

RANDOM PROCESSES

STOCHASTIC PROCESSES

CHAPTER: 5

2.1 what is stochastic process

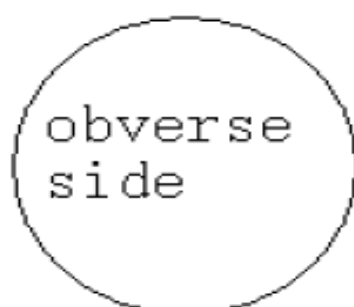
- Random event
- Random variable
- Stochastic process

What is the random event?

- The result that we can observe is not certain.

Example1:

A coin has two sides, the one is called obverse side and the other is called reverso.



- Throwing the coin , the obverse side or the reverso will be upturned.
- Before throwing it ,we can not make sure which side will be upturned.
- In a word, the result we can observe is not certain.

Example2:

How many guests will go into the supermarket , from 9 to 10 in the morning ?

The result maybe any positive integer $(0,1,2,\dots)$,which is not certain.

Random Variable

- Let $X(A)$ represent the functional relationship between a random event A and a real number. $X(A)$ is called a random variable.
- The random variable may be discrete or continuous. when $X(A)$ is countable , it is **discrete**; and $X(A)$ is an **interval of a real line**, it is **continuous**.

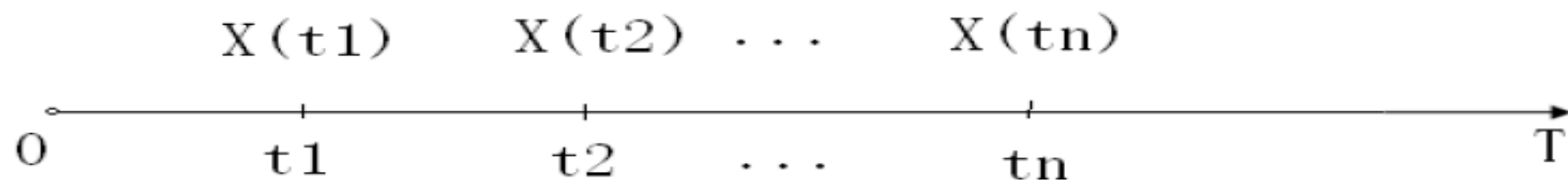
$$X(t) = \begin{cases} 1, & \text{if the obverse side} \\ & \text{is upturned} \\ 0, & \text{if the reverso} \\ & \text{is upturned} \end{cases}$$

- The probability of $X(t)=1$ is 0.5 ;
- The probability of $X(t)=0$ is 0.5 ;

The random variable $X(t)$ is as following :

$X(t)$	0	1
probability	0.5	0.5
$P\{X(t)\}$	0.5	0.5

- Let $X(t_i)$ denote state of throwing coin at time t_i .
- We observe the state of throwing coin at successive times



- The collection of random variables $X = \{X(t_i), t_i \in T\}$ can show process of throwing coin in any time period.
- The collection of random variables $X = \{X(t_i), t_i \in T\}$ is called Stochastic process.

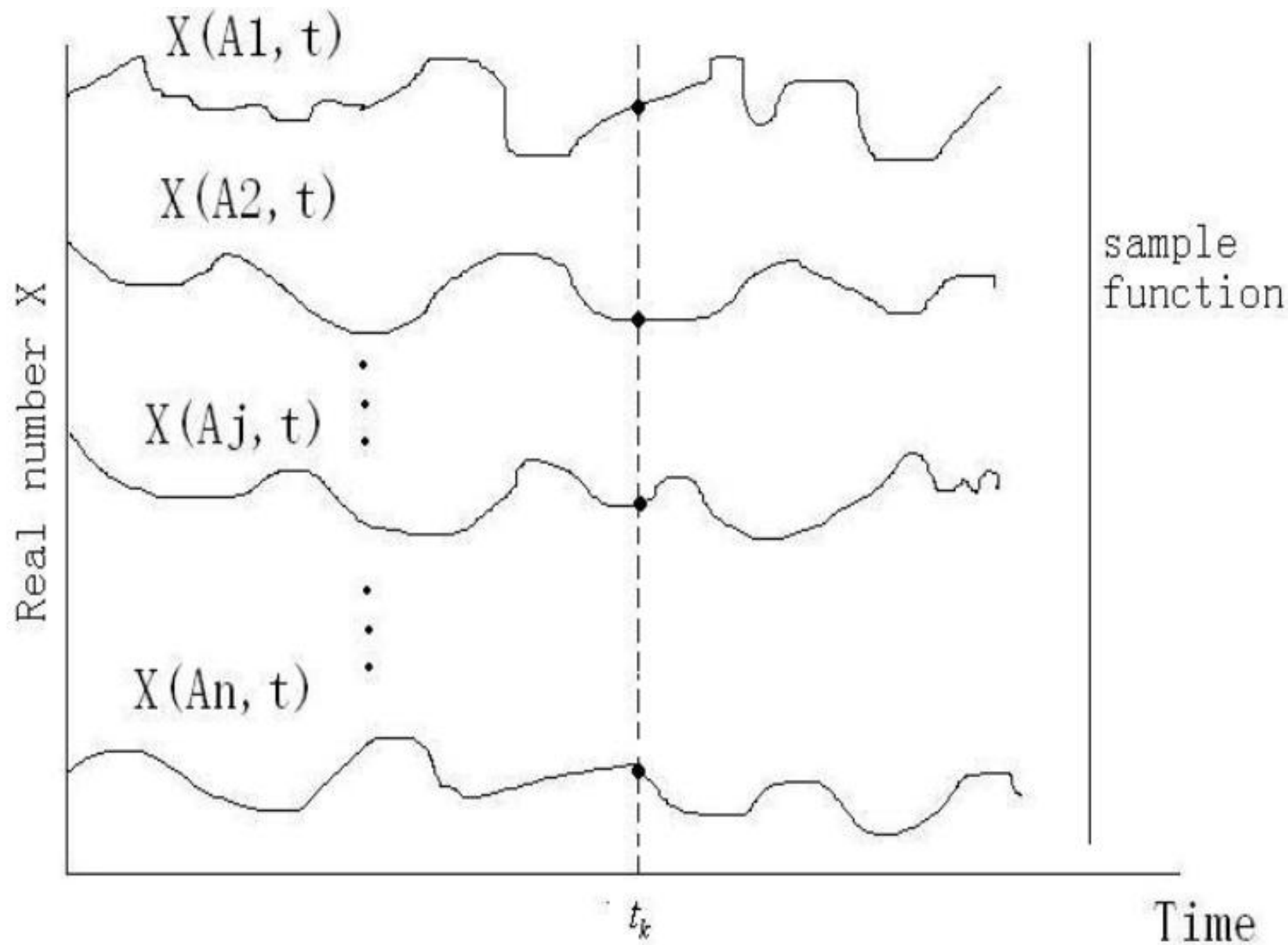
Definition of Stochastic Process .

