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Distributions, Models, and Applications

Marshall–Olkin Extended Lomax Distribution and Its Application to Censored Data

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This paper investigates properties of a new parametric distribution generated by Marshall and Olkin (1997) extended family of distributions based on the Lomax model. We show that the proposed distribution can be expressed as a compound distribution with mixing exponential model. Simple sufficient conditions for the shape behavior of the density and hazard rate functions are given. The limiting distributions of the sample extremes are shown to be of the exponential and Fréchet type. Finally, utilizing maximum likelihood estimation, the proposed distribution is fitted to randomly censored data.

Keywords Compound distribution; Extreme order statistics; Geometric extreme stability; Hazard rate; Kaplan–Meier estimator; Likelihood ratio test; Limiting distribution; Lomax distribution; Maximum likelihood; P–P plot.

Mathematics Subject Classification Primary 62N01, 62N02, 62N03, 62N05; Secondary 60E15.

1. Introduction

An absolutely continuous random variable (RV) is said to have the *Lomax distribution* (also known as *Pareto distribution of the second kind*) if its survival function (SF) is given by

$$\overline{F}_0(x) = 1 - F_0(x) = (1 + \beta x)^{-\gamma}, \quad x > 0, \ \beta, \gamma > 0, \tag{1}$$

where β, γ are, respectively, the scale and shape parameters.

Lomax (1954) used the distribution to analyze business failure data. Lingappaiah (1986) considers different estimation procedures for the Lomax

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distribution. Myhre and Saunders (1982) apply Lomax distribution to right-censored data. Ahsanullah (1991) and Balakrishnan and Ahsanullah (1994) consider, respectively, distributional properties and moments of record values from Lomax distribution. Childs et al. (2001) consider order statistics from nonidentical, right-truncated Lomax distributions and provide applications for this situation. Al-Awadhi and Ghitany (2001) use the Lomax distribution as a mixing distribution for the Poisson parameter and obtain the discrete Poisson–Lomax distribution. Howlader and Hossain (2002) consider Bayesian estimation of the survival function of the Lomax distribution.

Adding parameters to a well-established family of distributions is a time honored device for obtaining more flexible new families of distributions. Marshall and Olkin (1997) introduced an ingeneous general method of adding a parameter to a family of distributions. Their method: if we have a family of distributions with SF $\overline{F}_0(x)$, then a new family of distributions can be constructed based on the property:

$$\overline{F}(x) = \frac{\alpha \overline{F}_0(x)}{1 - \bar{\alpha}\overline{F}_0(x)}, \quad -\infty < x < \infty, \ \alpha > 0, \ \bar{\alpha} = 1 - \alpha. \tag{2}$$

They discovered that the method has a stability property, namely, if the method is applied twice, nothing new will be obtained the second time around. Moreover, they have shown that the newly generated family of distributions with the SF (2) is *geometric extreme stable*, that is, if $X_1, X_2, \ldots$ is a sequence of independent and identically distributed RVs with SF (2) and independent from a RV N which possess a geometric distribution with the probability mass function:

$$P(N = n) = (1 - p)^{n-1}p, \quad n = 1, 2, \ldots, \ 0 < p < 1,$$

then the RVs

$$U = \min(X_1, X_2, \ldots, X_N), \quad V = \max(X_1, X_2, \ldots, X_N),$$

also belong to the family of distributions with the SF (2).

In this article, we introduce a new variant of the Marshall–Olkin extended family of distributions by selecting in (2) the Lomax SF (1) which yields

$$\overline{F}(x) = \frac{\alpha}{(1 + \beta x)^{\gamma} - \bar{\alpha}}, \quad x > 0, \ \alpha, \beta, \gamma > 0, \ \bar{\alpha} = 1 - \alpha. \tag{3}$$

We shall write $X \sim MOEL(\alpha, \beta, \gamma)$ to denote an absolutely continuous RV X possessing the Marshall–Olkin extended Lomax distribution with parameters α, β, γ and SF given by (3).

