

REVIEW EXERCISE FOR UNIT 3

1. Write true for correct statement and false for the incorrect one.
 - a) The graph of the line $y = -2x$ passes through the II and III quadrants.
 - b) The graph of the equation $x = a$, $a \in \mathbb{Q}$, and $a > 0$ is a vertical line that lies to the right of the y-axis.
 - c) A horizontal line has the equation $x - b = 0$.
 - d) If $a < b$ and $c = -3$, then $ac > bc$
 - e) The inequality $8 \geq 5x$, $x \in \mathbb{W}$ has a finite solution set.
2. Plot the following points in a Cartesian coordinate plane:
 $(0, 6)$, $(2, -3)$, $(-4, 5)$ and $(-4, -5)$. Which point lies in neither of the quadrant?
3. Refer Figure 3.12 and answer the following questions.
 - a) Name the coordinates of the point D, E and F.
 - b) Which point has the coordinates $(-2, 1)$?
 - c) Which coordinate of B is 3?
 - d) In which quadrant does the point C, E lie?

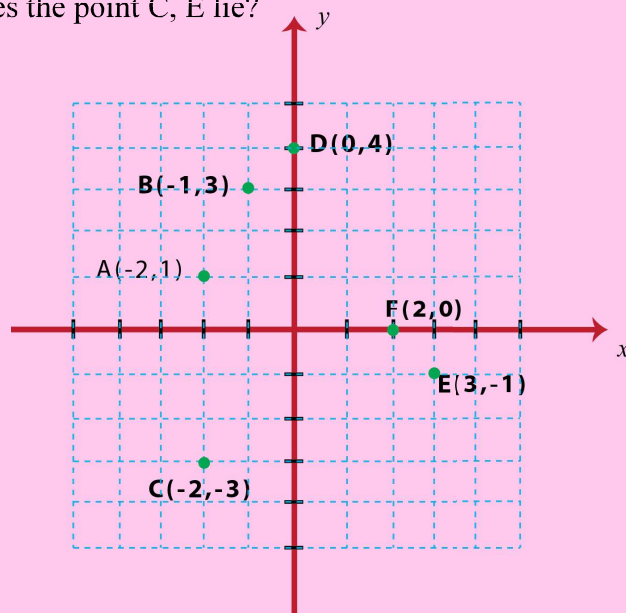


Figure 3.12

4. Fill in the blank with an appropriate inequality sign.
 - a) If $x \geq 3$, then $-4x$ ----- -12
 - b) If $x + 4 < 2x$, then $\frac{1}{2}x$ ----- 2
 - c) If $x < -2$, then $1 - x$ ----- 3
5. Determine whether the given points are on the graph of the equation $x - 2y - 1 = 0$
 - a) (0, 0) b) (1, 0) c) (-1, -1) d) (2, 1)
6. Sketch the graph of the following equations.
 - a) $y = -7x$ b) $y - 11x = 0$
 - c) $3y = -7x + 6$ d) $y - 7 = 0$
7. Find a and b , if the points P(3, 1) and Q(0,2) lie on the graph of $ax - by = 6$
8. Consider the equations $y = x + 1$ and $y = 1 - x$.
 - a) Determine the values of y for each equation when the values of x are $-1, 0$ and 1 .
 - b) Plot the ordered pairs on the Cartesian coordinate plane.
 - c) Which point is a common point, called intersection point?
9. Solve the following inequalities and graph the solution set.
 - a) $3x + 11 \leq 6x + 8$ c) $4 - 3x \leq -\frac{1}{2}(2 + 8x)$
 - b) $6 - 2x > x + 9$
10. Solve the following inequalities
 - a) $\frac{x}{7} \geq \frac{3}{14}x - \frac{1}{7}$
 - b) $2(\frac{1}{2} - x) < 3(1 + \frac{1}{2}x) + 5$
 - c) $\frac{1}{2}x + \frac{1}{3}x - x - 1 \geq 0$
 - d) $\frac{3}{4}y + \frac{1}{2}(y - 3) < \frac{y + 1}{4}$
 - e) $7(y + 1) - y > 2(3y + 4)$
 - f) $-4(y - 1) + 3y \geq 1 - y$

11. Solve the following inequalities in the given domain.

a) $\frac{2}{3}x < 4(4 - x), x \in \mathbb{Z}^+$

b) $\frac{1}{6} - \frac{1}{4}x \geq 2 + \frac{2}{3}x, x \in \mathbb{W}$

c) $2(3y - 7) - 14 \geq 3(2y - 11), x \in \mathbb{Q}$

12. Translate the following sentences in to mathematical expressions.

a) A year ago a father's age exceeded three times his son's age.

b) Three –fourth of a number is greater than 12.

c) The average mark of Saron is not smaller than 89.

13. A board with 2.5 m in length must be cut so that one piece is 30 cm more than the other piece. Find the length of each piece.



Figure 3.13

14. Rodas is 25 years old and her brother Mathanya is 10 years old. After how many years will Rodas be exactly twice as old as Mathanya.

15. Yoseph wants to surprise his wife with a birthday party at her favourite restaurant. It will cost Birr 56.50 per person for dinner, including tip and tax. His budget for the party is Birr 735. What is the maximum number of people Yoseph can have at the party?

16. Getaneh and Salhedin play in the same soccer team. Last Saturday Salhedin scored 3 more goals than Getaneh, but together they scored less than 7 goals. What are the possible number of goals Salhedin scored?

UNIT SIMILARITY OF FIGURES

4

Learning Outcomes: At the end of this unit, learners will be able to:

- Know the concept of similar figures and related terminologies.
- Understand the condition for triangles being similar.
- Apply tests to check whether two given triangles are similar or not.
- Apply real-life situations in solving geometric problems.

Main Contents

4.1 Similar plane figures

4.2 Perimeter and Area of Similar Triangles

Summary

Review Exercise

4.1. Similar Plane Figures

Competencies: At the end of this sub-topic, students should:

- Identify figures that are similar to each other.
- Apply the definition of similarity of two triangles to solve related problems.
- Determine the similarity of two triangles.

INTRODUCTION

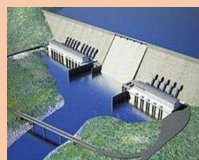
This unit focused on similarity of plane figures. In common language the word similar can have many meanings but in mathematics, the word similar and similarity have very specific meanings. In mathematics, we say that two objects are similar if they have the same shape, but are not necessarily the same size. For instance, an overhead projector forms an image on the screen which has the same shape as the image on the transparency but with the size altered. Two figures that have the same shape but not necessarily the same size are called similar figures.

Activity 4.1

1. Which of the following figures are similar?



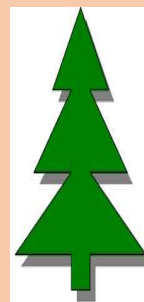
A



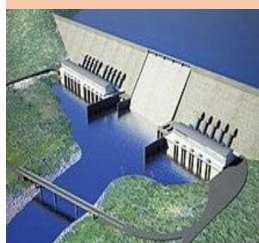
B



C



D



E



F



G



H

Figure 4.1

2. Are any two equilateral triangles always congruent?
3. Let ABCD be a square and $\overline{AC}$ be its diagonal. Are $\triangle ABC \cong \triangle ADC$?
4. In the triangles, $\triangle ABC$ and $\triangle PQR$, if $\angle A \cong \angle P$, and $\angle B \cong \angle Q$, then what is the relation between each $\angle C$ and $\angle R$?
5. Decide whether the polygons are similar.
 - a. Rectangle and parallelogram
 - b. Square and trapezium

4.1.1. Definition and Illustration of Similar Figures

In this section, we will discuss how to compare the size and shape of two given figures.

Recall that, one way to determine whether two geometric figures are congruent is to see if one figure can be moved on to the other figure in such a way that it “fits exactly”.

If two figures have the same shape, but not necessarily the same size (That is, one figure is an exact scale model of the other figure), then we say that the two geometric figures are similar. The symbol $\sim$ (read as “is similar to”) is used for the term similar.

Example 4.1:

A tree and its shadow are similar.

Example 4.2:

The two photograph of the same size of the same person, one at the age of 5 years and the other at the age of 50 years are not similar.

Because similar figures differ only in size, there is a test we can perform to make sure that our shapes are really similar.

Similar Polygons

Definition 4.1: Two polygons are said to be similar if there is a one – to – one correspondence between their vertices such that:

- i) all pairs of corresponding angles are congruent
- ii) the ratio of the lengths of all pairs of corresponding sides are equal.

In the diagram, ABCD is similar to EFGH.

That is, $ABCD \sim EFGH$, then

$$\text{i) } \angle A \cong \angle E, \angle B \cong \angle F, \angle C \cong \angle G, \angle D \cong \angle H$$

$$\text{ii) } \frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{AD}{EH}$$

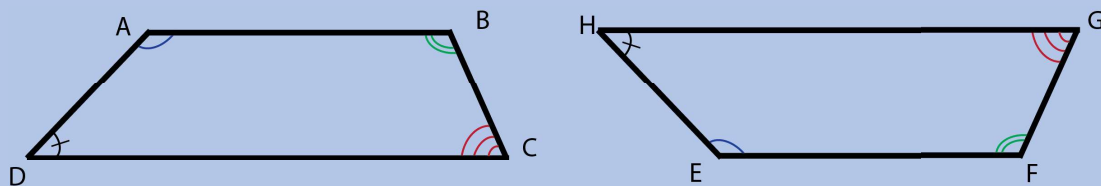


Figure 4.2

Example 4.3:

The following pairs of figures are always similar

- a) Any two squares.
- b) Any two equilateral triangles.

Example 4.4:

Consider the following polygons. Which of these are similar?

Rectangle

Square

Rhombus

Solution:

The square and the rectangle are not similar. Because their corresponding angles are congruent but their corresponding sides are not proportional. The square and the rhombus are not similar. Because their corresponding sides are proportional but their corresponding angles are not congruent. The rectangle and the rhombus are not similar. Because their corresponding angles are not congruent and corresponding sides are not proportional.

Definition: 4.2. The ratio of two corresponding sides of similar polygons is called the scale factor or constant of proportionality (k).

Example 4.5:

The sides of a quadrilateral are 2cm, 5cm, 6cm and 8cm. Find the sides of a similar quadrilateral whose shortest side is 3cm.

Solution:

Let the corresponding sides of a quadrilateral are 3, x , y , and z . Since the corresponding sides of similar polygons are proportional,

$$\frac{2}{3} = \frac{5}{x}, \quad \frac{2}{3} = \frac{6}{y}, \quad \frac{2}{3} = \frac{8}{z}$$

$$x = \frac{15}{2}, \quad y = 9, \quad z = 12$$

Therefore, the sides of the second quadrilateral are 3cm, 7.5cm, 9cm, and 12cm.

Example 4.6:

Decide whether the polygons are similar. If so find the scale factor of Figure A to Figure B.

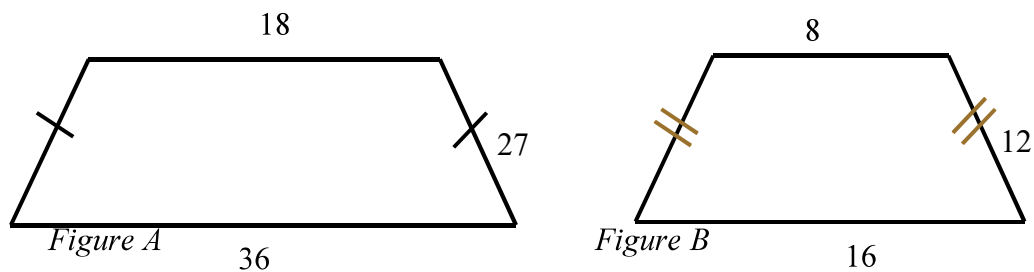


Figure 4.3

Solution: since the corresponding sides are proportional

(i.e. $\frac{18}{8} = \frac{27}{12} = \frac{36}{16} = \frac{9}{4}$) and the corresponding angles are congruent, the two polygons are similar with scale factor of $\frac{9}{4}$.

Exercise: 4.1.

- 1) Write true if the statement is true and false otherwise.
 - a) Any geometric figure is similar to itself.
 - b) Congruent polygons are not necessarily similar.
 - c) Any two regular polygons that have the same number of sides are similar.
 - d) Any two parallelograms are similar.
 - e) Two isosceles triangles are similar.
- 2) What must be the constant of proportionality of two similar polygons in order for the polygons to be congruent?
- 3) The sides of a quadrilateral measure 12cm, 9cm, 16cm and 20cm. The longest side of a similar quadrilateral measures 8cm. Find the measure of the remaining sides of this quadrilateral.

4.1.2. Similar Triangles

You have defined similar polygons in section 4.1.1. Also you know that, any polygon could be dividing into triangles by drawing the diagonals of the polygon. Thus the definition you gave for similar polygons could be used to define similar triangles.

Definition 4.3. ΔABC is similar to ΔDEF ($\Delta ABC \sim \Delta DEF$) if their

Corresponding:

- i) Sides are proportional.

