

## Real Analysis

June-2014

24. Let  $A \subseteq \mathbb{R}$  and  $f: A \rightarrow \mathbb{R}$  be given by  $f(x) = x^2$

Then  $f$  is uniformly continuous if

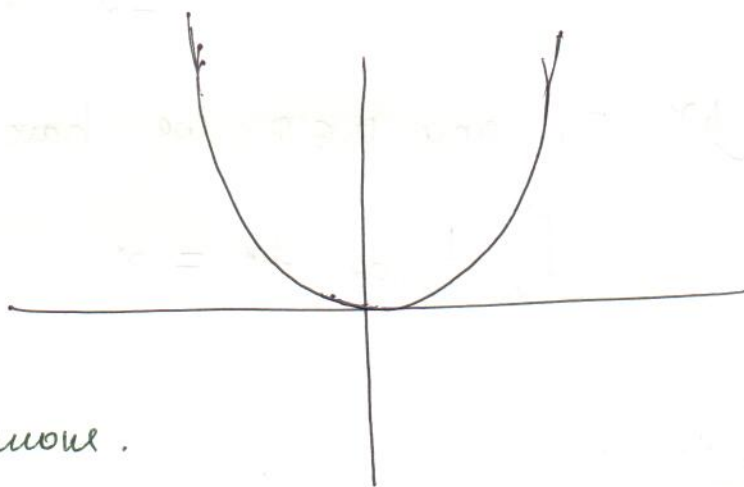
- 1)  $A$  is a bounded subset of  $\mathbb{R}$ .
- 2)  $A$  is a dense subset of  $\mathbb{R}$ .
- 3)  $A$  is an unbounded and connected subset of  $\mathbb{R}$ .
- 4)  $A$  is an unbounded and open subset of  $\mathbb{R}$ .

### Explanation:

2)  $A = \mathbb{R}$ .

$A$  is dense in  $\mathbb{R}$ .

but  $f(x) = x^2$  is  
not uniformly continuous.



3)  $A = [0, \infty)$

Then also  $f: A \rightarrow \mathbb{R}$  by  $f(x) = x^2$  is not uniformly continuous.

4)  $A = [0, \infty)$

Then  $f: A \rightarrow \mathbb{R}$  by  $f(x) = x^2$  is not uniformly continuous.

A is bdd

Then  $f: A \rightarrow \mathbb{R}$  by  $f(x) = x^\alpha$  is uniformly continuous.

25- Let  $\alpha, p$  be real no's and  $\alpha > 1$

1) If  $p > 1$  then  $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$ .

2) If  $p > \frac{1}{\alpha}$  then  $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$

3) If  $p < \frac{1}{\alpha}$  then  $\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx < \infty$ .

47 For any  $p \in \mathbb{R}$  we have

$$\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx = \infty.$$

Explanation:

$$\int_{-\infty}^{\infty} \frac{1}{|x|^{p\alpha}} dx = \int_{-\infty}^{-1} -\frac{1}{x^{p\alpha}} dx + \int_{-1}^0 -\frac{1}{x^{p\alpha}} dx + \int_0^1 \frac{1}{x^{p\alpha}} dx + \int_1^{\infty} \frac{1}{x^{p\alpha}} dx.$$

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{cgt} & \text{if } p > 1 \\ \text{dgt} & \text{if } p \leq 1 \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{cgt} & \text{if } p > 1 \\ \text{dgt} & \text{if } p \leq 1. \end{cases}$$

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0} \int_a^1 \frac{1}{x^p} dx.$$

$$= \lim_{a \rightarrow 0} \left( \frac{x^{-p+1}}{1-p} \right)_a^1$$

$$= \lim_{a \rightarrow 0} \left[ \frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right].$$

$$= \frac{1}{1-p} \left[ 1 - \lim_{a \rightarrow 0} a^{1-p} \right]$$

$$\lim_{a \rightarrow 0} a^{1-p} \text{ exist if } 1-p > 0.$$

$$\therefore \int_0^1 \frac{1}{x^p} dx \text{ cgt if } 1-p > 0$$

(i.e)  $p < 1$

$$\text{But } \int_{-\infty}^{-1} \frac{1}{x^p} dx = \int_1^{\infty} \frac{1}{x^p} dx.$$

$$\int_{-\infty}^0 \frac{1}{x^p} dx = \int_0^1 \frac{1}{x^p} dx.$$

$$\int_1^{\infty} \frac{1}{x^p} dx \quad \text{cys if } p > 1$$

$$\int_0^1 \frac{1}{x^p} dx \quad \text{cys if } p < 1$$

$$\int_1^{\infty} \frac{1}{x^p} dx \quad \text{dgs if } p \leq 1$$

$$\int_0^1 \frac{1}{x^p} dx \quad \text{dgs if } p \geq 1$$

$\therefore$  For any  $p \in \mathbb{R}$ , we have

$$\int_{-\infty}^{\infty} \frac{1}{|x|^p} dx = \infty.$$

26. Let  $f : X \rightarrow Y$  be a function from a metric space  $X$  to another metric space  $Y$ . For any Cauchy seq  $(x_n)$  in  $X$ .

1) if  $f$  is continuous then  $(f(x_n))$  is a Cauchy sequence in  $Y$ .

2) if  $(f(x_n))$  is Cauchy then  $(f(x_n))$  is always cys in  $Y$ .

3) if  $(f(x_n))$  is Cauchy in  $Y$  then  $f$  is continuous.

4)  $(x_n)$  is always cgt in  $X$ .

Explanation:

1)  $f: (0, \infty) \rightarrow \mathbb{R}$  by  $f(x) = 1/x$  be a continuous fun.

$x_n = 1/n$  is a Cauchy seq in  $(0, \infty)$ .

$f(x_n) = n$  is not a Cauchy seq in  $\mathbb{R}$ .

2) since  $Y$  is not complete,  $(f(x_n))$  is Cauchy does not implies  $(f(x_n))$  is cgt.

4) since  $X$  is not complete,  $(x_n)$  Cauchy does not implies  $(x_n)$  is cgt.

28.  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \dots + \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}} \right)$

equals

1)  $\sqrt{2}$

2)  $\frac{1}{\sqrt{2}}$

3)  $\sqrt{2} + 1$

4)  $\frac{1}{\sqrt{2} + 1}$

Explanation:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \dots + \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( \frac{\sqrt{1} - \sqrt{3}}{1 - 3} + \frac{\sqrt{3} - \sqrt{5}}{3 - 5} + \dots + \frac{\sqrt{2n-1} - \sqrt{2n+1}}{2n-1 - (2n+1)} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( -\frac{1}{2} \left( \frac{1 - \sqrt{2n+1}}{1} \right) \right)$$

$$= \lim_{n \rightarrow \infty} \frac{-1}{2\sqrt{n}} \sqrt{n} \left( \frac{1}{\sqrt{n}} - \sqrt{2+\frac{1}{n}} \right)$$

$$= -\frac{1}{2} \times -\sqrt{2} = \frac{1}{\sqrt{2}}$$

29. Consider the following sets of functions on  $\mathbb{R}$ .

$W$  = The set of constant functions on  $\mathbb{R}$

$X$  = The set of polynomial functions on  $\mathbb{R}$ .

$Y$  = The set of continuous functions on  $\mathbb{R}$ .

$Z$  = The set of all functions on  $\mathbb{R}$ .

Which of these sets has the same cardinality as that of  $\mathbb{R}$ ?

1) Only  $W$

2) Only  $W$  and  $X$

3) Only  $W$ ,  $X$  and  $Z$

~~4)~~ All of  $W$ ,  $X$ ,  $Y$  and  $Z$ .

30. Let  $p(x)$  be a polynomial in the real variable  $x$  of degree 5. Then  $\lim_{n \rightarrow \infty} \frac{p(n)}{2^n}$  is

- 1) 5      2) 1      3) 0      4)  $\infty$ .

Explanation:

$$\lim_{n \rightarrow \infty} \frac{p(n)}{2^n} = \lim_{n \rightarrow \infty} \frac{a_0 + a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4 + a_5 n^5}{2^n}.$$

$$O(\log n) \leq O(n) \leq O(n^2) \leq \dots \leq O(e^n)$$

True for any  $n$

$$\therefore \lim_{n \rightarrow \infty} \frac{p(n)}{2^n} \rightarrow 0.$$

31. For a continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  let  $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$ . Then  $Z(f)$  is always

- 1) Compact      2) Open      3) Connected      4) Closed

Explanation:

$$Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$$

$$= f^{-1}(\{0\})$$

$\{0\}$  is closed and  $f$  is continuous,

$f^{-1}(\{0\})$  is closed.

$\therefore Z(f)$  is closed.

65. Let  $l^2 = \{x = (x_1, x_2, \dots) : x_n \in \mathbb{C} \ \forall n \geq 1 \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ .

and  $e_n \in l^2$  be the sequence whose  $n$ -th element is 1 and all other elements are zero. Equip the space  $l^2$  with the norm  $\|x\| = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$ . Then the set  $S = \{e_n : n \geq 1\}$ .

- 1) is closed
- 2) is bdd
- 3) is compact
- 4) contains a convergent subsequence.

### Explanation

$S$  is closed unit ball in  $l^2$

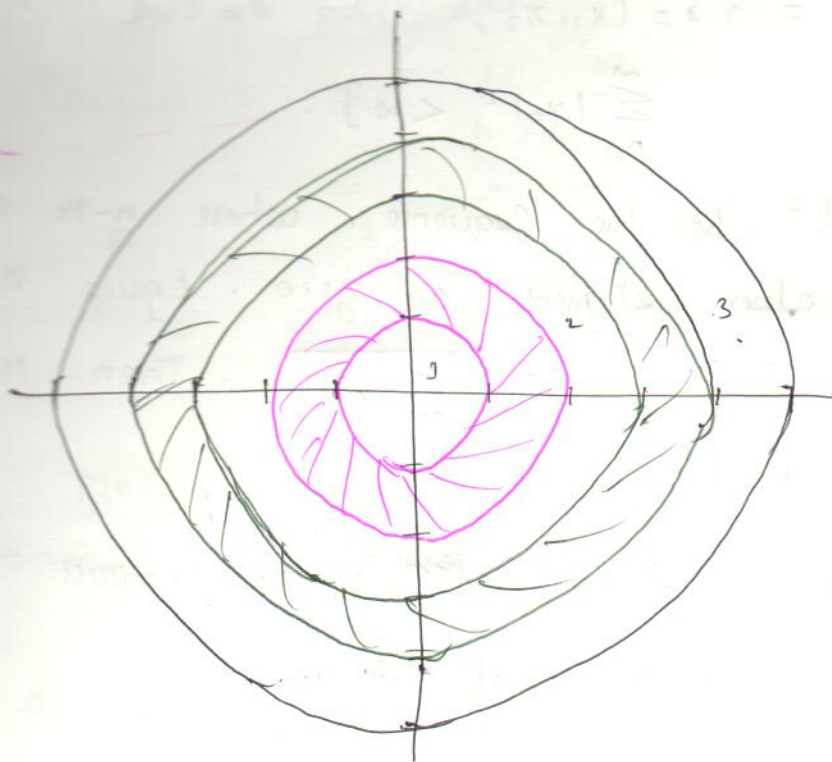
Unit ball is closed and bdd but not compact.

67. Let  $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5\}$  and

$$K = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2 \text{ or } 3 \leq x^2 + y^2 \leq 4\}$$

Then

- 1)  $X \setminus K$  has three connected components
- 2)  $X \setminus K$  has no relatively compact connected component in  $X$
- 3)  $X \setminus K$  has two relatively compact connected components in  $X$
- 4) All connected components of  $X \setminus K$  are relatively compact in  $X$ .



### Explanation:

A subset of a Topological space is relatively compact if its closure is compact.

$X \setminus K$  - has three connected components.

$$A_1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$$

$$A_2 = \{ (x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \}$$

$$A_3 = \{ (x, y) \in \mathbb{R}^2 \mid 3 < x^2 + y^2 < 5 \}$$

$\bar{A}_1, \bar{A}_2$  are closed and bdd in  $X \setminus K$

$\therefore \bar{A}_1, \bar{A}_2$  are relatively compact in  $X \setminus K$ .

$$\bar{A}_3 \not\subset X \setminus K.$$

$\bar{A}_3$  is not relatively compact in  $X \setminus K$ .

68. For two subsets  $X$  and  $Y$  of  $\mathbb{R}$ , let

$$X+Y = \{x+y : x \in X, y \in Y\}.$$

1) If  $X$  and  $Y$  are open sets then  $X+Y$  is open.

2) If  $X$  and  $Y$  are closed sets then  $X+Y$  is closed.

3) If  $X$  and  $Y$  are compact sets, then  $X+Y$  is compact.

4) If  $X$  is closed and  $Y$  is compact then  $X+Y$  is closed.

69. Let  $\{f_n\}$  be a sequence of continuous functions on  $\mathbb{R}$ .

1) If  $\{f_n\}$  conv to  $f$  pointwise on  $\mathbb{R}$ , then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} f(x) dx.$$

2) If  $\{f_n\}$  conv to  $f$  uniformly on  $\mathbb{R}$ , then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} f(x) dx.$$

3) If  $\{f_n\}$  conv to  $f$  uniformly on  $\mathbb{R}$ , then  $f$  is continuous.

4) There exist a seq of conts fun  $\{f_n\}$  on  $\mathbb{R}$  such that  $\{f_n\}$  conv to  $f$  uniformly on  $\mathbb{R}$ , but

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx \neq \int_{-\infty}^{\infty} f(x) dx.$$

Ex. Let  $\{a_n\}$ ,  $\{b_n\}$  be given bounded sequences of positive real numbers. Then:

1) if  $a_n \uparrow a$ , then  $\sup_{n \geq 1} (a_n b_n) = a \sup_{n \geq 1} b_n$

2) if  $a_n \uparrow a$ , then  $\sup_{n \geq 1} (a_n b_n) < a (\sup_{n \geq 1} b_n)$

3) if  $b_n \downarrow b$ , then  $\inf_{n \geq 1} (a_n b_n) = (\inf_{n \geq 1} a_n) b$

4) if  $b_n \downarrow b$ , then  $\inf_{n \geq 1} (a_n b_n) > (\inf_{n \geq 1} a_n) b$

75. Let  $f: (0,1) \rightarrow \mathbb{R}$  be continuous. Suppose that

$$|f(x) - f(y)| \leq |\sin x - \sin y| \quad \forall x, y \in (0,1). \text{ Then}$$

1)  $f$  is discontinuous at least at one point in  $(0,1)$

2)  $f$  is continuous everywhere on  $(0,1)$  but not uniformly continuous on  $(0,1)$

3)  $f$  is uniformly continuous on  $(0,1)$

4)  $\lim_{x \rightarrow 0+} f(x)$  exists.

Explanation.

$$|f(x) - f(y)| \leq |\sin x - \sin y|$$

$$= |\cos(c)| |x - y|$$

$$\leq 1 \cdot |x - y|$$

(By mean value thm)

$\therefore$  Satisfy Lipschitz conditions.

$\therefore f$  is uniformly continuous.

$\therefore \lim_{x \rightarrow 0+} f(x)$  exists.

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33. Let  $f_n: [1, 2] \rightarrow [0, 1]$  be given by

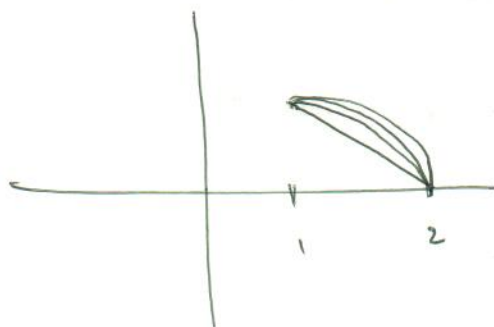
$$f_n(x) = (2-x)^n \text{ for all non-negative integers } n.$$

Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for  $1 \leq x \leq 2$ . Then which

of the following is true?

- 1)  $f$  is a continuous function on  $[1, 2]$ .
- 2)  $f_n$  converges uniformly to  $f$  on  $[1, 2]$  as  $n \rightarrow \infty$ .
- 3)  $\lim_{n \rightarrow \infty} \int_1^2 f_n(x) dx = \int_1^2 f(x) dx$ .
- 4) for any  $a \in (1, 2)$  we have  $\lim_{n \rightarrow \infty} f_n'(a) \neq f'(a)$ .

Explanation:



$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \\ & x \in (1, 2] \end{cases}$$

$$1 < x < 2$$

$$2 - x > 0$$

$$x > 1$$

$$2 - x < 2 - 1$$

$$\leq 1$$

$$2 - x < 1$$

$$(2-x)^n \rightarrow 0$$

$f$  is discontinuous.

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_1^2 f_n(x) dx &= \lim_{n \rightarrow \infty} \int_1^2 (2-x)^n dx \\&= \lim_{n \rightarrow \infty} \left( \frac{(2-x)^{n+1}}{n+1} \right)_1^2 \\&= \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.\end{aligned}$$

$$\int_1^2 f(x) dx = \int_1^2 0 dx = 0.$$

$$f'_n(x) = n(2-x)^{n-1}$$

$$a \in (1, 2), \quad \lim_{n \rightarrow \infty} f'_n(a) = \lim_{n \rightarrow \infty} n(2-a)^{n-1}$$

$$= 0 \quad ? \quad 0(n) < 0(2-a)^{n-1}$$

$$f'(a) = 0.$$

$$\therefore \lim_{n \rightarrow \infty} f'_n(a) = f'(a) \quad \forall a \in (1, 2)$$

38- Let  $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then  $f$  is

- 1) discontinuous
- 2) continuous but not differentiable.
- 3) differentiable only once
- 4) differentiable more than once.

Explanation:

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{1} \quad \left( \frac{0}{0}, \text{L' hopital rule} \right) \\ &= 1 \\ &= f(0). \end{aligned}$$

$\therefore f$  is continuous at  $x=0$ .

Hence  $f$  is continuous on  $\mathbb{R}$ .

$$x \neq 0, \quad f'(x) = \frac{x \cos x - \sin x}{x^2}.$$

$$x=0, \quad f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}.$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x} - 1}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{\sin x - x}{x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin x - x}{x^2}$$

$$= \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x}$$

$$= \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x}{2} = 0$$

$$\therefore f'(0) = 0$$

$$f'(x) = \begin{cases} x \frac{\cos x - \sin x}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\therefore f$  is differentiable.

$$f''(x) = \begin{cases} x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$x \neq 0$ ,  $f'$  is diff.

$x = 0$ , check whether  $f''$  exist.

$$f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x \cos x - \sin x}{x^2} - 0}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \rightarrow 0} \frac{-x \sin x + \cos x - \cos x}{3x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x}{3} = -\frac{1}{3}$$

$\therefore f''$  exist  $\forall x \in \mathbb{R}$ .

$\therefore f$  is differentiable twice.

39. Let  $(a_n), (b_n)$  be sequence of real numbers satisfying  $|a_n| \leq |b_n| \forall n \geq 1$ . Then

1)  $\sum_n a_n$  Cys whenever  $\sum_n b_n$  Cys

2)  $\sum_n a_n$  Cys absolutely whenever  $\sum_n b_n$  Converges absolutely (by comparison test).

3)  $\sum_n b_n$  Converges whenever  $\sum_n a_n$  Cys.

4)  $\sum_n b_n$  Cys absolutely whenever  $\sum_n a_n$  Cys absolutely.