The aim of this article is to reveal some statistical properties of the MOEL distribution. These properties include: (i) expressing this proposed distribution as a compounding process with exponential mixing model; (ii) shapes of the density function; (iii) moments and quantiles; (iv) shapes of the hazard rate function; and (v) limiting distributions of sample extremes. Finally, utilizing maximum likelihood estimation, the proposed distribution is fitted to random censored data set representing the remission times of bladder cancer patients.

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2. Compounding

Let $\overline{F}(x \mid \theta)$, $-\infty < x < \infty$, $-\infty < \theta < \infty$, be the conditional SF of a continuous RV X given a continuous RV Θ. Let Θ follows a distribution with the probability density function (PDF) $m(\theta)$. A distribution with the SF

$$\overline{F}(x) = \int_{-\infty}^{\infty} \overline{F}(x \mid \theta) m(\theta) d\theta, \quad -\infty < x < \infty,$$

is called a *compound distribution with mixing density* $m(\theta)$. Compound distributions provide a tool for obtaining new parametric families of distributions in terms of existing ones. They represent heterogeneous models where populations' items involve different risks.

The following theorem shows that the MOEL distribution can be expressed by the compounding argument.

Theorem 2.1. *Let the conditional SF of a continuous RV X given $\Theta = \theta$ be expressed as:*

$$\overline{F}(x \mid \theta) = \exp\{-[(1 + \beta x)^{\gamma} - 1]\theta\}, \quad x, \theta > 0, \ \beta, \gamma > 0.$$

Let Θ follows an exponential distribution with the PDF:

$$m(\theta) = \alpha\, e^{-\alpha\theta}, \quad \theta > 0, \ \alpha > 0.$$

Then the compound distribution of X becomes the $MOEL(\alpha, \beta, \gamma)$ distribution.

Proof. For all $x > 0$, $\alpha, \beta, \gamma > 0$, the unconditional SF of X is given by

$$\begin{aligned} \overline{F}(x) &= \int_0^{\infty} \overline{F}(x \mid \theta) m(\theta) d\theta \\ &= \alpha \int_0^{\infty} \exp\{-[(1 + \beta x)^{\gamma} - \bar{\alpha}]\theta\} d\theta \\ &= \frac{\alpha}{(1 + \beta x)^{\gamma} - \bar{\alpha}}. \end{aligned}$$

which is indeed the SF of the $MOEL(\alpha, \beta, \gamma)$ distribution. □

Theorem 2.1 provides another way to interpret the MOEL distribution in addition to the geometric-extreme stability property as given by Marshall and Olkin (1997).

3. Density, Moments, and Quantiles

The PDF of the $MOEL(\alpha, \beta, \gamma)$ distribution with SF (3), is given by

$$f(x) = \alpha\beta\gamma \frac{(1 + \beta x)^{\gamma - 1}}{[(1 + \beta x)^{\gamma} - \bar{\alpha}]^2}, \quad x > 0. \tag{4}$$

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The following theorem gives simple conditions under which the PDF (4) is decreasing or unimodal.

Theorem 3.1. *The PDF $f(x)$ of the MOEL(α, β, γ) distribution is decreasing (unimodal) if $\alpha + (2-\alpha)\gamma \geq 0$ (<0), independent of β.*

Proof. The first derivative of $f(x)$ is given by

$$f'(x) = -\frac{\alpha\beta^2\gamma(1+\beta x)^{\gamma-2}}{[(1+\beta x)^{\gamma} - \bar{\alpha}]^3}\xi\big((1+\beta x)^{\gamma}\big), \quad x > 0,$$

where

$$\xi(y) = (\gamma+1)y - \bar{\alpha}\bar{\gamma}, \quad y = (1+\beta x)^{\gamma} > 1, \quad \bar{\gamma} = 1-\gamma.$$

If $\xi(1) = \alpha + (2-\alpha)\gamma \geq 0$, then $\xi(y) \geq 0$ for all $y > 1$, and hence, $f'(x) \leq 0$ for all $x > 0$, i.e., $f(x)$ is decreasing.

