

Further Mathematics (9231)

COMPLEX NUMBER

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Chapter 7: De Moivre's Theorem and its Applications

7.1 De Moivre's theorem

7.2 Using de Moivre's theorem to evaluate powers of complex numbers

7.3 Application of de Moivre's theorem in establishing trigonometric identities

7.4 Exponential form of a complex number

7.5 The cube roots of unity

7.6 The n th roots of unity

7.7 The roots of $z^n = \alpha$, where α is a non-real number

z

This chapter introduces de Moivre's theorem and many of its applications. When you have completed it, you will:

- know the basic theorem;
 - be able to find shorter ways of working out powers of complex numbers;
 - discover alternative methods for establishing some trigonometric identities;
 - know a new way of expressing complex numbers;
 - know how to work out the n th roots of unity and, in particular, the cube roots;
 - be able to solve certain types of polynomial equations.
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7.1 De Moivre's theorem

De Moivre's theorem can be proved by induction method such that if n is a positive integer, then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

De Moivre's theorem holds not only when n is a positive integer, but also when it is negative and even when it is fractional.

Let n be a negative integer and suppose $n = -k$. Then k is a positive integer and

$$\begin{aligned} (\cos \theta + i \sin \theta)^n &= (\cos \theta + i \sin \theta)^{-k} \\ &= \frac{1}{(\cos \theta + i \sin \theta)^k} \\ &= \frac{1}{\cos k\theta + i \sin k\theta}. \end{aligned}$$

Some of the results obtained in Chapter 1 can now be put to use. In order to remove i from the denominator of the expression above, the numerator and denominator are multiplied by the complex conjugate of the denominator, in this case $\cos k\theta - i \sin k\theta$. Thus,

$$\begin{aligned} \frac{1}{\cos k\theta + i \sin k\theta} &= \frac{1}{\cos k\theta + i \sin k\theta} \times \frac{\cos k\theta - i \sin k\theta}{\cos k\theta - i \sin k\theta} \\ &= \frac{\cos k\theta - i \sin k\theta}{k\theta \cos k\theta - i^2 \sin^2 k\theta} \\ &= \frac{\cos k\theta - i \sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} \\ &= \cos k\theta - i \sin k\theta \\ &= \cos(-k\theta) + i \sin(-k\theta) \\ &= \cos n\theta + i \sin n\theta, \quad \text{as required.} \end{aligned}$$

If n is a fraction, say $\frac{p}{q}$ where p and q are integers, then

$$\begin{aligned} \left(\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta \right)^q &= \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} \\ &= \cos p\theta + i \sin p\theta. \end{aligned}$$

But p is also an integer and so

$$\cos p\theta + i \sin p\theta = (\cos \theta + i \sin \theta)^p.$$

Taking the q th root of both sides.

$$\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta = (\cos \theta + i \sin \theta)^{\frac{p}{q}}.$$

It is important to point out at this stage that $\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta$ is just one value of

$(\cos \theta + i \sin \theta)^{\frac{p}{q}}$. A simple example will illustrate this. If $\theta = \pi$, $p = 1$ and $q = 2$, then

$$\begin{aligned} (\cos \pi + i \sin \pi)^{\frac{1}{2}} &= \cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi \\ &= i \end{aligned}$$

But $(\cos \pi + i \sin \pi)^{\frac{1}{2}} = \sqrt{-1}$ ($\cos \pi = -1$ and $\sin \pi = 0$) and $\sqrt{-1} = \pm i$. So i is only one value of $(\cos \pi + i \sin \pi)^{\frac{1}{2}}$. There are, in fact, q different values of $(\cos \pi + i \sin \pi)^{\frac{p}{q}}$.

$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for positive and negative integers, and fractional values of n
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7.1 Using de Moivre's theorem to evaluate powers of complex numbers

One very important application of de Moivre's theorem is in the addition of complex numbers of the form $(a + ib)^n$. The method for doing this will be illustrated through examples.

Example 4.2.1

Simplify $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$.

Solution

It would, of course, be possible to multiply $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ by it self three times, but this would be laborious and time consuming – even more so had the power been greater than 3. Instead,

$$\begin{aligned}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 &= \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i \\ &= i\end{aligned}$$

Example 4.2.2

Find $(\sqrt{3} + i)^{10}$ in the form $a + ib$.

Solution

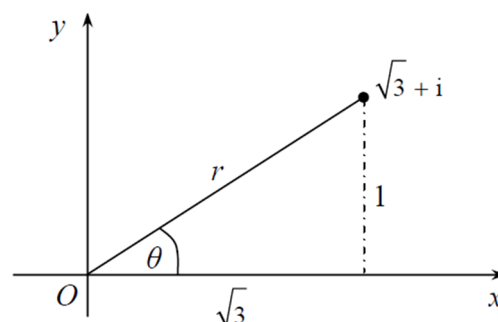
Clearly it would not be practical to multiply $(\sqrt{3} + i)$ by itself ten times. De Moivre's theorem could provide an alternative method but it can be used only for complex numbers in the form $\cos \theta + i \sin \theta$, and $\sqrt{3} + i$ is not in this form. A technique introduced in Chapter 1 (section 1.4) can be used to express it in polar form. #

On an Argand diagram, $\sqrt{3} + i$ is represented by the point whose Cartesian coordinates are $(\sqrt{3}, 1)$.

Now, $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan \theta = \frac{1}{\sqrt{3}}$ so that $\theta = \frac{\pi}{6}$.

Thus, $\sqrt{3} + i = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$

$$\begin{aligned} \text{and } (\sqrt{3} + i)^{10} &= 2^{10} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^{10} \\ &= 2^{10} \left(\cos \frac{10\pi}{6} + i \sin \frac{10\pi}{6} \right) \\ &= 1024 \left(\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) \\ &= 512(1 - i\sqrt{3}). \end{aligned}$$



[note that 2 is raised to the power 10 as well]

Example 4.2.3

Simplify $\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)^3$.

Solution

De Moivre's theorem applies only to expressions in the form $\cos \theta + i \sin \theta$ and not $\cos \theta - i \sin \theta$, so the expression to be simplified must be written in the form

$$\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right).$$

$$\begin{aligned} \left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)^3 &= \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]^3 \\ &= \cos\left(-\frac{3\pi}{6}\right) + i \sin\left(-\frac{3\pi}{6}\right) \\ &= \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \\ &= \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \\ &= -i. \end{aligned}$$

Note that it is apparent from this example that $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$.

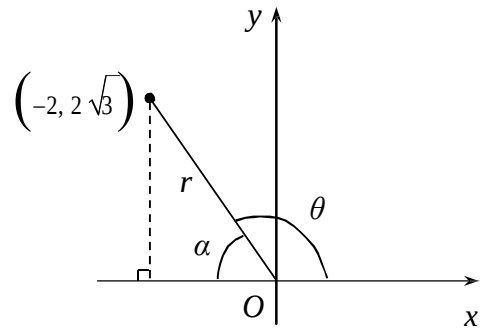
important to realise that this is a **deduction from** de Moivre's theorem and it must not be quoted **as** the theorem.