### Explanation

$$1) \quad a_n = \frac{1}{n} \quad b_n = \frac{(-1)^n}{n}$$

$$\sum \frac{1}{n} \text{ dgs} \quad \leq \quad \frac{(-1)^n}{n} \text{ cgt}$$

40. If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then which of the following is NOT true?

$$1) \quad \sum_{m=n}^{\infty} a_m \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$2) \quad \sum_{n=1}^{\infty} a_n \sin n \text{ is convergent.}$$

$$3) \quad \sum_{n=1}^{\infty} e^{a_n} \text{ is divergent.}$$

~~4)  $\sum_{n=1}^{\infty} a_n^2$  is divergent.~~ ( Given  $\sum a_n$  is absolutely cgt  
 $\exists m \in \mathbb{N}, |a_n| < 1 \quad \forall n \geq m$ .  
 $a_n^2 \leq |a_n| \cdot \forall n \geq m$   
By comparison test,  
 $\sum a_n^2$  cgt).

### Explanation:

$$1) \quad \sum_{n=1}^{\infty} a_n \text{ is absolutely cgt} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ cgt}$$

$$\Rightarrow \text{the tail of the series } \sum_{m=n}^{\infty} a_m \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$2) \quad |a_n \sin n| \leq |a_n|$$

$$\sum_{n=1}^{\infty} |a_n| \cdot \text{cgt} \Rightarrow \sum_{n=1}^{\infty} |a_n \sin n| \text{ cgt. (absolute)}$$

Complete metric space  
absolutely cgt  $\Rightarrow$  cgt

$$\therefore \sum_{n=1}^{\infty} a_n \sin n \text{ cgt.}$$

42. Let  $\lambda > 0$  and  $F(x) = 1 - e^{-\lambda x}$  for  $x > 0$ . Then for  $t > 0$   $\int_0^{\infty} e^{-tx} dF(x)$  equals.

- 1)  $\frac{\lambda}{\lambda+t}$       2)  $\frac{\lambda}{\lambda-t}$       3) 0      4)  $\infty$ .

Explanation:

$$\int_0^{\infty} e^{-tx} dF(x) = \int_0^{\infty} e^{-tx} (\lambda e^{-\lambda x}) dx.$$

$$= \lambda \int_0^{\infty} e^{-(t+\lambda)x} dx$$

$$= -\lambda \left[ \frac{e^{-(t+\lambda)x}}{(t+\lambda)} \right]_0^{\infty}$$

$$= \frac{-\lambda}{t+\lambda} (0 - 1)$$

$$= \frac{\lambda}{t+\lambda}.$$

43. Let  $f: [0,1] \rightarrow [0,1]$  be any twice differentiable function satisfying

$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$  for all  $x, y \in [0,1]$  and any  $\alpha \in [0,1]$ . Then for all  $a \in [0,1]$ ,

- 1)  $f'(x) \geq 0$     2)  $f''(x) \geq 0$     3)  $f'(x) \leq 0$     4)  $f''(x) \leq 0$ .

Explanation on  
Thm:

Let  $I$  be an open interval and let  $f: I \rightarrow \mathbb{R}$  have a second derivative on  $I$ . Then  $f$  is a convex function on  $I$  if and only if  $f''(x) \geq 0$  for all  $x \in I$ .

By Hm  $f''(x) \geq 0$ .

57. If  $f: [0, 1] \rightarrow (0, 1)$  is a continuous mapping then which of the following is Not true?

- 1)  $F \subseteq [0, 1]$  is a closed set implies  $f(F)$  is closed in  $\mathbb{R}$ .
- 2) If  $f(0) < f(1)$  then  $f([0, 1])$  must be equal to  $[f(0), f(1)]$ .
- 3) There must exist  $x \in (0, 1)$  such that  $f(x) = x$ .
- 4)  $f([0, 1]) \neq (0, 1)$ .

Explanation:

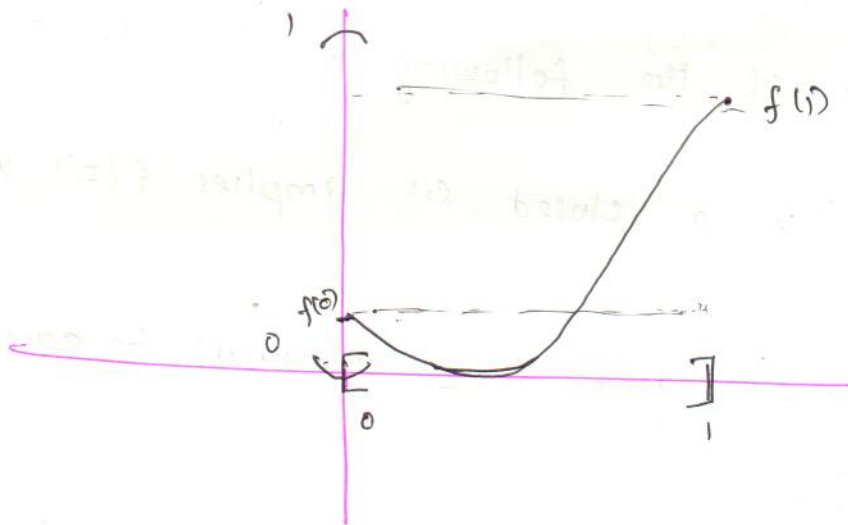
- 1)  $F \subseteq [0, 1]$  is closed set,  $F$  is closed and bdd  
 $\Rightarrow f$  is compact  
Since  $f$  is cont,  $f(F)$  is compact  
 $\Rightarrow f(F)$  is closed.

$f([0,1])$  is closed and compact, since  $f$  is continuous and  $[0,1]$  is compact.

But  $(0,1)$  is not compact

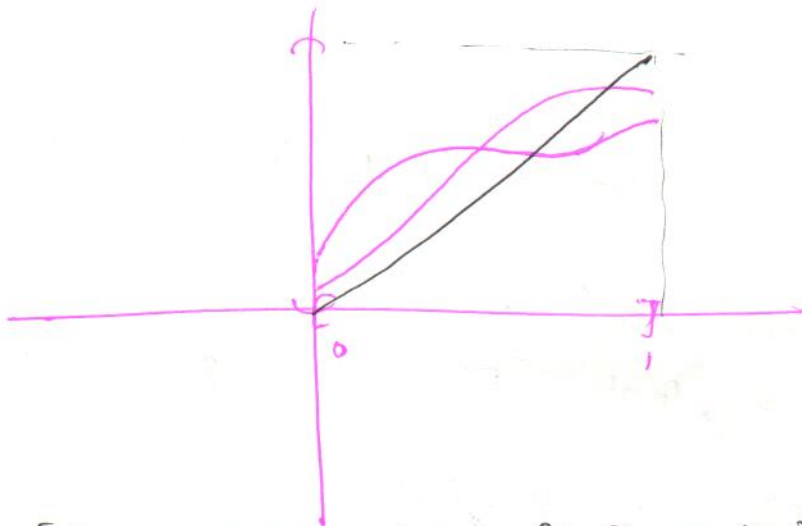
$$\therefore f([0,1]) \neq (0,1).$$

2)



$$\text{Here } f([0,1]) \neq [f(0), f(1)].$$

3)



Every graph from  $[0,1] \rightarrow (0,1)$  cuts the line  $y=x$ .

76. The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = e^{|x|+x^2} + |x|$

which of the following is true about the function

- a) It is not differentiable exactly at three points of  $\mathbb{R}$ .
- b) It is not differentiable at  $x=0$ .
- c) It is differentiable at  $x=2$ .
- d) It is not differentiable at  $x=1$  and  $x=-1$ .

Explanation:

Result: Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $c$  and that  $f(c) = 0$ . Show that Then  $g(x) = |f(x)|$  is differentiable at  $c$  if and only if  $f'(c) = 0$ .

Let  $C(a, r)$  be the subset of  $\mathbb{R}^2$  given by

$$C(a, r) = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + y^2 = r^2\}$$

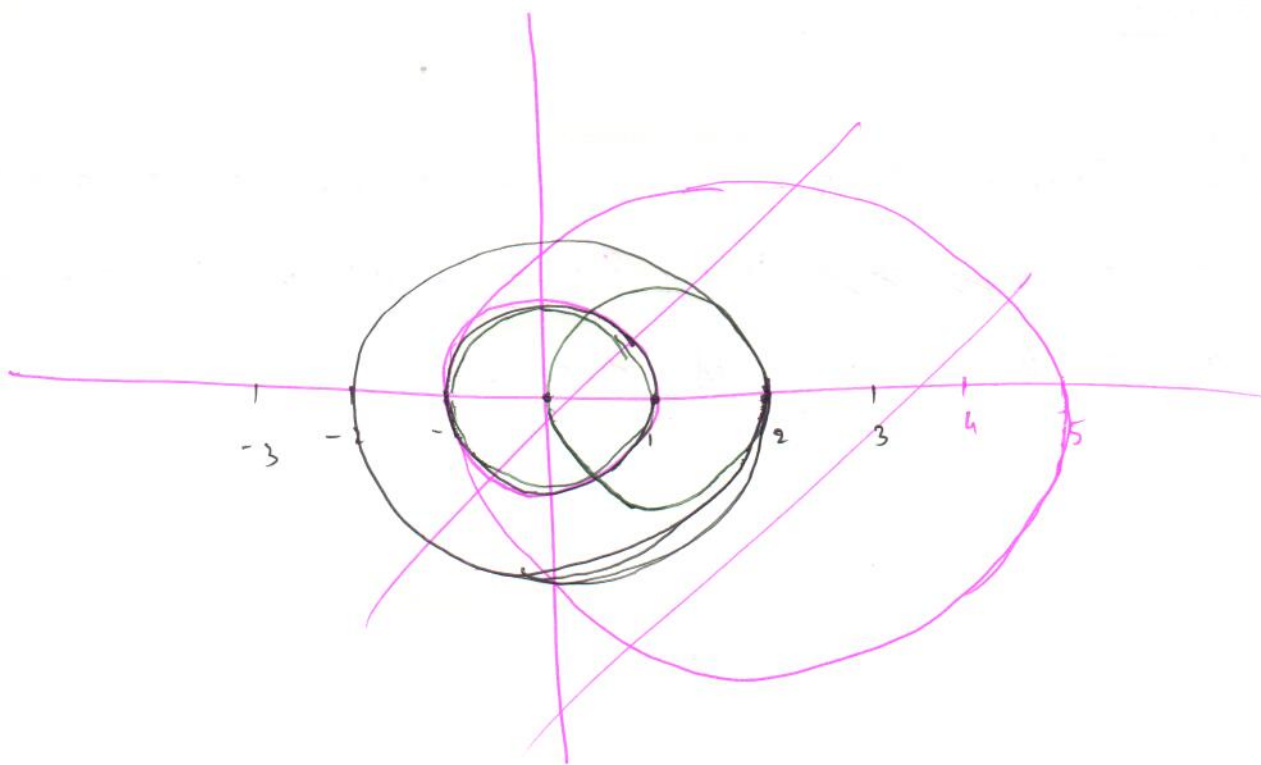
which of the following subsets of  $\mathbb{R}^2$  are connected?

1)  $C(0, 1) \cup C(0, 2)$

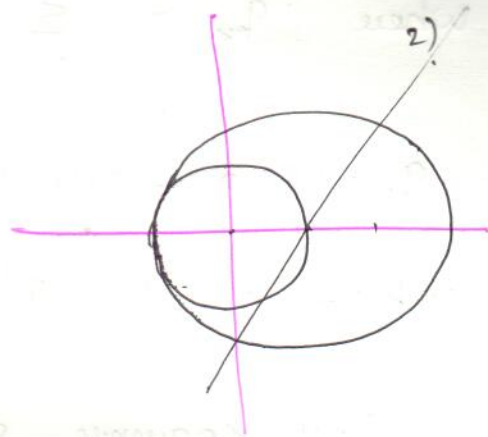
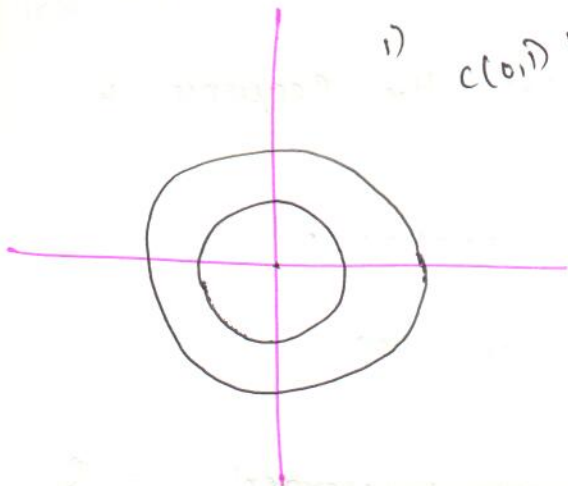
2)  $C(0, 1) \cup C(1, 3)$

3)  $C(0, 1) \cup C(1, 1)$

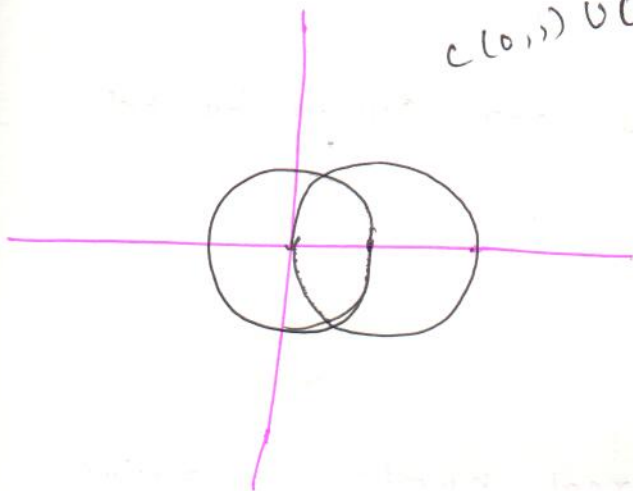
4)  $C(0, 1) \cup C(2, 1)$



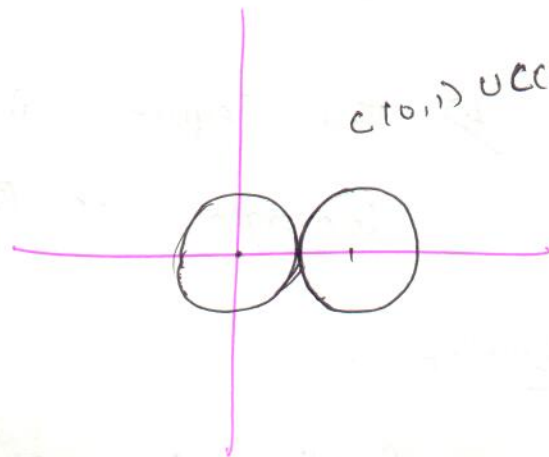
$$1) C(0,1) \cup C(0,2)$$



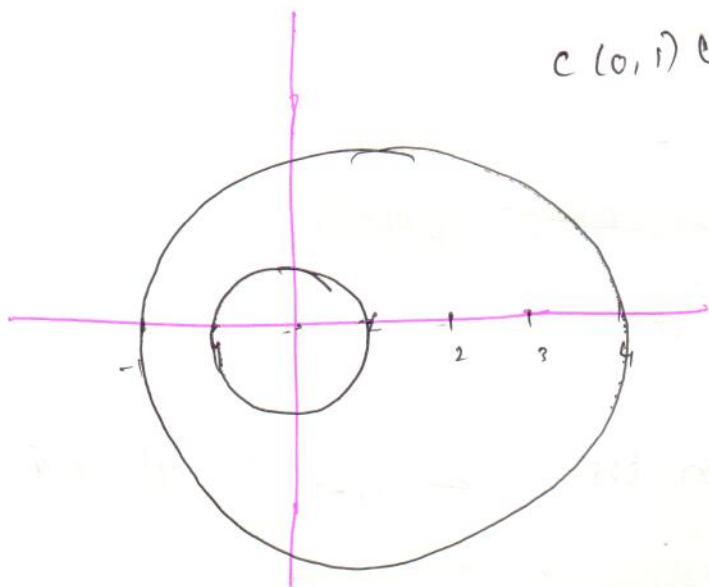
$$C(0,1) \cup C(1,1)$$



$$C(0,1) \cup C(2,1)$$



$$C(0,1) \cup C(1,3)$$



Q1. Consider the sequence of rational no's  $\{q_k\}_{k \geq 1}$   
where  $q_k = \sum_{n=1}^k \frac{1}{10^{n^2}}$ . (ie) the sequence is

$$q_1 = 1 \quad q_2 = 0.1001 \quad q_3 = 0.100100001 \text{ etc.}$$

which of the following is true?

- 1) This sequence is bdd and convergent in  $\mathbb{Q}$
- 2) This sequence is not bounded.
- 3) This sequence is bdd but not a Cauchy seq.
- 4) This sequence is bdd and Cauchy but not convergent in  $\mathbb{Q}$

### Explanation

Result: A positive real number is rational  
iff its decimal representation is periodic.

$$\text{Here } n^2 < 10^{n^2} \quad \forall n \in \mathbb{N}$$

$$\frac{1}{n^2} > \frac{1}{10^{n^2}}$$

By Comparison test,  $\sum \frac{1}{10^{n^2}}$  is  $\text{lyt}$  in  $\mathbb{R}$ .

$\therefore \sum \frac{1}{10^{n^2}}$  is bdd and Cauchy.

$$q_k \rightarrow \sum_{h=1}^{\infty} \frac{1}{10^{h2}} = 0.1001000010000001 \dots$$

This representation is never periodic.

$$\therefore q_k \mapsto x, x \in \mathbb{Q}.$$

82. Which of the following subsets of  $\mathbb{R}^2$  are uncountable?

1)  $\{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$

2)  $\{(a, b) \in \mathbb{R}^2 \mid a+b \in \mathbb{Q}\}$

3)  $\{(a, b) \in \mathbb{R}^2 \mid ab \in \mathbb{Z}\}$

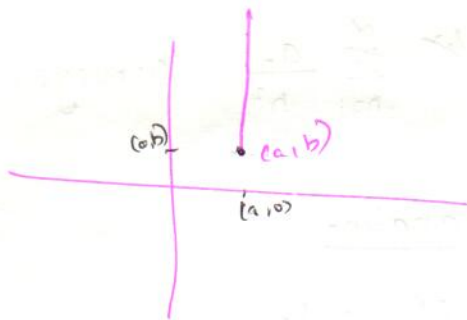
4)  $\{(a, b) \in \mathbb{R}^2 \mid a, b \in \mathbb{Q}\}.$

Explanation.

$$\{(a, b) \in \mathbb{R}^2 \mid a, b \in \mathbb{Q}\} = \mathbb{Q} \times \mathbb{Q}.$$

$$\{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$$

is uncountable



$$\{(a, b) \in \mathbb{R}^2 \mid a+b \in \mathbb{Q}\} \text{ is uncountable.}$$

$$\bigcup_{a,b \in \mathbb{Q}} \{(a+\sqrt{2}, -\sqrt{2}) \mid a \in \mathbb{R}\} \subseteq \{(a, b) \in \mathbb{R}^2 \mid a+b \in \mathbb{Q}\}$$

$\{(a, -a) \mid a \in \mathbb{R}\}$  is uncountable

$\therefore \{(a, b) \in \mathbb{R}^2 \mid a+b \in \mathbb{Q}\}$  is uncountable.

$\{(a, 1/a) \mid a \in \mathbb{R}, a \neq 0\} \subseteq \{(a, b) \in \mathbb{R}^2 \mid ab \in \mathbb{Z}\}$ .