If $\xi(1) < 0$, then $\xi(y)$ has a unique zero at $y_{\xi} = \frac{\bar{\alpha}\bar{\gamma}}{\gamma+1} > 1$. Since $y = (1+\beta x)^{\gamma}$ is 1-1 transformation, it follows that $f(x)$ has also a unique critical point at $x_f = \frac{1}{\beta}(y_{\xi}^{1/\gamma} - 1)$. Finally, since $f(0) = \frac{\beta\gamma}{\alpha}$ and $f(\infty) = 0$ then x_f must be a point of absolute maximum for $f(x)$. □

Remarks.

(i) When $\alpha \in (0, 2]$, $f(x)$ is decreasing in $x > 0$ for all values of $\beta, \gamma > 0$. In particular, when $\alpha = 1$, i.e., Lomax distribution, $f(x)$ is decreasing in x for all values of $\beta, \gamma > 0$.
(ii) When $\alpha > 2$, $f(x)$ may still exhibit a decreasing behavior, depending on the values of α, γ such that $\alpha + (2-\alpha)\gamma \geq 0$.
(iii) When $\gamma \in (0, 1]$, $f(x)$ is decreasing in $x > 0$ for all values of $\alpha, \beta > 0$.
(iv) The mode of the MOEL distribution is given by

$$mode(X) = \begin{cases} 0, & \text{if } \alpha + (2-\alpha)\gamma \geq 0, \\ \frac{1}{\beta}\left[\left(\frac{\bar{\alpha}\bar{\gamma}}{1+\gamma}\right)^{1/\gamma} - 1\right], & \text{otherwise.} \end{cases}$$

Figure 1 shows the PDF curves for the *MOEL*(α, β, γ) distribution for selected values of the parameters α, β, γ.

The rth moment of the MOEL distribution, using the substitution $y = (1+\beta x)^{\gamma}$, is given by

$$\begin{aligned} E(X^r) &= r\int_0^{\infty} x^{r-1}\overline{F}(x)dx \\ &= \frac{r\alpha}{\beta^r\gamma}\int_1^{\infty} \frac{(y^{1/\gamma}-1)^{r-1}y^{1/\gamma-1}}{y-\bar{\alpha}}dy, \\ &= \frac{r\alpha}{\beta^r\gamma}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^{r-1-i}\int_1^{\infty}\frac{y^{\frac{i+1}{\gamma}-1}}{y-\bar{\alpha}}dy, \quad \gamma > r. \end{aligned}$$

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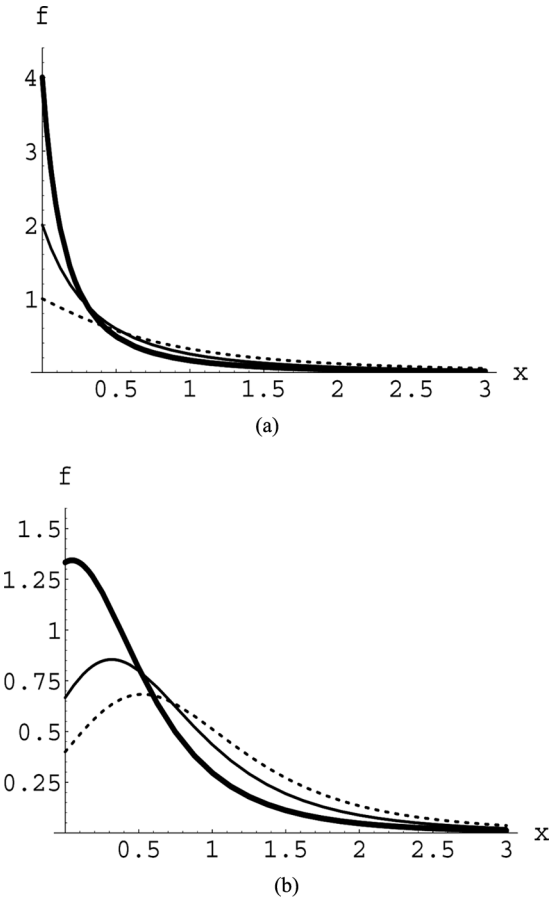