Example 4.2.4

Find $\frac{1}{(-2+2\sqrt{3}i)^3}$ in the form $a+ib$.

Solution

The complex number $-2+2\sqrt{3}i$ is represented by the point whose Cartesian coordinates are $(-2, 2\sqrt{3})$ on the Argand diagram shown here.



Hence, $r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$, and $\tan \theta = -\tan \alpha = -\frac{2\sqrt{3}}{2}$ so that $\theta = \frac{2\pi}{3}$. Thus

$$\begin{aligned} \frac{1}{(-2+2\sqrt{3}i)^3} &= (-2+2\sqrt{3}i)^{-3} \\ &= \left[4 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right]^{-3} \\ &= 4^{-3} \left[\cos \left(-3 \times \frac{2\pi}{3} \right) + i \sin \left(-3 \times \frac{2\pi}{3} \right) \right] \\ &= \frac{1}{64} [\cos(-2\pi) + i \sin(-2\pi)] \\ &= \frac{1}{64} (1+0) \\ &= \frac{1}{64}. \end{aligned}$$

Exercise 4A

1. Prove that $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$.

2. Express each of the following in the form $a+ib$:

(a) $(\cos 3\theta + i \sin 3\theta)^5$ (b) $\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right)^{10}$

(d) $(1+i)^6$

(e) $(2-2i)^4$

(c) $\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^2$

(f) $\frac{1}{(1+\sqrt{3}i)^5}$

(g) $(\sqrt{3}+3i)^9$

4.1 Application of de Moivre's theorem in establishing trigonometric identities

One way of showing how these identities can be derived is to use examples. The same principles are used whichever identity is required.

Example 4.3.1

Show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$.

Solution

There are several ways of establishing this result. The expansion of $\cos(A+B)$ can be used to express $\cos 2\theta$ in terms of $\cos \theta$ setting $A=\theta$ and $B=\theta$. Similarly, the expansion of $\cos(2\theta+\theta)$ can be used to give $\cos 3\theta$ in terms of $\cos \theta$. Using de Moivre's theorem gives a straightforward alternative method.

$$\begin{aligned}\cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 \\ &= \cos^3 \theta + 3\cos^2 \theta (i \sin \theta) + 3\cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 \\ &\quad \left[\text{using the binomial expansion of } (p+q)^3 \right] \\ &= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3\cos \theta \sin^2 \theta - i \sin^3 \theta \\ &\quad \left[\text{using } i^2 = -1 \right].\end{aligned}$$

Now $\cos 3\theta$ is the **real** part of the left-hand side of the equation, and the real parts of both sides can be equated,

$$\begin{aligned}\cos 3\theta &= \cos^3 \theta - 3\cos \theta \sin^2 \theta \\ &= \cos^3 \theta - 3\cos \theta (1 - \cos^2 \theta) \quad \left[\text{since } \cos^2 \theta + \sin^2 \theta = 1 \right] \\ &= 4\cos^3 \theta - 3\cos \theta.\end{aligned}$$

Note that this equation will also give $\sin 3\theta$ by equating the **imaginary** parts of both sides of the equation.

Example 4.3.2

Express $\tan 4\theta$ in terms of $\tan \theta$.

Solution

$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$ so expressions for $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$ must be established to start with. Using de Moivre's theorem,

$$\begin{aligned}\cos 4\theta + i \sin 4\theta &= (\cos \theta + i \sin \theta)^4 \\ &= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4 \\ &\quad \text{[using the binomial expansion]} \\ &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \\ &\quad \text{[using } i^2 = -1\text{]}\end{aligned}$$

Equating the real parts on both sides of the equation,

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta,$$

and equating the imaginary parts,

$$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta.$$

Now,

$$\begin{aligned}\tan 4\theta &= \frac{\sin 4\theta}{\cos 4\theta} \\ &= \frac{4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta}.\end{aligned}$$

Dividing every term by $\cos^4 \theta$ gives

$$\tan 4\theta = \frac{4 \frac{\sin \theta}{\cos \theta} - 4 \frac{\sin^3 \theta}{\cos^3 \theta}}{1 - 6 \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta}}.$$

But $\tan \theta = \frac{\sin \theta}{\cos \theta}$ so

$$\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$$

Exercise 4B

1. Express $\sin 3\theta$ in terms of $\sin \theta$.
 2. Express $\tan 3\theta$ in terms of $\tan \theta$.
 3. Express $\sin 5\theta$ in terms of $\sin \theta$.
 4. Show that $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$.
-

So far $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$ have been expressed in terms of $\sin\theta$, $\cos\theta$ and $\tan\theta$. De Moivre's theorem can be used to express powers of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of sines, cosines and tangents of multiple angles. First some important results must be established.

Suppose $z = \cos\theta + i\sin\theta$. Then

$$\begin{aligned} z^{-1} &= \frac{1}{z} = (\cos\theta + i\sin\theta)^{-1} \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta. \end{aligned}$$

So,

$$\begin{aligned} z &= \cos\theta + i\sin\theta \\ \frac{1}{z} &= \cos\theta - i\sin\theta. \end{aligned}$$

Adding,

$$z + \frac{1}{z} = 2\cos\theta,$$

and subtracting,

$$z - \frac{1}{z} = 2i\sin\theta.$$

If $z = \cos\theta + i\sin\theta$ $z + \frac{1}{z} = 2\cos\theta$ $z - \frac{1}{z} = 2i\sin\theta$
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Also,

$$\begin{aligned} z^n &= (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \\ z^{-n} &= \frac{1}{z^n} = (\cos\theta - i\sin\theta)^{-n} \\ z^n &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta. \end{aligned}$$

Combining z^n and $\frac{1}{z^n}$ as before,

$$z^n + \frac{1}{z^n} = 2\cos n\theta,$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

If $z = \cos\theta + i\sin\theta$, $z^n + \frac{1}{z^n} = 2\cos n\theta$ $z^n - \frac{1}{z^n} = 2i\sin n\theta$

A common mistake is to omit the i in $2i\sin n\theta$, so make a point of remembering this result carefully.

Example 4.3.3

Show that $\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$

Solution

Suppose $z = \cos \theta + i\sin \theta$.