$\{(a, 1/a) \mid a \in \mathbb{R}, a \neq 0\}$  is uncountable.

$\therefore \{(a, b) \in \mathbb{R}^2 \mid ab \in \mathbb{Z}\}$  is uncountable.

83. Let  $\{a_n\}_{n \geq 1}$  be a sequence of positive numbers such that  $a_1 > a_2 > a_3 > \dots$

Then which of the following is/are always true?

1)  $\lim_{n \rightarrow \infty} a_n = 0$

2)  $\lim_{n \rightarrow \infty} \frac{a_n}{n} = 0$

3)  $\sum_{n=1}^{\infty} \frac{a_n}{n}$  converges

4)  $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$  converges.

Explanation,

h) Abel's test

Let  $\sum a_n$  be a convergent series. Let  $(b_n)$  be bounded monotonic sequence. Then  $\sum a_n b_n$  is convergent

Since  $\sum \frac{1}{n^2}$  cgt and  $a_n$  is bdd monotonic seq

By Abel's test,  $\sum \frac{a_n}{n^2}$  cgt

1)  $a_n = 1 + \frac{1}{n}$ .

$a_n \rightarrow 1 \Rightarrow$  (1) is wrong.

2)  $0 \leq \frac{a_n}{n} \leq \frac{a_1}{n}$ .

By Sandwich lemma,  $\frac{a_n}{n} \rightarrow 0$ .

3)  $a_n = \frac{1}{\log n}$ .

Since  $\log n \uparrow$ ,  $a_n \downarrow$ .

But  $\sum \frac{1/\log n}{n}$  is divergent.

$$\sum \frac{1}{n \log n}$$

$\sum a_n$  be the series of positive terms and whose terms monotonic decreasing. Then the series  $\sum 2^n a_n$  cgt or dgt iff  $\sum a_n$  cgt or dgt.

$$\sum \frac{2^n}{2^n \log 2^n} = \frac{1}{\log 2} \sum \frac{1}{n}$$

$$\leq \frac{1}{n} dgs, \Rightarrow \leq \frac{1}{n \log n} dgs.$$

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21. Let  $f, g$  and  $h$  be bounded functions on the closed interval  $[a, b]$  such that  $f(x) \leq g(x) \leq h(x)$  for all  $x \in [a, b]$ . Let  $P = \{a = a_0 < a_1 < \dots < a_n = b\}$  be a partition of  $[a, b]$ . We denote  $U(f, P)$  and  $L(f, P)$  the upper and lower Riemann sums of  $f$  w.r.t the partition  $P$  and similarly for  $g$  and  $h$  which of the following statements is necessarily true?

1) If  $U(h, P) - U(f, P) < 1$ , then

$$U(g, P) - L(g, P) < 1$$

2) If  $L(h, P) - L(f, P) < 1$ , then  $U(g, P) - L(g, P) < 1$

3) If  $U(h, P) - L(f, P) < 1$ , then  $U(g, P) - L(g, P) < 1$

4) If  $L(h, P) - U(f, P) < 1$ , then  $U(g, P) - L(g, P) < 1$

Explanation:

$$L(f, P) \leq \int f(x) \leq \int g(x) \leq \int h(x) \leq U(f, P)$$

$$L(f, P) \leq L(g, P) \leq \int g(x) \leq U(g, P) \leq U(h, P)$$

$$U(h, p) - L(f, p) < 1 \Rightarrow U(g, p) - L(g, p) \neq 1.$$

30. The power series  $\sum_{n=0}^{\infty} \frac{[2 + (-1)^n]^n}{3^n} x^n$  converges

1) only for  $x=0$       2) for all  $x \in \mathbb{R}$

3) only for  $-1 < x < 1$       4) only for  $-1 < x \leq 1$

Explanation

$$\limsup_n |a_n|^{1/n} = \limsup_n \left| \frac{(2 + (-1)^n)^n}{3^n} \right|^{1/n}$$

$$= \limsup_n \frac{|2 + (-1)^n|}{3}$$

$$= \limsup_n \left\{ \frac{1}{3}, 1, \frac{1}{3}, 1, \dots \right\}$$

$$= 1$$

$$R = \frac{1}{\limsup_n |a_n|^{1/n}} = 1$$

$$|x| < R \Rightarrow |x| < 1$$

$$x=1$$

$$\sum_{n=0}^{\infty} \frac{(2 + (-1)^n)^n}{3^n} = 1 + \frac{1}{3} + 1 + \frac{1}{3} + 1 + \frac{1}{3} + \dots$$

which is divergent.

$$\therefore -1 < x < 1$$

31. In which of the following cases, there is no continuous function  $f$  from the set  $S$  onto the set  $T$ ?

1)  $S = [0, 1]$  ,  $T = \mathbb{R}$ .

2)  $S = (0, 1)$  ,  $T = \mathbb{R}$ .

3)  $S = (0, 1)$  ,  $T = [0, 1]$

4)  $S = \mathbb{R}$  ,  $T = (0, 1)$ .

Explanation

$[0, 1]$  and  $\mathbb{R}$  are homeomorphic.

Suppose  $\exists$  a continuous onto fun  $f: [0, 1] \rightarrow \mathbb{R}$ .

Then  $f([0, 1]) = \mathbb{R}$ .

$\therefore \mathbb{R}$  is compact  $\Rightarrow \Leftarrow$ .

$\therefore$  There is no continuous function from  $[0, 1]$  to  $\mathbb{R}$ .

32. The function  $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$  is differentiable at  $x=0$ .

1) for no values of  $a_0, a_1, a_2, a_3$

2) for any value of  $a_0, a_1, a_2, a_3$ .

3) only if  $a_1 = 0$

4) only if both  $a_1 = 0$  and  $a_3 = 0$

Explanation:

$$\begin{aligned} f(x) &= a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 \\ &= a_0 + a_1|x| + a_2|x^2| + a_3|x^3| \end{aligned}$$

At  $x=0$ ,  $g(x) = |x^2|$

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f'(0) = 0$$

$\Rightarrow g(x)$  is diff

$$h(x) = |x^3|$$

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$f'(0) = 0$$

$\Rightarrow h(x)$  is diff

Result: Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is diff at  $c$  and that  $f(c) = 0$ . Then  $g(x) = |f(x)|$  is diff at  $c \Leftrightarrow f'(c) = 0$

34. Consider the following subsets of  $\mathbb{R}^2$ , where  $a, b \in \mathbb{R}$

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \neq b \right\}$$

$$B = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, a \neq b \right\}$$

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid ax + by + 5 = 0 \right\}$$

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \right\}$$

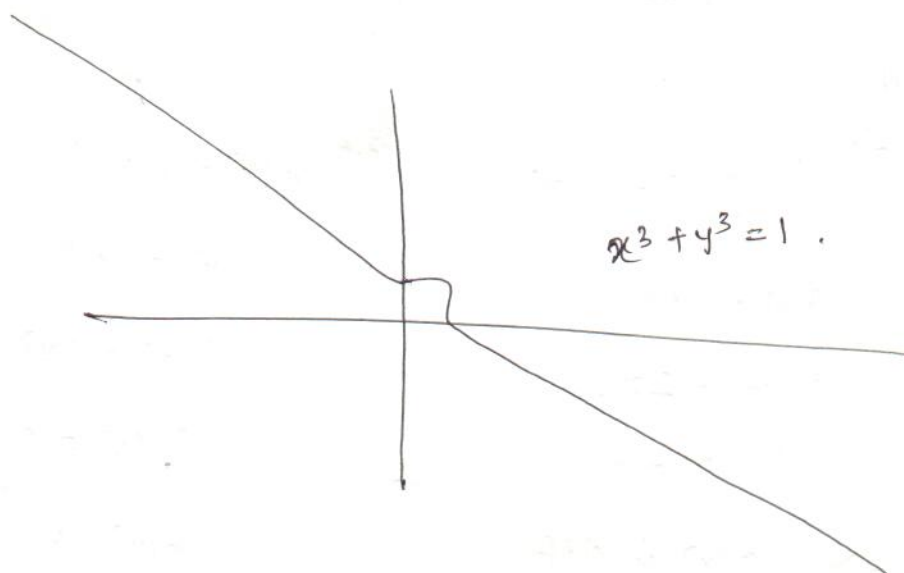
Then which of the following is correct?

1.  $C$  and  $D$  are compact but  $A, B, E$  are not compact

~~2.~~  $A$  and  $B$  are compact but  $C, D, E$  are not compact

3.  $A, B$  and  $E$  are compact but  $C, D$  are not compact

4.  $A$  and  $E$  are compact but  $B, C, D$  are not compact



35. The number of limit points of the set

$$\left\{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \right\} \text{ is}$$

1) 1      2) 2      3) finitely many      ~~4) infinitely many~~

Explanation.

limit point of  $\left\{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \right\}$

$$= \left\{ \frac{1}{k} \mid k \in \mathbb{N} \right\}.$$

$\therefore$  Infinitely many.

36. Area enclosed between  $x$ -axis from  $a$  to  $b$  and the curve  $f(x)$  is finite when

1)  $a=0$  ,  $b=\infty$  ,  $f(x) = e^{-5x^5}$

2)  $a=-\infty$   $b=\infty$  ,  $f(x) = e^{-5x^5}$

3)  $a=-7$   $b=\infty$  ,  $f(x) = \frac{1}{x^4}$

4)  $a=-7$   $b=7$  ,  $f(x) = \frac{1}{x^4}$

39- Which of the following Series is convergent?

1)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}}$

2)  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

3)  $\sum_{n=1}^{\infty} (-1)^n \log n$

4)  $\sum_{n=1}^{\infty} \frac{\log n}{n}$

Explanation:

$$\begin{aligned} 1) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}} &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \sum_{n=1}^{\infty} \sqrt{n+1} + \sqrt{n} \rightarrow \text{divergent} \end{aligned}$$

2)  $\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$

$\sum \frac{1}{n^2}$  cgt,  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  cgt.

3)  $\sum_{n=1}^{\infty} (-1)^n \log n$  is an oscillating series.

4)  $\sum_{n=1}^{\infty} \frac{\log n}{n}$

$\sum 2^n a_{2^n} = \sum \frac{\log 2^n}{2^n}$

$= \log 2 \leq n$  which is divergent

$\therefore \sum_{n=1}^{\infty} \frac{\log n}{n}$  divergent.

38. What is the smallest positive integer in the set  $\{24x + 60y + 2000z \mid x, y, z \in \mathbb{Z}\}$ ?

- 1) 2      2) 4      3) 6      4) 24.

Explanation:

Smallest div integer of the set  $\{24x + 60y + 2000z \mid x, y, z \in \mathbb{Z}\}$   
 $= \text{gcd}(24, 60, 2000)$   
 $= 4$

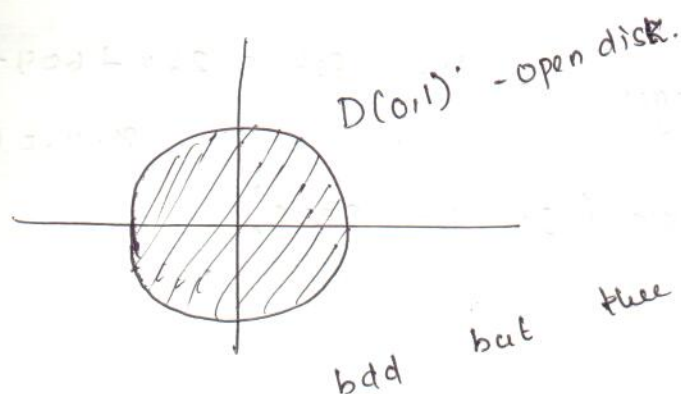
77. Let  $A$  be the set of rational numbers in the open interval  $(0, 1)$ . and  $f: A \rightarrow \mathbb{R}$  be a uniformly continuous function? Which of the following are true?

- 1)  $f$  is bdd  
 2)  $f$  is necessarily constant fun  
 3)  $f$  is diff on  $(0, 1)$   
 4)  $f$  is diff at all rational pts in  $(0, 1)$

74. Let  $A$  be a subset of  $\mathbb{R}^p$  and  $x \in \mathbb{R}^p$ .

Denote  $d(x, A) = \inf \{ d(x, y) \mid y \in A \}$ . There exists a point  $y_0 \in A$  with  $d(y_0, x) = d(x, A)$  if

- ~~1)~~  $A$  is any closed non-empty subset of  $\mathbb{R}^p$
- 2)  $A$  is any non-empty subset of  $\mathbb{R}^p$
- ~~3)~~  $A$  is any compact subset of  $\mathbb{R}^p$
- 4)  $A$  is any non-empty bdd subset of  $\mathbb{R}^p$ .



there is no  $y_0 \in D(0,1)$  s.t.  
 $d(y_0, x) = d(x, A)$ .

75. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with period  $p > 0$ . Then  $g(x) = \int_x^{x+p} f(t) dt$  is a

- ~~1)~~ constant function
- ~~2)~~ continuous function
- 3) continuous function but not differentiable
- 4) neither conts nor diff.

Explanation:

$$\frac{d}{dx} \int_{\varphi(x)}^{\psi(x)} g(t) dt = g(\psi(x)) \psi'(x) - g(\varphi(x)) \varphi'(x)$$

$$\frac{d}{dx} (g(x)) = \frac{d}{dx} \int_x^{x+p} f(t) dt.$$

$$= f(x+p) \cdot 1 - f(x) \cdot 1$$

$$= 0 \quad (\because f \text{ is a periodic fun with period } p)$$

$\therefore g(x)$  is constant.

79. Let  $A_n \subseteq \mathbb{R}$  for  $n \geq 1$ , and  $\chi_n: \mathbb{R} \rightarrow \{0, 1\}$  be the function  $\chi_n(x) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}$

Let  $g(x) = \limsup_{n \rightarrow \infty} \chi_n(x)$  and  $h(x) = \liminf_{n \rightarrow \infty} \chi_n(x)$

1) If  $g(x) = h(x) = 1$ , then there exists  $m$  such that  $\forall n \geq m$  we have  $x \in A_n$ .

2) If  $g(x) = 1$  and  $h(x) = 0$ , then there exists  $m$  such that for all  $n \geq m$  we have  $x \in A_n$ .

3) If  $g(x) = 1$  and  $h(x) = 0$  then there exists a seq.  $n_1, n_2, \dots$  of distinct integers such that  $x \in A_{n_k}$  for all  $k \geq 1$ .

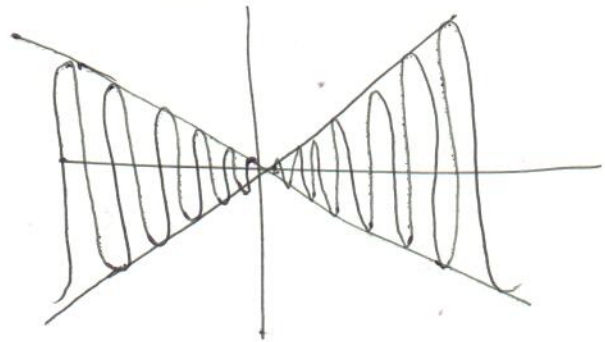
4) If  $g(x) = h(x) = 0$  then there exists  $m$  such that for all  $n \geq m$  we have  $x \notin A_n$ .

80. Which of the following functions  $f$  is uniformly continuous on the interval  $(0,1)$ ?

- 1)  $f(x) = 1/x$     2)  $f(x) = x \sin 1/x$     3)  $f(x) = \sin 1/x$     4)  $f(x) = \frac{\sin x}{x}$

Explanation:

$$f(x) = x \sin 1/x$$



$$f(x) = \frac{\sin x}{x}$$

$$\text{At } x \rightarrow 0, f(x) = 1.$$

$f$  is a continuous function on  $(0,1)$  [compact subset]

Then  $f$  is uniformly continuous

81. The minimum possible value of  $|z|^2 + |z-3|^2 + |z-b|^2$  where  $z$  is a complex number and  $i = \sqrt{-1}$  is

- 1) 15    2) 45    3) 30    4) 20

Explanation:

To find stationary pts  $f_x = 0$  and  $f_y = 0$

$$f_{xx} f_{yy} - f_{xy}^2 > 0$$



$f_{xx} > 0$   
 $f$  is minimum at stationary pts

$f_{xx} < 0$   
 $f$  is maximum at stationary pts.

$$f(x,y) = x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-6)^2$$

$$f_x = 2x + 2(x-3) + 2x$$

$$f_y = 2y + 2y + 2(y-6)$$

$$f_x = 0 \Rightarrow 6x - 6 = 0$$

$$\boxed{x=1}$$

$$f_y = 0$$

$$\Rightarrow 6y - 12 = 0$$

$$\boxed{y=2}$$

$$f_{xx} = 2 + 2 + 2 = 6$$

$$f_{yy} = 2 + 2 + 2 = 6$$

$$f_{xy} = 0$$

$$f_{xx} f_{yy} - f_{xy}^2 = 6 \times 6 - 0 = 12 > 0$$

$$f_{xx} = 6 > 0$$

$\therefore f$  has minimum at  $x=1$   $y=2$ .

$$f(1,2) = 1 + 4 + 4 + 4 + 1 + 16 = 30$$

83. Let  $p$  be a real polynomial of the real variable  $x$  of the form  $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x - 1$ . Suppose that  $p$  has no roots in the open unit disc and  $p(-1) = 0$ . Then

1)  $p(1) = 0$

2)  $\lim_{x \rightarrow \infty} p(x) = \infty$

3)  $p(2) > 0$

4)  $p(3) = 0$

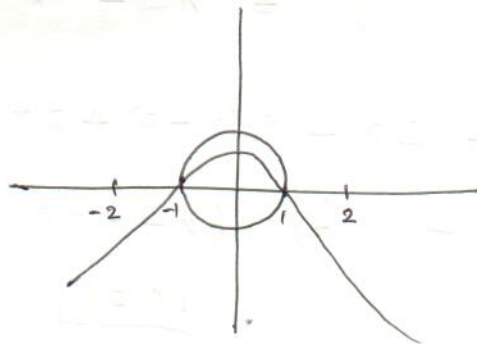
Explanation

Product of the roots  $= -1$

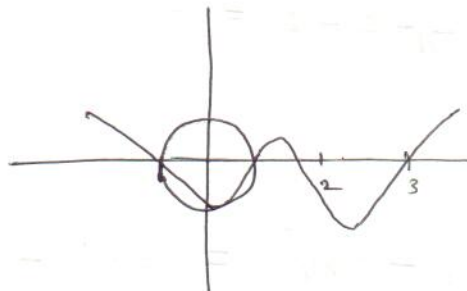
$$\therefore p(1) = 0$$

Suppose  $p(2) < 0$ .

Then  $\lim_{x \rightarrow \infty} p(x) \neq \infty$ .



From the figure,  
 $p(1/3) = 0$  But no  
roots occur in open disc



85. Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function that satisfies  $|f(x) - f(y)| \leq |x - y|^\beta$ ,  $\beta > 0$ . Which of the following is correct?