Figure 1. The PDF $f(x)$ of the MOEL distribution for selected values of the parameters. (a) $\alpha = 0.5$ (bold), 1 (plain), 2 (dashed), $\beta = 1, \gamma = 2$; (b) $\alpha = 3$ (bold), 6 (plain), 10 (dashed), $\beta = 1, \gamma = 4$.

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Remarks.

(i) When $\alpha = 1$, i.e., Lomax distribution, we have

$$E(X^r) = \frac{r}{\beta^r} \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-1-i}}{\gamma - 1 - i}, \quad \gamma > r.$$

(ii) For $\alpha \neq 1, \beta > 0, \gamma > r$, $E(X^r)$ cannot be expressed in a simple closed form in terms of r, α, β, γ and must be calculated numerically.

The qth quantile of the MOEL distribution is given by

$$x_q = F^{-1}(q) = \frac{1}{\beta}\left\{\left(1 + \frac{q\alpha}{1-q}\right)^{1/\gamma} - 1\right\}, \quad 0 \leq q \leq 1, \tag{5}$$

where $F^{-1}(\cdot)$ is the inverse distribution function. In particular, the median of the MOEL distribution is given by

$$median(X) = \frac{1}{\beta}\left[(1+\alpha)^{1/\gamma} - 1\right],$$

from which the median of the Lomax distribution is $\frac{1}{\beta}(2^{1/\gamma} - 1)$.

4. Hazard Rate Function

The hazard rate function (HRF) of the $MOEL(\alpha, \beta, \gamma)$ distribution is given by

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\beta\gamma}{(1+\beta x) - \bar{\alpha}(1+\beta x)^{1-\gamma}}, \quad x > 0. \tag{6}$$

The following theorem gives simple conditions under which the HRF (6) is decreasing or unimodal.

Theorem 4.1. *The HRF $h(x)$ of the $MOEL(\alpha, \beta, \gamma)$ distribution is decreasing (unimodal) if $\alpha + (1-\alpha)\gamma \geq 0$ (<0), independent of β.*

Proof. The first derivative of $h(x)$ with respect to x is given by

$$h'(x) = \frac{-\beta^2\gamma(1+\beta x)^{-\gamma}}{[(1+\beta x) - \bar{\alpha}(1+\beta x)^{1-\gamma}]^2}\eta((1+\beta x)^{\gamma}), \quad x > 0,$$

where

$$\eta(y) = y - \bar{\alpha}\bar{\gamma}, \quad y = (1+\beta x)^{\gamma} > 1.$$

The remaining of the proof is similar to that of Theorem 3.1. □

Remarks.

(i) For $\alpha \in (0, 1]$, $h(x)$ is decreasing for all values of $\beta, \gamma > 0$. In particular, when $\alpha = 1$, i.e., Lomax distribution, $h(x)$ is decreasing for all values of $\beta, \gamma > 0$.

(ii) For $\alpha > 1$, $h(x)$ may still exhibit decreasing behavior, depending on the values of α, γ such that $\alpha + (1-\alpha)\gamma \geq 0$.

(iii) The decreasing HRF implies other aging properties, e.g., the following implications are well known:

$$DHR \Rightarrow DHRA \Rightarrow NWU \Rightarrow NWUE \quad (\text{provided } \gamma > 1)$$

where *DHR, DHRA, NWU*, and *NWUE*, denote, respectively, *decreasing hazard rate, decreasing hazard rate average, new worse than used,* and *new worse than used in expectation*. For more details about the definitions of these aging concepts, see Barlow and Proschan (1981, p. 159).

(iv) For α, γ such that $\alpha + (1-\alpha)\gamma < 0$, the HRF $h(x)$ has its absolute maximum at the point $x_h = \frac{1}{\beta}\left\{[(1-\alpha)(1-\gamma)]^{1/\gamma} - 1\right\}$.