Then $z + \frac{1}{z} = 2\cos \theta$

$$\begin{aligned} \text{and} \quad (2\cos \theta)^5 &= \left(z + \frac{1}{z}\right)^5 \\ &= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5 \\ &= z^5 + 5z^3 + 10z + 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^5. \end{aligned}$$

$$\begin{aligned} \text{So} \quad 32\cos^5 \theta &= \left[z^5 + \left(\frac{1}{z}\right)^5\right] + \left[5z^3 + 5\left(\frac{1}{z}\right)^3\right] + \left[10z + 10\left(\frac{1}{z}\right)\right] \\ &= \left[z^5 + \left(\frac{1}{z}\right)^5\right] + 5\left[z^3 + \left(\frac{1}{z}\right)^3\right] + 10\left[z + \left(\frac{1}{z}\right)\right]. \end{aligned}$$

Using the results established earlier,

$$z^5 + \frac{1}{z^5} = 2\cos 5\theta,$$

$$z^3 + \frac{1}{z^3} = 2\cos 3\theta,$$

$$z + \frac{1}{z} = 2\cos \theta.$$

Hence $32\cos^5 \theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos \theta)$

$$\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta), \quad \text{as required.}$$

One very useful application of the example above would be in integrating $\cos^5 \theta$.

$$\begin{aligned} \int \cos^5 \theta &= \int \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta) \\ &= \frac{1}{16} \left[\left(\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin \theta \right) \right] + c, \quad \text{where } c \text{ is an arbitrary constant.} \end{aligned}$$

Example 4.3.4

(a) Show that $\cos^3 \theta \sin^3 \theta = \frac{1}{32}(3\sin 2\theta - \sin 6\theta)$

(b) Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 \theta \sin^3 \theta \, d\theta$.

Solution**(a)**

$$(2 \cos \theta)^3 = \left(z + \frac{1}{z}\right)^3$$

$$(2 \sin \theta)^3 = \left(z - \frac{1}{z}\right)^3.$$

Multiplying these,

$$\begin{aligned} 8 \cos^3 \theta \, 8i^3 \sin^3 \theta &= \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 \\ -64i \cos^3 \theta \sin^3 \theta &= \left[\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)\right]^3 \\ &= \left(z^2 - \frac{1}{z^2}\right)^3 \\ &= (z^2)^3 - 3(z^2)^2\left(\frac{1}{z^2}\right) + 3(z^2)\left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3 \\ &= z^6 - 3z^2 + 3\left(\frac{1}{z^2}\right) - \frac{1}{z^6} \\ &= \left(z^6 - \frac{1}{z^6}\right) - 3\left(z^2 - \frac{1}{z^2}\right). \end{aligned}$$

$$\text{Now } z^6 - \frac{1}{z^6} = 2i \sin 6\theta \quad \text{and} \quad z^2 - \frac{1}{z^2} = 2i \sin 2\theta.$$

$$\begin{aligned} \text{Thus,} \quad -64i \cos^3 \theta \sin^3 \theta &= 2i \sin 6\theta - 3(2i \sin 2\theta) \\ &= 2i \sin 6\theta - 6i \sin 2\theta. \end{aligned}$$

Dividing both sides by $-64i$,

$$\begin{aligned} \cos^3 \theta \sin^3 \theta &= -\frac{1}{32} \sin 6\theta + \frac{3}{32} \sin 2\theta \\ &= \frac{1}{32} (3 \sin 2\theta - \sin 6\theta), \quad \text{as required.} \end{aligned}$$

(b)

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^3 \theta \sin^3 \theta \, d\theta &= \frac{1}{32} \int_0^{\frac{\pi}{2}} (3 \sin 2\theta - \sin 6\theta) \, d\theta \\ &= \frac{1}{32} \left[-\frac{3 \cos 2\theta}{2} + \frac{\cos 6\theta}{6} \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{32} \left[\frac{3}{2} - \frac{1}{6} - \left(-\frac{3}{2} + \frac{1}{6} \right) \right] \\ &= \frac{1}{32} \times \frac{8}{3} = \frac{1}{12}. \end{aligned}$$

This section concludes with an example which uses the ideas introduced here and extends into other areas of mathematics.

Example 4.3.5

- (a) Show that $\cos 5\theta = \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5)$.
- (b) Show that the roots of the equation $16x^4 - 20x + 5 = 0$ are $\cos \frac{r\pi}{10}$ for $r = 1, 3, 7$ and 9 .
- (c) Deduce that $\cos^2 \frac{\pi}{10} \cos^2 \frac{3\pi}{10} = \frac{5}{16}$.

Solution

- (a)** Using the ideas introduced at the beginning of this section,

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5.$$

Using the binomial theorem for expansion, the right-hand side of this equation becomes

$$\cos^5 \theta + 5 \cos^4 \theta (i \sin \theta) + 10 \cos^3 \theta (i \sin \theta)^2 + 10 \cos^2 \theta (i \sin \theta)^3 + 5 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5.$$

Not every term of this expression has to be simplified. As $\cos 5\theta$ is the real part of the left-hand side of the equation, it equates to the real part of the right-hand side. The real part of the right-hand side of the equation comprises those terms with even powers of i in them, since $i^2 = -1$ and is real.

$$\text{Thus, } \cos 5\theta = \cos^5 \theta + 10 \cos^3 \theta (i \sin \theta)^2 + 5 \cos \theta (i \sin \theta)^4$$

$$= \cos^5 \theta + 10 \cos^3 \theta (-\sin^2 \theta) + 5 \cos \theta \sin^4 \theta$$

$$= \cos^5 \theta + 10 \cos^3 \theta (-1 + \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2 \quad \text{using } \cos^2 \theta + \sin^2 \theta = 1$$

$$= \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + 5 \cos^5 \theta$$

$$= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

$$= \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5).$$

- (b)** Now when $\cos 5\theta = 0$, either $\cos \theta = 0$ or $16\cos^4 \theta - 20\cos^2 \theta + 5 = 0$. So, putting $x = \cos \theta$, the roots of $16x^4 - 20x + 5 = 0$ are the values of $\cos \theta$ for which $\cos 5\theta = 0$, provided $\cos \theta \neq 0$.
-

But if $\cos 5\theta = 0$, $5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \dots$

so that $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, \frac{11\pi}{10}, \frac{13\pi}{10}, \dots$.

Also, $\cos \frac{11\pi}{10}$ is the same as $\cos \frac{9\pi}{10}$, and $\cos \frac{13\pi}{10}$ is the same as $\cos \frac{7\pi}{10}$, so that, although there is an infinite number of values of θ , there are only five **distinct** values of $\cos \theta$ and these are $\cos \frac{\pi}{10}, \cos \frac{3\pi}{10}, \cos \frac{5\pi}{10}, \cos \frac{7\pi}{10}$ and $\cos \frac{9\pi}{10}$.

Now $\cos \frac{5\pi}{10} = \cos \frac{\pi}{2} = 0$ and $\frac{\pi}{2}$ is, of course, a root of $\cos \theta = 0$, so that the roots of the equation $16x^4 - 20x + 5 = 0$ are $\cos \frac{\pi}{10}, \cos \frac{3\pi}{10}, \cos \frac{5\pi}{10}, \cos \frac{7\pi}{10}$ and $\cos \frac{9\pi}{10}$.

The roots may be written in a slightly different way as

$$\begin{aligned}\cos \frac{7\pi}{10} &= \cos \left(\pi - \frac{3\pi}{10} \right) \\ &= -\cos \frac{3\pi}{10},\end{aligned}$$

and

$$\begin{aligned}\cos \frac{9\pi}{10} &= \cos \left(\pi - \frac{\pi}{10} \right) \\ &= -\cos \frac{\pi}{10}.\end{aligned}$$

Thus the four roots of the quartic equation $16x^4 - 20x + 5 = 0$ can be written as $\pm \cos \frac{\pi}{10}$ and $\pm \cos \frac{3\pi}{10}$.