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = k$$

- ii) Angles are congruent.

$$\angle A \cong \angle D, \quad \angle B \cong \angle E, \text{ and } \angle C \cong \angle F$$

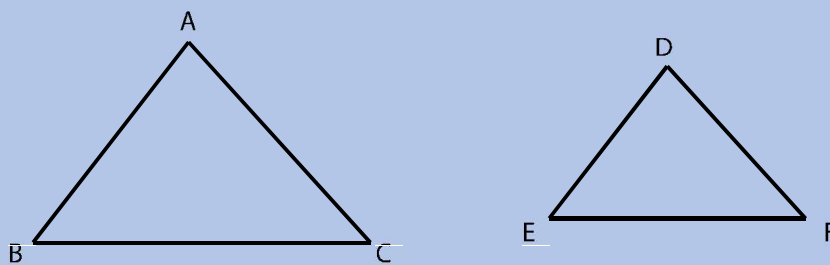


Figure 4.4

Example 4.7:

Show that $\Delta LMN \sim \Delta PQR$.

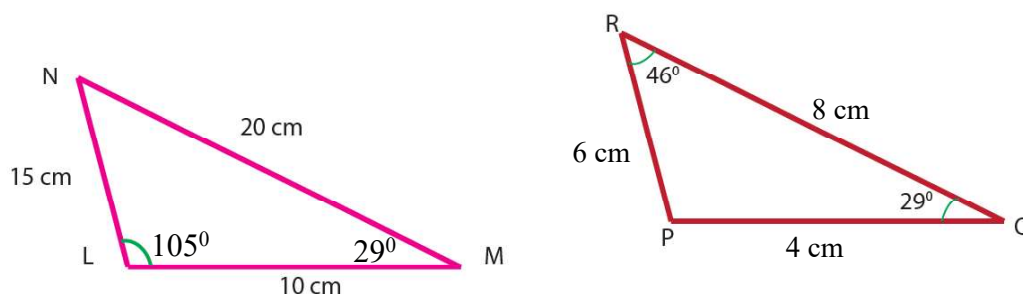


Figure 4.5

Solution:

To show the corresponding angles are congruent, first find the unknown angle.

$$\text{In } \Delta LMN, m\angle L + m\angle M + m\angle N = 180^\circ$$

$$105^\circ + 29^\circ + m\angle N = 180^\circ$$

$$m\angle N = 180^\circ - 134^\circ = 46^\circ$$

In ΔPQR , $m\angle P + m\angle Q + m\angle R = 180^\circ$

$$m\angle P + 29^\circ + 46^\circ = 180^\circ$$

$$m\angle P = 180^\circ - 75^\circ = 105^\circ$$

Hence, the corresponding angles are congruent.

To show whether corresponding sides are proportional or not, check that the ratio of the corresponding sides is the same.

$$\frac{LM}{PQ} = \frac{10}{4} = 2.5, \quad \frac{MN}{QR} = \frac{20}{8} = 2.5, \quad \frac{LN}{PR} = \frac{15}{6} = 2.5$$

Hence, the corresponding sides are proportional with proportionality constant 2.5.

Therefore, $\Delta LMN \sim \Delta PQR$.

Example 4.8:

Suppose $\Delta ABC \sim \Delta DEF$. Then

a) Find the values of x and y .

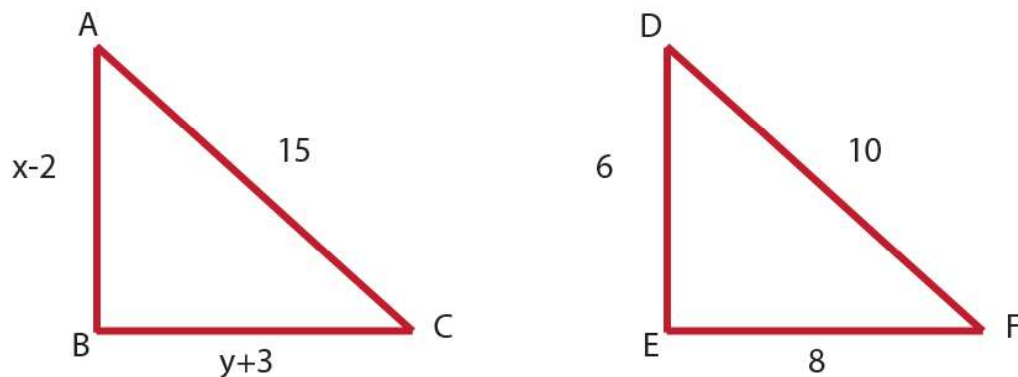


Figure 4.6

Solution:

a) Since $\Delta ABC \sim \Delta DEF$, then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

$$\frac{x-2}{6} = \frac{15}{10},$$

$$\frac{15}{10} = \frac{y+3}{8}$$

$$x - 2 = 6 \times \frac{3}{2}$$

$$y + 3 = 8 \times \frac{3}{2}$$

$$x = 11$$

$$y = 12$$

b) Find the values of a and b .

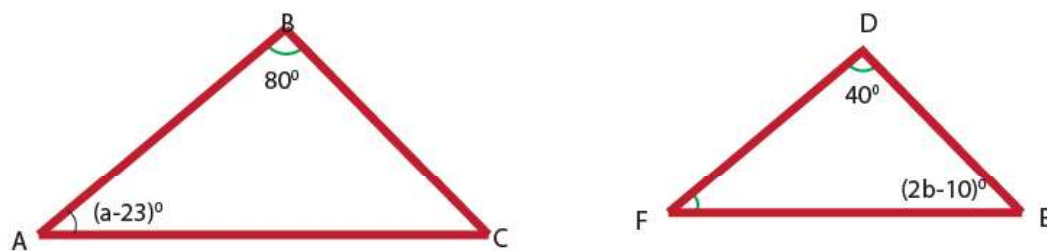


Figure 4.7

Solution:

b) Since $\triangle ABC \sim \triangle DEF$, then

$$m\angle A = m\angle D$$

$$(a - 23)^\circ = 40^\circ$$

$$a = 63^\circ$$

$$m\angle B = m\angle E$$

$$80^\circ = (2b + 10)^\circ$$

$$b = 35^\circ$$

Exercise 4.2:

1) Given that $\triangle ABC \sim \triangle DEF$. Then find x and y .

a) $\frac{x}{DE} = \frac{AC}{DF}$

c) $\angle B \cong \angle y$

b) a side corresponds to $\overline{CA}$ is x

2) Let ABCD be a square. Then show that $\triangle ABC \sim \triangle ADC$.

3) Challenge problem [You may use compass, straightedge, and protractor]

i. Draw any triangle, $\triangle ABC$

ii. Draw any line segment $\overline{DE}$ with $DE = 2AB$.

iii. Construct $\angle GDE \cong \angle A$ and $\angle HED \cong \angle B$.

Let F be the intersection of $\overrightarrow{DG}$ and $\overrightarrow{EH}$.

a) Is $DF = 2AC$? Is $EF = 2BC$?

b) Is $\triangle DEF \sim \triangle ABC$?

From the above challenge problem, you observe the following.

For any given triangle, there exists a similar triangle with any given constant of proportionality.

4.1.3. Tests for similarity of triangles [AA, SSS, and SAS]

We have proved triangles similar by proving that the corresponding angles are congruent and that the ratios of the lengths of corresponding sides are equal, which is long and tiresome. Hence we want to have the minimum requirements which will guarantee us that the triangles are similar. It is possible to prove that when some of these conditions exist, all of these conditions necessary for triangles to be similar exist. These short cut techniques are given as similarity theorems. These similarity theorems help us quickly find out whether two triangles are similar or not.

Activity 4.2.

Melaku said that if an acute angle of one right triangle is congruent to an acute angle of another right triangle, the triangles are similar. Do you agree with Melaku? Explain why or why not.

Theorem 4.1. [AA- Similarity Theorem]

If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.

Thus, $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$

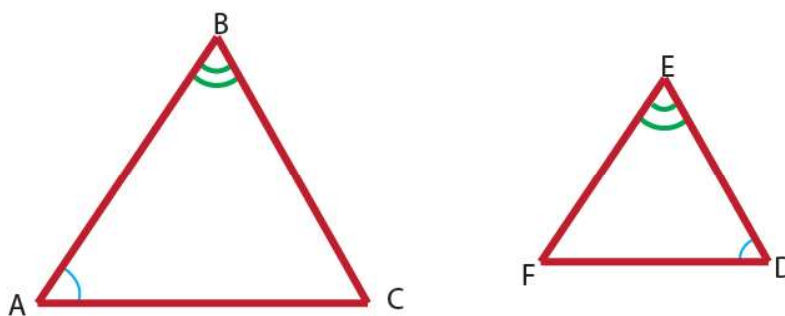


Figure 4.8

Example: 4.9:

Suppose $\overline{DE} \parallel \overline{BC}$ as shown in the figure.

Then show that $\triangle ADE \sim \triangle ABC$.

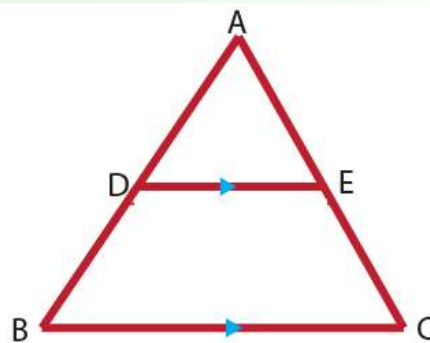


Figure 4.9

Proof:

	Statements	Reasons
1	$\angle A \cong \angle A$	Common angle
2	$\angle B \cong \angle D$	Corresponding angles
3	$\triangle ADE \sim \triangle ABC$	AA similarity theorem

From the above example we conclude that a line through two sides of a triangle parallel to the third side cut off a triangle similar to the given triangle.

Similar triangles are very useful for indirectly determining the size of items which are difficult to measure by hand.

Example 4.10:

Mustefa is a 170 cm tall. He is standing 350 cm away from a lamp post and his shadow is 80cm long. How high is the lamp post?

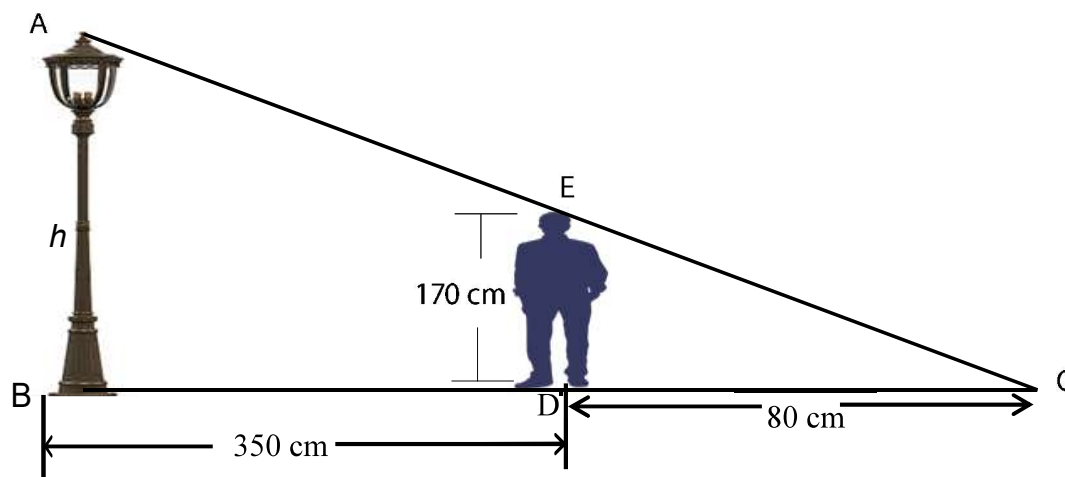


Figure 4.10

Solution:

Consider ΔABC and ΔEDC . We can see that $\angle B \cong \angle D = 90^\circ$ and $\angle C \cong \angle C$ (common angle). Hence, by AA similarity, $\Delta ABC \sim \Delta EDC$.

$$\text{So, } \frac{AB}{DE} = \frac{BC}{DC} = \frac{AC}{EC}$$

$$\frac{h}{170} = \frac{350+80}{80} \text{ (since } BC = BD + DC)$$

$$h = 913.75 \text{ cm}$$

Therefore, the height of the lamp post is 913.75 cm long.

Example 4.11:

Consider the figure below. Determine the values of x and y .

Solution: since $\overline{AB} \parallel \overline{CD}$,

$\angle B \cong \angle C$ [Alternate
Interior Angles]

$\angle O \cong \angle O$ [Vertically
Opposite Angles]

Then, by AA similarity theorem,

$\Delta ABO \sim \Delta DCO$.

$$\text{Thus, } \frac{AB}{DC} = \frac{BO}{CO} = \frac{AO}{DO}$$

$$\frac{15}{21} = \frac{y}{12} \quad \text{and} \quad \frac{15}{21} = \frac{7}{x}$$

$$y = \frac{60}{7} \quad \text{and} \quad x = \frac{49}{5}$$

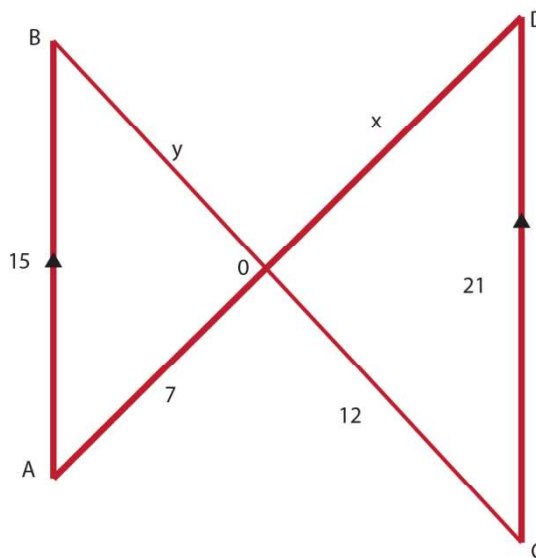


Figure 4.11

Theorem 4.2 [SAS similarity theorem]

If an angle of one triangle is congruent to an angle of another triangle and the sides including those angles are proportional, then the triangles are similar.