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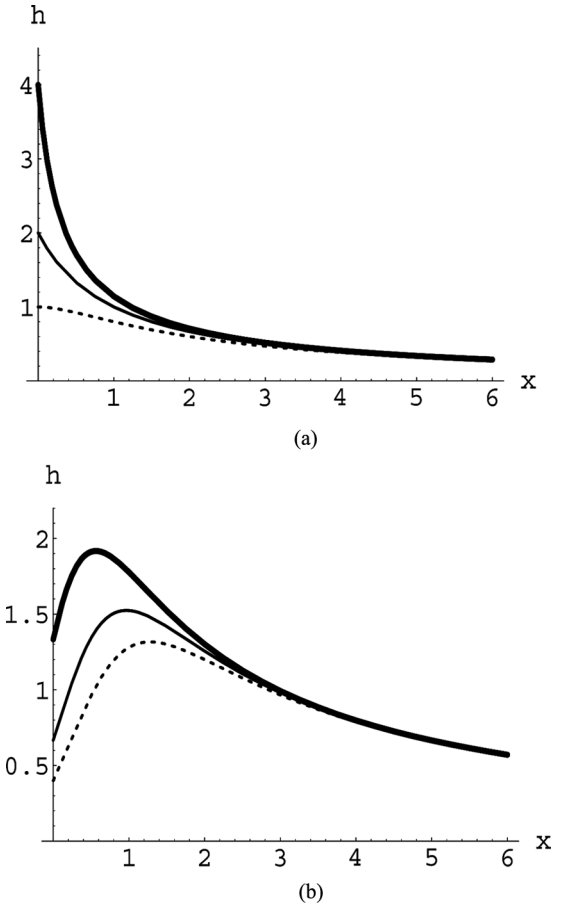

Figure 2. The HRF $h(x)$ of the MOEL distribution for selected values of the parameters. (a) $\alpha = 0.5$ (bold), 1 (plain), 2 (dashed), $\beta = 1, \gamma = 2$; (b) $\alpha = 3$ (bold), 6 (plain), 10 (dashed), $\beta = 1, \gamma = 4$.

(v) Since $\alpha + (1 - \alpha)\gamma \geq 0$ implies that $\alpha + (2 - \alpha)\gamma \geq 0$, then a decreasing HRF MOEL distribution implies decreasing PDF. The converse is not necessarily true, e.g., $\alpha = 3, \gamma = 2$ implies decreasing PDF but unimodal HRF.

(vi) Since $\alpha + (2 - \alpha)\gamma < 0$ implies that $\alpha + (1 - \alpha)\gamma < 0$, then a unimodal PDF MOEL distribution implies unimodal HRF. The converse is not necessarily true, e.g., $\alpha = 2, \gamma = 3$ implies unimodal HRF but decreasing PDF.

Figure 2 shows the HRF curves for the $MOEL(\alpha, \beta, \gamma)$ distribution for selected values of the parameters α, β, γ.

5. Limiting Distributions of Sample Extremes

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a MOEL distribution with SF (3). Limiting distributions of sample minima $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and maxima $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are a long stand area in applications of

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probability and statistics. These extreme order statistics $X_{1:n}$ and $X_{n:n}$ represent, respectively, the life of series and parallel systems.

First, we recall the following asymptotical results for $X_{1:n}$ and $X_{n:n}$ (see, e.g., Arnold et al., 1992, pp. 210–214).