- (c)** From the ideas set out in Chapter 2 (section 2.7), the product of the roots of the quartic equation $16x^4 - 20x + 5 = 0$ is $\frac{5}{16}$ so that

$$\cos \frac{\pi}{10} \left(-\cos \frac{\pi}{10} \right) \cos \frac{3\pi}{10} \left(-\cos \frac{3\pi}{10} \right) = \frac{5}{16}.$$

And hence, $\cos^2 \frac{\pi}{10} \cos^2 \frac{3\pi}{10} = \frac{5}{16}$.

Exercise 4C

1. If $z = \cos\theta + i\sin\theta$ write, in terms of z :

(a) $\cos 4\theta$ (b) $\cos 7\theta$ (c) $\sin 6\theta$ (d) $\sin 3\theta$

2. Prove the following results:

(a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(c) $\sin 6\theta = \sin\theta (32\cos^5\theta - 32\cos^3\theta + 6\cos\theta)$

(d) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

4.2 Exponential form of a complex number

Both $\cos \theta$ and $\sin \theta$ can be expressed as an infinite series in powers of θ , provided that θ is measured in radians. These are given by

$$\cos \theta = \theta - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots (-1)^{n-1} \frac{\theta^{2n-2}}{(2n-2)!} + \dots$$

and

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots (-1)^{n-1} \frac{\theta^{2n-1}}{(2n-1)!} + \dots$$

There is also a series for e^x given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^{n-1}}{(n-1)!} + \dots$$

If $i\theta$ is substituted for x in the series for e^x ,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots + \frac{(i\theta)^{n-1}}{(n-1)!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \end{aligned}$$

Regrouping,

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + i \left(\theta - \frac{\theta^3}{3!} + \dots \right),$$

and, using the previous results for $\sin \theta$ and $\cos \theta$,

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

It is also important to note that if $z = \cos \theta + i \sin \theta$, then

$$\begin{aligned} z^n &= (\cos \theta + i \sin \theta)^n \\ &= \cos n\theta + i \sin n\theta \\ &= e^{ni\theta}, \end{aligned}$$

and if $z = r(\cos \theta + i \sin \theta)$, then $z = re^{i\theta}$ and $z^n = r^n e^{ni\theta}$.

If $z = r(\cos \theta + i \sin \theta)$,
 then $z = re^{i\theta}$
 and $z^n = r^n e^{ni\theta}$

The form $re^{i\theta}$ is known as the exponential form of a complex number and is clearly linked to the polar form very closely.

Another result can be derived from the exponential form of a complex number:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

So,
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) \\ = \cos\theta - i\sin\theta.$$

Adding these

$$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) \\ = 2\cos\theta,$$

or
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtracting gives
$$e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) \\ = 2i\sin\theta,$$

or
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
--

Example 4.4.1

Express $2 - 2i$ in the form $re^{i\theta}$.

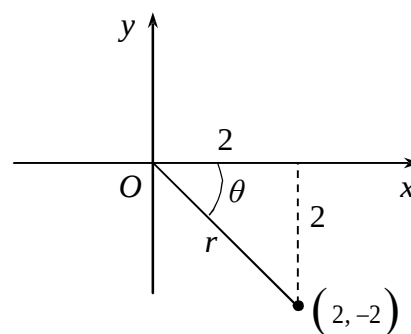
Solution

The complex number $2 - 2i$ is represented by the point with the coordinates $(2, -2)$ on an Argand diagram.

Hence,
$$r = \sqrt{2^2 + (-2)^2} = \sqrt{8},$$

and
$$\theta = -\tan^{-1} \frac{2}{2} = -\frac{\pi}{4},$$

so that
$$2 - 2i = \sqrt{8} e^{-\frac{\pi i}{4}}.$$



Exercise 4D

1. Express the following in the form $re^{i\theta}$:

- (a) $1 + i$ (b) $\sqrt{3} - i$ (c) $3 + \sqrt{3}i$ (d) $-2\sqrt{3} + 2i$

4.5 The cube roots of unity

The cube roots of 1 are numbers such that when they are cubed their value is 1. They must, therefore, satisfy the equation $z^3 - 1 = 0$. Clearly, one root of $z^3 - 1$ is $z = 1$ so that $z - 1$ must be a factor of $z^3 - 1$. Factorising,

$$z^3 - 1 = (z - 1)(z^2 + z + 1) = 0.$$

Now $z^3 - 1 = 0$ is a cubic equation and so has three roots, one of which is $z = 1$. The other two come from the quadratic equation $z^2 + z + 1 = 0$. If one of these is denoted by w , then w satisfies $z^2 + z + 1 = 0$ so that $w^2 + w + 1 = 0$. It can also be shown that if w is a root of $z^3 = 1$, then w^2 is also a root – in fact, the other root. Substituting w^2 into the left-hand side of $z^3 = 1$ gives $(w^2)^3 = w^6 = (w^3)^2 = 1^2 = 1$, as $w^3 = 1$ since w is a solution of $z^3 = 1$.

Thus the three cube roots of 1 are 1, w and w^2 , where w and w^2 are non-real. Of course, w can be expressed in the form $a + ib$ by solving $z^2 + z + 1 = 0$ using the quadratic formula:

$$\begin{aligned} z &= \frac{-1 \pm \sqrt{1^2 - (4 \times 1 \times 1)}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2}. \end{aligned}$$

It doesn't matter whether w is labelled as $\frac{-1 + i\sqrt{3}}{2}$ or as $\frac{-1 - i\sqrt{3}}{2}$ because each is the square of the other. In other words, if $w = \frac{-1 + i\sqrt{3}}{2}$ then

$$\begin{aligned} w^2 &= \left(\frac{-1 + i\sqrt{3}}{2} \right)^2 \\ &= \frac{1 - 2i\sqrt{3} + (i\sqrt{3})^2}{4} \\ &= \frac{1 - 2i\sqrt{3} - 3}{4} \\ &= \frac{-2 - 2i\sqrt{3}}{4} \\ &= \frac{-1 - i\sqrt{3}}{2}. \end{aligned}$$

If $w = \frac{-1 - i\sqrt{3}}{2}$, then $w^2 = \frac{-1 + i\sqrt{3}}{2}$.