Thus, if $\angle C \cong \angle F$, $\frac{AC}{DF} = \frac{BC}{EF}$, then $\Delta ABC \sim \Delta DEF$.

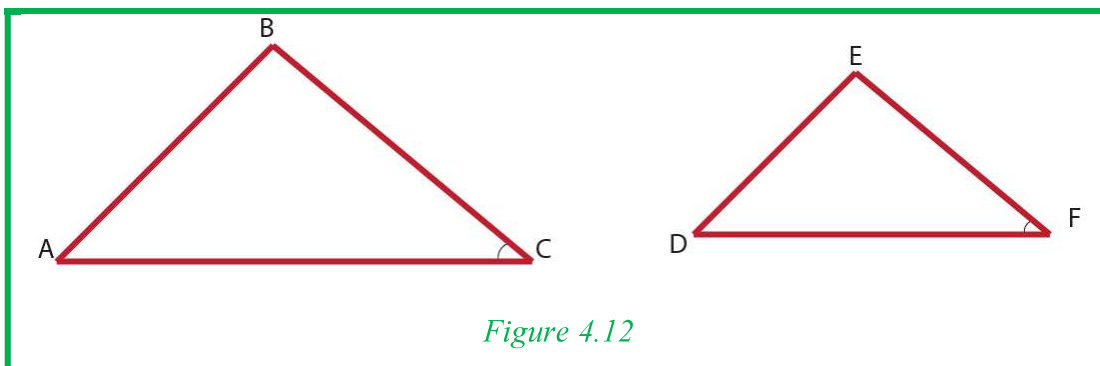


Figure 4.12

Example 4.12:

Is $\triangle ABC \sim \triangle DEF$? Explain.

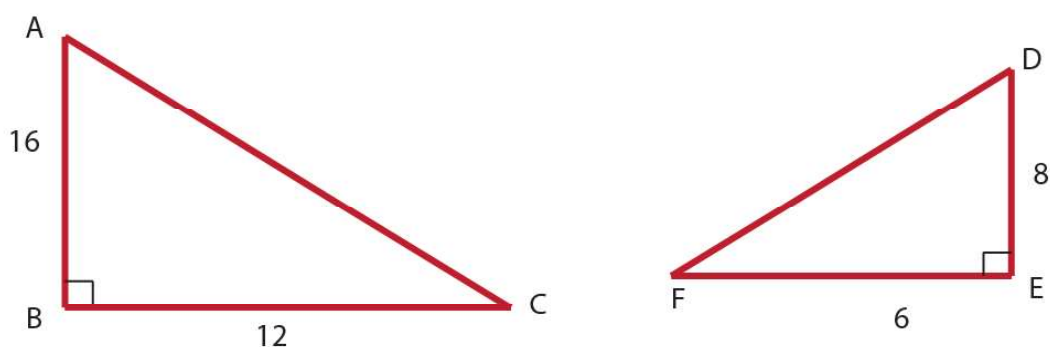


Figure 4.13

Solution:

$$\frac{AB}{DE} = \frac{16}{8} = 2 \text{ and } \frac{BC}{EF} = \frac{12}{6} = 2 \dots\dots \text{sides are proportional}$$

$$m(\angle B) = m(\angle E) = 90^\circ \dots\dots \text{included angles are congruent}$$

Therefore, by SAS similarity theorem $\triangle ABC \sim \triangle DEF$

Example 4.13:

In the figure below,

- i) Show that $\triangle ABC \sim \triangle EFD$.
- ii) Find the value of x .

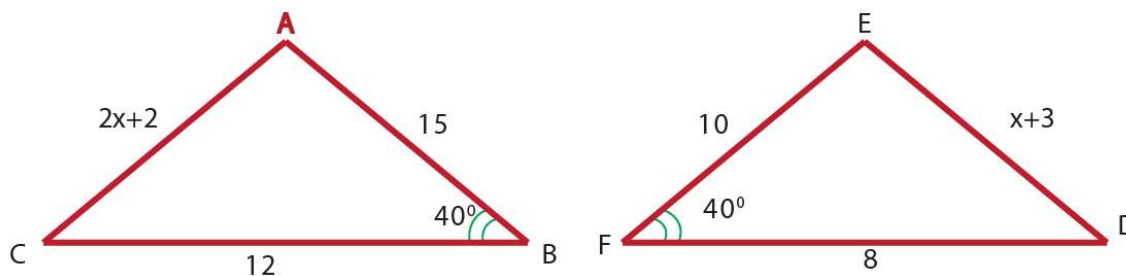


Figure 4.14

Solution:

$$\text{i) } \frac{AB}{EF} = \frac{15}{10} = \frac{3}{2} \quad \text{and} \quad \frac{BC}{FD} = \frac{12}{8} = \frac{3}{2}.$$

$\angle B \cong \angle F$ included angle

Therefore, $\triangle ABC \sim \triangle FED$ by SAS similarity theorem.

$$\text{ii) } \frac{AC}{ED} = \frac{BC}{FD}$$

$$\frac{2x+2}{x+3} = \frac{3}{2}$$

$$x = 5$$

Theorem 4.3 [SSS Similarity Theorem]

If the three sides of one triangle are in proportion to the three sides of another triangle, then the two triangles are similar.

Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then $\triangle ABC \sim \triangle DEF$

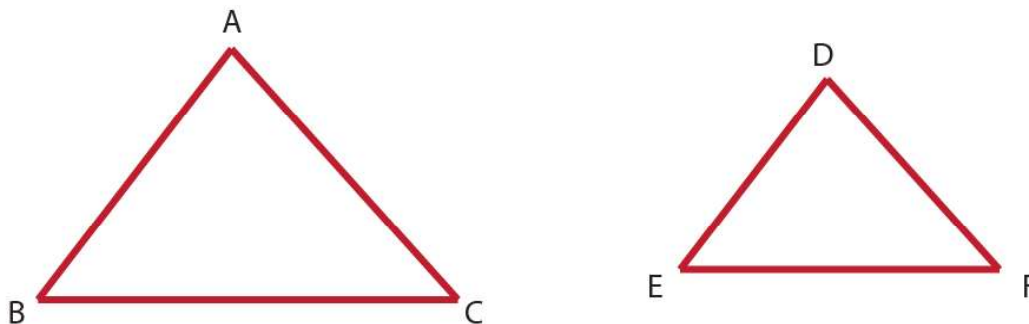


Figure 4.15

Example 4.14:

Based on the given figure show that $\Delta ABC \sim \Delta DEF$.

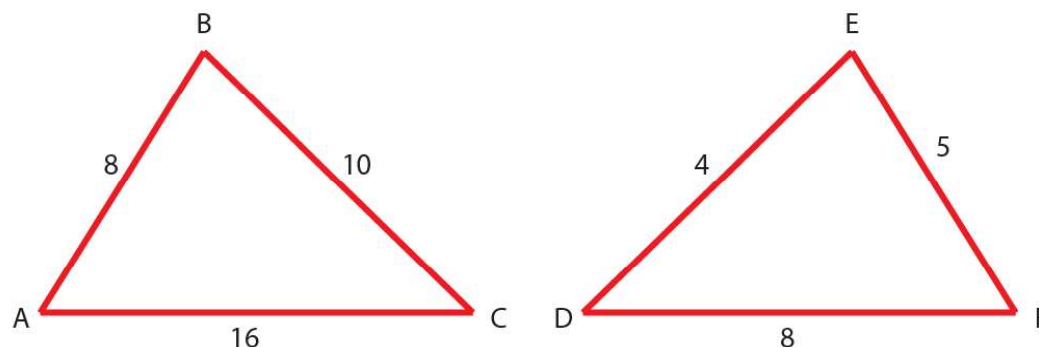


Figure 4.16

Solution:

$$\frac{AB}{DE} = \frac{8}{4} = 2, \quad \frac{BC}{EF} = \frac{10}{5} = 2, \quad \frac{AC}{DF} = \frac{16}{8} = 2$$

Hence, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ [the corresponding sides are proportional]

Therefore, $\Delta ABC \sim \Delta DEF$ SSS Similarity Theorem.

Example 4.15:

The length of the sides of ΔEFD are $EF = 15\text{cm}$, $FD = 8\text{cm}$ and $ED = 12\text{cm}$. If $\Delta EFD \sim \Delta LMN$ and the length of the smallest side of ΔLMN is 6cm , find the measures of the other two sides of ΔLMN .

Solution:

Since the smallest side of ΔEFD is $FD = 8\text{cm}$ and $\overline{FD}$ corresponds to $\overline{MN}$, $MN = 6\text{cm}$.

$$\text{Then, } \frac{FD}{MN} = \frac{EF}{LM},$$

$$\frac{8}{6} = \frac{15}{LM}$$

$$LM = 11\frac{1}{4}\text{cm}$$

$$\frac{FD}{MN} = \frac{ED}{LN},$$

$$\frac{8}{6} = \frac{12}{LN}$$

$$LN = 9\text{cm}$$

Therefore, the sides of ΔLMN are 6cm , $11\frac{1}{4}\text{cm}$ and 9cm .

Example 4.16:

A man who is 1.80m tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28m long; while his own shadow is 4.2m long. How tall is the building?

Solution:

Let h = the height of the building.

Draw a line from top of a building to the end of its shadow. Similarly draw a parallel line through the head of man and its shadow.

Ratio of height to base in large triangle = $\frac{h}{28}$

Ratio of height to base in small triangle = $\frac{1.8}{4.2}$

Since the large and small triangles are similar (Which similarity theorem?), we get the equation $\frac{h}{28} = \frac{1.8}{4.2}$

Now we solve for h ,

$$\begin{aligned} h &= \frac{1.8 \times 28}{4.2} \\ &= 12 \end{aligned}$$

So the building is 12 m tall.

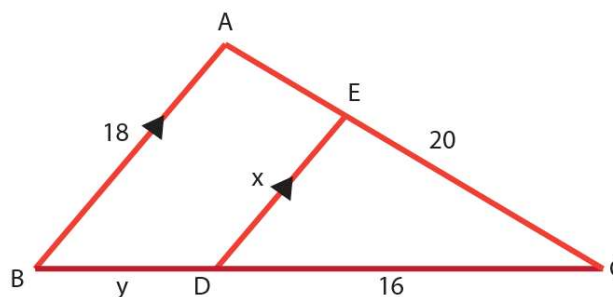
Exercise 4.3

1. In the figure below,

if $\overline{AB} \parallel \overline{ED}$, then

- a. Show that

$$\Delta ACB \sim \Delta ECD.$$



- b. Find the values of x and y .

Figure 4.17

2. Let $\Delta LMN \sim \Delta PQR$ and the sides of ΔLMN are $LM = 12$, $MN = 15$, and $LN = 18$. If the constant of proportionality is $\frac{2}{3}$, then find the length of the corresponding sides of ΔPQR .
3. When you shine a flashlight on a book that is 12cm tall and 8cm wide, it makes a shadow on the wall that is 18cm tall and 12cm wide. Then what is the scale factor of the book to its shadow?

4. Given the figure below.

- a) Show that

$$\Delta RSQ \sim \Delta RQT$$

- b) What is the scale factor of

$$\Delta RSQ \text{ to } \Delta RQT?$$

- c) Find the length of $\overline{QS}$.

- d) Let $RS = 9\text{cm}$. Is ΔRSQ

similar to ΔQST ? Explain

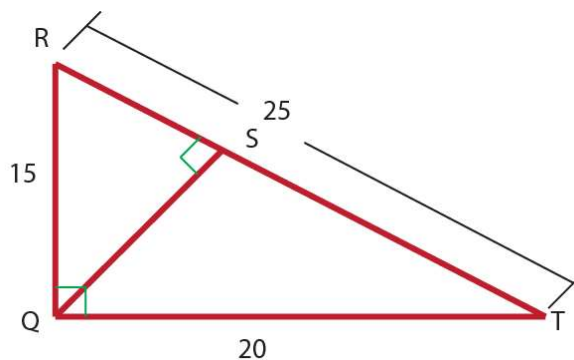


Figure 4.18

5. ΔPQR and ΔLMN are similar such that $\angle P \cong \angle L$, $\angle Q \cong \angle M$, $LM = 8$, $MN = 10$, $PQ = 12$, and $PR = 18$. Find the lengths of $\overline{LM}$ and $\overline{QR}$.
6. The sides of a triangle measures 4cm, 9cm and 11cm. If the shortest side of a similar triangle measures 12cm, find the measure of the remaining sides of this triangle.

4.2. Perimeter and Area of Similar Triangles

Competency: At the end of this sub-topic, students should:

- Explain the relation between the perimeters of two similar triangles.
- Explain the relation between the areas of two similar triangles.

Activity: 4.3

1. What does Perimeter of a triangle mean? Area of a triangle mean?
2. How do you calculate the area of a right- angled triangle?
3. Suppose $\Delta RPQ \sim \Delta DEF$ as shown in figure 4.19 below.
 - a) Find the perimeter P_1 of ΔRPQ and perimeter P_2 of ΔDEF .
 - b) Find $\frac{P_1}{P_2}$ [perimeter ratio]
 - c) Compare $\frac{P_1}{P_2}$ and $\frac{S_1}{S_2}$ [side ratio or scale factor]
 - d) Find the area A_1 of ΔRPQ and A_2 of ΔDEF
 - e) Find $\frac{A_1}{A_2}$ [area ratio]
 - f) Compare $\frac{A_1}{A_2}$ and $(\frac{S_1}{S_2})^2$
 - g) What do you observe from the comparison?