(i) For the minimal order statistic $X_{1:n}$, we have

$$\lim_{n\to\infty} P\{X_{1:n} \le a_n^* + b_n^* t\} = 1 - \exp[-t^c], \quad t > 0, \ \ c > 0, \tag{7}$$

(of the Weibull type) where $a_n^* = F^{-1}(0)$ and $b_n^* = F^{-1}(1/n) - F^{-1}(0)$ if and only if $F^{-1}(0)$ is finite and, for all $t > 0$ and $c > 0$,

$$\lim_{\epsilon\to 0+} \frac{F(F^{-1}(0) + \epsilon t)}{F(F^{-1}(0) + \epsilon)} = t^c. \tag{8}$$

(ii) For the maximal order statistic $X_{n:n}$, we have

$$\lim_{n\to\infty} P\{X_{n:n} \le a_n + b_n t\} = \exp[-t^{-k}], \quad t > 0, \ \ k > 0, \tag{9}$$

(of the Fréchet type) where $a_n = 0$, $b_n = F^{-1}(1 - 1/n)$ if and only if $F^{-1}(1) = \infty$ and there exists a constant $k > 0$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(xt)}{\overline{F}(x)} = t^{-k}. \tag{10}$$

The following theorem derives the limiting distributions of the smallest and largest order statistics from the MOEL distribution.

Theorem 5.1. *Let $X_{1:n}$ and $X_{n:n}$ be, respectively, the smallest and largest order statistics from the MOEL(α, β, γ) distribution. Then*

(i) $\lim_{n\to\infty} P\{X_{1:n} \le b_n^* t\} = 1 - \exp[-t], \quad t > 0$, *where* $b_n^* = \frac{1}{\beta}\big\{(1 + \frac{\alpha}{n-1})^{1/\gamma} - 1\big\}$.
(ii) $\lim_{n\to\infty} P\{X_{n:n} \le b_n t\} = \exp[-t^{-\gamma}], \quad t > 0$, *where* $b_n = \frac{1}{\beta}\big\{[1 + (n-1)\alpha]^{1/\gamma} - 1\big\}$.

Proof.

(i) For the MOEL distribution, $F^{-1}(0) = 0$ is finite and, using L'Hospital rule, we have

$$\lim_{\epsilon\to 0+} \frac{F(\epsilon t)}{F(\epsilon)} = t \lim_{\epsilon\to 0+} \frac{f(\epsilon t)}{f(\epsilon)} = t,$$

since $f(0) = \frac{\beta\gamma}{\alpha}$. Statement (i) follows from (7) and (8) where, using (5), we have $a_n^* = 0$ and $b_n^* = \frac{1}{\beta}\Big[(1 + \frac{\alpha}{n-1})^{1/\gamma} - 1\Big]$.

(ii) For the MOEL distribution, we have

$$\lim_{x\to\infty} \frac{\overline{F}(xt)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{(1+\beta x)^\gamma - \bar{\alpha}}{(1+\beta xt)^\gamma - \bar{\alpha}} = t^{-\gamma}.$$

Statement (ii) follows from (9) and (10) where, using (5), we have $b_n = \frac{1}{\beta}\big\{[1 + (n-1)\alpha]^{1/\gamma} - 1\big\}$. □

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Remarks. (i) When $\alpha = 1$, i.e., Lomax distribution, the norming constants b_n^*, b_n, respectively, are given by $b_n^* = \frac{1}{\beta}\left[\left(\frac{n}{n-1}\right)^{1/\gamma} - 1\right]$ and $b_n = \frac{1}{\beta}(n^{1/\gamma} - 1)$.

(ii) Let $G^*(t)$ and $G(t)$ denote, respectively, the limiting distributions of $(X_{1:n} - a_n^*)/b_n^*$ and $(X_{n:n} - a_n)/b_n$. Then, it is well known, see Arnold et al. (1992, p. 218–219), that, for any finite $i > 1$, the limiting distributions of $(X_{i:n} - a_n^*)/b_n^*$ and $(X_{n-i+1:n} - a_n)/b_n$, respectively, are given by

$$\lim_{n\to\infty} P\{X_{i:n} \le a_n^* + b_n^* t\} = 1 - \sum_{j=0}^{i-1}[1 - G^*(t)]\frac{\{-\log[1 - G^*(t)]\}^j}{j!},$$

$$\lim_{n\to\infty} P\{X_{n-i+1:n} \le a_n + b_n t\} = \sum_{j=0}^{i-1} G(t)\frac{\{-\log G(t)\}^j}{j!}.$$