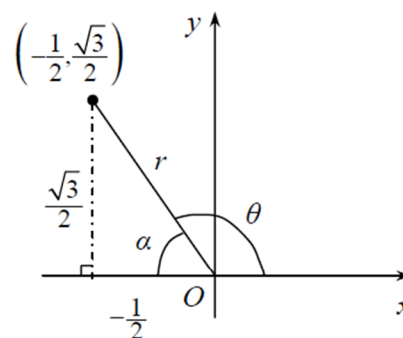
The cube roots of unity are 1, w and w^2 , where

$$w^3 = 1$$

$$1 + w + w^2 = 0$$

and the non-real roots are $\frac{-1 \pm i\sqrt{3}}{2}$

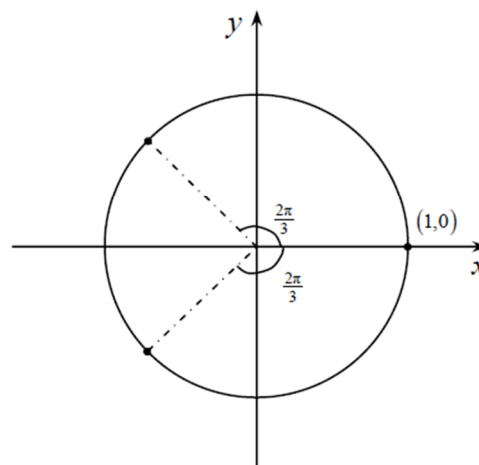
Both w and w^2 can be expressed in exponential form. Take $w = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$; w can be represented by the point whose Cartesian coordinates are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on an Argand diagram.



From the diagram, $r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$, and $\theta = \pi - \alpha$, where $\tan \alpha = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$. Thus,

$\alpha = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$ and $w = e^{\frac{2\pi i}{3}}$. The other root is $w^2 = \left(e^{\frac{2\pi i}{3}}\right)^2 = e^{\frac{4\pi i}{3}}$ and can also be written as $e^{\frac{2\pi i}{3}}$.

Plotting the three cube roots of unity on an Argand diagram shows three points equally spaced (at intervals of $\frac{2\pi}{3}$) round a circle of radius 1 as shown in the diagram alongside.



Example 4.5.1

Simplify $w^7 + w^8$, where w is a complex cube root of 1.

Solution

$$w^7 = w^6 \times w = (w^3)^2 \times w = 1^2 \times w = w \quad (\text{because } w^3 = 1),$$

$$w^8 = w^6 \times w^2 = (w^3)^2 \times w^2 = 1^2 \times w^2 = w^2.$$

$$\therefore w^7 + w^8 = w + w^2 = -1 \quad (\text{because } 1 + w + w^2 = 0).$$

Example 4.5.2

Show that $\frac{1}{1+w} + \frac{1}{1+w^2} + \frac{1}{w+w^2} = 0$.

Solution

$1 + w + w^2 = 0$ so $1 + w = -w^2$, and so on. So the denominators of the left-hand side of the equation can be replaced to simplify to $\frac{1}{-w^2} + \frac{1}{-w} + \frac{1}{-1}$.

Multiplying the first term of this expression by w in the numerator and denominator, and the second term by w^2 similarly gives

$$\begin{aligned} & \frac{w}{-w^3} + \frac{w^2}{-w^3} - 1 \\ &= -w - w^2 - 1 \quad (\text{as } w^3 = 1) \\ &= 0 \quad (\text{as } 1 + w + w^2 = 0). \end{aligned}$$

Exercise 4E

1. If w is a complex cube root of 1, find the value of

(a) $w^{10} + w^{11}$ (b) $(1 + 3w)(1 + 3w^2)$ (c) $(1 + 3w + w^2)^3$

4.6 The n th roots of unity

The equation $z^n = 1$ clearly has at least one root, namely $z = 1$, but it actually has many more, most of which (if not all) are complex. In fact, if n is odd $z = 1$ is the only real root, but if n is even $z = -1$ is also a real root because -1 raised to an even power is $+1$.

To find the remaining roots, the right-hand side of the equation $z^n = 1$ has to be examined. In exponential form, $1 = e^0$ because $e^0 = \cos 0 + i \sin 0 = 1 + i0 = 1$. But also, $1 = e^{2\pi i}$ because $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i0 = 1$. Indeed $1 = e^{2k\pi i}$ where k is any integer. Substituting the right-hand side of the equation $z^n = 1$ by this term gives $z^n = e^{2k\pi i}$. Taking the n th root of both sides gives $z = e^{\frac{2k\pi i}{n}}$. Different integer values of k will give rise to different roots, as shown below.

$$k = 0 \text{ gives } e^0 = 1,$$

$$k = 1 \text{ gives } e^{\frac{2\pi i}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},$$

$$k = 2 \text{ gives } e^{\frac{4\pi i}{n}} = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n},$$

and so on until

$$k = n-1 \text{ gives } e^{\frac{2(n-1)\pi i}{n}} = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}.$$

Thus, $z = e^{\frac{2k\pi i}{n}}$ $n = 0, 1, 2, \dots, n-1$ gives the n distinct roots of the equation $z^n = 1$.

There are no more roots because if k is set equal to n , $e^{\frac{2n\pi i}{n}} = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$, which is the same root as that given by $k = 0$.

Similarly, if k is set equal to $n+1$, $e^{\frac{2(n+1)\pi i}{n}} = e^{\frac{2n\pi i}{n}} \times e^{\frac{2\pi i}{n}} = e^{2\pi i} \times e^{\frac{2\pi i}{n}} = 1 \times e^{\frac{2\pi i}{n}} = e^{\frac{2\pi i}{n}}$ which is the same root as that given by $k = 1$, and so on.

The n roots of $z^n = 1$ can be illustrated on an Argand diagram. All the roots lie on the circle $|z| = 1$ because the modulus of every root is 1.

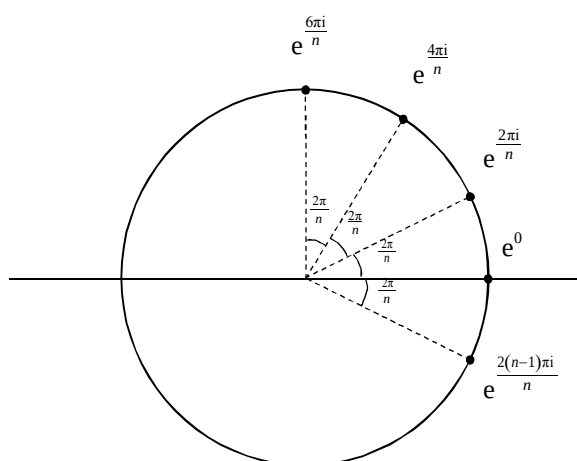
Also, the amplitudes of the complex numbers representing the roots are

$\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \dots, \frac{2(n-1)\pi}{n}$. In other words, the

roots are represented by n points equally spaced

around the unit circle at angles of $\frac{2\pi}{n}$ starting at

$(1, 0)$ – the point representing the real root $z = 1$.



The equation $z^n = 1$ has roots
 $z = e^{\frac{2k\pi i}{n}} \quad k = 0, 1, 2, \dots, (n-1)$

Example 4.6.1

Find, in the form $a + ib$, the roots of the equation $z^6 = 1$ and illustrate these roots on an Argand diagram.