- 1) if  $\beta = 1$  then  $f$  is differentiable
- ~~2) if  $\beta > 0$  then  $f$  is uniformly continuous~~
- ~~3) if  $\beta > 1$  then  $f$  is a constant function~~
- 4)  $f$  must be a polynomial

Explanation,

Suppose  $\beta > 1$ . Then  $\beta = 1 + h$   $h > 0$

$$|f(x) - f(y)| \leq |x - y|^\beta$$

$$|f(x) - f(y)| \leq |x - y|^{1+h}$$

$$\frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|^h$$

$$\lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{x \rightarrow y} |x - y|^h$$

$$\left| \lim_{x \rightarrow y} \frac{f(x) - f(y)}{x - y} \right| \leq 0$$

$$|f'(y)| \leq 0$$

$\therefore f'(y) = 0 \Rightarrow f$  is constant fun.

Let  $\epsilon > 0$ , To find  $\delta > 0$  s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$$|f(x) - f(y)| \leq |x - y|^p < \epsilon$$

$$|x - y| < \epsilon^{1/p} =$$

$$\text{Choose } \delta = \epsilon^{1/p}.$$

$$\begin{aligned} \text{Then } |x - y| < \delta &\Rightarrow |f(x) - f(y)| \leq |x - y|^p \\ &< \delta^p \\ &= \epsilon \end{aligned}$$

$\therefore f$  is uniformly continuous.

$f(x) = x^2$  is a polynomial.

$$|f(x) - f(y)| = |x^2 - y^2| \neq |x - y|^2$$

$$4) f(x) = \sin x.$$

$$|f(x) - f(y)| = |\sin x - \sin y| \leq |x - y|$$

But  $f(x)$  is not a polynomial.

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21. Let  $f$  be a twice differentiable function on  $\mathbb{R}$ .  
Given that  $f''(x) > 0 \quad \forall x \in \mathbb{R}$ .

1.  $f(x) = 0$  has exactly two solutions on  $\mathbb{R}$ .

2.  $f(x) = 0$  has a positive solution if  $f(0) = 0$  and  $f'(0) = 0$

3.  $f(x) = 0$  has no positive solution if  $f(0) = 0$  and  $f'(0) > 0$

4.  $f(x) = 0$  has no positive solution if  $f(0) = 0$  and  $f'(0) < 0$ .

Explanation,

1)  $f(x) = e^x$ , has no soln in  $\mathbb{R}$ .

2)  $f(x) = x^2$ ,  $f'(x) = 2x$   $f''(x) = 2$

$$f(0) = 0 \quad f'(0) = 0$$

But  $f$  has no positive soln

4)  $f(x) = x(x-1)$   $f(0) = 0$

$$f'(x) = 2x - 1 \quad f'(0) < 0$$

$$f''(x) = 2$$

But  $f$  has a positive soln.

$\therefore$  ① ② and ④ are wrong.

22. Let  $f$  be a continuously differentiable real-valued function on  $[a, b]$  such that  $|f'(x)| \leq K$  for all  $x \in [a, b]$ . For a partition  $P = \{a = a_0 < a_1 < \dots < a_n = b\}$

Let  $U(f, P)$  and  $L(f, P)$  denote the upper and lower Riemann sums of  $f$  with respect to  $P$ . Then

$$1. |L(f, P)| \leq K(b-a) \leq |U(f, P)|$$

$$2. U(f, P) - L(f, P) \leq K(b-a)$$

$$3. U(f, P) - L(f, P) \leq K \|P\| \text{ where}$$

$$\|P\| = \max_{0 \leq i \leq n-1} (a_{i+1} - a_i) \text{ is the norm of the partition}$$

$$4. U(f, P) - L(f, P) \leq K \|P\| (b-a)$$

23. Let  $f$  be a monotone nondecreasing real-valued function on  $\mathbb{R}$ . Then

1)  $\lim_{x \rightarrow a} f(x)$  exists at each point  $a$ .

2) If  $a < b$ , then  $\lim_{x \rightarrow a^+} f(x) \leq \lim_{x \rightarrow b^-} f(x)$

3)  $f$  is an unbounded function

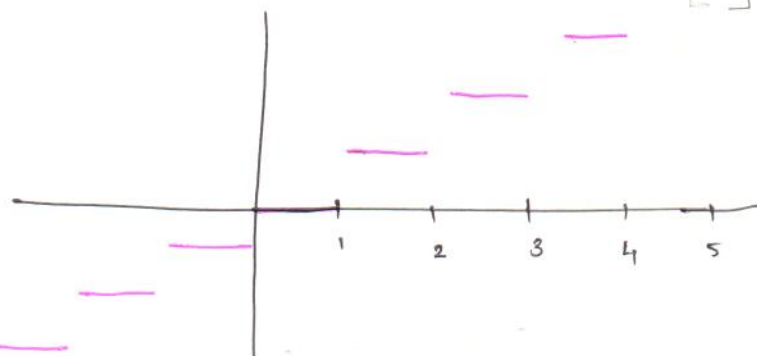
4) The function  $g(x) = e^{-f(x)}$  is bounded function.

Explanation:

1)  $f(x) = \lfloor x \rfloor$

$\lim_{x \rightarrow n} f(x)$  does not

exist for all  $n \in \mathbb{Z}$



2)  $a < b$

$$\{f(x) : a < x < b\}$$

$$f(x) \leq f(y) \quad \forall x, y \in [a, b]$$

$$\sup \{f(x) : a < x\} \leq \inf \{f(x) : x < b\}$$

$$\inf \{f(x) : a < x\} \leq \sup \{f(x) : x < b\}$$

$$\lim_{x \rightarrow a^+} f(x) \leq \lim_{x \rightarrow b^-} f(x)$$

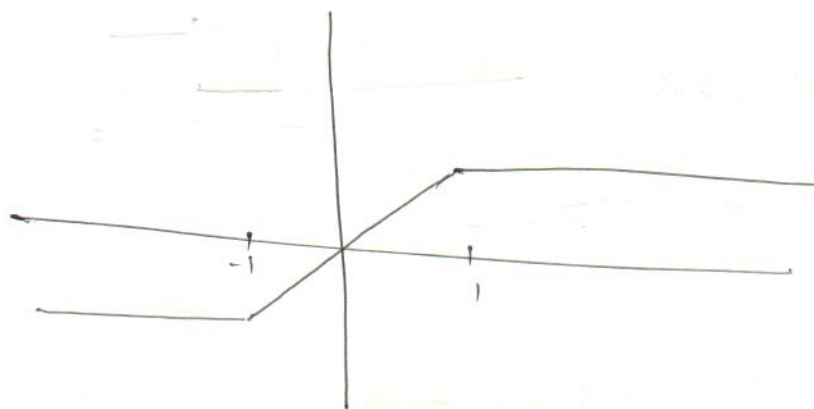
Thm: Let  $f: I \rightarrow \mathbb{R}$  be an increasing fun.  
and  $c \in I$  and  $c$  is not the end pt of  $I$ .

Then

$$\lim_{x \rightarrow c^-} f(x) = \text{lub } \{f(x) : x \in I, x < c\}$$

$$\lim_{x \rightarrow c^+} f(x) = \text{glb } \{f(x) : x \in I, x > c\}.$$

3.  $f(x) = \begin{cases} -1/x & -\infty < x < -1 \\ 0 & x = -1 \\ 1/x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$



$f$  is increasing but bdd.

4)  $f(x) = x$

$g(x) = e^{-x}$  is an unbdd fun.

25. Which of the following real-valued functions on  $(0,1)$  is uniformly continuous?

1.  $f(x) = 1/x$

2.  $f(x) = \sin x / x$

3.  $\sin 1/x$

4.  $\frac{\cos x}{x}$

Explanation:

$$f(x) = \frac{\sin x}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$\therefore f(x)$  is uniformly continuous.

But  $\lim_{x \rightarrow 0} \frac{\cos x}{x}$  does not exist

26. Let  $X$  be a metric space and  $A \subseteq X$  be a connected set with at least two distinct points. Then the number of distinct points in  $A$  is

1. 2      2. More than 2 but finite  
3. Countably infinite      4. Uncountable.

Explanation:

Take  $X = \mathbb{R}$ .

If  $A$  has only two points or more than two but finite or countably infinite then  $A$  will not be connected.

$\therefore A$  is uncountable.

61. Let  $(f_n)$  be a seq of continuous real-valued functions defined on  $[0, \infty)$ . Suppose  $f_n(x) \rightarrow f(x)$  for all  $x \in [0, \infty)$  and that  $f$  is integrable. Then

1.  $\int_0^{\infty} f_n(x) dx \rightarrow \int_0^{\infty} f(x) dx$  as  $n \rightarrow \infty$

2. if  $f_n \rightarrow f$  uniformly on  $[0, \infty)$ , then

$$\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$$

3. if  $f_n \rightarrow f$  uniformly on  $[0, \infty)$ , then

$$\int_0^{\infty} f_n(x) dx \rightarrow \int_0^{\infty} f(x) dx.$$

4. if  $\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0$ , then  $f_n \rightarrow f$

uniformly on  $[0, 1]$ .

Explanation

$$f_n \rightarrow f \text{ uniformly} \Rightarrow \int_a^b f_n \rightarrow \int_a^b f$$

$$\therefore \int_0^{\infty} f_n(x) dx \rightarrow \int_0^{\infty} f(x) dx.$$

(2). Thm.

3. fails

4.  $f_n(x) = e^{-nx}$

$$f(n) = \lim_{n \rightarrow \infty} e^{-nx} = 0.$$

$$|f_n(x) - f(x)| = |e^{-nx}| = e^{-nx}.$$

$$\int_0^1 |f_n(x) - f(x)| dx = \int_0^1 e^{-nx} dx = \left( \frac{e^{-nx}}{-n} \right)_0^1 = \frac{1}{n} (1 - e^{-1}).$$

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n(x) - f(x)| dx \rightarrow 0.$$

but  $f_n \not\rightarrow f$  uniformly on  $[0, 1]$

62. Let  $X$  be a compact topological space and let  $f: X \rightarrow \mathbb{R}$  be a function. The graph of  $f$  is the set  $G = \{(x, f(x)) : x \in X\} \subseteq X \times \mathbb{R}$ . Which of the following are necessarily true?

1.  $G$  is a closed set if and only if  $f$  is continuous
2. If  $f$  is continuous then  $G$  is closed
3. If  $f$  is continuous, then  $G$  is connected
4. If  $f$  is a bounded continuous function, then  $G$  is compact.

### Explanation:

2)  $f$  is continuous. T.p  $G$  is closed

Let  $(x_n, f(x_n))$  be a seq in  $G$  s.t

$$(x_n, f(x_n)) \rightarrow (x, y)$$

$$\Rightarrow x_n \rightarrow x \quad \text{and} \quad f(x_n) \rightarrow y.$$

Since  $f$  is continuous,  $f(x_n) \rightarrow f(x)$

$$\therefore y = f(x).$$

$$\therefore (x, y) \in G$$

Hence  $G$  is closed.

$$1) \quad f: [0, 1] \rightarrow \mathbb{R} \quad \text{by} \quad f(x) = \begin{cases} 1/x & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

$$G = \{ (x, f(x)) \mid x \in [0, 1] \}.$$

$$= \{ (0, 0) \} \cup \{ (x, 1/x) \mid x \in (0, 1] \}$$

which is a closed subset of  $[0, 1] \times \mathbb{R}$ .

w.r.t  $\|\cdot\|$ ,

3).

4)  $f$  is bdd continuous fun,  $f(x)$  is compact  
 since  $X$  is compact.

$$\therefore G = \{(x, f(x)) \mid x \in X\} \subseteq X \times f(X).$$

since  $X \times f(X)$  is compact and  $G$  is closed,  
 $G$  is compact.

64. Which of the conditions below imply that a  
 function  $f: [0, 1] \rightarrow \mathbb{R}$  is necessarily of bounded variation?

1.  $f$  is a monotone function on  $[0, 1]$

2.  $f$  is a continuous and monotone function on  $[0, 1]$

3.  $f$  has a derivative at each  $x \in (0, 1)$

4.  $f$  has a bounded derivative on the interval  $(0, 1)$ .

Explanation:  $f$  is monotone fun. Assume  $f$  is  $\uparrow$ .

$$V(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

$$\text{Let } P \in \mathcal{D} \equiv x_0 < x_1 < x_2 \dots < x_n = 1$$

$$f(x_0) < f(x_1) < f(x_2) \dots < f(x_n)$$

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots + |f(x_n) - f(x_{n-1})|$$

$$= f(x_n) - f(x_0)$$

$$= f(1) - f(0) < \infty.$$

$$\therefore V(f, P) < \infty.$$

Hence  $f$  is bdd variation.

4. Suppose  $f$  has a bdd derivative.

$$(i.e) \exists M > 0 \ni |f'(x)| \leq M \quad \forall x \in (0, 1).$$

$f$  is diff on  $(0, 1)$  and conts on  $[0, 1]$ .

By Meanvalue theorem,

$$\exists c \in (0, 1) \ni |f(x) - f(y)| \leq |f'(c)| |x - y|.$$

$$\text{Let } P: 0 = x_0 < x_1 < \dots < x_n = 1$$

$$V(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^n |f'(c_i)| |x_i - x_{i-1}|$$

$$\leq \sum_{i=1}^n M |x_i - x_{i-1}|$$

$$= M (1 - 0)$$

$$= M < \infty.$$

$\therefore f$  is bdd variation.

65. Let  $f(x) = \sin x - x + \frac{x^3}{3!}$  and  $g(x) = \cos x - 1 + \frac{x^2}{2!}$  for  $x \in \mathbb{R}$ . Which of the following statements are correct?

1.  $f(x) \geq 0$  for all  $x > 0$

2.  $g$  is an increasing function on  $[0, \infty)$

3.  $g$  is a decreasing function on  $[0, \infty)$

4.  $f$  is a decreasing function on  $[0, \infty)$

Explanation:

Since,  $\sin x \geq x - \frac{x^3}{3!} \quad \forall x \in \mathbb{R}$ .

$$\sin x - x + \frac{x^3}{3!} \geq 0$$

$$\therefore f(x) \geq 0 \quad \forall x > 0$$

Result:

Let  $J$  be an interval and  $f: J \rightarrow \mathbb{R}$  be differentiable with  $f'(x) > 0$  for  $x \in J$ .  
( $f'(x) < 0$ )

Then  $f$  is increasing (decreasing) on  $J$ .

$$J = \mathbb{R}.$$

$$f'(x) = \cos x - 1 + \frac{x^2}{2!}$$

$$\geq 0 \quad \left( \because \cos x \geq 1 - \frac{x^2}{2!} \right)$$

$\therefore f$  is increasing

$$g'(x) = -\sin x - 0 + \frac{2x^2}{2!} = -\sin x + x$$

$$\text{Since } \sin x < x, \quad x - \sin x > 0$$

$$\therefore g'(x) > 0$$

Hence  $g$  is increasing on  $(0, \infty)$

69. Let  $X = [-1, 1] \times [-1, 1]$ ;  $A = \{(x, y) \in X : x^2 + y^2 = 1\}$

$$B = \{(x, y) \in X : |x| + |y| = 1\}$$

$$C = \{(x, y) \in X : xy = 0\}$$

$$D = \{(x, y) \in X : x \neq \pm y\}. \text{ Then}$$

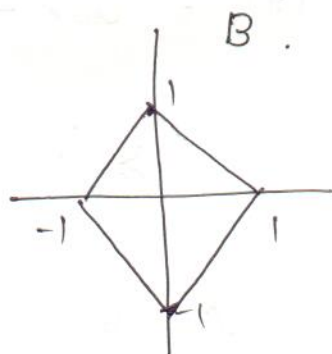
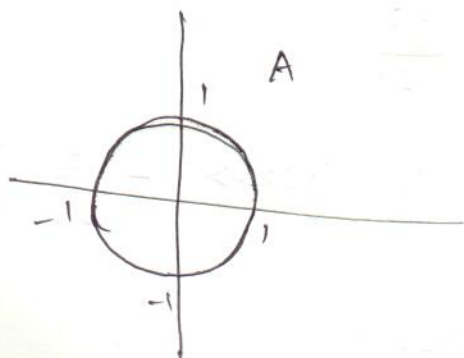
1.  $A$  is homeomorphic to  $B$ .

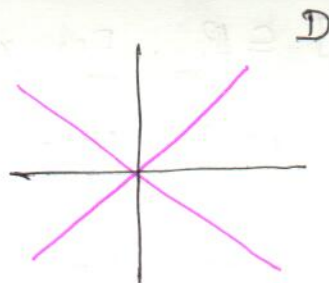
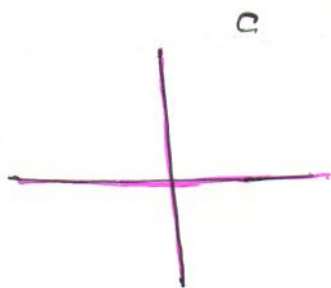
2.  $B$  is homeomorphic to  $C$ .

3.  $C$  is homeomorphic to  $D$ .

4.  $D$  is homeomorphic to  $A$ .

Explanation:





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24. Let  $f_n(x) = \begin{cases} 1 - nx & \text{for } x \in [0, 1/n] \\ 0 & \text{for } x \in [1/n, 1] \end{cases}$

Then

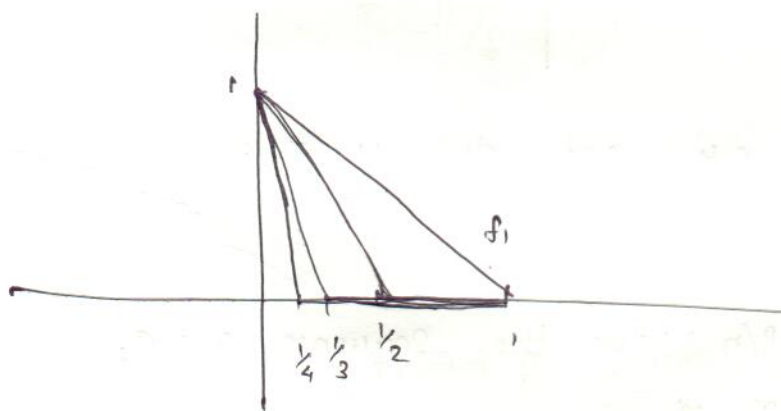
1.  $\lim_{n \rightarrow \infty} f_n(x)$  defines a continuous function on  $[0, 1]$

2.  $(f_n)$  converges uniformly on  $[0, 1]$ .

3.  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in [0, 1]$

4.  $\lim_{n \rightarrow \infty} f_n(x)$  exists for all  $x \in [0, 1]$ .

Explanation:



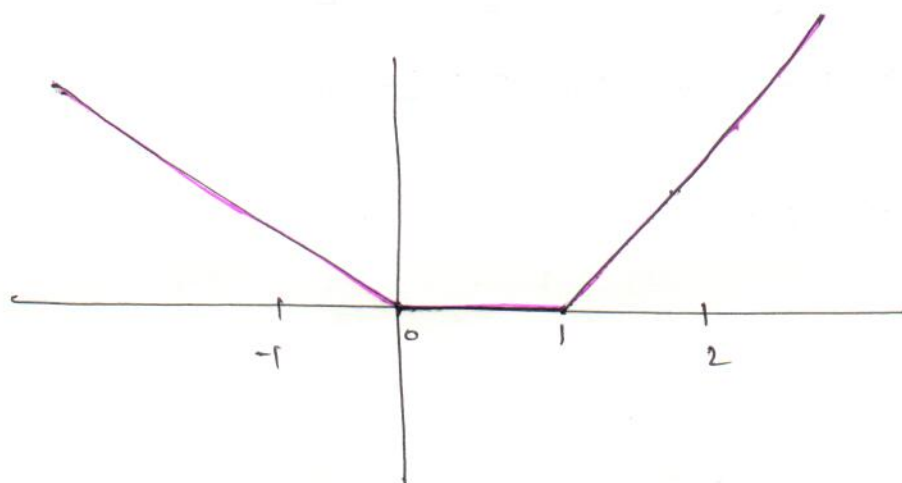
$$\lim_{n \rightarrow \infty} f_n(x) = f(x) = \begin{cases} 0 & x \in (0, 1] \\ 1 & x = 0 \end{cases}$$

So. Let  $I = [0, 1] \subseteq \mathbb{R}$ , For  $x \in \mathbb{R}$ , let  $\varphi(x) = \text{dist}(x, I)$   
 $= \inf \{ |x - y| : y \in I \}$ . Then

1.  $\varphi(x)$  is discontinuous somewhere on  $\mathbb{R}$ .
2.  $\varphi(x)$  is continuous on  $\mathbb{R}$  but not uniformly continuously differentiable exactly at  $x=0$
3.  $\varphi(x)$  is continuous on  $\mathbb{R}$  but not continuously differentiable exactly at  $x=0$  and at  $x=1$
4.  $\varphi(x)$  is differentiable on  $\mathbb{R}$ .