Now, using Theorem 5.1, it is straightforward to show that, for any finite $i > 1$, the limiting distributions of the ith and $(n - i + 1)$st order statistics from the MOEL distribution, respectively, are given by

$$\begin{aligned}\lim_{n\to\infty} P\{X_{i:n} \le b_n^* t\} &= 1 - \sum_{j=0}^{i-1}\exp[-t]\frac{t^j}{j!}, \quad t > 0,\\ &= 1 - P(Y < i),\\ \lim_{n\to\infty} P\{X_{n-i+1:n} \le b_n\ t\} &= \sum_{j=0}^{i-1}\exp[-t^{-\gamma}]\frac{t^{-\gamma j}}{j!}, \quad t, \gamma > 0,\\ &= P(Z < i),\end{aligned}$$

where Y and Z follow, respectively, the Poisson distributions with means t and $t^{-\gamma}$.

6. Estimation and Application to Censored Data

6.1. *Maximum Likelihood Estimation*

Let $X_i, i = 1, 2, \ldots, n$, be the failure time of individual i and let Y_i be the corresponding censoring time. The X_i's and Y_i's are assumed independent. The X_i's are assumed to have the $MOEL(\alpha, \beta, \gamma)$ distribution and the Y_i's are assumed to have a non-informative distribution, i.e., a distribution which does not involve the parameters α, β, γ. One observes only the pairs (T_i, Δ_i) where $T_i = \min(X_i, Y_i)$ and $\Delta_i = 1(0)$ if $T_i = X_i$ $(T_i = Y_i)$ is the censoring indicator. Given the data $(t_1, \delta_1), (t_2, \delta_2), \ldots, (t_n, \delta_n)$, the log-likelihood function is given by

$$\begin{aligned}L(\alpha, \beta, \gamma) &= \sum_{i=1}^{n}\left\{\delta_i \log[f(t_i)] + (1 - \delta_i)\log[\overline{F}(t_i)]\right\}\\ &= \sum_{i=1}^{n}\left\{\delta_i\big[\log(\beta) + \log(\gamma) + (\gamma - 1)\log(1 + \beta t_i)\big]\right.\\ &\qquad \left. + \log(\alpha) - (\delta_i + 1)\log[(1 + \beta t_i)^{\gamma} - \bar{\alpha}]\right\}. \qquad (11)\end{aligned}$$

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The first derivatives of $L(\alpha, \beta, \gamma)$ with respect to α, β, γ, respectively, are given by

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} - \frac{\delta_i + 1}{(1+\beta t_i)^{\gamma} - \bar{\alpha}} \right\},$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \left\{ \delta_i \left[\frac{1}{\beta} + \frac{(\gamma-1)t_i}{1+\beta t_i} \right] - \frac{(\delta_i+1)\gamma(1+\beta t_i)^{\gamma-1} t_i}{(1+\beta t_i)^{\gamma} - \bar{\alpha}} \right\},$$

$$\frac{\partial L}{\partial \gamma} = \sum_{i=1}^{n} \left\{ \delta_i \left[\frac{1}{\gamma} + \log(1+\beta t_i) \right] - \frac{(\delta_i+1)(1+\beta t_i)^{\gamma} \log(1+\beta t_i)}{(1+\beta t_i)^{\gamma} - \bar{\alpha}} \right\}.$$

Maximum likelihood estimates (MLEs) $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ of the parameters α, β, γ are obtained by solving the equations: $\partial L/\partial \alpha = 0$, $\partial L/\partial \beta = 0$, $\partial L/\partial \gamma = 0$. This can be done using an iterative numerical procedure such as Newton–Raphson method.

To test the null hypothesis $H_0 : \alpha = 1$ (Lomax distribution), we use the likelihood ratio test (LRT). Under H_0, the likelihood ratio statistic: $\Lambda = -2[L(1, \tilde{\beta}, \tilde{\gamma}) - L(\hat{\alpha}, \hat{\beta}, \hat{\gamma})]$, has approximately a chi-square distribution with 1 degree of freedom.

