$

ii) $z_1 - z_2 = (2 + 5i) - (7 - 11i) = \mathbf{-5 + 16i}$

iii) $z_1 . z_2 = (2 + 5i).(7 - 11i) = 2.7 - 2.11i + 7.5i - 5.11i.i = \mathbf{69 - 13i}$

iv) $\frac{z_1}{z_2} = \frac{(2+5i)}{(7-11i)} = \frac{(2.7-11.5)}{7^2+(-11)^2} + i\frac{(7.5-(-11).2)}{7^2+(-11)^2} = \mathbf{\frac{-41}{170} + i\frac{57}{170}}$.

EXERCISE:

Perform the following operations and write your answers in the form of $\boldsymbol{a + ib}$ where *a* and *b* are real numbers.

a) $\frac{1}{2+3i}$ b) $i^{30} + i^{121} - 3i^{49}$ c) $\frac{7+2i}{2+3i}(3 - 5i)$ d) $\left(\frac{7-11i}{2-3i}\right)\left(\frac{5+2i}{3+3i}\right)$

COMPLEX CONJUGATE AND MODULUS

Definition

1. The complex conjugate (or conjugate) of a complex number $z = x + iy$, denoted by $\bar{z}$, is given by

$\bar{z} = x - iy$

2. The absolute value (or modulus) of a complex number $z = x + yi$, denoted by $|z|$, is defined to be $|z| = \sqrt{x^2 + y^2}$

Example: write the conjugate and modulus of the following complex numbers.

complex number	*conjugate*	*modulus*
$z = 3 + 4i$	$\bar{z} = 3 - 4i$	$\lvert z\rvert = \sqrt{3^2 + 4^2} = 5$
$z = -5 - 3i$	$\bar{z}$	$\lvert z\rvert = \sqrt{(-5)^2 + (-3)^2} = \sqrt{34}$
$z = 4i$	$\bar{z} = -4i$	$\lvert z\rvert = \sqrt{0^2 + (-4)^2} = 4$
$z = 3$	$\bar{z} = 3$	$\lvert z\rvert = \sqrt{3^2 + 0^2} = 3$
$z = \frac{5 + 2i}{7}$	$\bar{z} = \frac{5 - 2i}{7}$	$\lvert z\rvert = \sqrt{\left(\frac{5}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = \frac{\sqrt{29}}{7}$

Properties of conjugates and modulus

For any complex numbers z_1 and z_2, the following properties hold true.

1. $\overline{\bar{z}_1} = z_1$
2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
3. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
4. $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$
5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
6. $z_1 + \bar{z}_1 = 2Re(z_1)$
7. $z_1 - \bar{z}_1 = 2iIm(z_1)$
8. $z_1 . \bar{z}_1 = |z_1|^2$
9. $|z_1| = |\bar{z}_1|$
10. $|z_1 . z_2| = |z_1| . |z_2|$
11. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
12. $|z_1 + z_2| \leq |z_1| . |z_2|$
13. $|z_1 - z_2| \geq |z_1| . |z_2|$
14. $|Re(z_1)| \leq |z_1|$ and $|Im(z_1)| \leq |z_1|$

Example:

1. Express the following in the form of $a + bi$

   a) $\frac{14}{i} = \frac{14}{i} \times \frac{i}{i} = \frac{14i}{-1} = \mathbf{-14i}$

   b) $\overline{(2 - 3\imath).(8 + 7\imath)} = \overline{(2 - 3\imath)}.\overline{(8 + 7\imath)} = (2 + 3i).(8 - 7i) = 37 + 10i$

2. Solve the equation

   a) $z^2 + 4 = 0$ b) $3z^2 - 2z + 1 = 0$ c) $z^3 = -1$

solution

a) $z^2 + 4 = 0$

$z^2 = -4$

$z = \pm\sqrt{-4}$

$z = \pm 2i$

b) $3z^2 - 2z + 1 = 0$

$z = \frac{2 \pm \sqrt{4-12}}{6} = \frac{2 \pm \sqrt{-8}}{6}$

$z = \frac{1 \pm i\sqrt{2}}{3}$

c) $z^3 = -1$

$z^3 + 1 = 0$

$(z + 1)(z^2 - z + 1) = 0$

$\Rightarrow z = -1$ or $z^2 - z + 1 = 0$

But $z^2 - z + 1 = 0 \Rightarrow z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