Explanation:

$$\varphi: \mathbb{R} \rightarrow \mathbb{R}.$$



At  $x=0, 1$ ,  $\varphi$  is not differentiable.

31. Let  $a_n = \sin \pi/n$ . For the sequence  $a_1, a_2, \dots$   
 Then Supremum is

1. 0 and it is attained
2. 0 and it is not attained
3. 1 and it is attained
4. 1 and it is not attained.

32. Using the fact that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$  equals

- 1)  $\pi^2/12$     2)  $\pi^2/12 - 1$     3)  $\pi^2/8$     4)  $\pi^2/8 - 1$

Explanation:

$$\frac{1}{2n^2}$$

$$(2n+1)^2 \neq$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + 1$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} = \frac{1}{n^2} - 1$$

$$= \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} - 1$$

$$= \frac{3\pi^2}{24} - 1$$

$$= \frac{\pi^2}{8} - 1$$

61. Consider the function

$$f(x) = \cos(|x-5|) + \sin(|x-3|) + |x+10|^3 - (|x|+4)^2$$

At which of the following points is  $f$  not differentiable?

- 1)  $x = 5$     2)  $x = 3$     3)  $x = -10$     4)  $x = 0$

Explanation:

$$\cos(|x-5|) = \cos(x-5)$$

$$\sin(|x-3|) = \begin{cases} \sin(x-3) & \text{if } x-3 \geq 0 \\ -\sin(x-3) & \text{if } x-3 < 0 \end{cases}$$

$\therefore x=3$   $\sin|x-3|$  is not differentiable.

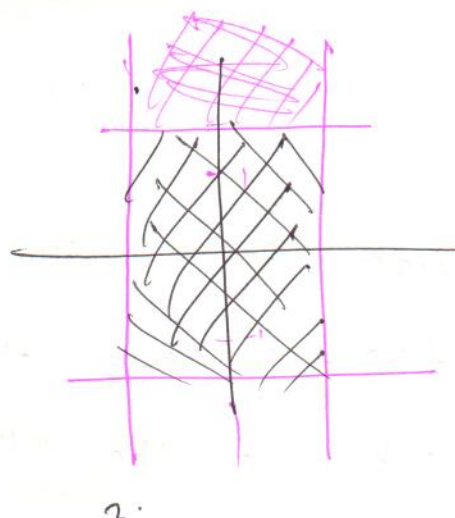
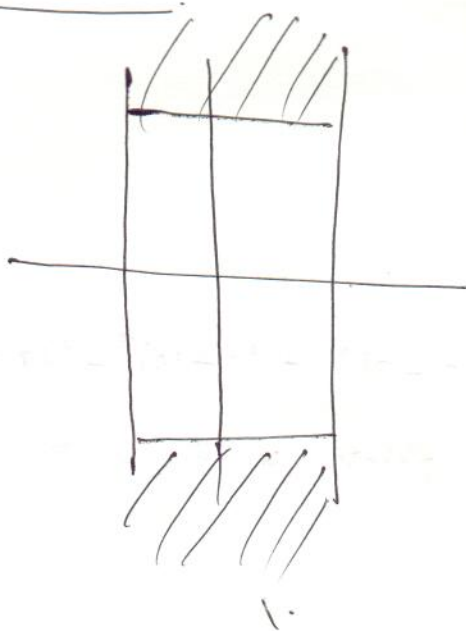
$$(|x|+4)^2 = |x|^2 + 8|x| + 16.$$

$|x|$  is not differentiable at  $x=0$

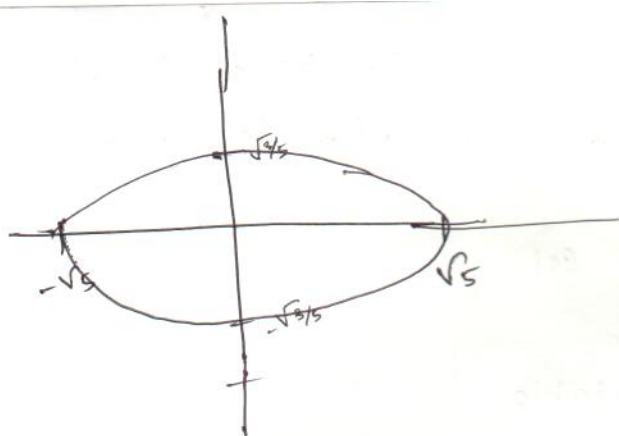
62. Which of the following subsets of  $\mathbb{R}^2$  are compact?

1.  $\{(x, y) : |x| \leq 1, |y| \geq 2\}$     2.  $\{(x, y) : |x| \leq 1, |y|^2 \leq 2\}$   
3.  $\{(x, y) : x^2 + 3y^2 \leq 5\}$     3.  $\{(x, y) : x^2 \leq y^2 + 5\}$

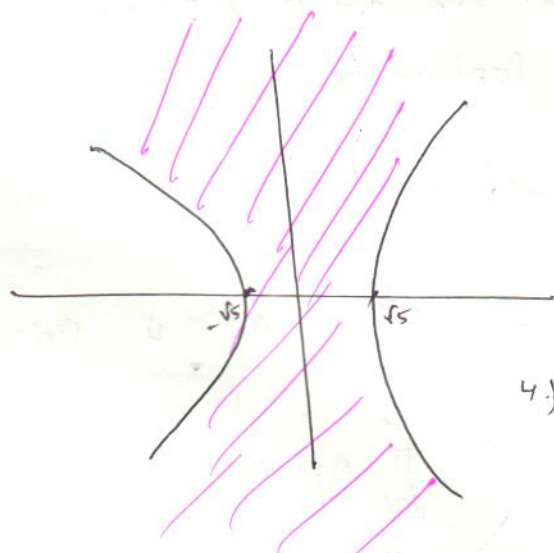
Explanation.



3)



$$\frac{x^2}{5} + \frac{y^2}{5} \leq 1$$



4.)

$$\frac{x^2}{5} - \frac{y^2}{5} \leq 1$$

63. Which of the following are metrics on

$C = \{f: [0, 1] \rightarrow \mathbb{R} \text{ is a continuous function}\}$

1.  $d(f, g) = \sup \{ |f(x) - g(x)| : x \in [0, 1] \}$

2.  $d(f, g) = \inf \{ |f(x) - g(x)| : x \in [0, 1] \}$

3.  $d(f, g) = \int_0^1 |f(x) - g(x)| dx$

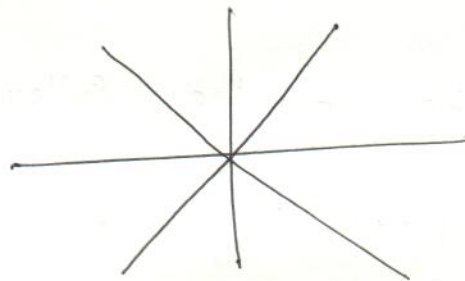
4.  $d(f, g) = \sup \{ |f(x) - g(x)| : x \in [0, 1] \} + \int_0^1 |f(x) - g(x)| dx$

2)

$$f(x) = x \quad g(x) = -x$$

$$d(f, g) = 0$$

$$\text{but } f \neq g$$



64. For each  $j = 1, 2, 3, \dots$  let  $A_j$  be a finite set containing at least two distinct elements. Then

1.  $\bigcup_{j=1}^{\infty} A_j$  is a countable set

2.  $\bigcup_{n=1}^{\infty} \prod_{j=1}^n A_j$  is uncountable

3.  $\prod_{j=1}^{\infty} A_j$  is uncountable

4.  $\bigcup_{n=1}^{\infty} A_j$  is uncountable.

Explanation:

$\bigcup_{j=1}^{\infty} A_j$  is a countable union of finite set

$\therefore \bigcup_{j=1}^{\infty} A_j$  is countable.

(2)  $\bigcup_{n=1}^{\infty} \prod_{j=1}^n A_j$  is a countable union of countable sets

$\therefore \bigcup_{n=1}^{\infty} \prod_{j=1}^n A_j$  is countable.

$\prod_{j=1}^{\infty} A_j$  is uncountable.

65. Which of the following is/are correct?

1.  $(1 + 1/n)^{n+1} \rightarrow e$  as  $n \rightarrow \infty$

2.  $(1 + 1/n)^n \rightarrow e$  as  $n \rightarrow \infty$

3.  $(1 + 1/n)^{n^2} \rightarrow e$  as  $n \rightarrow \infty$

4.  $(1 + 1/n^2)^n \rightarrow e$  as  $n \rightarrow \infty$ .

Explanation:

$$1. \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^{n+1} = \lim_{n \rightarrow \infty} \left[ (1 + \frac{1}{n})^n (1 + \frac{1}{n}) \right]$$

$$= e \cdot 1$$

$$2. \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n+1} \right)^{n+1-1}$$

$$= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n+1} \right)^{n+1} \cdot \left( 1 + \frac{1}{n+1} \right)^{-1}$$

$$= e \cdot 1$$

$$3. \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^{n^2} \geq \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e$$

$$4. \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n^2} \right)^{n^2} = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n^2} \right)^{n^2} \leq \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n^2} \right)^n = e$$

66. Which of the following is/are true?

$$1. \log \frac{x+y}{2} \leq \frac{\log x + \log y}{2} \quad \text{for all } x, y > 0$$

$$2. e^{\frac{x+y}{2}} \leq \frac{e^x + e^y}{2} \quad \text{for all } x, y > 0$$

$$3. \sin \frac{x+y}{2} \leq \frac{\sin x + \sin y}{2} \quad \text{for all } x, y > 0$$

$$4. \frac{(x+y)^k}{2^k} \leq \max \{x^k, y^k\} \quad \text{for all } x, y > 0 \text{ and all } k \geq 1$$

Explanation

$$\sqrt{xy} \leq \frac{x+y}{2}$$

$$(G \cdot M \leq A \cdot M)$$

$$\log \sqrt{xy} \leq \log \frac{x+y}{2}$$

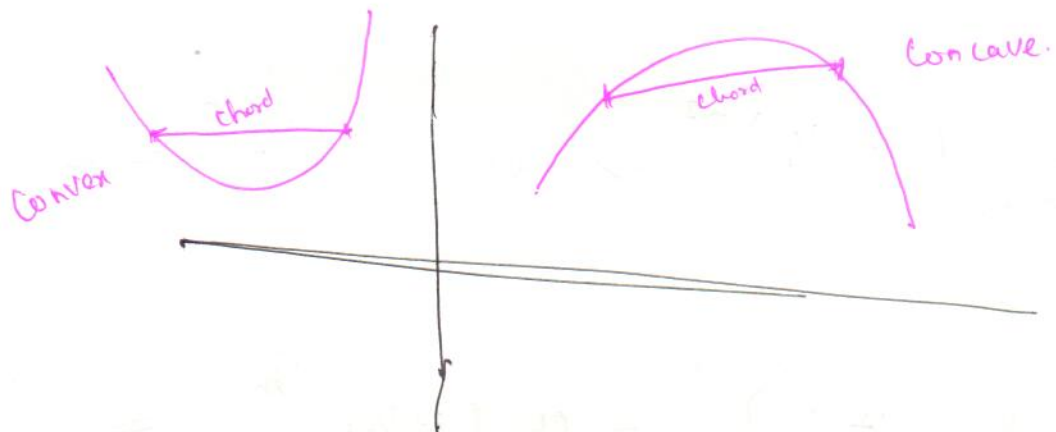
$$\frac{1}{2} \log xy \leq \log \frac{x+y}{2}$$

$$\frac{1}{2} (\log x + \log y) \leq \log \frac{x+y}{2}$$

Convex function:

Convex function  $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

$\forall x, y \in D(f)$  and  $t \in [0, 1]$ .



$\log x$  is not convex.

$e^x$  is convex.

$\sin x$  is not convex.

4)  $K \neq 1$

$$\frac{x+y}{2} \leq \max\{x, y\}$$

68. For  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  in  $\mathbb{R}^n$

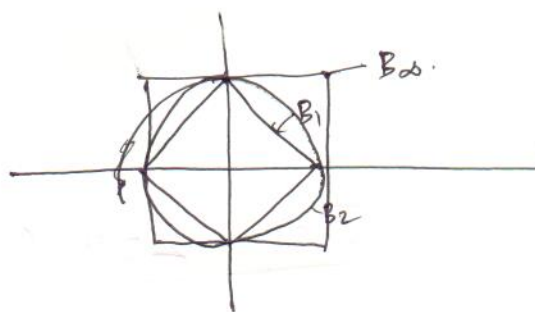
let  $d_p(x, y) = \left( \sum_{j=1}^n |x_j - y_j|^p \right)^{1/p}$  for  $1 \leq p < \infty$  and

$$d_\infty(x, y) = \max \{ |x_i - y_i| : i = 1, 2, \dots, n \}$$

Let  $B_p = \{ x \in \mathbb{R}^n : d_p(x, 0) < 1 \}$ ,  $1 \leq p \leq \infty$

Which of the following are correct?

1.  $B_1$  is open in the  $d_\infty$  metric
2.  $B_2$  is open in the  $d_\infty$  metric
3.  $B_1$  is not open in the  $d_2$ -metric
4.  $B_2$  is not open in the  $d_2$ -metric.



70. The space  $C[0, 1]$  of continuous functions on  $[0, 1]$  is complete with respect to the norm

1.  $\|f\|_\infty = \sup \{ |f(x)| : x \in [0, 1] \}$

2.  $\|f\|_1 = \int_0^1 |f(x)| dx$

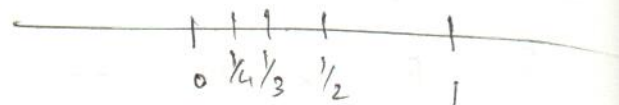
3.  $\|f\|_\infty^{(0,1)} = \|f\|_\infty + |f(0)| + |f(1)|$

4.  $\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 dx}$

72. Let  $X = \{x \in [0,1] : x \neq 1/n, n \in \mathbb{N}\}$  be given the subspace topology. Then

1.  $X$  is connected but not compact
- ☒ 2.  $X$  is neither compact nor connected
3.  $X$  is compact and connected
4.  $X$  is compact but not connected.

$$X = \bigcup_{n=1}^{\infty} \left( \frac{1}{n+1}, \frac{1}{n} \right)$$



$X$  is not closed,  $\Rightarrow$  not compact

Dec 2011 (c)

22. Which of the following statements is true?

1.  $\lim_{x \rightarrow \infty} \frac{\log x}{x^{1/2}} = 0$  and  $\lim_{x \rightarrow \infty} \frac{\log x}{x} = \infty$ .
2.  $\lim_{x \rightarrow \infty} \frac{\log x}{x^{1/2}} = \infty$  and  $\lim_{x \rightarrow \infty} \frac{\log x}{x} = 0$
- ☒ 3.  $\lim_{x \rightarrow \infty} \frac{\log x}{x^{1/2}} = 0$  and  $\lim_{x \rightarrow \infty} \frac{\log x}{x} = 0$
4.  $\lim_{x \rightarrow \infty} \frac{\log x}{x^{1/2}} = 0$  but  $\lim_{x \rightarrow \infty} \frac{\log x}{x}$  does not exist.

### Explanation

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\log x}{x^{1/2}} &= \lim_{x \rightarrow \infty} \frac{1/x}{1/2 x^{-1/2}} \\&= \lim_{x \rightarrow \infty} 2x \times \frac{1}{x} \times \frac{\sqrt{x}}{\sqrt{x}} \\&= 0\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\log x}{x} &= \lim_{x \rightarrow \infty} \frac{1/x}{1} \\&= 0\end{aligned}$$

26. Let  $X$  be a connected subset of real numbers. If every element of  $X$  is irrational, then the cardinality of  $X$  is

- 1) infinite    2) countably infinite    3) 2    4) 1

### Explanation:

Since connected subsets of  $\mathbb{R}$  are only intervals.