- Let $X(t)$ denote state of throwing coin at time t .
- The collection of random variables $X = \{X(t), t \in T\}$ is called Stochastic process .

- The set T is called index set.
- When the index set T is countable, X is called a discrete-time process.
- When the index set T is an interval of the real line, X is called a continuous-time process.
- For example, we observe the state of throwing coin every minute, the index set T is $\{0, 1, 2, \dots\}$ which is countable, thus the collection of random variable $X = \{X(t), t \in T\}$ can be called a discrete-time process.

A Stochastic process $X(A,t)$ can be viewed as a function of two variables: random event A and time t .

- For a specific event A_j , we have a single time function $X(A_j,t)$ which is called a sample function.
- For a specific time t_k , $X(A,t_k)$ is a random variable whose value depends on the event A .
- For notational convenience we shall designate the random process by $X(t)$, and let the functional dependence upon A be implicit.



1.2 number character

Stochastic process $X = \{X(t), t \in T\}$ Let $X(t)$ denote state of X at time t ; $X(t)$ is the random variable obtained by observing the Stochastic process at time t .

Definition1 : we define the mean of the Stochastic process X as

$$m_X(t) = E\{X(t)\} = \begin{cases} \int_{-\infty}^{+\infty} xf_{X_t} dx & \text{when } X(t) \text{ is a continuous random variable} \\ \sum_{i=0}^{+\infty} X_i P\{X(t) = X_i\} & \text{when } X(t) \text{ is a discrete random variable} \end{cases}$$

Where $E\{\bullet\}$ is called the *expected value operator*
 $f_{X_t}(x)$ is the probability density function of $X(t)$;
 $P\{X(t) = X_i\}$ is the probability distribution of $X(t)$.

Average of random process

random process $X = \{X(t), t \geq 0\} = \{X(t, A), t \geq 0\}$

is a function with two variable , the time t and the random event A .

- Ensemble average

Expectation operator $E\{\cdot\}$

The Ensemble average of $X(t)$: $E\{X(t)\}$

The Ensemble average of the autocorrelation :

$$R_X(t, t + \tau) = E\{X(t) \cdot X(t + \tau)\}$$

- Time average

Operator : $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \{ \cdot \} dt$

The time average of $X(t)$:

$$\langle X(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$$

where $x(t)$ is any specific sample function.

The time average of the autocorrelation :

$$\langle X(t)X(t + \tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t + \tau) dt$$

Example:

Consider process $X(t) = a \cdot \cos(\omega t + \varphi)$,

a, ω are constant. RV φ is uniform In the interval $(-\pi, \pi)$

(1) Find the Ensemble average of $X(t)$ and its autocorrelation function.

(2) Find the time average of $X(t)$ and its autocorrelation function.

Solution :

(1) The Ensemble average and autocorrelation function

● Solution:

The probability density function of RV φ

$$f(\varphi) = \begin{cases} \frac{1}{\pi - (-\pi)}, & -\pi < \varphi \leq \pi \\ 0 & \text{other} \end{cases}$$

$$E\{X(t)\} = E\{a \cdot \cos(\omega t + \varphi)\}$$

$$= \int_{-\infty}^{+\infty} a \cdot \cos(\omega t + \varphi) \cdot f(\varphi) d\varphi$$

$$= \int_{-\pi}^{\pi} a \cdot \cos(\omega t + \varphi) \cdot \frac{1}{2\pi} d\varphi$$

$$= 0$$

$$R_X(t, t + \tau) = E\{X(t) \cdot X(t + \tau)\}$$

$$= E\{a \cdot \cos(\omega t + \varphi) \cdot a \cdot \cos(\omega t + \omega \tau + \varphi)\}$$

$$= E\left\{\frac{a^2}{2} \cdot [\cos(2\omega t + 2\omega \tau + \varphi) + \cos(\omega \tau)]\right\}$$