$\Rightarrow S.S = \{-1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$

POLAR REPRESENTATION OF COMPLEX NUMBERS

Let $\boldsymbol{z = x + yi}$ be a non – zero complex number, $r = |z| = \sqrt{x^2 + y^2}$. Then you have $\boldsymbol{x = r\cos\theta, y = r\sin\theta}$, where θ is the angle made by the vector corresponding to $\boldsymbol{z}$ with the positive *x*-axis. ($\boldsymbol{\theta}$ is unique up to addition of a multiple of $2\boldsymbol{\pi}$ radians.)

From the above you have:

$$z = x + yi = r\cos\boldsymbol{\theta} + \boldsymbol{i}\, r\sin\boldsymbol{\theta} = \boldsymbol{r}(\cos\boldsymbol{\theta} + \boldsymbol{i}\sin\boldsymbol{\theta})$$

This is called the polar representation of *z*.

DEFINITION

When a complex number is written in the form $z = \boldsymbol{r}(\cos\boldsymbol{\theta} + \boldsymbol{i}\sin\boldsymbol{\theta})$, θ is called an argument of *z* and is denoted by $arg\ z$. The particular argument of *z* lying in the range $-\pi < \theta \leq \pi$ is called the ***principal argument*** of *z* and is denoted by $Arg\ z$.

The principal argument of *z*: $Arg\ z = \theta$, you also have $arg\ z = \theta + 2\pi$

In general,

$$\boldsymbol{r}(cos\,\boldsymbol{\theta} + \boldsymbol{i}sin\,\boldsymbol{\theta}) = \boldsymbol{r}(cos\,(\boldsymbol{\theta} + \boldsymbol{2n}\pi) + \boldsymbol{i}sin(\boldsymbol{\theta} + \boldsymbol{2n}\pi))$$

for any integer *n*, $\theta + 2n\pi$ is also an argument of *z*, whenever $\theta = arg\,(z)$.

Note that $\frac{y}{x} = \boldsymbol{tan\theta}$ ***if*** $\boldsymbol{x \neq 0}$***, so*** $\boldsymbol{\theta}$ ***is determined by this equation up to a multiple of*** $\boldsymbol{\pi}$***.***

Complex number	**Cartesian form**	**Polar form**
$\boldsymbol{z = x + iy}$	$\boldsymbol{(x, y)}$	$\boldsymbol{z = r}(cos\,\boldsymbol{\theta} + \boldsymbol{i}sin\,\boldsymbol{\theta})$, $r = \sqrt{x^2 + y^2}$

Example

1. $Arg\,(1) = 0$, $Arg\,(-1) = \pi$, $Arg\,(i) = \frac{\pi}{2}$, $Arg\,(-i) = -\frac{\pi}{2}$

2. Express each of the following complex numbers in polar form.

a) $z = 4 + 4\sqrt{3}i$ **b)** $z = 3\sqrt{2} - 3\sqrt{2}\,i$ $z = -2\sqrt{6} - 2\sqrt{2}\,i$

SOLUTION

a) $z = 4 + 4\sqrt{3}i$, $x = 4\ and\ y = 4\sqrt{3}$ and corresponds to the first quadrant.

$tan\theta = \frac{y}{x} = \frac{4\sqrt{3}}{4} = \sqrt{3}$ $r = \sqrt{x^2 + y^2} = \sqrt{4^2 + (4\sqrt{3})^2} = \boldsymbol{8}$

$\theta = tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$z = r(cos\,\theta + isin\,\theta) = \boldsymbol{8}(\boldsymbol{cos}\frac{\boldsymbol{\pi}}{\boldsymbol{3}} + \boldsymbol{isin}\frac{\boldsymbol{\pi}}{\boldsymbol{3}})$ is the polar form.

b) $z = 3\sqrt{2} - 3\sqrt{2}\,i$, $x = 3\sqrt{2}$, $y = -3\sqrt{2}$ and corresponds to the third quadrant.