28. Define  $f: [0, 1] \rightarrow [0, 1]$  by  $f(x) = \frac{2^k - 1}{2^k}$  for

$x \in \left[ \frac{2^{k-1} - 1}{2^{k-1}}, \frac{2^k - 1}{2^k} \right]$ ,  $k \geq 1$ . Then  $f$  is a

Riemann-integrable function such that

1.  $\int_0^1 f(x) dx = 2/3$

$$2. \quad 1/2 < \int_0^1 f(x) dx < 2/3$$

$$3. \quad \int_0^1 f(x) dx = 1$$

$$4. \quad 2/3 < \int_0^1 f(x) dx < 1$$

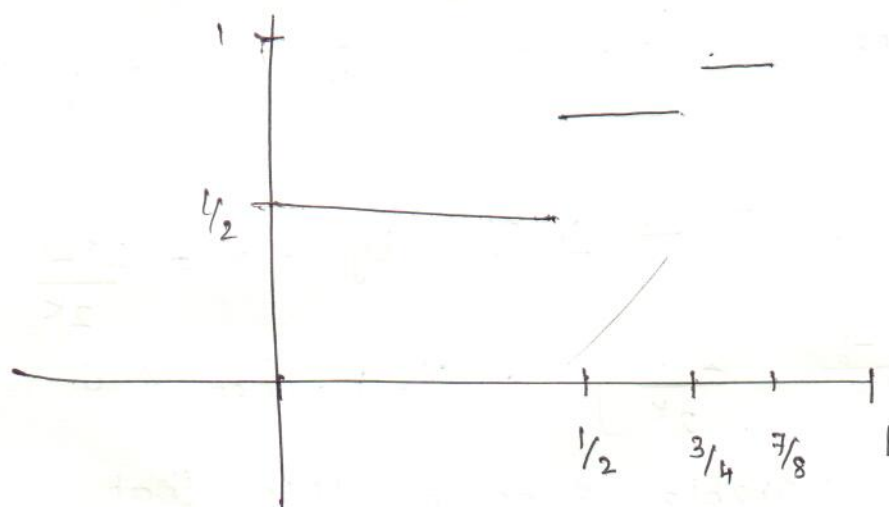
Explanation:

$$K=1 \quad \left[ \frac{2^{1-1}-1}{2^{1-1}}, \frac{2^1-1}{2^1} \right] = [0, 1/2]$$

$$K=2 \quad , \quad [1/2, 3/4]$$

$$K=3 \quad , \quad [3/4, 7/8]$$

$$P = 0 < 1/2 < 3/4 < 7/8 < \dots < 1$$



$$U(f, p) = \sum_{i=1}^n M_i |x_{i+1} - x_i| \quad M_i = \max_{x_i < x_K < x_{i+1}} f(x_K)$$

$$= \sum_{K=1}^{\infty} \frac{2^{K-1}}{2^K} \left( \frac{2^{K-1}-1}{2^K} - \frac{2^{K-1}-1}{2^{K-1}} \right)$$

$$= \sum_{K=1}^{\infty} \frac{2^{K-1}}{2^{2K}} (2^K - 1 - 2 \cdot 2^{K-1} + 2)$$

$$= \sum_{K=1}^{\infty} \frac{2^{K-1}-1}{2^{2K}} (1)$$

$$= \sum_{K=1}^{\infty} \frac{1}{2^{K+1}}$$

$1 + \frac{1}{2} + \frac{1}{4} + \dots$

$$= \frac{1}{2} \sum_{K=1}^{\infty} \frac{1}{2^K} = \frac{1}{2} \sum_{K=1}^{\infty} \frac{1}{2^{K-1}}$$

$$= \frac{1}{4} \left( \frac{1}{1 - \frac{1}{2}} \right)$$

$$= \frac{1}{4} \times \frac{2}{1}$$

$$= \frac{1}{2}$$

$$= \sum_{K=1}^{\infty} \frac{2^K}{2^{2K}} - \sum_{K=1}^{\infty} \frac{1}{2^{2K}}$$

$$= \sum_{K=1}^{\infty} \frac{1}{2^K} - \sum_{K=1}^{\infty} \frac{1}{2^{2K}}$$

$$\sum_{K=0}^{\infty} \frac{1}{2^K} = \frac{1}{1 - \frac{1}{2}} = 2$$

$\sum_{K=1}^{\infty} \frac{1}{2^K} = 1$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} - \frac{\cancel{\frac{1}{2}}}{2^{\cancel{2}} \cancel{2}^2} \sum_{k=1}^{\infty} \frac{1}{2^{2(k-1)}}$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{1}{2}} - \frac{1}{4} \times \frac{1}{1-\frac{1}{4}}$$

$$= \frac{1}{2} \times 2 - \frac{1}{4} \times \frac{4}{3}$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\int_0^1 f(x) dx = \inf_P U(f, P) = \sup_P L(f, P)$$

$$= \frac{2}{3}$$

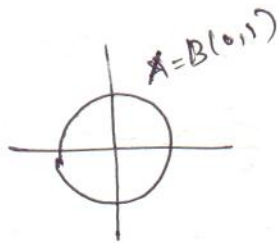
34. Let  $(X, d)$  be a metric space and  $A \subseteq X$ . For  $x \in X$ , define

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

If  $d(x, A) = 0 \quad \forall x \in X$ , then which of the following assertions must be true.

1)  $A$  is compact      2)  $A$  is closed

~~3)~~  $A$  is dense in  $X$     4)  $A = X$ .



$$d(x, A) > 0 \text{ if } x \notin B(0,1)$$

$B(0,1)$  is closed also compact.

6). Which of the following real-valued functions defined on  $\mathbb{R}$  have the property that  $f$  is continuous and  $f \circ f = f$ .

$$1) f(x) = \begin{cases} |x| & \text{if } x \in [-1, 1] \\ x^2 & \text{if } x \in [-1, 1] \end{cases}$$

$$2) f(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ 1 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$3) f(x) = \begin{cases} x & \text{if } x \in [-1, 1] \\ 1 & \text{if } x \geq 1 \\ -1 & \text{if } x \leq -1 \end{cases}$$

$$4) f(x) = \begin{cases} 1 & \text{if } x \in [-23, 27] \\ 22+x & \text{if } x \leq -23 \\ -26+x & \text{if } x \geq 27 \end{cases}$$

Explanation

$$x = 5$$

$$\begin{aligned} 1) f(f(5)) &= f(25) & \text{but } f(5) &= 25 \\ &= 625 \\ &\neq f(5) \end{aligned}$$

4)  $f$  is not continuous at  $x = -23$ .

65. It is given that the series  $\sum_{n=1}^{\infty} a_n$  is convergent but not absolutely convergent and  $\sum_{n=1}^{\infty} a_n = 0$ . Denote by  $S_k$  the partial sum  $\sum_{n=1}^k a_n$ ,  $k=1, 2, \dots$ . Then

1.  $S_k = 0$  for infinitely many  $k$ .

2.  $S_k > 0$  for infinitely many  $k$ , and  $S_k < 0$  for infinitely many  $k$ .

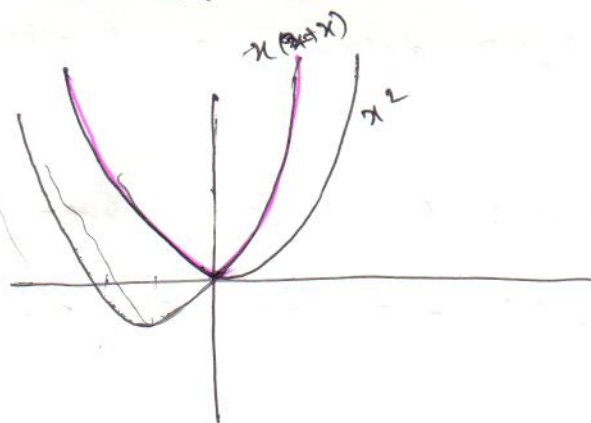
3. It is possible that  $S_k > 0$  for all  $k$ .

4. It is possible that  $S_k > 0$  for all but a finite number of values of  $k$ .

86. Define  $f: \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2x + x^2 & \text{if } x \geq 0 \end{cases}$ .

Then which of the following statements are correct?

- 1)  $f''(x) = 2$  for all  $x \in \mathbb{R}$     2)  $f''(0)$  does not exist  
 3)  $f'(x)$  exists for each  $x \neq 0$     4)  $f'(0)$  does not exist



$$y = x(2+x)$$

Explanation:

$f$

$$f'(0-) = \lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{h^2 - 0}{h}$$

$$= 0$$

$$f'(0+) = \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{2h + h^2 - 0}{h}$$

$$= 2$$

$$\therefore f'(0-) \neq f'(0+)$$

$f'(0)$  does not exist.

$\Rightarrow f''(0)$  also does not exist

89. Consider

June 2011

24. Let  $f_n(x) = x^n/n$ , for  $x \in [0,1]$ . Then

1)  $\lim_{n \rightarrow \infty} f_n(x)$  exists for all  $x \in [0,1]$ .

2)  $\lim_{n \rightarrow \infty} f_n(x)$  defines a continuous function on  $[0,1]$ .

3)  $f_n$  converges uniformly on  $[0,1]$ .

4)  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in [0,1]$ .

Explanation:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

25. Let  $A = \{x^2 : 0 < x < 1\}$  and  $B = \{x^3 : 1 < x < 2\}$ .

Which of the following statements is true?

1. There is a one to one, onto function from A to B

2. There is no one to one onto function from A to B taking rationals to rationals.

3. There is no one-one function from A to B which is onto.

4. There is no onto function from A to B which is one to one.

Explanation:

$$A = (0, 1) \quad B = (0, 8)$$

$A$  and  $B$  are homeomorphic.

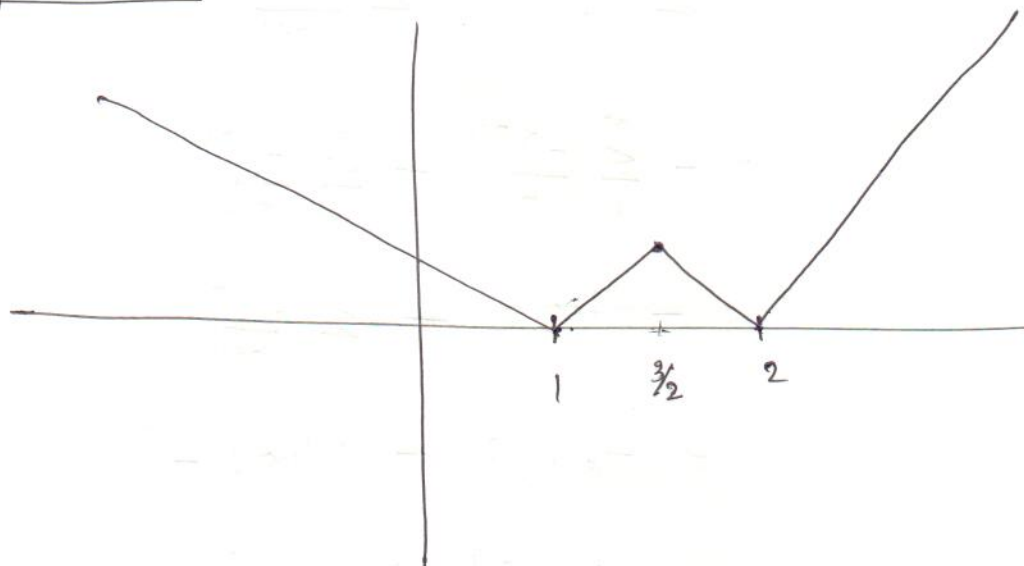
$\therefore$

30. Let  $I = \{1\} \cup \{2\} \subseteq \mathbb{R}$ . For  $x \in \mathbb{R}$ , let

$$\varphi(x) = \text{dist}(x, I) = \inf \{ |x - y| : y \in I \}. \text{ Then}$$

1.  $\varphi$  is discontinuous somewhere on  $\mathbb{R}$ .
2.  $\varphi$  is continuous on  $\mathbb{R}$  but not differentiable only at  $x=1$ .
3.  $\varphi$  is continuous on  $\mathbb{R}$  but not differentiable only at  $x=1$  and  $x=2$ .
4.  $\varphi$  is continuous on  $\mathbb{R}$  but not differentiable only at  $x=1, \frac{3}{2}$  and  $2$ .

Explanation



31. The set  $\{ \frac{1}{n} \sin \frac{1}{n} : n \in \mathbb{N} \}$  has

~~1.~~ one limit point and it is 0

2. one limit point and it is 1

3. one limit point and it is -1

4. three limit points and these are -1, 0 and 1

32. Using the fact that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$  equals.

$$1 - \frac{1}{2} + \frac{1}{3}$$

~~1)~~  $1 - 2 \log 2$

2)  $1 + \log 2$

3)  $(\log 2)^2$

4)  $-(\log 2)^2$

Explanation:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^n \left[ \frac{1}{n} - \frac{1}{n+1} \right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

$$= - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= - \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= - \log 2 - (\log 2 - 1)$$

$$= 1 - 2 \log 2.$$

61. Consider the function  $f(x) = |\cos x| + |\sin(2-x)|$

At which of the following points is  $f$  not differentiable?

1.  $\{ (2n+1)\pi/2 : n \in \mathbb{Z} \}$

2.  $\{ n\pi : n \in \mathbb{Z} \}$

3.  $\{ n\pi + 2 : n \in \mathbb{Z} \}$

4.  $\{ n\pi/2 : n \in \mathbb{Z} \}$

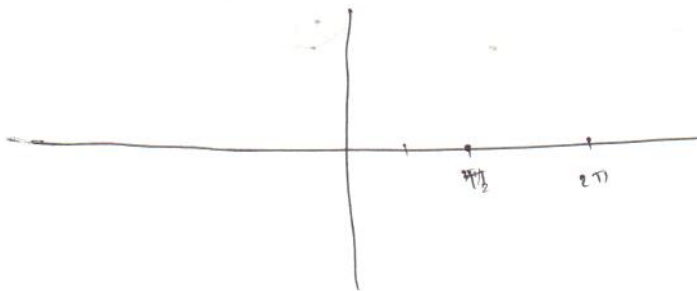
Explanation:

$\cos x = 0 \Rightarrow x = (2n+1)\pi/2$

$\sin(2-x) = 0 \Rightarrow 2-x = k\pi, k \in \mathbb{Z}$

$x = 2 - k\pi$

$= n\pi + 2, n \in \mathbb{Z}$



62. Which of the following subsets of  $\mathbb{R}^2$  are convex?

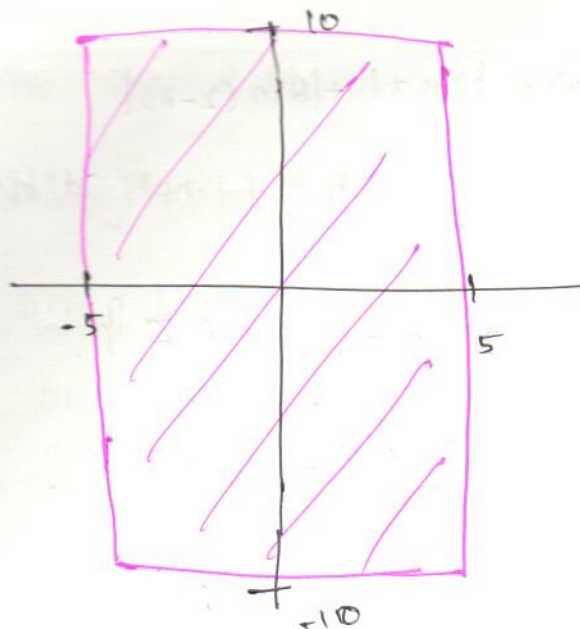
1.  $\{ (x, y) : |x| \leq 5, |y| \leq 10 \}$

2.  $\{ (x, y) : x^2 + y^2 = 1 \}$

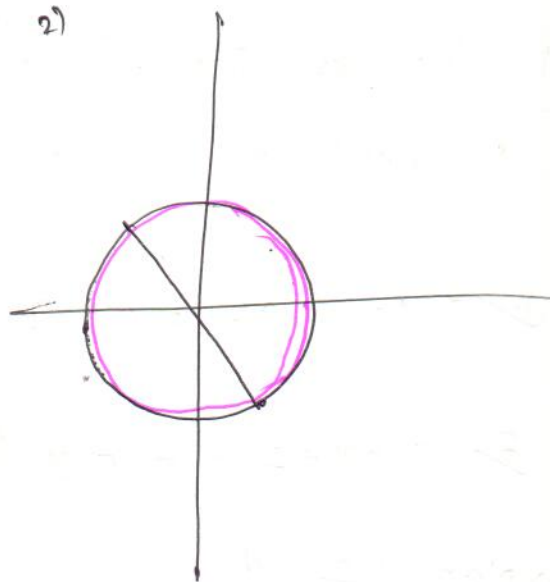
3.  $\{ (x, y) : y \geq x^2 \}$

4.  $\{ (x, y) : y \leq x^2 \}$

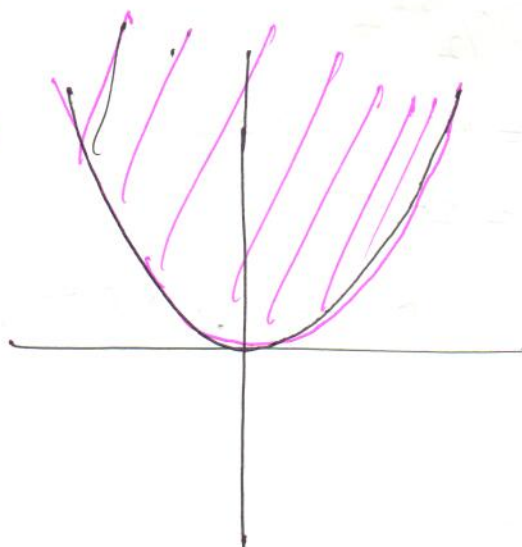
1)



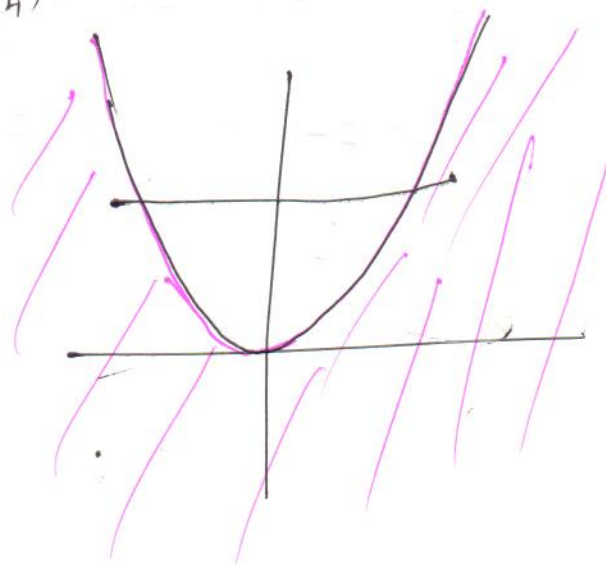
2)



3)



4)



63. Which of the following  $d$  are metrics on  $\mathbb{R}$ ?

1.  $d(x, y) = \min(x, y)$

2.  $d(x, y) = |x - y|$

3.  $d(x, y) = |x^2 - y^2|$

4.  $d(x, y) = |x^3 - y^3|$

Explanation:

1)  $d(1, 0) = 0$  But  $1 \neq 0$

2)  $d(x, y) = |x - y|$  is the usual metric

3) for  $x=2$  ,  $y=-2$   $x \neq y$ .

$$d(x, y) = |x^2 - y^2|$$
$$= 0$$

4)  $d(x, y) = 0$  (???)

$$|x^3 - y^3| = 0$$

$$|(x-y)(x^2+xy+y^2)| = 0$$

$$x-y=0 \text{ or } x^2+xy+y^2=0$$

But  $x^2+xy+y^2 \neq 0$  , since  $x^2+y^2 = -xy$  (?)

64. Let  $X$  denote the two-point set  $\{0, 1\}$  and write  $x_j = \{0, 1\}$  for every  $j=1, 2, 3, \dots$ . Let  $Y = \prod_{j=1}^{\infty} x_j$  which of the following is true?

1.  $Y$  is a countable set

2.  $\text{Card } Y = \text{Card } [0, 1]$

3.  $\bigcup_{n=1}^{\infty} \left( \prod_{j=1}^n x_j \right)$  is uncountable

4.  $Y$  is uncountable.