$$= \int_{-\infty}^{+\infty} \frac{a^2}{2} \cdot [\cos(2\omega t + 2\omega \tau + \varphi) + \cos(\omega \tau)] \cdot f(\varphi) d\varphi$$

$$= \frac{a^2}{2} \cdot \cos(\omega \tau)$$

(2) The time average of $X(t)$

$$\langle X(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a \cos(\omega t + \varphi) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \frac{a}{\omega} \cdot [\sin(\omega T + \varphi) - \sin(-\omega T + \varphi)]$$

$$(-2 \leq |\sin(\omega T + \varphi) - \sin(-\omega T + \varphi)| \leq 2)$$

$$= 0$$

- The time average of the autocorrelation :

$$\begin{aligned}
 \langle X(t) \cdot X(t + \tau) \rangle &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \cdot x(t + \tau) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a \cos(\omega t + \varphi) \cdot a \cos(\omega t + \omega \tau + \varphi) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T a \cos(\omega t + \varphi) \cdot a \cos(\omega t + \omega \tau + \varphi) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left\{ \frac{a^2}{4\omega} [\sin(2\omega T + \omega \tau + 2\varphi) + \sin(-2\omega T + \omega \tau + 2\varphi)] + a^2 \cdot T \cdot \cos(\omega \tau) \right\} \\
 &\quad (-2 \leq |\sin(2\omega T + \omega \tau + 2\varphi) + \sin(-2\omega T + \omega \tau + 2\varphi)| \leq 2) \\
 &= \frac{a^2}{2} \cdot \cos(\omega \tau)
 \end{aligned}$$

- random process $\{X(t), t \in T\}$ is ergodic

If it satisfies following condition .

$$E\{X(t)\} = \langle X(t) \rangle$$

$$\begin{aligned} R_X(t, t + \tau) &= E\{X(t)X(t + \tau)\} \\ &= \langle X(t)X(t + \tau) \rangle \end{aligned}$$

Example:

Consider process $X(t) = a \cdot \cos(\omega t + \varphi)$,

a, ω are constant. RV φ is uniform In the interval $(-\pi, \pi)$

$X(t)$ is ergodic ?

$$E\{X(t)\} = 0 = \langle X(t) \rangle$$

$$R_X(t, t + \tau) = E\{X(t)X(t + \tau)\}$$

$$= \frac{a^2}{2} \cos(\omega \tau)$$

$$= \langle X(t)X(t + \tau) \rangle$$

$X(t)$ is ergodic.

- Definition 2 :we define the variance function of the Stochastic process X as

$$\begin{aligned} D_X(t) &= D\{X(t)\} = E\{|X(t) - m_X(t)|^2\} \\ &= E\{|X(t)|^2\} - |m_X(t)|^2 \end{aligned}$$

- It is a measure of the “randomness” of the Stochastic process X at time t (designated by $X(t)$).
- The smaller is $D_X(t)$, the more the samples of $X(t)$ concentrate .
- It is equal to the difference between the mean-square value and the square of the mean.

- **Definition 3:** Define the autocorrelation function of the Stochastic process $X(t)$ as:

$$R_X(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\}$$

Where $X(t_1)$ and $X(t_2)$ are random variables obtained by observing $X(t)$ at times t_1 and t_2 , respectively.

- It is a function of two variables, t_1 and t_2 ;
- It is the measure of the degree to which two time samples of the same Stochastic process are related.

- **Definition 4:** The Covariance function of the Stochastic process $X(t)$ is defined as:

$$\begin{aligned}C_X(t_1, t_2) &= E\{[X(t_1) - m_X(t_1)] \cdot [X(t_2) - m_X(t_2)]\} \\&= E\{X(t_1) \cdot X(t_2)\} - m_X(t_1) \cdot m_X(t_2)\end{aligned}$$

where $X(t_1)$ and $X(t_2)$ are random variables obtained by observing $X(t)$ at times t_1 and t_2 , respectively.

- It is a function of two variables, t_1 and t_2 ;
- It is the measure of the degree to which two time samples of the same Stochastic process are related.

Example:

Consider a process $X(t) = a \cdot \cos(\omega t + \varphi)$ a, ω are constant. R.V. φ is uniform in the interval $(-\pi, \pi)$. Find the mean function, variance function and autocorrelation function.

● Solution:

The probability density function of RV φ

$$f(\varphi) = \begin{cases} \frac{1}{\pi - (-\pi)}, & -\pi < \varphi \leq \pi \\ 0 & \text{other} \end{cases}$$

$$E\{X(t)\} = E\{a \cdot \cos(\omega t + \varphi)\}$$

$$= \int_{-\infty}^{+\infty} a \cdot \cos(\omega t + \varphi) \cdot f(\varphi) d\varphi$$

$$= \int_{-\pi}^{\pi} a \cdot \cos(\omega t + \varphi) \cdot \frac{1}{2\pi} d\varphi$$

$$= 0$$

Continuing...