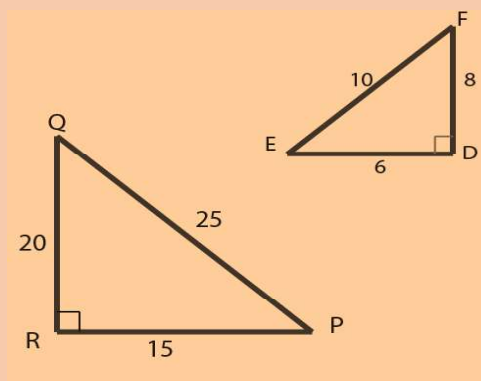


Figure: 4.19

From grade 7 mathematics, you recall that the triangle and the set of points inside of it is called a triangular region, and the area of this triangular region refer the area of a triangle.

Example 4.17:

Let $\Delta ABC \sim \Delta PQR$ and $\overline{CD}$ and $\overline{RE}$ are the respective altitudes from vertex C and vertex R of the triangles.

Then determine the relationship between:

- The altitudes of the triangles.
- The perimeters of the triangles.
- The areas of the triangles.

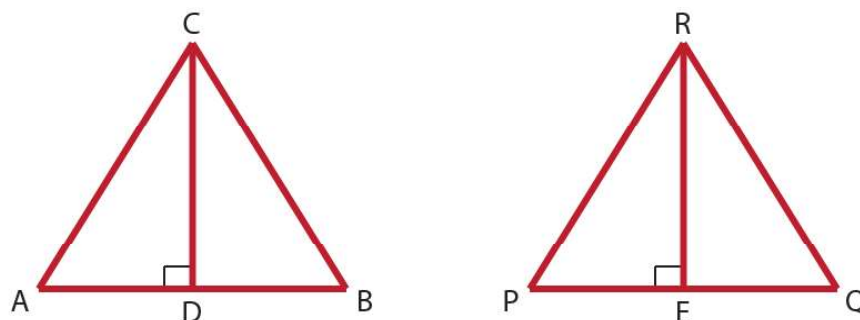


Figure 4.20

Solution:

$$\text{a) } \triangle ABC \sim \triangle PQR$$

Let k be the constant of proportionality between the corresponding

sides. That is, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = k$.

$$AB = kPQ, \quad BC = kQR, \quad AC = kPR$$

Since $\overline{CD}$ and $\overline{RE}$ are the altitudes, $\angle D \cong \angle E$ [right angles] and $\angle B \cong \angle Q$ [similarity definition].

Hence $\triangle CDB \sim \triangle REQ$ by AA similarity.

$$\text{Therefore, } \frac{CD}{RE} = \frac{CB}{RQ} = k.$$

$$CD = k \cdot RE$$

$$\text{b) } p(\triangle ABC) = AB + BC + CA = P_1$$

$$= kPQ + kQR + kRP$$

$$= k(PQ + QR + RP) \text{ and}$$

$$p(\triangle PQR) = PQ + QR + RP = P_2$$

$$\text{Then, } p(\triangle ABC) = k p(\triangle PQR)$$

$$P_1 = k P_2$$

$$\begin{aligned} \text{c) } a(\Delta ABC) &= \frac{1}{2} \cdot AB \cdot CD = A_1 \\ &= \frac{1}{2} \cdot kPQ \cdot kRE \\ &= k^2 \cdot \frac{1}{2} \cdot PQ \cdot RE \text{ and} \end{aligned}$$

$$a(\Delta PQR) = \frac{1}{2} \cdot PQ \cdot RE = A_2$$

$$\text{Then, } a(\Delta ABC) = k^2 \cdot a(\Delta PQR)$$

$$A_1 = k^2 A_2$$

The above example will lead us to the following generalizations.

Theorem 4.4. If the ratios of the corresponding sides of two similar triangles is k , then the ratio of their perimeters is given by:

$$\frac{P_1}{P_2} = \frac{S_1}{S_2} = k$$

Theorem 4.5. If the ratios of the corresponding sides of two similar triangles is k , then the ratio of their areas is given by:

$$\frac{A_1}{A_2} = \left(\frac{S_1}{S_2} \right)^2 = k^2$$

Example 4.18:

The corresponding sides of two similar triangles are 8cm and 6cm. Find the ratio of

- The perimeters of the triangles
- The areas of the triangles.

Solution:

$$\frac{S_1}{S_2} = \frac{8}{6} = \frac{4}{3}$$

$$\text{a) } \frac{P_1}{P_2} = \frac{S_1}{S_2} = \frac{4}{3}$$

$$\text{b) } \frac{A_1}{A_2} = \left(\frac{S_1}{S_2} \right)^2 = \left(\frac{4}{3} \right)^2 = \frac{16}{9}$$

Example 4.19:

The areas of two similar triangles are 64 cm^2 and 100 cm^2 . If one side of the smaller triangle is 4 cm , then find the corresponding side of the second triangle.

Solution: Let $A_1 = 64 \text{ cm}^2$, $A_2 = 100 \text{ cm}^2$, and $S_1 = 4 \text{ cm}$

$$\begin{aligned}\text{Then, } \frac{A_1}{A_2} &= \left(\frac{S_1}{S_2}\right)^2 \\ \frac{64}{100} &= \left(\frac{4}{S_2}\right)^2 \\ S_2^2 &= \frac{100 \times 16}{64} \\ S_2 &= 5 \text{ cm}\end{aligned}$$

Example 4.20:

The ratio of the sides of two similar triangles is $\frac{2}{3}$. The area of the smaller triangle is 36 cm^2 . Find the area of the larger triangle.

Solution:

Let S_1 be the sides of the smaller triangle whose corresponding area be A_1 .

$$\frac{S_1}{S_2} = \frac{2}{3}, \text{ and } A_1 = 36 \text{ cm}^2. \text{ Then, } \frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2$$

$$\frac{36}{A_2} = \left(\frac{2}{3}\right)^2$$

$$A_2 = \frac{36 \times 9}{4} = 81 \text{ cm}^2$$

Example 4.21:

The lengths of sides of a triangle are 16 cm , 23 cm , and 31 cm . If the perimeter of a similar triangle is 280 cm , find

- The length of the longest side of the second triangle
- The ratio of the area of the largest triangle to the smaller.

Solution:

a) Let P_1 be the perimeter of the first triangle. Then

$$P_1 = 16 + 23 + 31 = 70 \text{ cm}, \quad P_2 = 280 \text{ cm}$$

$$\frac{P_1}{P_2} = \frac{S_1}{S_2}$$

$$\frac{70}{280} = \frac{31}{S_2}$$

$$S_2 = \frac{280 \times 31}{70} = 124 \text{cm}$$

$$\text{b) } \frac{A_2}{A_1} = \left(\frac{P_2}{P_1}\right)^2 = \left(\frac{280}{70}\right)^2 = 16, \text{ Therefore, } A_2 : A_1 = 16:1$$

Exercise: 4.4

1. In Figure 4.21 given below, find the ratio of

- Perimeters of $\triangle ABE$ to $\triangle ACD$
- Areas of $\triangle ABE$ to $\triangle ACD$

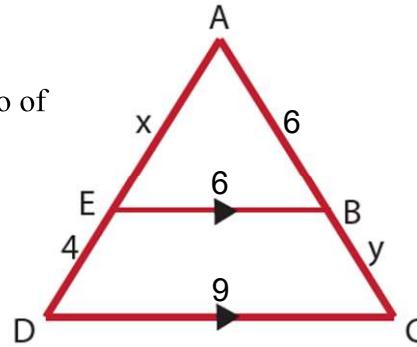


Figure 4.21

- The scale factor of $\triangle ABC$ to $\triangle DEF$ is $4 : 3$. The area of $\triangle ABC$ is x and the area of $\triangle DEF$ is $x - 7$. The perimeter of $\triangle ABC$ is $8 + y$ and the perimeter of $\triangle DEF$ is $3y - 12$.
 - Find the perimeter of $\triangle DEF$.
 - Find the area of $\triangle ABC$.
- $\triangle ABC \sim \triangle DEF$. The length of altitude $\overline{BP}$ exceeds the length of altitude $\overline{EQ}$ by 7. If $AC = \frac{5}{4} DF$, then find the length of each altitude.

SUMMARY FOR UNIT 4

1. Similar geometric figures are figures which have the same shape, but not necessarily the same size.
2. Two polygons are similar if:
 - I. Their corresponding sides are proportional.
 - II. Their corresponding angles are congruent.
3. Congruent figures are always similar.
4. The ratio of two corresponding sides of similar polygons is called the scale factor or constant of proportionality (k).
5. ΔABC is similar to ΔDEF , $\Delta ABC \sim \Delta DEF$ if
 - I. Their corresponding sides are proportional.
 - II. Their corresponding angles are congruent.
6. AA similarity theorem
If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.
7. SAS similarity theorem
If an angle of one triangle is congruent to an angle of another triangle and the sides including those angles are proportional, then the triangles are similar.
8. SSS similarity theorem
If the three sides of one triangle are in proportion to the three sides of another triangle, then the two triangles are similar.
9. If $\Delta ABC \sim \Delta DEF$ with constant of proportionality k , then
 - I. $p(\Delta ABC) = k.p(\Delta DEF)$
 - II. $a(\Delta ABC) = k^2.a(\Delta DEF)$

REVIEW EXERCISE FOR UNIT 4

- Write true if the statement is correct and false if it is not.
 - Two congruent polygons are necessarily similar.
 - Two similar polygons are necessarily congruent.
 - Congruence is a similarity when the constant of proportionality is 1.
 - Rectangle ABCD is similar to rectangle EFGH if $\frac{AB}{EF} = \frac{BC}{FG}$
 - Doubling the side length of a triangle doubles the area.
 - Two triangles with the same area are similar.
- Let $\triangle ABC \sim \triangle DEF$ and the ratio of the respective altitudes is 1:3. If the measures of the sides of $\triangle ABC$ are 3cm, 5cm and 7cm, then find the measures of the sides of the larger triangle, $\triangle DEF$.
- In $\triangle ABC$, the midpoint of $\overline{AC}$ is M and the midpoint of $\overline{BC}$ is N.
 - Show that $\triangle ABC \sim \triangle MNC$.
 - What is the ratio of the sides of $\triangle MNC$ to $\triangle ABC$.
- In $\triangle ABC$ shown below, $\overline{DE} \parallel \overline{AC}$. Find the values of x and y

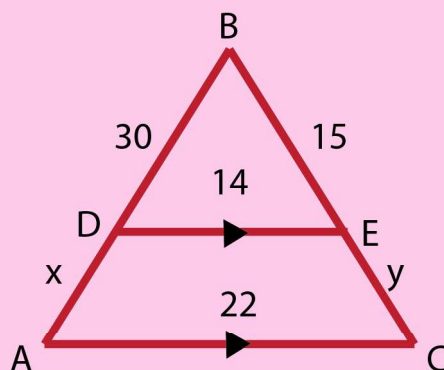


Figure 4.22

- The 1.80m tall lady makes a 9m long shadow, and the palm tree makes a 26m long shadow. Find the height of the tree. [Draw a diagram by yourself]
- ABCD is a trapezium with parallel sides $\overline{BC}$ and $\overline{AD}$ and its diagonals $\overline{BD}$

and $\overline{AC}$ intersect at E. Show that $\triangle BEC \sim \triangle AED$

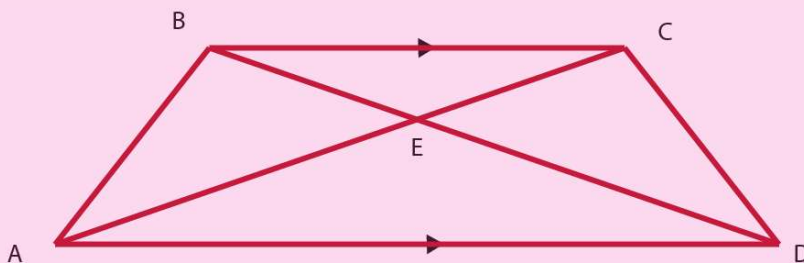


Figure 4.23

7. In the figure below, two pairs of angles can be used to prove $\triangle ABC \sim \triangle DEF$. Determine the congruent angles.