Explanation:

$Y$  is uncountable.

$$\therefore |Y| = |[0, 1]|$$

$\prod_{j=1}^n x_j$  is countable for each  $n$ ,  $\therefore \bigcup_{n=1}^{\infty} \left( \prod_{j=1}^n x_j \right)$  is countable

65. Which of the following is/are correct?

1.  $n \log \left(1 + \frac{1}{n+1}\right) \rightarrow 1$  as  $n \rightarrow \infty$

2.  $(n+1) \log \left(1 + \frac{1}{n}\right) \rightarrow 1$  as  $n \rightarrow \infty$

3.  $n^2 \log \left(1 + \frac{1}{n}\right) \rightarrow 1$  as  $n \rightarrow \infty$

4.  $n \log \left(1 + \frac{1}{n^2}\right) \rightarrow 1$  as  $n \rightarrow \infty$

Explanation:

$$\begin{aligned} 1) \lim_{n \rightarrow \infty} n \log \left(1 + \frac{1}{n+1}\right) &= \lim_{n \rightarrow \infty} \log \left(1 + \frac{1}{n+1}\right)^{n+1-1} \\ &= \lim_{n \rightarrow \infty} \left[ \log \left(1 + \frac{1}{n+1}\right)^{n+1} - \log \left(1 + \frac{1}{n+1}\right) \right] \\ &= \log e - \log 1 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} 2) \lim_{n \rightarrow \infty} (n+1) \log \left(1 + \frac{1}{n}\right) &= \lim_{n \rightarrow \infty} \log \left(1 + \frac{1}{n}\right)^{n+1} \\ &= \lim_{n \rightarrow \infty} \left[ \log \left(1 + \frac{1}{n}\right)^n - \log \left(1 + \frac{1}{n}\right) \right] \\ &= \log e - \log 1 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$3) \lim_{n \rightarrow \infty} n \log(1 + \frac{1}{n}) = \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n})^{n^2}$$

$$\geq \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n})^n.$$

$$= \log e = 1$$

$$4) \lim_{n \rightarrow \infty} n \log(1 + \frac{1}{n^2}) = \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n^2})^n$$

$$\leq \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n^2})^{n^2}$$

$$\leq 1$$

66. If  $(x_n)$  and  $(y_n)$  are sequences of real numbers which of the following is/are true?

1.  $\limsup_n (x_n + y_n) \leq \limsup_n x_n + \limsup_n y_n$

2.  $\limsup_n (x_n + y_n) \geq \limsup_n x_n + \limsup_n y_n$

3.  $\liminf_n (x_n + y_n) \leq \liminf_n x_n + \liminf_n y_n$

4.  $\liminf_n (x_n + y_n) \geq \liminf_n x_n + \liminf_n y_n$

68. For  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ , and  $p \geq 1$ , define

$$\|x\|_p = \left( \sum_{j=1}^d |x_j|^p \right)^{1/p} \text{ and}$$

$\|x\|_\infty = \max \{ |x_j| : j=1, 2, \dots, d \}$ . Which of the following inequalities hold for all  $x \in \mathbb{R}^d$ ?

1.  $\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty$

2.  $\|x\|_1 \leq d \|x\|_\infty$

3.  $\|x\|_1 \leq \sqrt{d} \|x\|_\infty$

4.  $\|x\|_1 \leq \sqrt{d} \|x\|_2$ .

Explanation:

All norms are equivalent in a finite dimensional space.  $\therefore \|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty$ .

$$4. \|x\|_1 = \sum_{j=1}^d |x_j| = \sum_{j=1}^d |x_j| \|e_j\|$$

$(x_1, x_2)$

$= x_1 e_1 + x_2 e_2$

$$\leq \left( \sum_{j=1}^d |x_j|^2 \right)^{1/2} \left( \sum_{j=1}^d \|e_j\|^2 \right)^{1/2}$$

(By Cauchy's inequality)

$$= \|x\|_2 d^{1/2}$$

$$\therefore \|x\|_1 \leq \sqrt{d} \|x\|_2.$$

$$2. \|x\|_1 = \sum_{j=1}^d |x_j|$$

$$= |x_1| + |x_2| + \dots + |x_d|$$

$$\leq \max_i |x_i| + \max_i |x_i| + \dots + \max_i |x_i|$$

$$= d \max_i |x_i|$$

$$= d \|x\|_\infty$$

$$\therefore \|x\|_1 \leq d \|x\|_\infty.$$

70. Which of the following sets are dense in  $\mathbb{R}^2$  with respect to the usual topology.

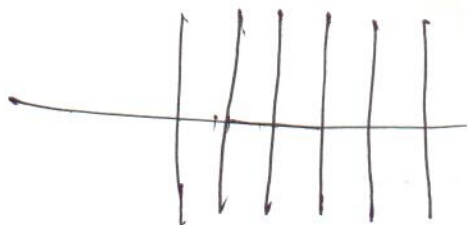
1.  $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{N}\}.$

2.  $\{(x, y) \in \mathbb{R}^2 : x+y \text{ is a rational number}\}.$

3.  $\{(x, y) \in \mathbb{R}^2 : x + y^2 = 5\}$

4.  $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}.$

1.



2.

$$A = \{(x, y) \in \mathbb{R}^2 : x+y \text{ is a rational number}\}$$

$$\cong \mathbb{Q} \times \mathbb{Q}.$$

$$\therefore \overline{\mathbb{Q} \times \mathbb{Q}} = \mathbb{R}^2$$

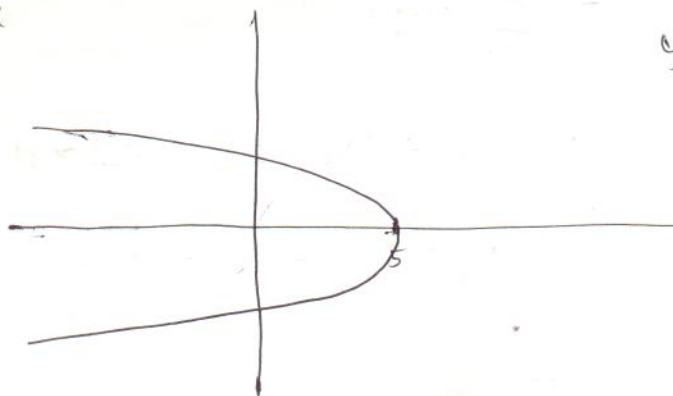
$$\therefore \overline{\{(x, y) \in \mathbb{R}^2 : x+y \text{ is a rational number}\}} = \mathbb{R}^2.$$

$\therefore A$  is dense in  $\mathbb{R}^2$

3.  $y^2 = 5 - x$

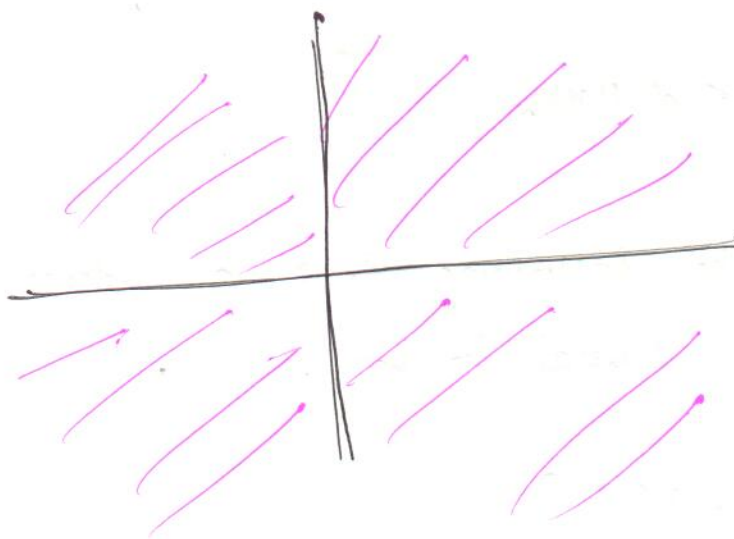
$y^2 = x$

$y^2 = -x$



$\{(x, y) \in \mathbb{R}^2 : x + y^2 = 5\}$  is not dense.

4.  $\mathbb{R}$



$\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$  is dense in  $\mathbb{R}^2$

71. Let  $F = \{f: \mathbb{R} \rightarrow \mathbb{R} : |f(x) - f(y)| \leq K|x - y|^\alpha \text{ for all } x, y \in \mathbb{R} \text{ for some } \alpha > 0 \text{ and } K > 0\}$ . Which of the following is/are true?

1. every  $f \in F$  is continuous
2. every  $f \in F$  is uniformly continuous
3. every differentiable function  $f$  is in  $F$ .
4. Every  $f \in F$  is differentiable.

Suppose  $\alpha \in (0, 1)$

$$\alpha = 1 - \beta, \quad \beta \in (0, 1)$$

$$|f(x) - f(y)| \leq K |x - y|^{1-\beta}$$

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \frac{K}{|x - y|^\beta}, \quad \beta \in (0, 1)$$

$$\lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} \leq K \lim_{x \rightarrow y} \frac{1}{|x - y|^\beta} \rightarrow \infty.$$

$\therefore$  (4) does not exist

## Multivariable calculus

Dec-2014 (B)

22. Let  $\Omega \subseteq \mathbb{R}^n$  be an open set and  $f: \Omega \rightarrow \mathbb{R}$  be a differentiable function such that  $(Df)(x) = 0$  for all  $x \in \Omega$ . Then which of the following is true?
1.  $f$  must be a constant function
  2.  $f$  must be constant on connected components of  $\Omega$
  3.  $f(x) = 0$  or  $1$  for  $x \in \Omega$
  4. The range of the function  $f$  is a subset of  $\mathbb{Z}$ .

### Explanation.

Result: Let  $\Omega \subseteq \mathbb{R}^n$  be an open set and  $f: \Omega \rightarrow \mathbb{R}$  be a differentiable function s.t.  $(Df)(x) = 0$   $\forall x \in \Omega$ . If  $\Omega$  is connected, then  $f$  is constant on  $\Omega$ .

June - 2014 (B)

68. Let  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{C}$  be the map  $\varphi(x, y) = z$  where  $z = x + iy$ . Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be the function  $f(z) = z^2$  and  $F = \varphi^{-1} \circ f \circ \varphi$ . Which of the following are correct?

1. The linear transformation  $T(x, y) = 2 \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$  represents the derivative of  $F$  at  $(x, y)$ .

2. The linear transformation  $T(x, y) = 2 \begin{pmatrix} x & y \\ y & x \end{pmatrix}$  represents the derivative of  $F$  at  $(x, y)$

3. The linear transformation  $T(z) = 2z$  represents the derivative of  $f$  at  $z \in \mathbb{C}$ .

4. The linear transformation  $T(z) = 2z$  represents the derivative of  $f$  only at 0.

Explanation:

$$\varphi: \mathbb{R}^2 \rightarrow \mathbb{C} \quad f: \mathbb{C} \rightarrow \mathbb{C}$$

$$\begin{aligned} F = \varphi^{-1} \circ f \circ \varphi: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \quad \text{By } F(x, y) = (\varphi^{-1} \circ f \circ \varphi)(x, y) \\ &= \varphi^{-1}(f(\varphi(x, y))) \\ &= \varphi^{-1}(f(z)) \\ &= \varphi^{-1}(z^2) \\ &= \varphi^{-1}(x^2 - y^2 + i2xy) \\ &= (x^2 - y^2, 2xy) \end{aligned}$$

$$f_1 = x^2 - y^2 \quad f_2 = 2xy$$

$$\frac{\partial f_1}{\partial x} = 2x \quad \frac{\partial f_2}{\partial x} = 2y$$

$$\frac{\partial f_1}{\partial y} = -2y \quad \frac{\partial f_2}{\partial y} = 2x$$

$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 2 \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

75 Let  $A = \{(x, y) \in \mathbb{R}^2 : x + y \neq -1\}$ . Define  $f: A \rightarrow \mathbb{R}^2$  by

$$f(x, y) = \left( \frac{x}{1+x+y}, \frac{y}{1+x+y} \right). \text{ Then }$$

1. The Jacobian matrix of  $f$  does not vanish on  $A$ .

2.  $f$  is infinitely differentiable on  $A$ .

3.  $f$  is injective on  $A$ .

4.  $f(A) = \mathbb{R}^2$ .

Explanation:

$$f_1 = \frac{x}{1+x+y} \quad f_2 = \frac{y}{1+x+y}$$

$$\frac{\partial f_1}{\partial x} = \frac{(1+x+y) \cdot 1 - x \cdot 1}{(1+x+y)^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{(1+x+y) \cdot 0 - y \cdot 1}{(1+x+y)^2}$$

$$= \frac{1+y}{(1+x+y)^2}$$

$$= \frac{-y}{(1+x+y)^2}$$

$$\frac{\partial f_1}{\partial y} = \frac{(1+x+y) \cdot 0 - x \cdot 1}{(1+x+y)^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{(1+x+y) \cdot 1 - y \cdot 1}{(1+x+y)^2}$$

$$= \frac{-x}{(1+x+y)^2}$$

$$= \frac{1+x}{(1+x+y)^2}$$

$$J = \begin{pmatrix} \frac{1+y}{(1+x+y)^2} & \frac{-y}{(1+x+y)^2} \\ \frac{-x}{(1+x+y)^2} & \frac{1+x}{(1+x+y)^2} \end{pmatrix}$$

$$J = \frac{1}{(1+x+y)^2} \begin{pmatrix} 1+y & -y \\ -x & 1+x \end{pmatrix}$$

Suppose  $J$  vanishes on  $A$

$$1+y = 0 \quad -y \geq 0$$

$$-x \geq 0 \quad 1+x = 0$$

This have no soln.

$\therefore J$  does not vanish on  $A$ .

$$f(x_1, y_1) = f(x_2, y_2) \quad \frac{1}{(1+x+y)^2} \begin{pmatrix} 1+x+y+yx \\ (1+y)(1+x) - xy \end{pmatrix}$$

$$\frac{x_1}{1+x_1+y_1} = \frac{y_2}{1+x_2+y_2}$$

$$x_1(1+x_2+y_2) = y_2(1+x_1+y_1)$$

$$x_1 + x_1x_2 + x_1y_2 = y_2 + y_2x_1 + y_2y_1$$

$$\left( \frac{x_1}{1+x_1+y_1}, \frac{y_1}{1+x_1+y_1} \right) = \left( \frac{x_2}{1+x_2+y_2}, \frac{y_2}{1+x_2+y_2} \right)$$

$$\frac{x_1}{1+x_1+y_1} = \frac{x_2}{1+x_2+y_2} \quad \text{and} \quad \frac{y_1}{1+x_1+y_1} = \frac{y_2}{1+x_2+y_2}$$

$$x_1 + x_1x_2 + x_1y_2 = x_2 + x_1x_2 + x_2y_1 \quad \& \quad y_1 + y_1x_2 + y_1y_2 = y_2 + x_1y_2 + y_2x_2$$

$$x_1 - x_2 = x_2y_1 - x_1y_2$$

$$\& \quad y_1 - y_2 = x_1y_2 - y_1x_2$$

$$= -(x_2y_1 - x_1y_2)$$

$$y_1 - y_2 = -(x_1 - x_2)$$

$$x_1 + y_1 = x_2 + y_2$$

Since  $f$  is differentiable and  $f'$  is a linear op

Also linear operator is differentiable  
infinitely.

$\therefore f$  is infinitely differentiable.

7b. Define  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $f(x, y) = (x + 2y + y^2 + |xy|, 2x + y + x^2 + |xy|)$

for  $(x, y) \in \mathbb{R}^2$ . Then

1.  $f$  is discontinuous at  $(0, 0)$ .

2.  $f$  is continuous at  $(0, 0)$  but not differentiable at  $(0, 0)$

3.  $f$  is differentiable at  $(0, 0)$  and the derivative

$Df(0, 0)$  is invertible.

4.  $f$  is differentiable at  $(0, 0)$

Explanation:

$$f_1(x, y) = \begin{cases} x + 2y + y^2 + xy & , xy \geq 0 \\ x + 2y + y^2 - xy & , xy < 0 \end{cases}$$

$$f_2(x, y) = \begin{cases} 2x + y + x^2 + xy & , xy \geq 0 \\ 2x + y + x^2 - xy & , xy < 0 \end{cases}$$

$f = (f_1, f_2)$  is continuous at  $(0, 0)$

$$\frac{\partial f_1}{\partial x} = \begin{cases} 1 + y & , xy \geq 0 \\ 1 - y & , xy < 0 \end{cases}$$

$$\frac{\partial f_1}{\partial y} = \begin{cases} 2 + 2y + x & , xy \geq 0 \\ 2 + 2y - x & , xy < 0 \end{cases}$$

$$\frac{\partial f_2}{\partial x} = \begin{cases} 2 + 2x + y & , xy \geq 0 \\ 2 + 2x - y & , xy < 0 \end{cases}$$

$$\frac{\partial f_2}{\partial x_j} = \begin{cases} 1+x & , xy \geq 0 \\ 1-x & , xy < 0 \end{cases}$$

Result:

All the partial derivatives <sup>exist and.</sup> are continuous at  $a$ ,  
then  $f$  is differentiable at  $a$ .

By result,  $\frac{\partial f}{\partial x_i}$  is continuous at  $(0,0)$ .

By result,  $f$  is differentiable at  $(0,0)$ .

$$Df(x,y) = J = \begin{bmatrix} 1+y & 2+2y+x \\ 2+2x+y & 1+x \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = J|_{(0,0)}$$

$$\det J|_{(0,0)} = 1 - 4 = -3 \neq 0.$$

$\therefore Df(0,0)$  is invertible.

June 2013 (A)

105. Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be the map  $f(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ , where  $a = (a_1, a_2, \dots, a_n)$  is a fixed non-zero vector. Let  $Df(0)$  denote the derivative of  $f$  at 0. Which of the following are correct?

1.  $(Df)(0)$  is a linear map from  $\mathbb{R}^n \rightarrow \mathbb{R}$

2.  $[(Df)(0)](a) = \|a\|^2$

3.  $[(Df)(0)](a) = 0$

4.  $[(Df)(0)](b) = a_1 b_1 + \dots + a_n b_n$  for  $b = (b_1, b_2, \dots, b_n)$

Explanation:

$$(Df)(0) = f$$

$$[(Df)(0)](a) = f(a) = a_1 a_1 + a_2 a_2 + \dots + a_n a_n = \|a\|^2$$

$$\text{where } a = (a_1, a_2, \dots, a_n)$$

$$[(Df)(0)](b) = a_1 b_1 + \dots + a_n b_n \text{ for } b = (b_1, b_2, \dots, b_n)$$

106. Let  $F_1, F_2: \mathbb{R}^2 \rightarrow \mathbb{R}$  be the functions  $F_1(x_1, x_2) = \frac{-x_2}{x_1^2 + x_2^2}$  and  $F_2 = \frac{x_1}{x_1^2 + x_2^2}$ . Which of the following are correct?

1.  $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$

2. There exists a function  $f: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$  s.t.  $\frac{\partial f}{\partial x_1} = F_1$  and

and  $\frac{\partial f}{\partial x_2} = F_2$

3. There exists no function  $f: \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$  such that

$$\frac{\partial f}{\partial x_1} = F_1 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = F_2.$$

4. There exists a function  $f: D \rightarrow \mathbb{R}$  where  $D$  is the open disc of radius 1 centred at  $(2,0)$ , which satisfies  $\frac{\partial f}{\partial x_1} = F_1$  and  $\frac{\partial f}{\partial x_2} = F_2$  on  $D$ .

Explanation

$$\frac{\partial F_1}{\partial x_2} = \frac{(x_1^2 + x_2^2)x_1 + x_2 \times 2x_2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

$$\frac{\partial F_2}{\partial x_1} = \frac{(x_1^2 + x_2^2)x_1 - x_2 \times 2x_1}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

$$\therefore \frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$$

109. Let  $z = x+iy$  and  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $f(x, y) = f(z) = z^2$   
 $= (x^2 - y^2, 2xy) \in \mathbb{R}^2$ . Let  $(Df)(a)$  denote the derivative  
of  $f$  at  $a$ . Which of the following are true?

1.  $(Df)(a)h = 2ah$  where  $a = a_1 + ia_2$  and  $h = h_1 + ih_2$

2.  $(Df)(a) = 2 \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$   $a = (a_1, a_2) \in \mathbb{R}^2$ .