$$\begin{aligned}R_X(t, t + \tau) &= E\{X(t) \cdot X(t + \tau)\} \\&= E\{a \cdot \cos(\omega t + \varphi) \cdot a \cdot \cos(\omega t + \omega \tau + \varphi)\} \\&= E\left\{\frac{a^2}{2} \cdot [\cos(2\omega t + 2\omega \tau + \varphi) + \cos(\omega \tau)]\right\} \\&= \int_{-\infty}^{+\infty} \frac{a^2}{2} \cdot [\cos(2\omega t + 2\omega \tau + \varphi) + \cos(\omega \tau)] \cdot f(\varphi) d\varphi \\&= \frac{a^2}{2} \cdot \cos(\omega \tau)\end{aligned}$$

$$D_X(t) = E\{|X(t)|^2\} - |E\{X(t)\}|^2$$

For $R_X(t, t)$ Put
 $\tau = 0$

$$= R_X(t, t) - 0 = \frac{a^2}{2}$$

$$C_X(t, t + \tau) = E\{X(t) \cdot X(t + \tau)\} - m_X(t) \cdot m_X(t + \tau)$$

$$= R_X(t, t + \tau) - m_X(t) \cdot m_X(t + \tau)$$

$$= \frac{a^2}{2} \cdot \cos(\omega \tau) - 0$$

$$= \frac{a^2}{2} \cdot \cos(\omega \tau)$$

Example:

Give a RV ω with density $f(\omega)$, and an RV is φ uniform In the interval $(-\pi, \pi)$ and independent of ω , we form the process:

$$X(t) = a \cdot \cos(\omega t + \varphi)$$

Find the mean function 、 variance function and autocorrelation function.

Solution1:

$$X(t) = a \cdot \cos(\omega t + \varphi)$$

$$= a[\cos(\omega t)\cos\varphi - \sin(\omega t)\sin\varphi]$$

$$E\{X(t)\} = E\{a[\cos(\omega t)\cos\varphi - \sin(\omega t)\sin\varphi]\}$$

$$= a \cdot E\{\cos(\omega t) \cdot \cos\varphi\} - a \cdot E\{\sin(\omega t) \cdot \sin\varphi\}$$

$$= a \cdot E\{\cos(\omega t)\} \cdot E\{\cos\varphi\} - a \cdot E\{\sin(\omega t)\} \cdot E\{\sin\varphi\}$$

$$= 0$$

Continue:

The uniform probability density function of RV (ω, φ)

$$f(\varphi) = \begin{cases} \frac{1}{\pi - (-\pi)}, & -\pi < \varphi \leq \pi \\ 0 & \text{others} \end{cases}$$

$$f(\omega, \varphi) = f(\omega) \cdot f(\varphi) = \begin{cases} \frac{1}{2\pi} \cdot f(\omega), & -\pi < \varphi \leq \pi \\ 0 & \text{others} \end{cases}$$

$$E\{X(t)\} = E\{a \cdot \cos(\omega t + \varphi)\}$$

$$= \iint a \cdot \cos(\omega t + \varphi) \cdot f(\omega, \varphi) d\omega d\varphi$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\pi}^{+\pi} a \cdot \cos(\omega t + \varphi) \cdot \frac{1}{2\pi} \cdot f(\omega) d\varphi \right] d\omega$$

$$= \int_{-\infty}^{+\infty} \frac{a}{2\pi} \cdot f(\omega) \cdot [\sin(\omega t + \pi) - \sin(\omega t - \pi)] d\omega$$

$$= 0$$

- Continuing:

$$\begin{aligned}R_X(t, t + \tau) &= E\{X(t) \cdot X(t + \tau)\} \\&= E\{a \cdot \cos(\omega t + \varphi) \cdot a \cdot \cos(\omega t + \omega \tau + \varphi)\} \\&= \iint a^2 \cdot \cos(\omega t + \varphi) \cdot \cos(\omega t + \omega \tau + \varphi) \cdot f(\omega, \varphi) d\omega d\varphi \\&= \iint a^2 \cdot \frac{1}{2} [\cos(2\omega t + \omega \tau + 2\varphi) + \cos(\omega \tau)] \cdot f(\omega, \varphi) d\omega d\varphi \\&= \frac{a^2}{2} \cdot E\{\cos(\omega \tau)\} = R_X(\tau) = R_X(0, 0 + \tau)\end{aligned}$$

$$\begin{aligned}D_X(t) &= D\{X(t)\} = E\{|X(t)|^2\} - |E\{X(t)\}|^2 \\&= R_X(t, \tau = 0) - 0 \\&= \frac{a^2}{2}\end{aligned}$$

Example:

Consider random series $\{X(n), n = 0, \pm 1, \pm 2, \dots\}$,

for all i, j , $X(i)$ and $X(j)$ are independent to each other.

and for all i , $E\{X(i)\} = 0$, $D\{X(i)\} = 1$

Let $Y_n = \sum_{i=1}^n X(i)$ then $\{Y_n, n = 1, 2, \dots\}$ is a random series.

Find the mean function and autocorrelation function.

Solution:

for all i, j

$$E\{X(i)\} = 0, D\{X(i)\} = 1$$

$$E\{X(i) \cdot X(j)\} = \begin{cases} E\{X(i)\} \cdot E\{X(j)\} = 0 & i \neq j \\ E\{|X(i)|^2\} = 1 & i = j \end{cases}$$

$$E\{Y_n\} = E\left\{\sum_{i=1}^n X(i)\right\} = \sum_{i=1}^n E\{X(i)\} = 0 \cdot n = 0$$

- Continuing...