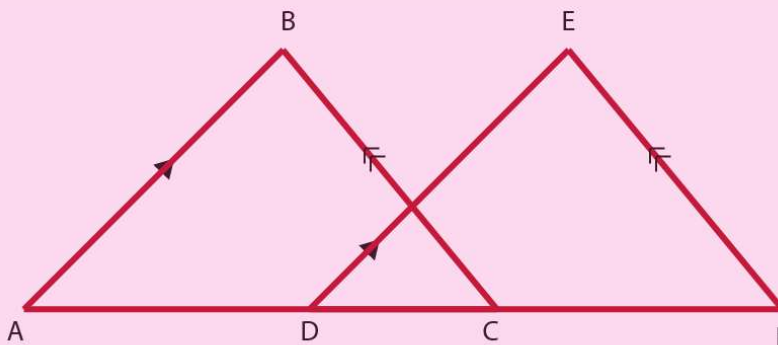


Figure 4.24

8. What pair of congruent angles and what proportion are needed to prove $\triangle ADE \sim \triangle ABC$?

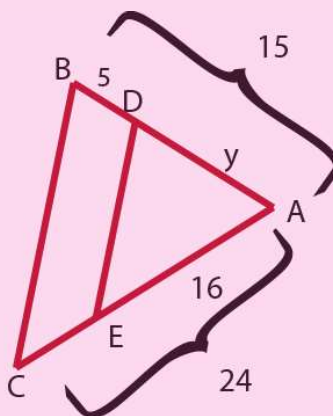


Figure 4.25

9. Indicate the proportion needed to prove

$$\triangle ADE \sim \triangle ABC.$$

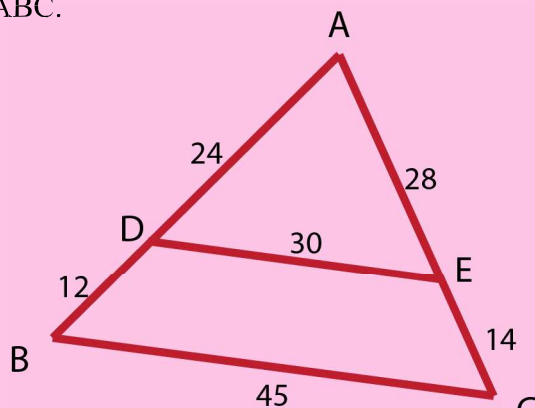


Figure 4.26

10. What angles can be used to prove

$$\triangle AED \sim \triangle FGB \text{ (ABCD is a parallelogram)}$$

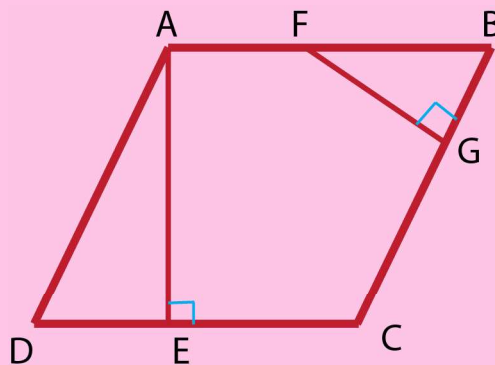


Figure 4.27

11. The areas of two similar triangles are 64 cm^2 and 144 cm^2 .
- Find the ratio of their altitudes.
 - If the perimeter of the smaller triangle is 49 cm , find the perimeter of the larger triangle.

UNIT THEOREMS ON TRIANGLES

5

Learning Outcomes: At the end of this unit, learners will be able to:

- Understand basic concepts about right angled triangles
- Apply some important theorems on right angled triangles.
- Apply real-life situations in solving geometric problems

Main contents

5.1 The three angles of a triangle add up to 180°

5.2 The exterior angle of a triangle equals the sum of the two remote interior angles

5.3 Theorems on the right angled triangle

Summary

Review Exercise

INTRODUCTION

In geometry, one of the most used shapes is a triangle. A triangle has three sides and three angles. These sides and angles are the elements of the triangle. All the polygons have two types of angles which are interior angles and exterior angles. As the triangle is the smallest polygon, it has three interior angles and six exterior angles. There are various kinds of triangles with different angles and sides, but all of them follow the triangle sum properties. The two most important properties are the angle sum property of a triangle and the exterior angle property of a triangle. In this unit you will learn some basic theorems about triangles.

5.1 The three angles of a triangle add up to 180°

Competencies: At the end of this sub-topic, students should:

- Describe the angle sum theorem of a triangle
- Apply the angle sum theorem of a triangle in solving related problems

Group work 5.1

1. Define each of the following terms in your own words.
 - a. Vertically opposite angles
 - b. Alternate interior angles
 - c. Alternate Exterior angles
 - d. Corresponding angles
2. Calculate the measures of marked angles if $l_1 \parallel l_2$ and l_3 is a transversal.

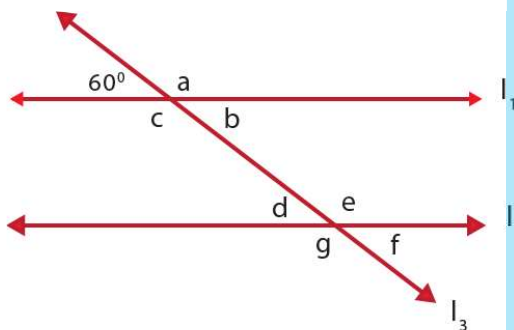


Figure 5.1

3. a) Take $\overline{AB}$. Fold $\overline{AB}$ to get the midpoint at D. Similarly take $\overline{AC}$ and fold to get the midpoint at E.
- b) From D and E draw perpendicular lines to $\overline{BC}$ whose respective intersection points are F and G.
- c) Fold in wards $\triangle ADE$ at $\overline{DE}$, $\triangle DBF$ at $\overline{DF}$ and $\triangle CEG$ at $\overline{EG}$
- d) Is the three folded triangles meet at a single point? If so say H.
- e) Show that $m\angle DHF + m\angle DHE + m\angle EHG = 180^\circ$
- f) Show that $m\angle DHE + m\angle HDE + m\angle DEH = 180^\circ$
- g) What do you conclude from f ?
4. From question number 3 observation, Answer the following questions.
 - a) What is the value of x in the figure?

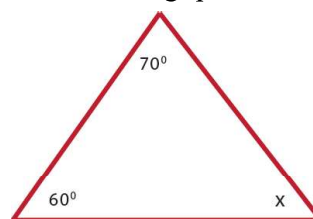


Figure 5.2

- b) What is the measure of $\angle B$ in the figure?

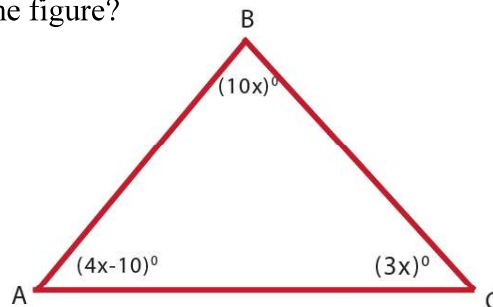


Figure 5.3

- c) Calculate the measure of $\angle A$ in the figure.

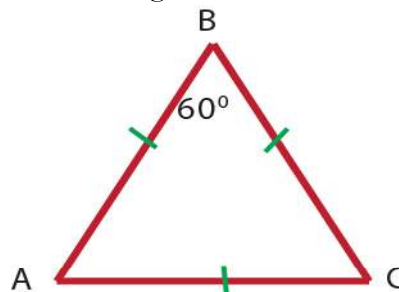


Figure 5.4

Theorem 5.1: (Angle- Sum Theorem)

The sum of the degree measures of the interior angles of a triangle is equal to 180°

Proof:

Consider the following figure:

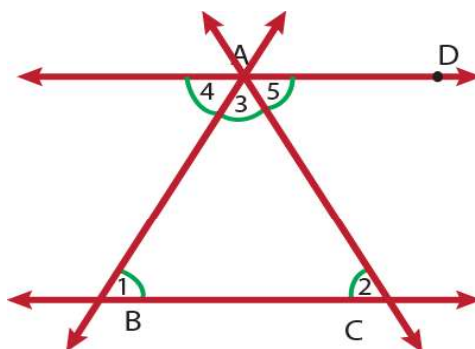


Figure 5.5

Given: $\triangle ABC$ with $\overline{AD} \parallel \overline{BC}$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

Statements	Reasons
1. $\triangle ABC$ With $\overline{AD} \parallel \overline{BC}$	Given
2. $\angle 4 \cong \angle 1$, $\angle 5 \cong \angle 2$	Alternate interior angles
3. $m\angle 4 + m(\angle BAD) = 180^\circ$	Definition of straight angle.
4. $m\angle 3 + m\angle 5 = m(\angle BAD)$	Addition
5. $m\angle 4 + m\angle 3 + m\angle 5 = 180^\circ$	Substitution
6. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	Substitution

Example 5.1:

Find the values of x for each of the following triangles

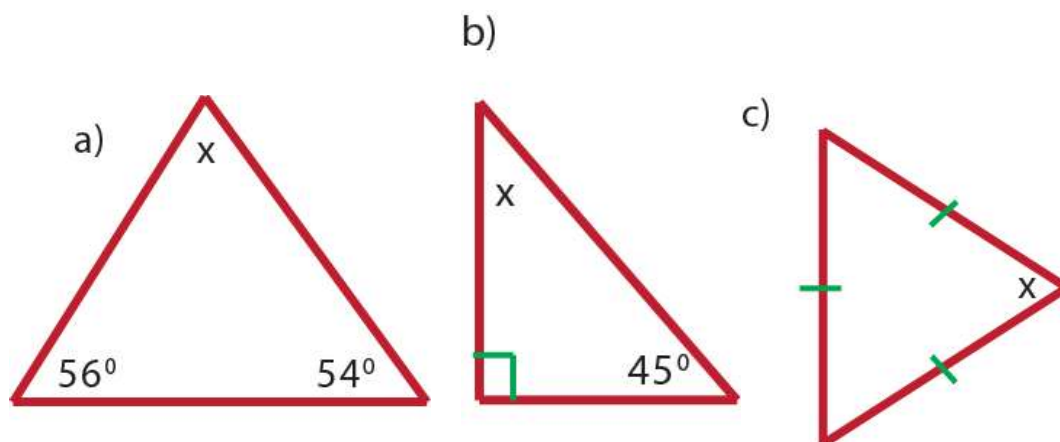


Figure 5.6

Solution:

- a. $x + 56^\circ + 54^\circ = 180^\circ$ (Angle Sum Theorem)
 $x + 110^\circ = 180^\circ$
 $x + 110^\circ - 110^\circ = 180^\circ - 110^\circ$
 $x = 70^\circ$
- b. $x + 90^\circ + 45^\circ = 180^\circ$ (Angle Sum Theorem)
 $x + 135^\circ = 180^\circ$
 $x = 45^\circ$
- c. Since the triangle is an equilateral triangle
 $x + x + x = 180^\circ$
 $3x = 180^\circ$
 $x = 60^\circ$

Example 5.2:

If the measure of the angles of a triangle are $3x$, $4x$, and $5x$, then find the measure of each angle.

Solution:

Consider $\triangle ABC$ with interior angle measures $3x$, $4x$, and $5x$

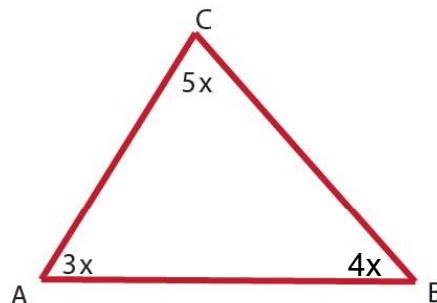


Figure 5.7

Then, $m\angle ABC + m\angle BCA + m\angle CAB = 180^\circ$ (Angle Sum Theorem)

$$4x + 5x + 3x = 180^\circ$$

$$12x = 180^\circ$$

$$x = 15^\circ$$

$$\text{Hence, } m\angle ABC = 4x = 60^\circ$$

$$m\angle BCA = 5x = 75^\circ$$

$$m\angle CAB = 3x = 45^\circ$$

Therefore, the angles are 45° , 60° , and 75°

Exercise 5.1

- Find the values of x for each of the following triangles

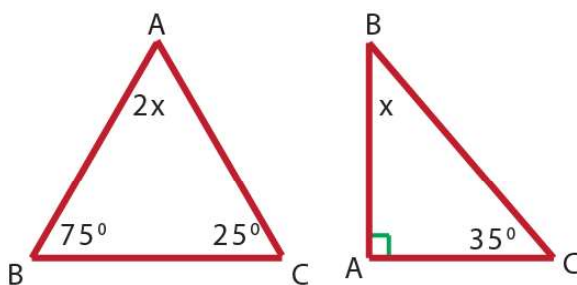


Figure 5.8

- The measures of the three interior angles of a triangle are in the ratio 2: 3: 4. What is the measure of the largest interior angle of a triangle?

3. In a right angled triangle, the measure of one acute angle of a triangle is 6 more than two times the measure of the other acute angle. Find the measure of each acute angle.
4. Determine the values of x, y and z in the figure below

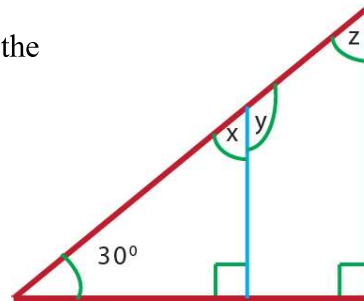


Figure 5.9

5. Find the measures of the missing angles
 - a.

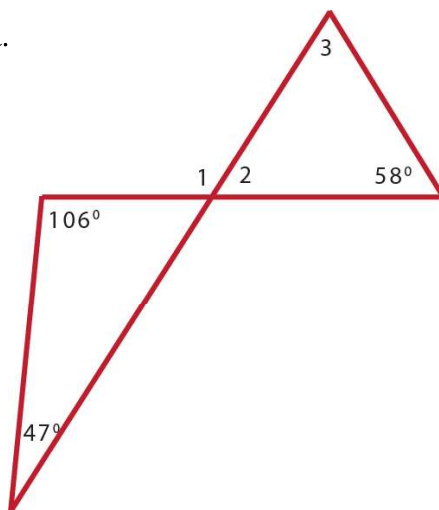


Figure 5.10

- b.