$r = \sqrt{x^2 + y^2} = \sqrt{(3\sqrt{2})^2 + (-3\sqrt{2})^2} = \boldsymbol{6}$

$tan\theta = \frac{-3\sqrt{2}}{3\sqrt{2}} = -1$ $\Rightarrow \theta = tan^{-1}(-1) = -\frac{\pi}{4} + 2\pi = \frac{7}{4}\pi$

$z = r(cos\,\theta + isin\,\theta) = \boldsymbol{6}(\boldsymbol{cos}\frac{\boldsymbol{7}}{\boldsymbol{4}}\boldsymbol{\pi} + \boldsymbol{isin}\frac{\boldsymbol{7}}{\boldsymbol{4}}\boldsymbol{\pi})$ is the polar form.

Theorem

Suppose $z_1 = r_1(cos\,\theta_1 + isin\,\theta_1)$ and $z_2 = r_2(cos\,\theta_2 + isin\,\theta_2)$. Then the following holds true.

i) $z_1 z_2 = r_1(cos\,\theta_1 + isin\,\theta_1)\ .r_2(cos\,\theta_2 + isin\,\theta_2)$

$= r_1 r_2(cos\,(\theta_1 + \theta_2) + isin\,(\theta_1 + \theta_2))$

ii) $\frac{z_1}{z_2} = \frac{r_1(cos\,\theta_1 + isin\,\theta_1)}{r_2(cos\,\theta_2 + isin\,\theta_2)} = \frac{r_1}{r_2}\,(cos\,(\theta_1 - \theta_2) + isin\,(\theta_1 - \theta_2))$

iii) $z^n = r^n(cos\,n\theta + isin\,n\theta)$; for any integer *n*.

proof: exercise

Example: Find the modulus and principal argument of the following complex numbers and express **z** in polar form.

a) $\boldsymbol{z = (1+i)^{40}}$

i) $|z| = \left|(1+i)^{40}\right| = |1+i|^{40} = \left(\sqrt{2}\right)^{40} = (2)^{20}$

ii) in the complex number $z = 1 + i;\ x = 1\ and\ y = 1$, then $r = \sqrt{2}$

$tan\theta = \frac{1}{\sqrt{2}}$, $\Rightarrow \theta = tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ is the principal argument of z.

iii) The polar form of $z = 1 + i = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})$

Then the polar form of $\boldsymbol{z = (1+i)^{40}} = \left(\sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4})\right)^{40}$

$= \left(\sqrt{2}\right)^{40}(cos40.\frac{\pi}{4} + isin40.\frac{\pi}{4})$

$= (2)^{20}(cos10\pi + isin10\pi)$

b) $\boldsymbol{z = \left(\frac{\sqrt{3}-i}{1+i}\right)^{80}}$

i) $|z| = \left|\left(\frac{\sqrt{3}-i}{1+i}\right)^{80}\right| = \left|\left(\frac{\sqrt{3}-i}{1+i}\right)\right|^{80} = \left(\frac{|\sqrt{3}-i|}{|1+i|}\right)^{80} = \left(\frac{\sqrt{\sqrt{3}^2+(-1)^2}}{\sqrt{2}}\right)^{80}$

$= \left(\frac{\sqrt{4}}{\sqrt{2}}\right)^{80} = \left(\sqrt{2}\right)^{80} = (2)^{40}$

ii) in the complex number $z = \frac{\sqrt{3}-i}{1+i}$; let $z_1 = \sqrt{3} - i$, then

$r_1 = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2$ and θ_1 a principal argument corresponding z_1 is

$\theta_1 = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6} + 2\pi = \frac{11}{6}\pi$

$z_2 = 1 + i$, then $r_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$ and θ_2 a principal argument corresponding z_2 is

$\theta_2 = tan^{-1}(1) = \frac{\pi}{4}$

Thus Arg$z = \left(\frac{\sqrt{3}-i}{1+i}\right)^{80} = 80\left(Arg(\sqrt{3} - i) - Arg(1 + i)\right) = 80\left(\frac{11}{6}\pi - \frac{\pi}{4}\right) = \frac{380}{3}\pi$

iii) the polar form of $z = \left(\frac{\sqrt{3}-i}{1+i}\right)^{80} = (2)^{40}\left(cos\frac{380}{3}\pi + isin\frac{380}{3}\pi\right)$.