3.  $f$  is one to one on  $\mathbb{R}^2$

4. For any  $a \in \mathbb{R}^2 \setminus \{(0,0)\}$ ,  $f$  is 1-1 on some neighbourhood of  $a$ .

Explanation:

$$(Df)(a) = f'(a) = 2a$$

$$(Df)(a)h = 2ah$$

$$f_1 = x^2 - y^2 \quad f_2 = 2xy$$

$$\frac{\partial f_1}{\partial x} = 2x \quad \frac{\partial f_2}{\partial x} = 2y$$

$$\frac{\partial f_1}{\partial y} = -2y \quad \frac{\partial f_2}{\partial y} = 2x$$

$$a = (a_1, a_2)$$

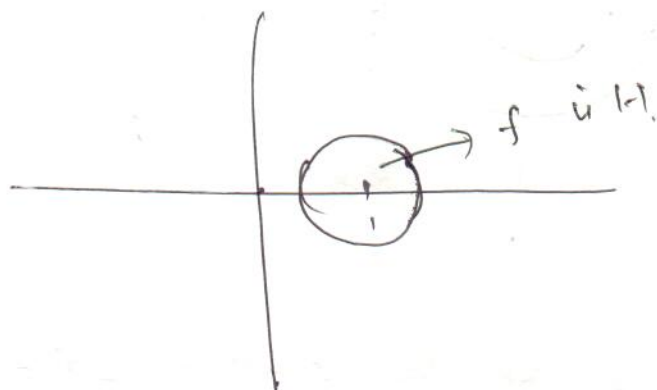
$$(Df)(a) = \begin{pmatrix} 2a_1 & -2a_2 \\ 2a_2 & 2a_1 \end{pmatrix}$$

$$= 2 \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

$$f(1,0) = f(1) = 1^2 = (1,0)$$

$$f(-1,0) = f(-1) = (-1)^2 = (1,0)$$

$\therefore f$  is not 1-1.



§

Dec 2012 (A)

66. Let  $f: A \cup E \rightarrow \mathbb{R}^2$  be differentiable, where

$$A = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1\} \text{ and}$$

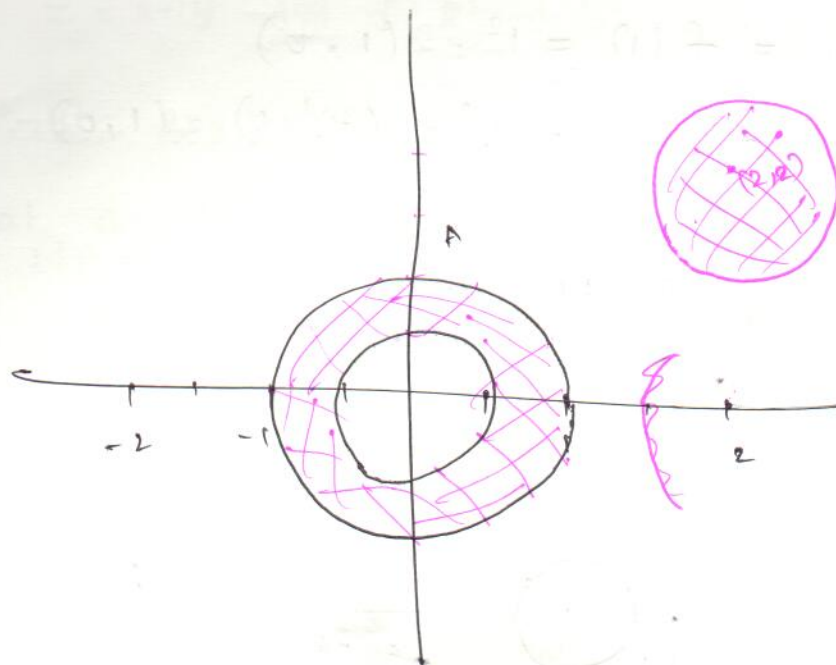
$$E = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-2)^2 < \frac{1}{2}\}. \text{ Which of the following are necessarily correct?}$$

1. If  $(Df)(x,y) = 0$  for all  $(x,y) \in A \cup E$ , then  $f$  is constant

2. If  $(Df)(x,y) = 0$  for all  $(x,y) \in A$ , then  $f$  is constant on  $A$ .

3. If  $(Df)(x,y) = 0$  for all  $(x,y) \in E$ , then  $f$  is constant on  $E$ .

4. If  $(Df)(x,y) = 0$  for all  $(x,y) \in A \cup E$ , then for some  $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ ,  $f(x,y) = (x_0, y_0) \forall (x,y) \in A$  and  $f(x,y) = (x_1, y_1) \forall (x,y) \in E$ .



67. Let  $L: \mathbb{R}^n \rightarrow \mathbb{R}$  be the function  $L(x) = \langle x, y \rangle$ .  $\langle \cdot, \cdot \rangle$  is some inner product on  $\mathbb{R}^n$  and  $y$  is a fixed vector in  $\mathbb{R}^n$ . Further denote by  $DL$ , the derivative of  $L$ . Which of the following are necessarily correct?

1.  $DL(u) = DL(v)$  for all  $u, v \in \mathbb{R}^n$ .

2.  $DL(0, 0, 0, \dots, 0) = L$ .

3.  $DL(x) = \|x\|^2$  for all  $x \in \mathbb{R}^n$ .

4.  $DL(1, 1, \dots, 1) = 0$ .

Explanation:

$$DL = L.$$

$$\therefore DL(x) = L(x) = \langle x, y \rangle$$

$$\text{Hence } DL(1, 1, \dots, 1) \neq 0.$$

68. Let  $f: [\pi, 2\pi] \rightarrow \mathbb{R}^2$  be the function  $f(t) = (\cos t, \sin t)$ .  
Which of the following are necessarily correct?

1. There exists  $t_0 \in [\pi, 2\pi]$  s.t.  $f'(t_0) = \frac{1}{\pi} (f(2\pi) - f(\pi))$

2. There does not exist any  $t_0 \in [\pi, 2\pi]$  such that  
$$f'(t_0) = \frac{1}{\pi} (f(2\pi) - f(\pi))$$

3. There exists  $t_0 \in [\pi, 2\pi]$  such that  $\|f(2\pi) - f(\pi)\| \leq \pi \|f'(t_0)\|$

4.  $f'(t) = (-\sin t, \cos t)$  for all  $t \in [\pi, 2\pi]$ .

Explanation:

Meanvalue theorem:

Suppose  $f$  is a continuous mapping of  $[a, b]$  into  $\mathbb{R}^K$  and  $f$  is differentiable in  $(a, b)$ .  
Then there exist  $c \in (a, b)$  s.t.

$$\|f(b) - f(a)\| \leq \|f'(c)\| (b-a)$$

$$f'(t) = (-\sin t, \cos t)$$

$$\|f'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\|f(2\pi) - f(\pi)\| = \|(1, 0) - (-1, 0)\|$$

$$= \|(2, 0)\|$$

$$= \sqrt{2^2 + 0} = 2$$

$$\therefore \text{no } t_0 \in [\pi, 2\pi] \text{ s.t. } 1 \neq \frac{1}{\pi} 2$$

June 2012 (A)

69. Consider the map  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$f(x, y) = (7x + x^4, 3x + 4y + y^4)$$

Then

1.  $f$  is discontinuous at  $(0,0)$
2.  $f$  is continuous at  $(0,0)$  and all directional derivatives exist at  $(0,0)$
3.  $f$  is differentiable at  $(0,0)$  but the derivative  $Df(0,0)$  is not invertible.
4.  $f$  is differentiable at  $(0,0)$  and the derivative  $Df(0,0)$  is invertible.

Explanation:

$$\text{D.D, } Df_u(0,0) = \nabla f(0,0) \cdot u$$

$$\nabla f|_{(0,0)} = \begin{bmatrix} 7+4x^3 & 0 \\ 3 & 4+4y^3 \end{bmatrix} \Big|_{(0,0)}$$

$$= \begin{bmatrix} 7 & 0 \\ 3 & 4 \end{bmatrix}$$

$$u = (u_1, u_2)$$

$$Df_u(0,0) = \begin{bmatrix} 7 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$2 \times 2$     $2 \times 1$

$$= [7u_1, 3u_1 + 4u_2]$$

$\therefore Df_u(0,0)$  exist for all direction

$$\det (Df|_{(0,0)}) \neq 0.$$

$\therefore$  derivative of  $Df(0,0)$  is invertible.

June 2013.

Dec — 2013 (c)

107. Let  $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a differentiable function. Let  $(Df)(x)$  be the derivative of  $f$  at  $x \in \mathbb{R}^m$ . Which of the following is/are correct?

1.  $Df(0)(u) = 0 \quad \forall u \in \mathbb{R}^m$

2.  $Df(x)(u) = 0 \quad \forall u \in \mathbb{R}^m$  and some  $x \in \mathbb{R}^m$  only if  $f$  is a constant.

3.  $Df(x)(u) = 0 \quad \forall u \in \mathbb{R}^m$  and  $\forall x \in \mathbb{R}^m$  only if  $f$  is a constant.

4. If  $f$  is not a constant function, then  $Df(x)$  is a one to one function for some  $x \in \mathbb{R}^m$ .

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad f(x, y) = (x^2, 0)$$

$$Df(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & 0 \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow Df(0,0)(u) = 0 \quad \forall u \in \mathbb{R}^m$$

But  $f$  is not a constant fun.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ by } f(x, y) = (x^2, 0)$$

$f$  is not a constant fun.

But  $Df(x)$  is not 1-1 since  $\det(Df(x)) = 0$

108. If  $f: S \rightarrow S$  is a function, then we denote by  $f^k$ , the function  $f \circ f \circ \dots \circ f$  ( $k$  times). Let  $f_1$  and  $f_2$  be two functions defined on  $\mathbb{R}^2$  as follows.

$$f_1(x, y) = (x+1, y+3)$$

$$f_2(x, y) = (x-3, y-2)$$

Then

1. For any positive integer  $k$ , there exists a unique  $(a, b) \in \mathbb{R}^2$ , such that  $f_1^k(0, 0) = f_2^k(a, b)$ .

2. For any real number  $\alpha$  and any positive integer  $k$ , there is at most one solution  $y$  for  $f_1^k(0, 0) = f_2^k(\alpha, y)$ .

3. There exists  $(a, b) \in \mathbb{R}^2$  such that  $f_1^k(a, b) \neq f_2^k(a, b)$  for any  $(x, y) \in \mathbb{R}^2$  and any positive integer  $k$ .

4.  $f_1$  is a linear transformation.

Explanation:

$$(f_1 \circ f_1 \circ \dots \circ f_1)_{(K \text{ times})} = f_1(f_1(\dots f_1(x, y)))$$

$$f_1(x, y) = (x+1, y+3)$$

$$f_1(f_1(x, y)) = (x+2, y+3+3)$$

$$f_1(f_1(f_1(x, y))) = (x+1+1+1, y+3+3+3)$$

$$f_1(f_1(\dots f_1)_{(K \text{ times})}(x, y)) = (x+k, y+3k)$$

$$f_2(x, y) = (x-3, y-2)$$

$$(f_2 \circ f_2 \circ \dots \circ f_2)_{(K \text{ times})} = (x-3k, y-2k)$$

$$f_1^k(x, y) = (x+k, y+3k) \quad f_2^k(x, y) = (x-3k, y-2k)$$

$$f_1^k(0, 0) = (k, 3k) \quad f_2^k(a, b) = (a-3k, b-2k)$$

$$(k, 3k) = (a-3k, b-2k)$$

$$a = 4k \quad b = 5k.$$

Dec 2011 (C)

63. Let  $f(r, \theta) = (r \cos \theta, r \sin \theta)$  for  $(r, \theta) \in \mathbb{R}^2$  with  $r \neq 0$ . Which of the following statements are correct? (Here  $Df$  denotes the derivative of  $f$ ).

1. the linear transformation  $Df(r, \theta)$  is not zero for any  $(r, \theta) \in \mathbb{R}^2$  with  $r \neq 0$ .

2.  $f$  is one-one on  $\{(r, \theta) \in \mathbb{R}^2 \mid r \neq 0\}$

3. For any  $(r, \theta) \in \mathbb{R}^2$  with  $r \neq 0$ ,  $f$  is 1-1 on a neighbourhood of  $(r, \theta)$

4.  $Df(r, \theta) = r^2 I$  for any  $(r, \theta) \in \mathbb{R}^2$  with  $r \neq 0$ .

Explanation:

$$Df(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \neq r^2 I.$$

$$f(r, 2\pi) = (r, 0) = f(r, 4\pi)$$

$\therefore f$  is not 1-1

But  $f$  is 1-1 on a nbd of  $(r, \theta)$ .

64. The map  $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $L(x, y) = (x, -y)$  is

1) differentiable everywhere on  $\mathbb{R}^2$

2) differentiable only at  $(0, 0)$

3)  $DL(0, 0) = L$ .

4)  $DL(x, y) = L$  for all  $(x, y) \in \mathbb{R}^2$ .

Explanation:

Since  $L$  is a linear transformation,  $DL = L$ .

$$\therefore DL(x, y) = L(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

Hence  $L$  is differentiable everywhere on  $\mathbb{R}^2$ .

68. Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function defined by  $f(x) = x \|x\|^2$  for  $x \in \mathbb{R}^n$ . Which of the following are correct?

1)  $(Df)(0) = 0$       2)  $(Df)(x) = 0$  for all  $x \in \mathbb{R}^n$ .

3)  $f$  is one-one      4)  $f$  has an inverse

Explanation

$$f(x_1, x_2, \dots, x_n) = \|x\|^2 (x_1, x_2, \dots, x_n)$$

$$f_1 = x_1 \|x\|^2 = x_1 (x_1^2 + x_2^2 + \dots + x_n^2)$$

$$f_2 = x_2 \|x\|^2 = x_2 (x_1^2 + x_2^2 + \dots + x_n^2)$$

$$\vdots$$
$$f_n = x_n \|x\|^2 = x_n (x_1^2 + x_2^2 + \dots + x_n^2)$$

$$Df(x) = \begin{bmatrix} \|x\|^2 & 0 & \dots & 0 \\ 0 & \|x\|^2 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \|x\|^2 \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = (x_1^2 + x_2^2 + \dots + x_n^2) + 2x_1^2$$

$$\frac{\partial f_2}{\partial x_1} = (\cancel{x_1^2} + \cancel{x_2^2} + \dots + \cancel{x_n^2}) + 2x_1^2 \quad 0 + 2x_1 x_2$$

$$\frac{\partial f_3}{\partial x_1} = 2x_1 x_3$$

$$\frac{\partial f_n}{\partial x_1} = 2x_1 x_n$$

$$\frac{\partial f_1}{\partial x_2} = 2x_1 x_2$$

$$\frac{\partial f_2}{\partial x_2} = (x_1^2 + x_2^2 + \dots + x_n^2) + 2x_2^2$$

,

$$\frac{\partial f_n}{\partial x_2} = 2x_2 x_n$$

③

$Df(x) =$

$$\begin{bmatrix} \|x\|^2 + 2x_1^2 & 2x_1x_2 & \dots & 2x_1x_n \\ 2x_1x_2 & \|x\|^2 + 2x_2^2 & \dots & 2x_2x_n \\ 2x_1x_3 & 2x_2x_3 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 2x_1x_n & 2x_2x_n & \dots & \|x\|^2 + 2x_n^2 \end{bmatrix}$$

June - 2011

69. Consider the map  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$f(x, y) = (3x - 2y + x^2, 4x + 5y + y^2). \text{ Then}$$

1.  $f$  is discontinuous at  $(0, 0)$
- ~~2.~~  $f$  is continuous at  $(0, 0)$  and all directional derivatives exist at  $(0, 0)$
3.  $f$  is differentiable at  $(0, 0)$  but the derivative  $Df(0, 0)$  is not invertible.
- ~~4.~~  $f$  is differentiable at  $(0, 0)$  and the derivative  $Df(0, 0)$  is invertible.

Explanation:

$$Df(x, y) = \begin{bmatrix} 3+2x & -2 \\ 4 & 5+2y \end{bmatrix}$$

$$Df(0, 0) = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}$$

$$|Df(0, 0)| = 15 + 8 = 23 \neq 0.$$

$\therefore Df(0, 0)$  is invertible.

Algebra  
Dec - 2014

33. Find the degree of the field extension

$$\mathbb{Q}(\sqrt{2}, 4\sqrt{2}, 8\sqrt{2}) \text{ over } \mathbb{Q}.$$

- 1) 4    2) 8    3) 14    4) 32

Explanation:

$$(8\sqrt{2})^4 = 2^{1/2}$$

$$(8\sqrt{2})^2 = 2^{1/4}$$

$$[\mathbb{Q}(\sqrt{2}, 4\sqrt{2}, 8\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(8\sqrt{2}) : \mathbb{Q}]$$

$$= \deg(x^8 - 2) = 8.$$

$$[\mathbb{Q}(\sqrt{2}, 4\sqrt{2}, 8\sqrt{2}) : \mathbb{Q}]$$

$$= [\mathbb{Q}(8\sqrt{2}, 4\sqrt{2}, \sqrt{2}) : \mathbb{Q}(4\sqrt{2}, 8\sqrt{2})] \times$$

$$[\mathbb{Q}(4\sqrt{2}, \sqrt{2}) : \mathbb{Q}(\sqrt{2})] \times [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$$

$$= 2 \times 2 \times 2$$

$$= 8.$$

34. The number of conjugacy classes in the permutation group  $S_6$  is

- 1) 12      2) 11      3) 10      4) 6.

Explanation:

1. 6
2. 5+1
3. 4+2
4. 4+1+1
5. 3+3
6. 3+2+1
7. 3+1+1+1
8. 2+2+2
9. 2+2+1+1
10. 2+1+1+1+1
11. 1+1+1+1+1+1

Conjugacy classes of  $S_n$  = Partition of  $n$ .

39. In the group of all invertible  $4 \times 4$  matrices with entries in the field of 3 elements, any 3-sylow subgroup has cardinality.

- 1) 3      2) 81      3) 243      4) 729.

Explanation:

$$|GL_4(\mathbb{F}_3)| = (3^4 - 1)(3^4 - 3)(3^4 - 3^2)(3^4 - 3^3)$$

$$= 80 \times 78 \times$$

$$= 80 \times 3(3^3-1) 3^2(3^2-1) 3^3(3-1)$$

$$= 3^6 \times 80 \times 26 \times 8 \times 2$$

$$= 3^6 \times 5 \times 13 \times 2^9$$

$\therefore$  any 3-Sylow subgroup of order  $3^6 = 729$ .

79. Let  $G$  be a non-abelian group. Then its order can be

- 1) 25      2) 55      3) 125      4) 35

Explanation.

$|G| = p^2$  abelian.

$|G| = p^3$  non-abelian (Eg  $Q_8$ )

$|G| = pq$  and  $p \nmid q-1$   $G$  is abelian (cyclic)

$|G| = pq$  and  $p \mid q-1$   $G$  is non-abelian.

84. Let  $G$  be group of order 45. Then

1)  $G$  has an element of order 9

2)  $G$  has a subgroup of order 9

3)  $G$  has a normal subgroup of order 9.

4)  $G$  has a normal subgroup of order 5.

Explanation

$$|G| = 45 = 9 \times 5 = 3^2 \times 5$$

$$n_3 = 1 + 3K \mid 9$$

$$n_5 = 1 + 5K \mid 9$$

$$n_3 = 1$$

$$n_5 = 1,$$

$\therefore H$  be the 3-Sylow subgroup and is unique

$K$  be the 5-Sylow subgroup and is unique  
 $\therefore K$  is normal.

$\therefore H$  is normal

$$\therefore |H| = 9.$$

$H$  is abelian group

But cannot be cyclic.

$\Rightarrow$  There is no guarantee that an elt of  $H$

86. Let  $R[x]$  be the polynomial ring over  $R$  in one variable. Let  $I \in R[x]$  be an ideal. Then

1.  $I$  is a maximal ideal iff  $I$  is a non-zero