6.2. *Application to Censored Data*

In the following we fit the proposed MOEL model to randomly censored data set representing the remission times (in months) of a random sample of 137 bladder cancer patients (Lee and Wang, 2003, p. 231). The data set has minimum (maximum) uncensored remission time at 0.08 (79.05) months. Also, the data set contains 9 censored remission times at 0.87, 3.02, 4.33, 4.65, 4.70, 8.60, 10.86, 19.36, 24.80 months.

Fitting the MOEL distribution to the given data, the MLEs of α, β, γ, and corresponding log-likelihood, respectively, are given by

$$\hat{\alpha} = 2.959, \quad \hat{\beta} = 0.058, \quad \hat{\gamma} = 4.115, \quad L(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = -419.842.$$

Similarly, fitting the Lomax distribution, i.e., when $\alpha = 1$, to the given data, the MLEs of β, γ, and corresponding log-likelihood, respectively, are given by

$$\tilde{\beta} = 0.007, \quad \tilde{\gamma} = 14.923, \quad L(1, \tilde{\beta}, \tilde{\gamma}) = -422.348.$$

For the given data, the likelihood ratio statistic is $\Lambda = 5.012$ and hence, the p-value of the LRT is $P(\chi^2_{(1)} > 5.012) = 0.025$. Hence, the LRT rejects the assumption that the Lomax model is suitable for the given data.

Let $t_{(i)}$'s be the ordered survival times and $\delta_{(i)}$'s be their corresponding censoring indicators. The Kaplan–Meier (1958) estimator (KME), also known as the *product-limit estimator*, of a survival function is defined as:

$$\overline{F}_n(t) = \prod_{\substack{i:t_{(i)} \le t \\ i=1,2,\ldots,n}} \left\{ 1 - \frac{\delta_{(i)}}{n-i+1} \right\}, \quad t > 0. \tag{12}$$

Figure 3 shows the probability–probability (P–P) plot of the KME versus the fitted MOEL survival function for the given data. The plot indicates the

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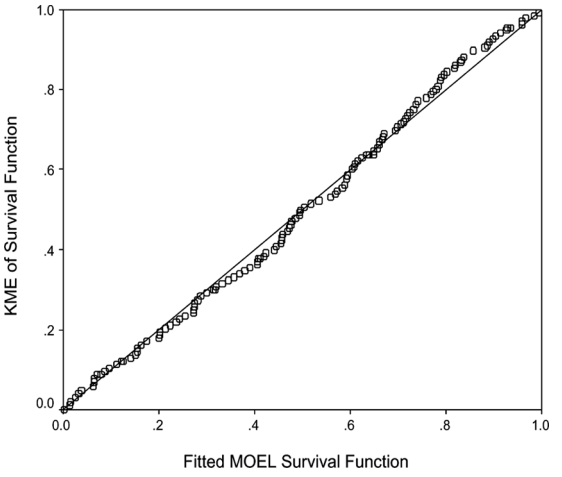

Figure 3. P–P plot of KME versus fitted MOEL survival function.

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goodness-of-fit of the fitted MOEL distribution to the given data. Note that the Pearson correlation coefficient between KME of survival function, $\overline{F}_n(t)$, and the fitted MOEL survival function, $\overline{F}(t; \hat{\alpha}, \hat{\beta}, \hat{\gamma})$, is 0.998, indicating strong linear relationship between the empirical and fitted survival function.

Since $\hat{\alpha} + (2 - \hat{\alpha})\hat{\gamma} = -0.987$, then, by Remark (vi) in Sec. 4, both the estimated PDF and HRF of the MOEL distribution are unimodal with absolute maximum values, respectively, at $x_f = 0.756$ month and $x_h = 9.517$ months. The estimated mean (median) survival time is 9.869 (6.846) months.

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