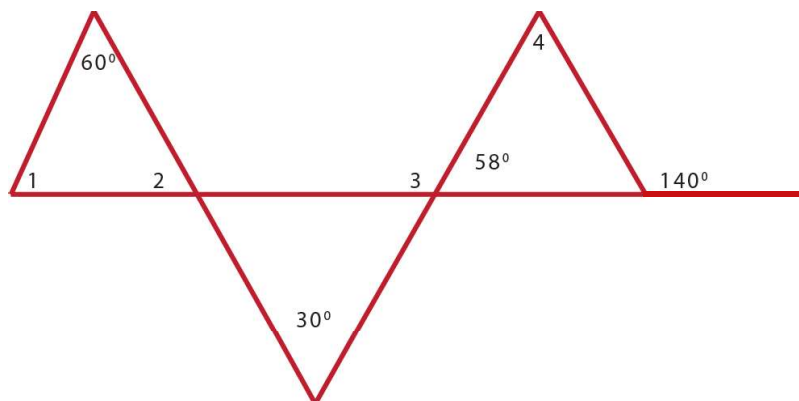


Figure 5.11

5.2. The exterior angle of a triangle equals to the sum of two remote interior angles

Competencies: At the end of this sub-topic, students should:

- Describe the relation between the exterior angle and the two remote interior angles of a triangle
- Prove the exterior angle of a triangle equals the sum of the two remote interior angles

Activity 5.1

1. Define straight angle in your own words.
2. What does an exterior angle of a triangle mean?
3. How do you get an exterior angle of a triangle?
4. Prove that the exterior angle of a triangle equals to the sum of two remote interior angles.

Exterior angle is the angle between one side of the triangle and the extended adjacent side. Each sides of a triangle can form two exterior angles with the two of its extended adjacent sides. Thus, a triangle has six exterior angles.

Theorem 5.2: The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

Proof: Let ABC be a triangle with $\overline{AC}$ extended to form an exterior angle. Let x, y and z be the degree measures of the interior angles of $\triangle ABC$ and d be degree measure of the formed exterior angle.

We want to show that: $x + y = d$

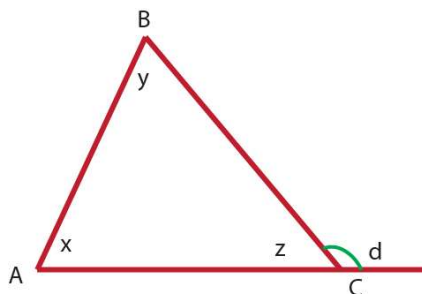


Figure 5.12

Statements	Reasons
1. $z + d = 180^\circ$	Straight angle
2. $x + y + z = 180^\circ$	Angle Sum Theorem
3. $x + y + z = z + d$	Substitution
4. $x + y = d$	Subtracting z from both sides

Therefore, the exterior angle of a triangle is equal to the sum of the two remote interior angles.

Example 5.3:

Given that for a triangle, the two interior angles 25° and $(x + 15)^\circ$ are remote to an exterior angle $(3x - 10)^\circ$. Find the value of x

Solution:

Apply the triangle exterior angle theorem

$$(3x - 10)^\circ = 25^\circ + (x + 15)^\circ,$$

$$(3x - 10)^\circ = x + 25^\circ + 15^\circ$$

$$(3x - 10)^\circ = x + 40^\circ$$

$$2x = 50^\circ$$

$$x = 25^\circ$$

Example 5.4:

Calculate values of x and y in the following triangle.

Solution:

It is clear from the figure that y is an interior angle and x is an exterior angle.

So, by triangle exterior angle theorem

$$x = 80^\circ + 60^\circ$$

$$x = 140^\circ \text{ and}$$

$$y + x = 180^\circ$$

$$y + 140^\circ = 180^\circ$$

$$y = 40^\circ$$

Therefore, the values of x and y are respectively 140° and 40° .

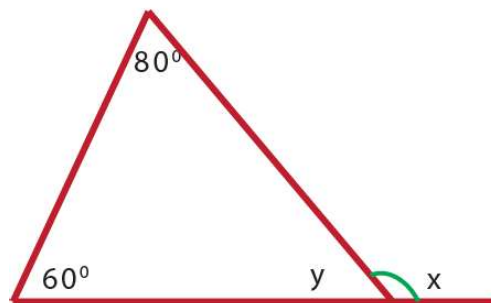
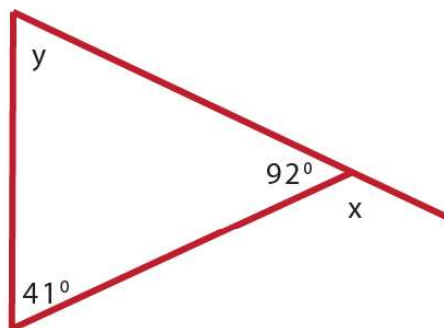


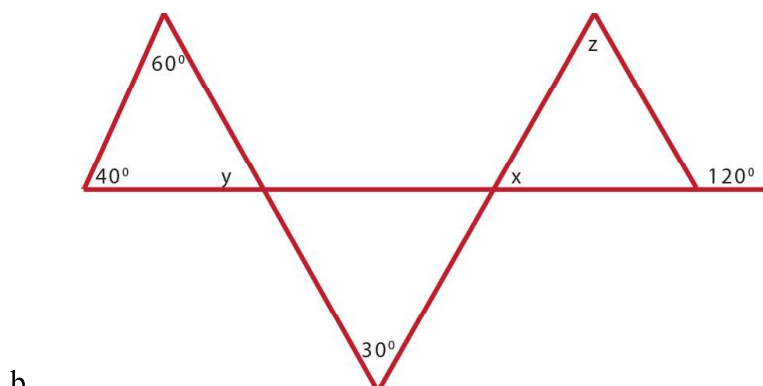
Figure 5.13

Exercise 5.2

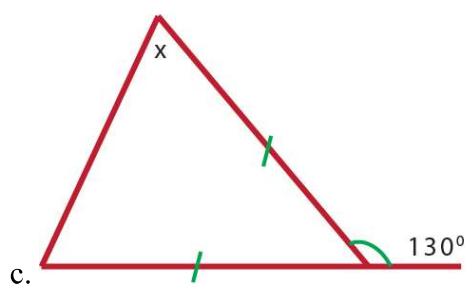
1. The exterior angle of a triangle is 120° . Find the value of x if the interior remote angles are $(4x + 40)^\circ$ and 60° .
2. Calculate the values of x , y and z in the following triangles.



a.



b.



c.

Figure 5.14

3. Fill in the missing steps of the following exterior angle theorem proof.

Given: $\angle 1$ is an exterior angle of $\triangle ABC$

Prove: $m\angle 1 = m\angle A + m\angle B$

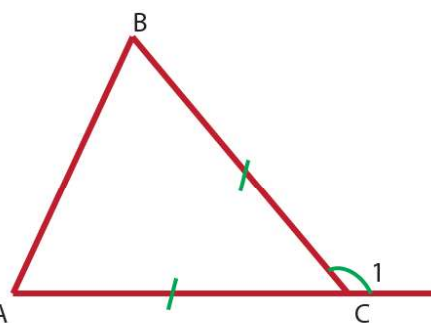


Figure 5.15

Statements	Reasons
1. $\angle 1$ is an exterior angle of $\triangle ABC$	Given
2. $m\angle ACB + m\angle 1 = 180^\circ$	?
3. ?	Triangle Sum Theorem
4. $m\angle ACB + m\angle 1 = m\angle A + m\angle B + m\angle ACB$	?
5. $m\angle 1 = m\angle A + m\angle B$	?

5.3. Theorems on the right angled triangle

5.3.1. Euclid's Theorem and its Converse

Competencies: At the end of this sub-section, students should:

- Describe the right angle triangle, the altitude and the hypotenuse
- Apply Euclid's theorem and its converse for solving related problems

Euclid's Historical note

Euclid is a famous mathematician, very little is known about his life. Euclid was born around 365 B.C. Euclid's most famous work is his collection of 13 books, dealing with geometry, called the Elements. Euclid is often referred to as the "Father of Geometry".



Figure 5.16 (Euclid's)

Activity 5.2:

1. Name the altitude, hypotenuse and the right angle triangle from the given triangle

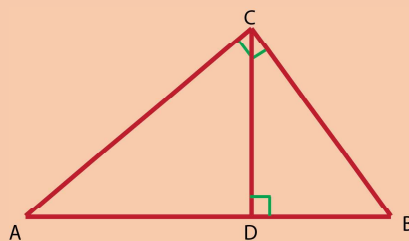


Figure 5.17

2. In figure below Find three similar triangles

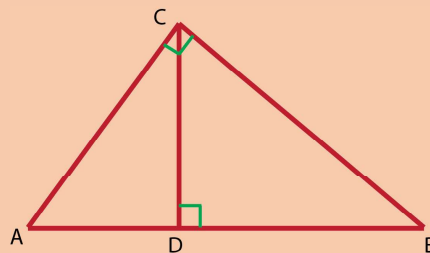


Figure 5.18

In a right triangle, the side opposite the right angle is called the hypotenuse; each of the remaining sides is called a leg. It is sometimes convenient to refer to the length of a side of a triangle by using the lower case letter of the vertex that lies opposite the side. If we draw an altitude to the hypotenuse of the right triangle, we notice that the altitude divides the hypotenuse into two segments.

Theorem 5.3 (Euclid's Theorem)

In a right angled triangle with an altitude to the hypotenuse, the square of the length of each leg of a triangle is equal to the product of the hypotenuse and the length of the adjacent segment into which the altitude divides the hypotenuse

Symbolically

$$1. (BC)^2 = AB \times BD$$

$$\text{or } a^2 = c \times c_2$$

$$2. (AC)^2 = AB \times AD$$

$$\text{or } b^2 = c \times c_1$$

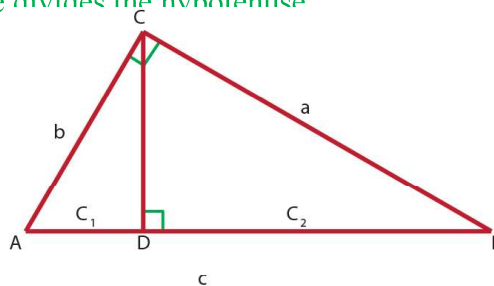


Figure 5.19

Let us consider Figure 5.19 to prove Euclid's theorem.

The altitude $\overline{CD}$ of $\triangle ABC$ divides the triangle into two right angled triangles: $\triangle CBD$ and $\triangle ACD$. How many right angled triangles do we see in figure 5.19?

There are three right angled triangles: $\triangle ACB$, $\triangle CDB$ and $\triangle CDA$. As a result of the similarity relationships that exist between these pairs of triangles the Euclid's theorem will be proved.

- i. $\triangle CBD \sim \triangle ABC$ (by AA similarity theorem)

$$\frac{CB}{AB} = \frac{BD}{BC}$$

$$\frac{a}{c} = \frac{c_2}{a}$$

$$a^2 = cc_2$$

- ii. $\triangle ACD \sim \triangle ABC$ (by AA similarity theorem)

$$\frac{AC}{AB} = \frac{AD}{AC}$$

$$\frac{b}{c} = \frac{c_1}{b}$$

$$b^2 = cc_1$$

Hence the theorem is proved.

Example 5.5:

In Figure 5.20 to the right,

$\triangle ABC$ a right angled triangle with $\overline{CD}$

the altitude to the hypotenuse $\overline{AB}$.

Determine the lengths of $\overline{AC}$ and $\overline{BC}$

if $AD = 4\text{cm}$ and $DB = 12\text{cm}$

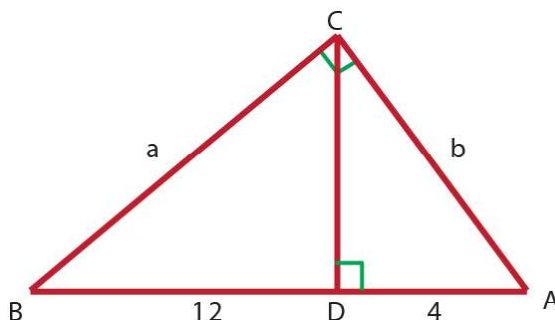


Figure 5.20

Solution:

- i. $(AC)^2 = AB \times AD$ Euclid's Theorem

$$(AC)^2 = 16\text{cm} \times 4\text{cm} \dots \text{ Since } AB = AD + BD$$

$$(AC)^2 = 64\text{cm}^2$$

$$AC = \sqrt{64\text{cm}^2}$$

$$AC = 8\text{cm}$$

- ii. $(BC)^2 = AB \times BD$ Euclid's Theorem

$$(BC)^2 = 16\text{cm} \times 12\text{cm}$$

$$(BC)^2 = 192\text{cm}^2$$

$$BC = \sqrt{192\text{cm}^2}$$

$$BC = 8\sqrt{3}\text{cm}$$

Example 5.6:

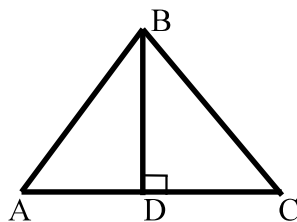
What is the length of the leg of an isosceles triangle if the length of the base is

18cm and its vertex angle is 90° ?

Solution:

Consider an isosceles triangle ABC with vertex angle B.
 since the altitude $\overline{BD}$ divides the base $\overline{AC}$ into two equal parts,

$$\begin{aligned}(AB)^2 &= AD \times AB \\ &= 9 \times 18 \\ AB &= 9\sqrt{2} \text{ cm}\end{aligned}$$



Theorem 5.4 (converse of Euclid's Theorem)

In a triangle, if square of each shorter side of a triangle is equal to the product of the length of the longest side of the triangle and the adjacent segment into which the altitude to the longest sides divides this side, then the triangle is right angled.