$$R_Y(n, n+m) = E\{Y_n \cdot Y_{n+m}\}$$

$$= E\left\{\sum_{i=1}^n X(i) \cdot \sum_{j=1}^{n+m} X(j)\right\} = E\left\{\sum_{i=1}^n \sum_{j=1}^{n+m} [X(i)X(j)]\right\}$$

$$= \sum_{i=1}^n \sum_{j=1}^{n+m} E[X(i)X(j)]$$

$$= \begin{cases} \sum_n E[X(i)X(j)] = 1 \cdot n = n, & i = j \\ + \\ \sum_{n(n+m)-n} E[X(i)X(j)] = 0 \cdot [n(n+m) - n] = 0, & i \neq j \end{cases}$$

$$= n$$

Properties of the autocorrelation function

For convenience of notation , we redefine

$$R_X(\tau) = E[X(t)X(t+\tau)] \text{ ,} \quad \text{for all } t$$

Property 1: The mean – square value

$$R_X(0) = E[X^2(t)] \text{ ,} \quad \tau = 0$$

Property 2:

$R_X(\tau)$ is an even function of τ

$$R_X(\tau) = R(-\tau)$$

$$R(-\tau) = E[X(t)X(t-\tau)]$$

put $t - \tau = u$

$$\begin{aligned} \therefore R(-\tau) &= E[X(u)X(u+\tau)] \\ &= R_X(\tau) \end{aligned}$$

Property 3:

The maximum value of $R_X(\tau)$ is attained at $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

Consider $E[(X(t+\tau) \pm X(t))^2] \geq 0$

$$\Rightarrow E[X^2(t+\tau)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

$$\Rightarrow 2E[X^2(t)] \pm 2R_X(\tau) \geq 0$$

$$\Rightarrow 2R_X(0) \pm 2R_X(\tau) \geq 0$$

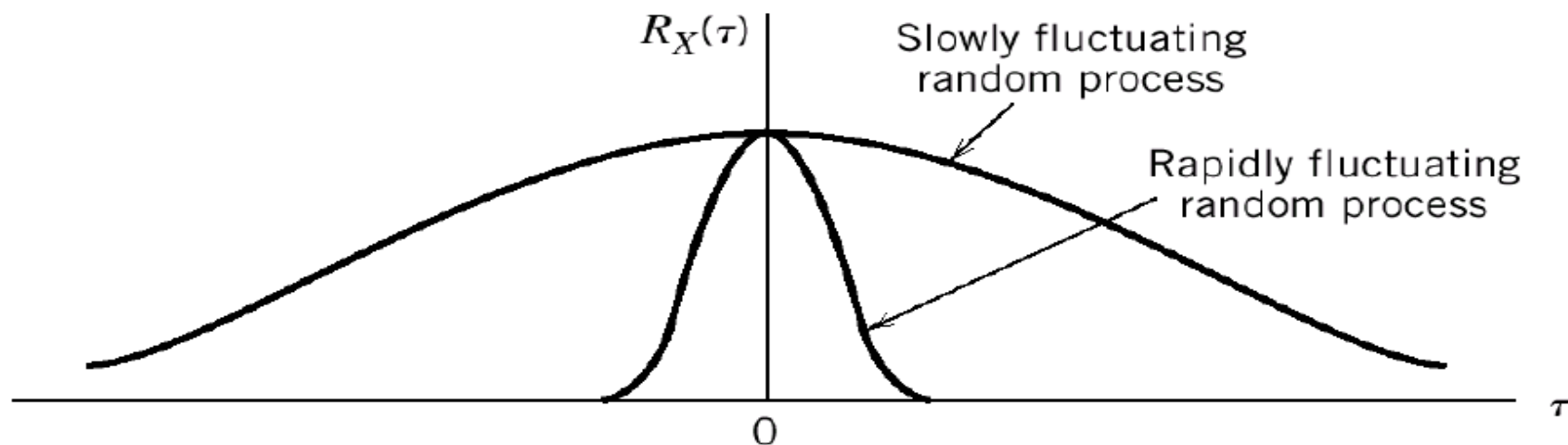
$$\Rightarrow -R_X(0) \leq R_X(\tau) \leq R_X(0)$$

$$\therefore |R_X(\tau)| \leq R_X(0)$$

Property 5:

The auto correlation function of a random process cannot have an arbitrary shape.

The $R_X(t)$ provides the interdependence information of two random variables obtained from $X(t)$ at times t seconds apart



Cross correlation and its properties

Let $\{X(t)\}$ and $\{Y(t)\}$ be two random process.

Then the cross correlation between them is defined by

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)$$

$$R_{YX}(t, t + \tau) = E[Y(t)X(t + \tau)] = R_{YX}(\tau)$$

Note: Cross correlation function of two R.P is defined as a measure of the similarity between a signal and a time delayed version of a second signal

Property 1:

Cross correlation function is not an even function

$$R_{XY}(\tau) \neq R_{YX}(\tau)$$

$$\text{But } R_{XY}(\tau) = R_{YX}(-\tau)$$

Proof:

$$\text{Let } t - \tau = \mu,$$

$$\begin{aligned}\Rightarrow R_{YX}(-\tau) &= E[Y(t)X(t - \tau)] \\ &= E[Y(\mu + \tau)X(\mu)] \\ &= E[X(\mu)Y(\mu + \tau)] \\ &= R_{XY}(\tau)\end{aligned}$$

Property 2:

If $\{X(t)\}$ and $\{Y(t)\}$ are 2 R.P's then

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

Proof:

We know that $E[(X(t) + KY(t + \tau))^2] \geq 0$ for any real K .