Solution

$$z^6 = 1 = e^{2k\pi i}$$

Therefore

$$\begin{aligned} z &= e^{\frac{2k\pi i}{6}} \\ &= e^{\frac{k\pi i}{3}} \quad k = 0, 1, 2, 3, 4, 5. \end{aligned}$$

Hence the roots are

$$k = 0, \quad z = 1$$

$$k = 1, \quad z = e^{\frac{\pi i}{3}} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$k = 2, \quad z = e^{\frac{2\pi i}{3}} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

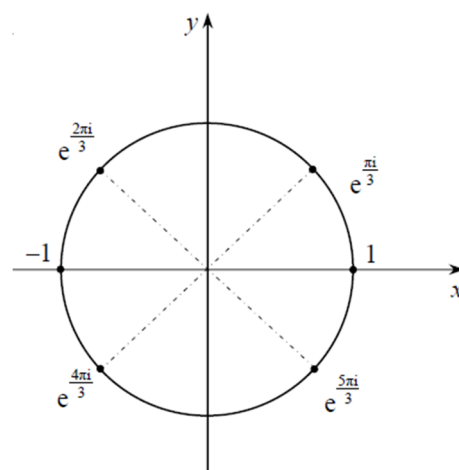
$$k = 3, \quad z = e^{\pi i} = \cos \pi + i \sin \pi = -1$$

$$k = 4, \quad z = e^{\frac{4\pi i}{3}} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$k = 5, \quad z = e^{\frac{5\pi i}{3}} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

To summarise, the six roots are

$z = \pm 1, \quad z = \pm \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ and these are illustrated on the Argand diagram alongside.



Two further points are worth noting. Firstly, you may need to give the arguments of the roots between $-\pi$ and $+\pi$ instead of between 0 and 2π . In example 4.6.1, the roots would be given as $z = e^{\frac{k\pi i}{3}}$ for $k = 0, \pm 1, \pm 2, 3$. Secondly, a given equation may not involve unity – for example, if example 4.6.1 had concerned $z^6 = 64$, the solution would have been written

$$z^6 = 64$$

$$z^6 = 2^6 e^{2k\pi i}$$

$$z = 2e^{\frac{2k\pi i}{6}} \quad k = 0, 1, 2, 3, 4, 5$$

and the only difference would be that the modulus of each root would be 2 instead of 1, with the consequence that the six roots of $z^6 = 64$ would lie on the circle $|z| = 2$ instead of $|z| = 1$.

Of course, there are variations on the above results. For example, you may need to find the roots of the equation $1 + z + z^2 + z^3 + z^4 + z^5 = 0$. This looks daunting but if you can recognise the left-hand side as a geometric progression with common ratio z , it becomes more straightforward. Summing the left-hand side of the equation,

$$1 + z + z^2 + z^3 + z^4 + z^5 = \frac{z^6 - 1}{z - 1} = 0,$$

so that the five roots of $1 + z + z^2 + z^3 + z^4 + z^5 = 0$ are five of the roots of $z^6 - 1 = 0$. The

root to be excluded is the root $z = 1$ because $\frac{z^6 - 1}{z - 1}$ is indeterminate when $z = 1$. So the roots

of $1 + z + z^2 + z^3 + z^4 + z^5 = 0$ are $z = \pm \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ and -1 , when written in the form $a + ib$.

Exercise 4F

1. Write, in the form $a + ib$, the roots of:

(a) $z^4 = 1$ (b) $z^5 = 32$ (c) $z^{10} = 1$.

In each case, show the roots on an Argand diagram.

2. Solve the equation $z^4 + z^3 + z^2 + z + 1 = 0$.

3. Solve the equation $1 - 2z + 4z^2 - 8z^3 = 0$.

4. By considering the roots of $z^5 = 1$, show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} + \cos \frac{6\pi}{5} + \cos \frac{8\pi}{5} = -1$.

4.3 The roots of $z^n = \alpha$ where α is a non-real number

Every complex number of the form $a + ib$ can be written in the form $re^{i\theta}$, where r is real and θ lies in an interval of 2π (usually from 0 to 2π or from $-\pi$ to $+\pi$). Suppose that

$$\alpha = re^{i\theta}.$$

Now
$$\begin{aligned} e^{i\theta+2\pi i} &= e^{i\theta} \times e^{2\pi i} && \left(\text{using } e^{p+q} = e^p \times e^q \right) \\ &= e^{i\theta} && \left(\text{because } e^{2\pi i} = 1 \right). \end{aligned}$$

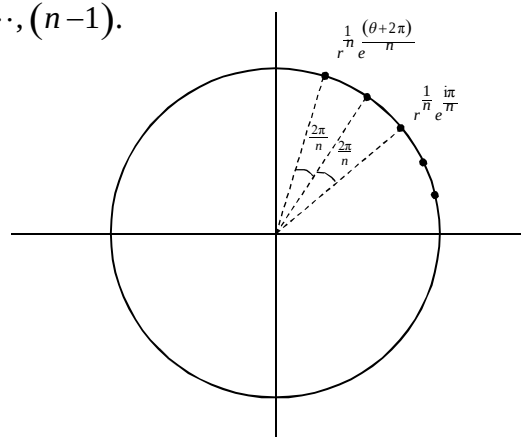
Similarly
$$\begin{aligned} e^{i\theta+2k\pi i} &= e^{i\theta} \times e^{2k\pi i} \\ &= e^{i\theta} \text{ also.} \end{aligned}$$

So,
$$z^n = \alpha = re^{i\theta+2k\pi i}$$
 and, taking the n th root of both sides,

$$\begin{aligned} z &= r^{\frac{1}{n}} e^{\frac{i(\theta+2k\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{\frac{i(\theta+2k\pi)}{n}} \quad k = 0, 1, 2, 3, \dots, (n-1). \end{aligned}$$

These roots can be illustrated on an Argand

diagram as before. All lie on the circle $|z| = r^{\frac{1}{n}}$ and are equally spaced around the circle at intervals of $\frac{2\pi}{n}$. When $k = 0$, $z = r^{\frac{1}{n}} e^{\frac{i\theta}{n}}$ and this could be taken as the starting point for the intervals of $\frac{2\pi}{n}$.



The equation $z^n = \alpha$, where $\alpha = re^{i\theta}$,
has roots $z = r^{\frac{1}{n}} e^{\frac{i(\theta+2k\pi)}{n}} \quad k = 0, 1, 2, \dots, (n-1)$

Example 4.7.1

Find the three roots of the equation $z^3 = 2 + 2i$.

Solution

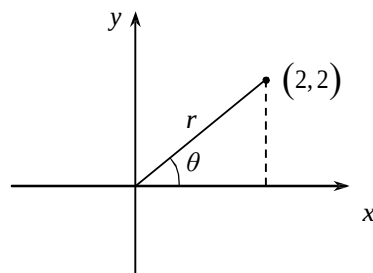
First, $2 + 2i$ must be expressed in exponential form.