Symbolically 1. $a^2 = c \times c_2$ and

2. $b^2 = c \times c_1$

then, $\triangle ABC$ is right angled triangle.

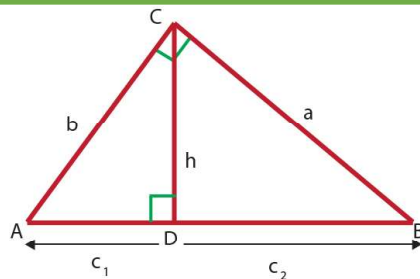


Figure 5.21

Example 5.7:

In figure 5.22 to the right

$AD = 4\text{cm}$, $BD = 8\text{cm}$

$AC = 4\sqrt{3}\text{cm}$ and $BC = 4\sqrt{6}\text{cm}$ and
 $m\angle ADC = 90^\circ$

Is $\triangle ABC$ a right angled?

Solution:

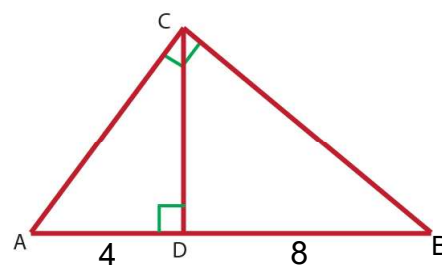


Figure 5.22

i. $(AC)^2 ? AB \times AD$

$(4\sqrt{3}\text{cm})^2 ? 12\text{cm} \times 4\text{cm}$ Since $AB = AD + BD$

$48\text{cm}^2 = 48\text{cm}^2$

Hence $b^2 = c \times c_2$

ii. $(BC)^2 ? AB \times BD$

$(4\sqrt{6}\text{cm})^2 ? 12\text{cm} \times 8\text{cm}$

$96\text{cm}^2 = 96\text{cm}^2$

Hence $a^2 = c \times c_1$

Therefore, from (i) and (ii), $\triangle ABC$ is a right angled triangle.

Exercise 5.3

1. In Figure 5.23 to the right if

$DB = 8\text{cm}$ and $AD = 4\text{cm}$

Then find the lengths of

- a. $\overline{AB}$ b. $\overline{BC}$
c. $\overline{AC}$ d. $\overline{DC}$

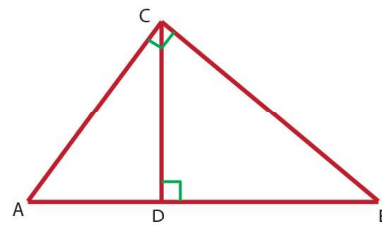


Figure 5.23

2. In Figure 5.24, find the length

of x , a and c If $AD = 4\text{cm}$ and

$BD = 8\text{cm}$

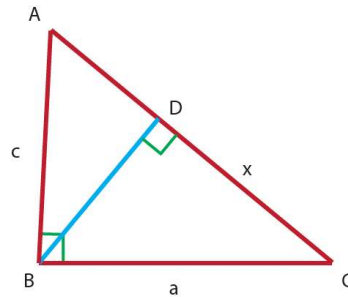


Figure 5.24

3. How long is the altitude of an equilateral triangle if a side of a triangle is

- a. 4cm ? b. 8cm ?

4. In figure 5.25, $AD = 3.2\text{cm}$,

$DB = 1.8\text{cm}$, $AC = 4\text{cm}$ and

$BC = 3\text{cm}$. Is $\triangle ABC$ is a right angled?

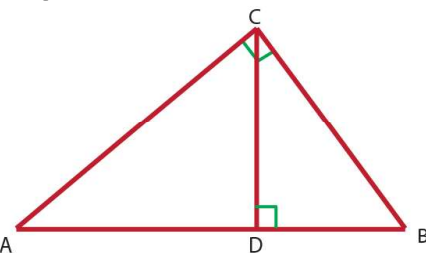


Figure 5.25

5.3.2. The Pythagoras' Theorem and its converse

Competencies: At the end of this sub-topic, students should:

- Derive the Pythagoras theorem by using Euclid's theorem and paper folding
- Apply Pythagoras' Theorem and its converse for solving related problem

Pythagoras's Historical note

Pythagoras was a Greek religious leader and a philosopher who made developments in astronomy, mathematics and music theories.

Pythagoras theorem was introduced by the Pythagoras. He started a group of Mathematicians who works religiously on numbers and lived like monks.

Finally, the Greek Mathematician stated the theorem hence it is called by his name as the "Pythagoras theorem".

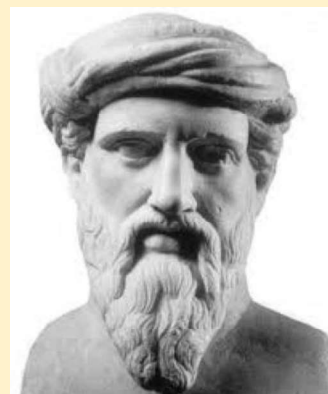


Figure 5.26 (Pythagoras's)

Activity 5.3

- Pythagorean triples consists of three whole numbers a, b , and c which obey the rule: $a^2 + b^2 = c^2$
 - When $a = 15$ and $b = 20$, then find the value of c
 - When $a = 24$ and $c = 40$, then find the value of b
- Pythagorean Theorem states that $a^2 + b^2 = c^2$ for the sides a, b , and c of a right-angled triangle. When $a = 5$, $b = 12$ then $c = 13$. Find three more sets of a rational numbers which satisfy Pythagoras' theorem

One of the most useful theorems in mathematics provides a means for finding the length of any side of a right triangle, given the lengths of the other two sides. The sides are related by the equation

$$(\text{hypotenuse})^2 = (\text{leg1})^2 + (\text{leg2})^2.$$

This relationship is known as the Pythagoras theorem.

Theorem 5.5 (Pythagoras' Theorem)

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

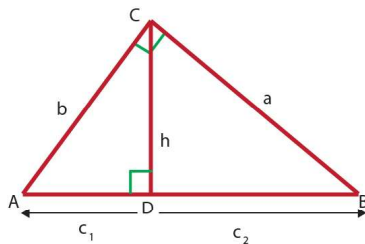


Figure 5.27

That is, if legs of lengths a and b , hypotenuse of length c , then

$$a^2 + b^2 = c^2$$

There are different methods to prove Pythagoras Theorem.

Algebraic Method of proof of Pythagoras theorem

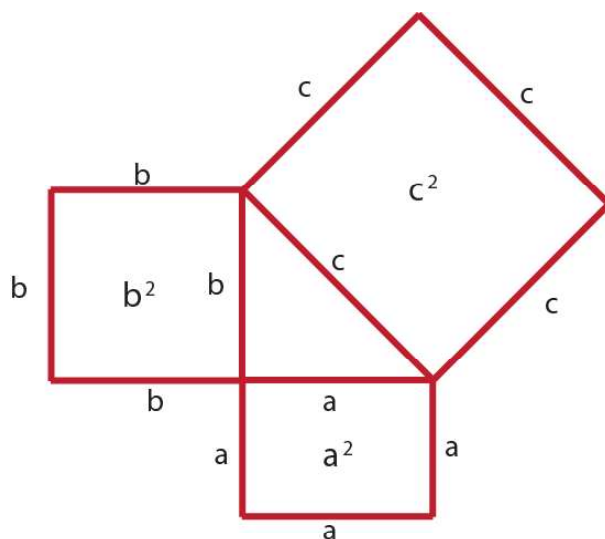


Figure 5.28

Consider the above figure; three squares and one right angled triangle

- The area of square with length $a = a^2$
- The area of square with length $b = b^2$
- The area of square with length $c = c^2$ but the area of square with side c is equal to the square of the length of hypotenuse.

The *Pythagoras theorem* says that the square of the two smaller squares add up to the same as the area of the largest square.

That is, $a^2 + b^2 = c^2$ which is proved.

Algebraic Method proof of Pythagoras theorem will help us in deriving the proof of the Pythagoras theorem by using of a, b and c (values of the measures of the sides corresponding to sides BC, AC, and AB respectively)

Consider four right triangles where b is base, a is altitude and c is hypotenuse. Arrange these four congruent right triangles in the given square, whose side is $a + b$.

The area of the square so formed by arranging the four triangles is c^2 .

The area of a square with side

$$a + b = \text{Area of 4 triangles} + \text{Area of square}$$

with side c .

This implies $(a + b)^2 = 4 \times \frac{1}{2}(a \times b) + c^2$ but

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

$$a^2 + b^2 + 2ab = 2ab + c^2$$

$$\text{Therefore, } a^2 + b^2 = c^2$$

Pythagoras theorem proof using similar triangles

Proof: Let $\triangle ACB$ be right angled triangle, the right angle at C .

Given: $\triangle ACB$ is a right triangle and $\overline{CD} \perp \overline{AB}$.

We want to show that: $a^2 + b^2 = c^2$

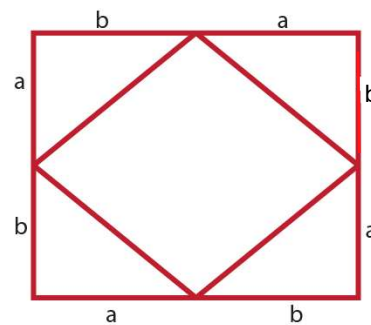


Figure 5.29

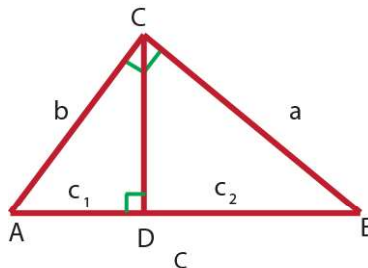


Figure 5.30

Statements	Reasons
1. $a^2 = c_2 \times c$	Euclid's Theorem
2. $b^2 = c_1 \times c$	Euclid's Theorem
3. $a^2 + b^2 = (c_2 \times c) + (c_1 \times c)$	Sum of step1 and step2
4. $a^2 + b^2 = c(c_2 + c_1)$	Taking c as a common factor
5. $a^2 + b^2 = c(c)$	$c = c_2 + c_1$
6. $a^2 + b^2 = c^2$	Proved

Example 5.8:

What is the length of the hypotenuse of a right angle triangle with leg lengths are 3cm and 4cm.

Solution:

Let $a = 3cm$, $b = 4cm$ and c be the length of the hypotenuse. Then

$$a^2 + b^2 = c^2 \text{ [Pythagorean theorem]}$$

$$(3cm)^2 + (4cm)^2 = c^2$$

$$9cm^2 + 16cm^2 = c^2$$

$$25cm^2 = c^2$$

$$c = \sqrt{25cm^2}$$

$$c = 5cm$$

Therefore, the hypotenuse is 5cm long.

Example 5.9:

Consider a right angle triangle. The measure of its hypotenuse is 13cm. One of the sides of a triangle is 5cm. Find measure of the third side.

Solution:

Let the unknown leg be b . Then

$$(Hypotenuse)^2 = (leg1)^2 + (leg2)^2$$

$$(13cm)^2 = b^2 + (5cm)^2$$

$$169cm^2 = b^2 + 25cm^2$$

$$b^2 = 169cm^2 - 25cm^2$$

$$b^2 = 144cm^2$$

$$b = \sqrt{144cm^2}$$

$$b = 12cm$$

Therefore, the base of the right angled triangle is 12cm.

Example 5.10:

Find the length of the leg of the given right angle triangle

Solution:

Let x be the length of the required leg.

Then $(\text{hypotenuse})^2 = (\text{leg1})^2 + (\text{leg2})^2$

$$(10\text{cm})^2 = (8\text{cm})^2 + (x)^2$$

$$100\text{cm}^2 = 64\text{cm}^2 + x^2$$

$$x^2 = 100\text{cm}^2 - 64\text{cm}^2$$

$$x^2 = 36\text{cm}^2$$

$$x = \sqrt{36\text{cm}^2}$$

$$x = 6\text{cm}$$

Therefore, the other leg is 6cm long.

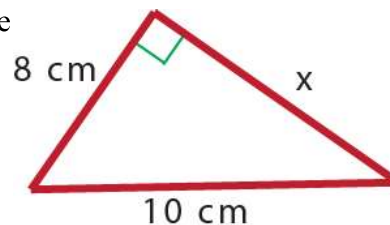


Figure 5. 31

Note:

When three positive integers can be the lengths of the sides of a right triangle, this set of numbers is called Pythagorean triple. The most common Pythagorean triples are 3, 4, and 5. If we multiply each number of a Pythagorean triples by some positive integer x , then the new triple created is also a Pythagorean triple because it will satisfy the relation $a^2 + b^2 = c^2$.

In general patterns of $3x, 4x, 5x$ for x is a positive integer forms a Pythagorean triples. There are many other families of Pythagorean triples including $5x, 12x, 13x$ or $8x, 15x, 17x$ where x is a positive integer.

Example 5.11:

Let a, b and c be a Pythagorean triple numbers:

- if $a = 15$ and $b = 20$, then find the value of c
- if $a = 24$ and $c = 40$, then find the value of b

Solution:

Since a , b and c are a Pythagorean triple numbers

$$\text{a. } a^2 + b^2 = c^2$$

$$(15)^2 + (20)^2 = c^2$$

$$225 + 400 = c^2$$

$$c^2 = 625$$

$$c = \sqrt{625}$$

$$c = 25$$

$$\text{b. } a^2 + b^2 = c^2$$

$$(24)^2 + b^2 = (40)^2$$

$$576 + b^2 = 1600$$

$$b^2 = 1600 - 576$$

$$c^2 = 1024$$

$$c = \sqrt{1024}$$

$$c = 32$$

If the lengths of the sides of a triangle are known, then we can determine whether the triangle is a right triangle by applying the converse of Pythagoras Theorem.