$$\Rightarrow E[X^2(t)] + 2KE[X(t)Y(t + \tau)] + K^2E[Y^2(t + \tau)] \geq 0$$

$$\Rightarrow R_{XX}(0) + 2R_{XY}(\tau) + K^2R_{YY}(0) \geq 0$$

If $a\lambda^2 + 2b\lambda + c \geq 0$ for all real λ

then $b^2 \leq ac$

$$\Rightarrow R_{XY}(\tau)^2 \leq R_{XX}(0)R_{YY}(0)$$

$$\therefore |R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

Property 3:

If $\{X(t)\}$ and $\{Y(t)\}$ are 2 R.P's then

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

Proof:

We know that the G.M of 2 positive numbers does not exceed their A.M. So we've

$$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

So from Property 2 we've

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

Property 4:

If the process $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal then

$$R_{XY}(\tau) = 0$$

Property 5:

If the process $\{X(t)\}$ and $\{Y(t)\}$ are independent then

$$R_{XY}(\tau) = \mu_x \mu_y$$

Problem:

Two R.P's $\{X(t)\}$ and $\{Y(t)\}$ are given by

$$X(t) = A \cos(\omega t + \theta) \text{ and } Y(t) = A \sin(\omega t + \theta)$$

where A and ω are constants and θ is uniformly distributed over $(0, 2\pi)$. Find the cross correlation.

Solution:

$$\begin{aligned} R_{XY}(\tau) &= E[X(t).Y(t+\tau)] \\ &= E[A \cos(\omega t + \theta).A \sin(\omega t + \omega \tau + \theta)] \\ &= \frac{A^2}{2} [\sin(2\omega t + \omega \tau + 2\theta) + \sin(\omega \tau)] \end{aligned}$$

Since θ is uniform in $(0, 2\pi)$, $f(\theta) = \frac{1}{2\pi}$

$$\begin{aligned} R_{XY}(\tau) &= \frac{A^2}{4\pi} \int_0^{2\pi} [\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega\tau)] d\theta \\ &= \frac{A^2}{4\pi} \left[\frac{\cos(2\omega t + \omega\tau + 2\theta)}{2} + \sin(\omega\tau)(\theta) \right]_0^{2\pi} \\ &= \frac{A^2}{4\pi} [2\pi \sin(\omega\tau)] \\ &= \frac{A^2}{2} \sin(\omega\tau) \end{aligned}$$

Stationary process

- Strict sense stationary process (SSS)
- Wide sense stationary process (WSS)

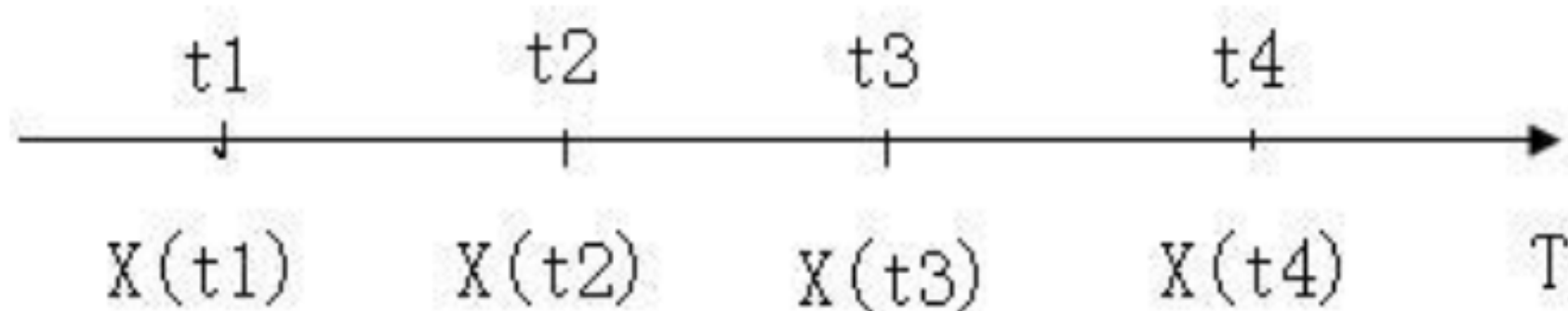
Independent-increment Process

Definition

$\{X(t), t \in T\}$ is a given process.

$$t_1 < t_2 \leq t_3 < t_4 \in T$$

$RV(X(t_2) - X(t_1))$ and $RV(X(t_4) - X(t_3))$
are independent to each other.



Stationary processes (strict-sense)

- Definition

A stochastic process $X(t)$ is called strict-sense stationary (abbreviated SSS). This means that the processes $X(t)$ and $X(t+c)$ have the same statics for any c .

from the definition it follows that the n th-order density of an SSS process must be such that

$$f(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \\ f(x_1, x_2, \dots, x_n; t_1 + c, t_2 + c, \dots, t_n + c)$$

For any c and n .

Stationary processes (wide-sense)

- A stochastic process $X(t)$ is called wide-sense stationary (abbreviated WSS) if it satisfy the following condition:

1. The mean function is constant

$$m_X(t) = E\{X(t)\} = \text{constant}$$

2. its autocorrelation depends on only $\tau = t_2 - t_1$

$$R(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\} = R(\tau = t_2 - t_1)$$

Note in particular that :

$$\text{Mean} = \lim_{|\tau| \rightarrow \infty} R_{XX}(\tau)$$

$$E\{|X(t)|^2\} = R(0)$$

$$E[X^2(t)] = R_{XX}(0)$$

Thus the average power of a stationary process is independent to t and it equal $R(0)$.