From the diagram alongside,

$$r = \sqrt{2^2 + 2^2} = \sqrt{8}, \quad \tan \theta = 1, \quad \text{and} \quad \theta = \frac{\pi}{4}.$$

So, $2 + 2i = \sqrt{8} e^{\frac{\pi i}{4}}.$

Hence, $z^3 = \sqrt{8} e^{\left(\frac{\pi i}{4} + 2k\pi i\right)}.$



Taking the cube root of each side,

$$\begin{aligned} z &= \sqrt[3]{2} e^{\frac{(\frac{\pi i}{4} + 2k\pi i)}{3}} \\ &= \sqrt[3]{2} e^{\frac{(1+8k)\pi i}{12}} \quad k = 0, 1, 2. \end{aligned}$$

So the roots are

$$\begin{aligned} k = 0, \quad z &= \sqrt[3]{2} e^{\frac{\pi i}{12}} \\ k = 1, \quad z &= \sqrt[3]{2} e^{\frac{9\pi i}{12}} \\ k = 2, \quad z &= \sqrt[3]{2} e^{\frac{17\pi i}{12}} \quad \left(\text{or } \sqrt[3]{2} e^{-\frac{7\pi i}{12}} \right). \end{aligned}$$

The roots can also be written $\sqrt[3]{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$ when $k = 0$, and so on.

This chapter closes with one further example of the use of the principles discussed.

Example 4.7.2

Solve the equation $(z+1)^5 = z^5$ giving your answers in the form $a + ib$.

Solution

At first sight, it is tempting to use the binomial expansion on $(z+1)^5$ but this generates a quartic equation (the terms in z^5 cancel) which would be difficult to solve. Instead, because $e^{2k\pi i} = 1$, the equation can be written as

$$(z+1)^5 = e^{2k\pi i} z^5.$$

Taking the fifth root of each side,

$$z + 1 = e^{\frac{2k\pi i}{5}} z \quad k = 1, 2, 3, 4.$$

Note that $k = 0$ is excluded because this would give $z + 1 = z$, and in any case as the equation is really a quartic equation it will have only four roots.

Solving the equation for z ,

$$1 = z \left(e^{\frac{2k\pi i}{5}} - 1 \right) \quad k = 1, 2, 3, 4$$

or

$$z = \frac{1}{e^{\frac{2k\pi i}{5}} - 1}.$$

The next step is new to this section and is well worth remembering. The term $e^{\frac{2k\pi i}{5}}$ can be written as $\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$ making the denominator have the form $p + iq$. The numerator and denominator of the right-hand side of the equation can then be multiplied by $p - iq$ to remove i from the denominator. As p would then equal $\cos \frac{2k\pi}{5} - 1$ and q would equal

$\sin \frac{2k\pi}{5}$, this would be a rather cumbersome method. Instead, the numerator and

denominator of the right-hand side of the equation are multiplied by $e^{-\frac{k\pi i}{5}}$ (for reasons which will be apparent later).

Thus,

$$z = \frac{1}{e^{\frac{2k\pi i}{5}} - 1},$$

So

$$\begin{aligned} z &= \frac{e^{-\frac{k\pi i}{5}}}{\left(e^{\frac{2k\pi i}{5}} e^{-\frac{k\pi i}{5}} \right) - e^{-\frac{k\pi i}{5}}} \\ &= \frac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}}. \end{aligned}$$

But $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin \theta$ so that $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i \sin \frac{k\pi}{5}$ and so,

$$\begin{aligned} z &= \frac{e^{-\frac{k\pi i}{5}}}{2i \sin \frac{k\pi}{5}} \\ &= \frac{\cos\left(-\frac{k\pi}{5}\right) + i \sin\left(-\frac{k\pi}{5}\right)}{2i \sin \frac{k\pi}{5}} \\ &= \frac{1}{2i} \cot\left(\frac{k\pi}{5}\right) - \frac{1}{2} \\ &= -\frac{1}{2} - \frac{1}{2}i \cot\left(\frac{k\pi}{5}\right) \quad k = 1, 2, 3, 4 \text{ as required} \end{aligned}$$

Exercise 4G

1. Solve the following equations:

(a) $z^4 = 16i$

(b) $z^3 = 1 - i$

(c) $z^8 = 1 - \sqrt{3}i$

(d) $z^2 = -1$

(e) $(z+1)^3 = 8i$

(f) $(z-1)^5 = z^5$

Miscellaneous exercises 4

1. (a) Write down the modulus and argument of the complex number -64 .
- (b) Hence solve the equation $z^4 + 64 = 0$ giving your answers in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.
- (c) Express each of these four roots in the form $a + ib$ and show, with the aid of a diagram, that the points in the complex plane which represent them form the vertices of a square.

2. (a) Express each of the complex numbers

$$1 + i \quad \text{and} \quad \sqrt{3} - i$$

in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

- (b) Using your answers to part (a),

(i) show that
$$\frac{(\sqrt{3} - i)^5}{(1 + i)^{10}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

- (ii) solve the equation $z^3 = (1 + i)(\sqrt{3} - i)$
giving your answers in the form $a + ib$, where a and b are real numbers to be determined to two decimal places.

3. (a) By considering $z = \cos\theta + i\sin\theta$ and using de Moivre's theorem, show that

$$\sin 5\theta \equiv \sin\theta(16\sin^4\theta - 20\sin^2\theta + 5).$$

- (b) Find the exact values of the solutions of the equation

$$16x^4 - 20x^2 + 5 = 0.$$

- (c) Deduce the exact values of $\sin\frac{\pi}{5}$ and $\sin\frac{2\pi}{5}$, explaining clearly the reasons for your answers.
-

4. (a) Show that the non-real cube roots of unity satisfy the equation

$$z^2 + z + 1 = 0.$$

- (b) The real number a satisfies the equation

$$\frac{1}{a - \omega + \omega^2} + \frac{1}{a + \omega - \omega^2} = \frac{1}{2},$$

where ω is one of the non-real cube roots of unity. Find the possible values of a .

5. (a) Verify that $z_1 = 1 + e^{\frac{\pi i}{5}}$
is a root of the equation $(z - 1)^5 = -1$.

- (b) Find the other four roots of the equation.

- (c) Mark on an Argand diagram the points corresponding to the five roots of the equation. Show that these roots lie on a circle, and state the centre and radius of the circle.

- (d) By considering the Argand diagram, find

- (i) $\arg z_1$ in terms of π ,
(ii) $|z_1|$ in the form $a \cos \frac{\pi}{b}$, where a and b are integers to be determined.

6. (a) (i) Show that $w = e^{\frac{2\pi i}{5}}$ is one of the fifth roots of unity.

- (ii) Show that the other fifth roots of unity are 1 , w^2 , w^3 and w^4 .

- (b) Let $p = w + w^4$ and $q = w^2 + w^3$, where $w = e^{\frac{2\pi i}{5}}$.

- (i) Show that $p + q = -1$ and $pq = -1$.

- (ii) Write down the quadratic equation, with integer coefficients, whose roots are p and q .

- (iii) Express p and q as integer multiples of $\cos \frac{2\pi}{5}$ and $\cos \frac{4\pi}{5}$, respectively,

- (iv) Hence obtain the values of $\cos \frac{2\pi}{5}$ and $\cos \frac{4\pi}{5}$ in surd form.
-

7. (a) (i) Use de Moivre's theorem to show that if $z = \cos \theta + i \sin \theta$, then

$$z^n + \frac{1}{z^n} = 2 \cos n\theta.$$

- (ii) Write down the corresponding result for $z^n - \frac{1}{z^n}$.