Theorem 5.6 (Converse of Pythagoras' Theorem)

If the square of the length of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle.

Consider the following figure.

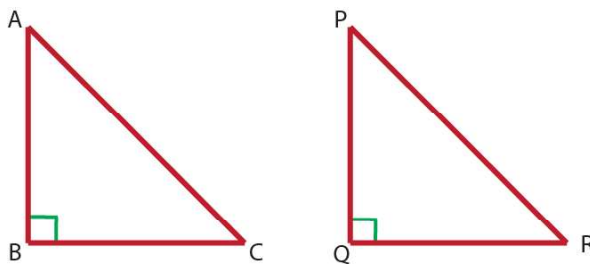


Figure 5.32

Given: $\triangle ABC$ in which $(AC)^2 = (AB)^2 + (BC)^2$

To prove: $\angle B = 90^\circ$

Construction: Draw a right angle triangle

PQR in which $\angle Q = 90^\circ$ and $PQ = AB$, $RQ = BC$

Proof: In a right angle triangle PQR

$$(PR)^2 = (PQ)^2 + (RQ)^2 \text{ [Pythagoras Theorem]}$$

Since, $PQ = AB$, $RQ = BC$

$$(PR)^2 = (AB)^2 + (BC)^2. \text{ But } (AC)^2 = (AB)^2 + (BC)^2$$

So, $PR = AC$

In $\triangle ABC$ and $\triangle PQR$ we have $AB = PQ$, $BC = QR$, and $AC = PR$

So, $\triangle ABC \cong \triangle PQR$ by SSS.

Thus $\angle B = \angle Q$. But $\angle Q = 90^\circ$

So $\angle B = 90^\circ$.

Example 5.12:

Determine whether the triangle whose sides have lengths 11, 60, and 61 is a right triangle.

Solution:

We want to show that $a^2 + b^2 = c^2$.

$$(11)^2 + (60)^2 ? (61)^2$$

$$121 + 3600 ? 3721$$

$$3721 = 3721$$

Hence, the triangle is a right triangle.

Exercise 5.4

1. State the Pythagoras theorem in your own words.
2. Find the lengths of the sides of a rhombus whose diagonals are 6cm and 8cm.
3. Find the unknown sides length of each of the following figures.

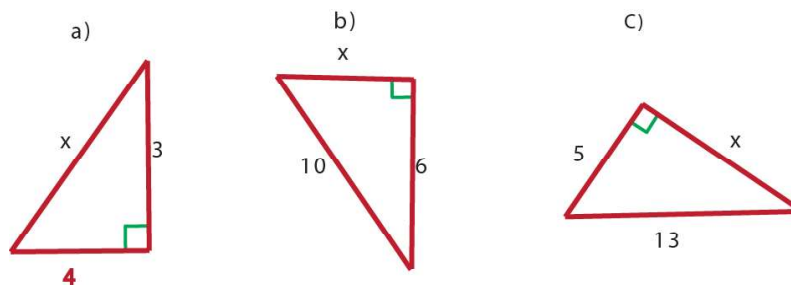


Figure 5.33

4. What is the length of the hypotenuse of a right angle triangle with leg lengths of 27cm and 36cm?
5. If diagonal of a square is 12cm long, then how long is each sides of a square?
6. A 15m ladder lean against the side of the house, and the base of the ladder is 9m away. How high above the ground is the top of the ladder?
7. A man travels 24km due west and then 10km due north. How far is the man now from the starting place?

8. In the figure to the right

i) find the length of

a. $\overline{AE}$

b. $\overline{DF}$

c. $\overline{EF}$

d. $\overline{AB}$

ii) Is $\triangle CEF$ right- angled?

iii) Is $\triangle BDF$ right- angled?

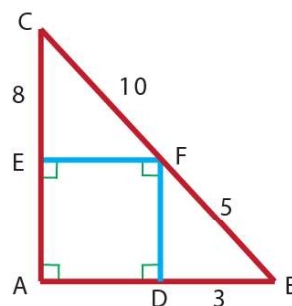


Figure 5.34

SUMMARY FOR UNIT 5

1. The sum of the degree measures of the interior angles of a triangle is equal to 180°
2. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

3. Theorem (Euclid's Theorem)

In a right angled triangle with an altitude to the hypotenuse, the square of the length of each leg of a triangle is equal to the product of the hypotenuse and the length of the adjacent segment into which the altitude divides the hypotenuse:

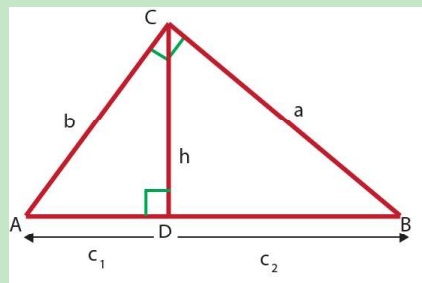


Figure 5.35

Symbolically

1. $(BC)^2 = AB \times BD$ or $a^2 = c \times c_2$
2. $(AC)^2 = AB \times AD$ or $b^2 = c \times c_1$

4. Theorem (converse of Euclid's Theorem)

In a triangle, if a square of each shorter side of a triangle is equal to the product of the length of the longest side of the triangle and the adjacent segment into which the altitude to the longest sides divides this side, then the triangle is right angled.

Symbolically 1. $a^2 = c \times c_2$ and

2. $b^2 = c \times c_1$, then $\triangle ABC$ is right angled triangle

5. Theorem (Pythagoras' Theorem)

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of other two sides. That is legs of lengths a and b hypotenuse of length c , then $a^2 + b^2 = c^2$

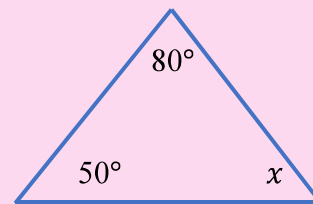
6. When three positive integers can be the length of the sides of a right triangle, this set of numbers is called Pythagoras triple. The most common Pythagoras triple is 3, 4, and 5

7. Theorem 5.3.2 (converse of Pythagoras' Theorem)

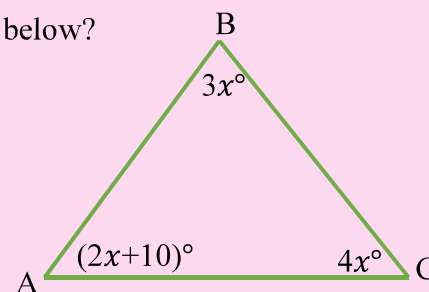
In a triangle, if square of one side is equal to the sum of the square of the other two sides, then angle opposite the first side is a right angle.

REVIEW EXERCISE FOR UNIT 5

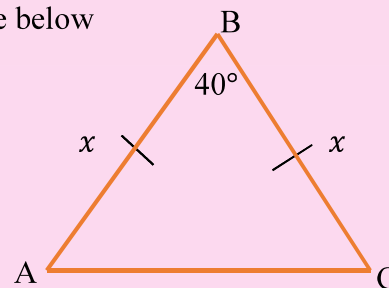
1. If the measure of the angles of a triangle are $6x$, $8x$ and $10x$, then give the measures of each angle.
2. One acute angle of a right triangle measure 37° . Find the measures of the other acute angle?
3. Prove that the sum of the degree measures of the interior angles of a triangle is equal to 180° .
4. What is the value of x in figure below?

*Figure 5.38*

5. What is the measure of angle B in the figure below?

*Figure 5.39*

6. Calculate the measure of angle A in the figure below

*Figure 5.40*

7. Prove that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.
8. Given that for a triangle the two interior angles $(x + 10)^\circ$ and $(x + 20)^\circ$ are remote to an exterior angle $(6x - 30)^\circ$, then find the value of x ?

9. Calculate the values of x and y in the following triangle.

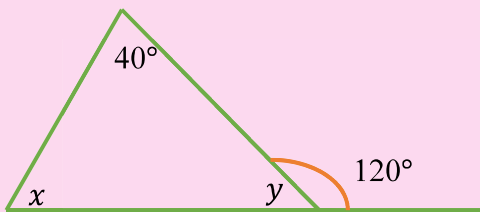


Figure 5.41

10. State Pythagoras Theorem
 11. State Euclid's Theorem
 12. In figure 5.42 to the right if $DB = 8\text{cm}$ and $AD = 4\text{cm}$

Then find the lengths of

- a. $\overline{AB}$ b. $\overline{BC}$
 c. $\overline{AC}$ d. $\overline{DC}$

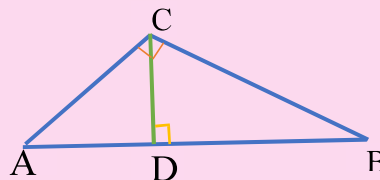


Figure 5.42

13. A triangle has sides of lengths 16cm, 48cm and 50cm respectively. Is the triangle a right angled triangle?
 14. In figure 5.43 below. What is the value of x ?

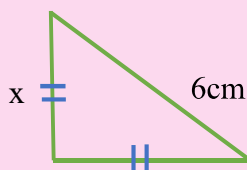


Figure 5.43

15. A tree 18 meter-high is broken off 3 meter from the ground. How far from the foot of the tree will the top strike the ground?
 16. A rectangle has its sides 5cm and 12cm long. What is the length of its diagonal?
 17. What is the length of the hypotenuse of a right angle triangle with leg lengths are 9cm and 12cm?

UNIT CIRCLES

6

Learning outcomes: At the end of this unit, learners will be able to

- Have a better understanding of circles.
- Realize the relationship between lines and circles.
- Apply basic facts about central and inscribed angles and angles formed by intersecting chords to compute their measures.
- Apply real-life situations in solving geometric problems.

Main Contents

6.1. Circles

6.2. Applications of circle

Summary

Review Exercise

INTRODUCTION

In lower grades especially in grade 6 mathematics you had learnt about circles and its parts like center and radius. In this unit we consider the concepts of chord, circumference of a circle, classification of arcs, sector of a circle, segment of a circle, central and inscribed angle and positional relations between a line and a circle. Of all simple geometric figures, a circle is perhaps the most appealing. Have you ever considered how useful a circle is? Without circles there would be no watches, wagon, automobiles, electricity or many other modern conveniences.

6.1. Circles

Group Work 6.1

Discuss with your partner.

- 1 i) Take a compass and set a fixed point on the paper.
ii) Draw a closed figure. Then answer the following questions.
 - a) The small puncture hole (the impression that the metal compass point makes on the paper) is called -----
 - b) The compass setting distance is called -----

6.1.1. Lines and circles

Competencies: At the end of this section, students should:

- Identify the different types of arcs, sectors, segments.
- Describe the concepts “tangent” and “secant” of a circle.

Activity 6.1

- 1) Name any three parts of a circle.
- 2) The diameter of a circle is 18cm. what is the radius of the circle?
What do you conclude?
- 3) How are chords and secants of circles alike? How are they different?

Definition 6.1. A circle is a set of points in a plane having the same distance from a fixed point.

- ✓ The fixed point, O is called the center of the circle.
- ✓ The distance between the center of the circle and any point of the circle, r is called the radius (plural: radii) of the circle.

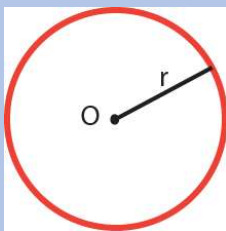


Figure 6.1

Note:

- ✓ A capital letter is used to designate the center of the circle. If the center of a circle is designated by the letter O, the circle is referred to as circle O.

Several basic terms with which you should be familiar with are associated with circles. These terms are defined below and illustrated in Figure 6.2

Definition 6.2

- a) A chord of a circle is a line segment joining two points of the circle. Thus, in the figure, $\overline{AB}$ is a chord.
- b) A diameter of a circle is a chord through the center. Thus, $\overline{CD}$ is a diameter of circle O.
- c) Secant of a circle is a line that intersects the circle at two points. Thus, $\overleftrightarrow{EF}$ is a secant.
- d) A tangent of a circle is a line that touches the circle at one and only one point. Thus, $\overleftrightarrow{GH}$ is a tangent to the circle at P. The point P is called the point of tangency or point of contact.

e) The distance around a circle is called the circumference of the circle.

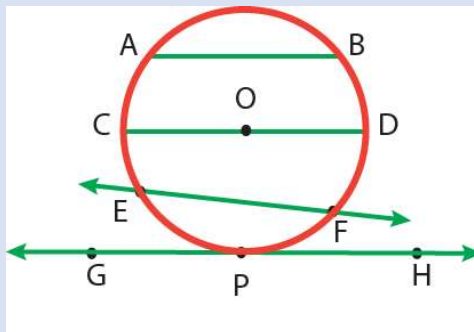


Figure 6.2

Note:

- ✓ All radii of a circle are congruent.
- ✓ A diameter of a circle consists of two radii that lie on the same straight line.

Example 6.1:

In circle O, radius $OA = 3x - 10$ and radius $OB = x + 2$. Find the length of a diameter of circle O.

Solution:

Since all radii of a circle are congruent,

$$OA = OB$$

$$3x - 10 = x + 2$$

$$3x = x + 12$$

$$x = 6$$

$$OA = OB = x + 2 = 6 + 2 = 8$$

Therefore, the length of a diameter of circle O is $2 \times 8 = 16$.