● Example

Consider random process $X(t) = a \cos(\omega t + \varphi)$

A and ω are constant, RV φ is uniform

In the interval $(-\pi, \pi)$.

$$(1) \quad m_X(t) = E\{X(t)\} = 0$$

$$(2) \quad R_X(t, t + \tau) = E\{X(t) \cdot X(t + \tau)\} = \frac{a^2}{2} \cos(\omega \tau)$$

$X(t) = a \cos(\omega t + \varphi)$ is WSS

● Example

Consider the random process $X(t) = A \cdot \cos(\pi t)$,
R.V A follows normal distribution , the mean of
 A is 0 , the variance is σ^2 .

$$EA = 0, \quad DA = \sigma^2$$

$$E\{A^2\} = DA + |EA|^2 = \sigma^2 \quad D(A) = E\{A^2\} - E|A|^2$$

$$(1) m_X(t) = E\{X(t)\}$$

$$= E\{A \cdot \cos(\pi t)\}$$

$$= \cos(\pi t) \cdot E\{A\} = 0$$

$$(2) R_X(t, t + \tau) = E\{X(t) \cdot X(t + \tau)\}$$

$$= E\{A \cos(\pi t) \cdot A \cos(\pi t + \pi \tau)\}$$

$$= \cos(\pi t) \cos(\pi t + \pi \tau) E\{A^2\}$$

$$= \frac{1}{2} \cdot [\cos(2\pi t + \pi \tau) + \cos(\pi \tau)] \cdot \sigma^2$$

$X(t) = A \cdot \cos(\pi t)$ is not WSS.

Problem:

1. A stationary process has an autocorrelation function given by

$R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$. Find the mean and variance of the process.

Solution:

$$\left(\overline{X}\right)^2 = \lim_{|\tau| \rightarrow \infty} R_{XX}(\tau) = \lim_{|\tau| \rightarrow \infty} \frac{\tau^2 \left(25 + \frac{36}{\tau^2} \right)}{\tau^2 \left(6.25 + \frac{4}{\tau^2} \right)}$$

$$= 4 \Rightarrow \text{Mean} = 2$$

$$E[X^2(t)] = R_{XX}(0) = 9$$

$$\text{Variance} = E[X^2(t)] - E[X(t)]^2 = 5$$

Weiner Process

$W = \{W(t), -\infty < t < +\infty\}$ is called a Wiener process if it satisfy the following condition:

- (1) $W(0)=0$;
- (2) Independent-increment process;
- (3) For all t, s , $W(t) - W(s) \sim N(0, \sigma^2 \cdot |t - s|)$, $\sigma^2 > 0$

W is also called Brown motion which is a model of noise in communication and molecule motion .

● Property of weiner process

- (1) For all t ,
- $$W(t) \sim N(0, \sigma^2 \cdot |t|)$$

$$f_W(w) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left\{-\frac{w^2}{2\sigma^2}\right\}$$

- (2) The autocorrelation :

$$0 < t < s$$

$$R_W(t, s) = E\{W(t) \cdot W(s)\} = \sigma^2 \cdot t$$

● Example:

$W = \{W(t), -\infty < t < +\infty\}$ is a weiner process
with parameter σ^2 , let $X(t) = e^{-\alpha t}W(e^{2\alpha t}), \alpha > 0$
 α is a constant.

Find the mean $m_X(t)$ and the autocorelation

$$R_X(t, t + \tau), \tau > 0, t > 0$$

Solution:

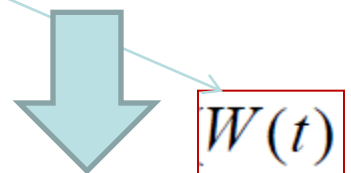
$$W(t) \sim N(0, \alpha^2 \cdot |t|)$$

$$E\{W(t)\} = 0, D\{W(t)\} = \alpha^2 \cdot |t|$$

$$m_X(t) = E\{X(t)\}$$

$$= E\{e^{-\alpha t}W(e^{2\alpha t})\} = e^{-\alpha t} \cdot E\{W(e^{2\alpha t})\}$$

$$= 0$$



$$\begin{aligned}
 R_X(t, t + \tau) &= E\{X(t) \cdot X(t + \tau)\} \\
 &= E\{e^{-\alpha t} W(e^{2\alpha t}) \cdot e^{-\alpha(t+\tau)} W(e^{2\alpha t + 2\alpha\tau})\} \\
 &= e^{-\alpha t} \cdot e^{-\alpha(t+\tau)} \cdot E\{W(e^{2\alpha t}) \cdot W(e^{2\alpha t + 2\alpha\tau})\} \\
 &= e^{-2\alpha t - 2\alpha\tau} \cdot \sigma^2 \cdot e^{2\alpha t} \\
 &= \sigma^2 \cdot e^{-2\alpha\tau}
 \end{aligned}$$

USE AUTOCORRELATION FORMULA

$$0 < t < s$$

$$R_W(t, s) = E\{W(t) \cdot W(s)\} = \sigma^2 \cdot t$$

Assignment : question1

- Random process $X(t) = Vt + b$, b is a constant , R.V V follows normal distribution , and $EV = 0, DV = 1$

- (1) Find the mean function $m_X(t)$ 、 variance function $D_X(t)$ and autocorrelation function $R_X(t, t + \tau), t > 0, \tau > 0$.
- (2) Find the first-order probability density function $f_X(t)$.