- (b) (i) Show that

$$\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 = A \left(z^6 - \frac{1}{z^6}\right) + B \left(z^2 - \frac{1}{z^2}\right),$$

where A and B are numbers to be determined.

- (ii) By substituting $z = \cos \theta + i \sin \theta$ in the above identity, deduce that

$$\cos^3 \theta \sin^3 \theta = \frac{1}{32} (3 \sin 2\theta - \sin 6\theta).$$

8. (a) (i) Express $e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}$ in terms of $\sin \frac{\theta}{2}$.

- (ii) Hence, or otherwise, show that

$$\frac{1}{e^{i\theta} - 1} = -\frac{1}{2} - \frac{i}{2} \cot \frac{\theta}{2}, \quad (e^{i\theta} \neq 1).$$

- (b) Derive expressions, in the form $e^{i\theta}$ where $-\pi < \theta \leq \pi$, for the four non-real roots of the equation $z^6 = 1$.

- (c) The equation
$$\left(\frac{w+1}{w}\right)^6 = 1 \quad (*)$$

has one real root and four non-real roots.

- (i) Explain why the equation has only five roots in all.

- (ii) Find the real root.

- (iii) Show that the non-real roots are

$$\frac{1}{z_1 - 1}, \quad \frac{1}{z_2 - 1}, \quad \frac{1}{z_3 - 1}, \quad \frac{1}{z_4 - 1},$$

where z_1, z_2, z_3 and z_4 are the non-real roots of the equation $z^6 = 1$.

- (iv) Deduce that the points in an Argand diagram that represents the roots of equation (*) lie on a straight line.

9. (a) Express the complex number $2 + 2i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Show that one of the roots of the equation

$$z^3 = 2 + 2i$$

is $\sqrt{2} e^{i\frac{\pi}{12}}$, and find the other two roots giving your answers in the form $re^{i\theta}$, where r is a surd and $-\pi < \theta \leq \pi$.

(c) Indicate on an Argand diagram points A , B and C corresponding to the three roots found in part (b).

(d) Find the area of the triangle ABC , giving your answer in surd form.

(e) The point P lies on the circle through A , B and C . Denoting by w , α , β and γ the complex numbers represented by P , A , B and C , respectively, show that

$$\left| (w - \alpha)^2 + (w - \beta)^2 + (w - \gamma)^2 \right| = 6.$$

Answers to Exercises – Further Pure 1

Chapter 4

Exercise 4A

2. (a) $\cos 15\theta + i \sin 15\theta$ (b) 1 (c) i (d) $-8i$
 (e) -64 (f) $\frac{1}{64}(1 + \sqrt{3}i)$ (g) $-41472\sqrt{3}$

Exercise 4B

1. $3\sin\theta - 4\sin^3\theta$
 2. $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$
 3. $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Exercise 4C

1. (a) $\frac{1}{2}\left(z^4 + \frac{1}{z^4}\right)$ (b) $\frac{1}{2}\left(z^7 + \frac{1}{z^7}\right)$ (c) $\frac{1}{2i}\left(z^6 - \frac{1}{z^6}\right)$ (d) $\frac{1}{2i}\left(z^3 - \frac{1}{z^3}\right)$

Exercise 4D

1. (a) $\sqrt{2}e^{\frac{\pi i}{4}}$ (b) $2e^{-\frac{\pi i}{6}}$ (c) $\sqrt{12}e^{\frac{\pi i}{6}}$ (d) $4e^{\frac{5\pi i}{6}}$

Exercise 4E

1. (a) -1 (b) 7 (c) 8

Exercise 4F

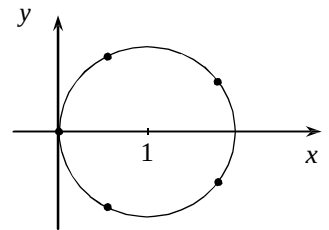
1. (a) $\pm 1, \pm i$ (b) $2\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right), \quad k = 0, \pm 1, \pm 2$
 (c) $\cos\frac{k\pi}{5} + i\sin\frac{k\pi}{5}, \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, 5$
 2. $\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, \quad k = \pm 1, \pm 2$
 3. $\frac{1}{2}, \pm\frac{1}{2}i$
-

Exercise 4G

1. (a) $2e^{\frac{(1+4k)\pi i}{8}}$ $k = 0, 1, 2, 3$ (b) $2^{\frac{1}{6}}e^{\frac{(8k-1)\pi i}{12}}$ $k = 1, 2, 3$
 (c) $2^8e^{\frac{1}{24}(6k-1)\pi i}$ $k = 1, 2, 3, \dots, 8$
 (d) $\pm i$
 (e) $2e^{\frac{(4k+1)\pi i}{6}} - 1$ $k = 0, 1, 2$ (f) $\frac{1}{2}\left(1 + i \cot \frac{k\pi}{5}\right)$

Miscellaneous exercises 4

1. (a) $64, +\pi$
 (b) $2\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right), 2\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right),$
 $2\sqrt{2}\left(\cos \frac{-\pi}{4} + i \sin \frac{-\pi}{4}\right), 2\sqrt{2}\left(\cos \frac{-3\pi}{4} + i \sin \frac{-3\pi}{4}\right)$
 (c) $\pm 2(1+i), \pm 2(1-i)$
2. (a) $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right), 2\left(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6}\right)$
 (b)(ii) $1.41 + 0.12i, -0.81 + 1.16i, -0.60 - 1.28i$
3. (b) $\pm \sqrt{\frac{5 \pm \sqrt{5}}{8}}$ (c) $\sqrt{\frac{5 - \sqrt{5}}{8}}, \sqrt{\frac{5 + \sqrt{5}}{8}}$
4. (b) 1, 3
5. (b) $z = 1 + e^{\frac{\pi i(1+2k)}{5}}$ $k = \pm 1, \pm 2$ (c) centre $z = 1$, radius 1
 (d)(i) $\frac{\pi}{10}$ (ii) $2 \cos \frac{\pi}{10}$
6. (b)(ii) $x^2 + x - 1 = 0$ (ii) $2 \cos \frac{2\pi}{5}, 2 \cos \frac{4\pi}{5}$ (iv) $\frac{-1 + \sqrt{5}}{4}, \frac{-1 - \sqrt{5}}{4}$
7. (a)(ii) $2i \sin n\theta$ (b)(i) $A = 1, B = -3$
8. (a)(i) $2 \sin \frac{\theta}{2}$ (b) $e^{\frac{k\pi i}{3}}$ $k = \pm 1, \pm 2$
 (c)(i) coefficients of w^6 cancel (ii) $-\frac{1}{2}$



9. (a) $2\sqrt{2}e^{\frac{\pi i}{4}}$

(b) $\sqrt{2}e^{\frac{3\pi i}{4}}, \sqrt{2}e^{-\frac{7\pi i}{12}}$

(c) y

(d) $\frac{3\sqrt{3}}{2}$