Question 2

- Random process $X(t) = A \cdot \cos \omega t + B \cdot \sin \omega t$, ω is constant , R.V A and B are mutually independent. $EA = EB = 0, DA = DB = \sigma^2$. Both A and B follows normal distribution.

- (1) Find the mean function $m_X(t)$ 、 variance function $D_X(t)$ and autocorrelation function $R_X(t, t + \tau)$.
- (2) Find the first-order probability density function $f_X(t)$
- (3) Is the Random process $X(t)$ WSS ?

power spectral density for random processes

- Let $\{X(t), t \in T\}$ be a wide-sense stationary processes (WSS) ,

$$P_{Xav} = E \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} |X(t)|^2 dt \right\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E \left[|X(t)|^2 \right] dt$$

P_{Xav} is the total average power of $X(t)$.

P_{Xav} can be denoted as

$$P_{Xav} = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega$$

$S_X(\omega)$ is the power spectral density of $X(t)$.

Spectral Property

If $X(t)$ is WSS ,

$$(1) \quad P_{X_{av}} = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega = R_X(\tau = 0)$$

(2) $R_X(\tau)$ and $S_X(\omega)$ are Fourier transform pair.

if $X(t)$ is a continuous WSS

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$R_X(\tau) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) \cdot e^{j\omega\tau} d\omega$$

if $X(t)$ is a discrete WSS

$$S_X(\omega) = \sum_{k=-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega k}$$

$$R_X(\tau) = \frac{1}{2\pi} \cdot \int_{-\pi}^{+\pi} S_X(\omega) \cdot e^{j\omega k} d\omega$$

(3) $S_X(\omega)$ is a even function and more than zero.

$$S_X(\omega) = S_X(-\omega); \quad S_X(\omega) > 0$$

if $X(t)$ is a discrete WSS

$$S_X(\omega) = \sum_{k=-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega k}$$

$$R_X(\tau) = \frac{1}{2\pi} \cdot \int_{-\pi}^{+\pi} S_X(\omega) \cdot e^{j\omega k} d\omega$$

(3) $S_X(\omega)$ is a even function and more than zero.

$$S_X(\omega) = S_X(-\omega); \quad S_X(\omega) > 0$$

(1) Showing :
$$P_{X_{av}} = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega = R_X(\tau = 0)$$

$$X(t) \xleftrightarrow{\text{Fourier transform}} G(\omega)$$

According to Parseval's theorem , the total energy of $X(t)$

$$Q_X = \int_{-\infty}^{+\infty} |X(t)|^2 dt = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} |G(\omega)|^2 d\omega$$

Let $X_T(t)$ be defined as :
$$X_T(t) = \begin{cases} X(t), & |t| \leq T \\ 0, & |t| > T \end{cases}$$

$$X_T(t) \xleftrightarrow{\text{Fourier transform}} G_T(\omega)$$

$$G_T(\omega) = \int_{-\infty}^{+\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^T X(t) e^{-j\omega t} dt$$

the total energy of $X_T(t)$

$$Q_{X_T} = \int_{-\infty}^{+\infty} |X_T(t)|^2 dt = \int_{-T}^{+T} |X(t)|^2 dt = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} |G_T(\omega)|^2 d\omega$$

$$\lim_{T \rightarrow \infty} Q_{X_T} = Q_X$$

The time average of Q_{X_T}

$$\frac{Q_{X_T}}{2T} = \frac{1}{2T} \cdot \int_{-T}^{+T} |X(t)|^2 dt = \frac{1}{2T} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G_T(\omega)|^2 d\omega$$

Then the ensemble average of $\frac{Q_{X_T}}{2T}$

$$E \left\{ \frac{Q_{X_T}}{2T} \right\} = E \left\{ \frac{1}{2T} \cdot \int_{-T}^{+T} |X(t)|^2 dt \right\} = E \left\{ \frac{1}{2T} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G_T(\omega)|^2 d\omega \right\}$$

- Continuing...

the total average power of $X(t)$:

$$\begin{aligned}
 P_{Xav} &= \lim_{T \rightarrow \infty} E \left\{ \frac{Q_{X_T}}{2T} \right\} = \lim_{T \rightarrow \infty} E \left\{ \frac{1}{2T} \cdot \int_{-T}^{+T} |X(t)|^2 dt \right\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \int_{-T}^{+T} E \left[|X(t)|^2 \right] dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \int_{-T}^{+T} R_X(t, t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \int_{-T}^{+T} R_X(\tau = 0) dt = R_X(\tau = 0)
 \end{aligned}$$

$$P_{Xav} = R_X(\tau = 0)$$

$$\begin{aligned}
 P_{Xav} &= \lim_{T \rightarrow \infty} E \left\{ \frac{Q_{X_T}}{2T} \right\} = \lim_{T \rightarrow \infty} E \left\{ \frac{1}{2T} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G_T(\omega)|^2 d\omega \right\} \\
 &= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[|G_T(\omega)|^2 \right] \right\} d\omega
 \end{aligned}$$

Let $S_X(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[|G_T(\omega)|^2 \right]$, then

$$P_{Xav} = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega$$

(2) Showing: $R_X(\tau) \xleftrightarrow{\text{Fourier transform}} S_X(\omega)$

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[|G_T(\omega)|^2 \right]$$

$$G_T(\omega) = \int_{-T}^T X(t) e^{-j\omega t} dt$$

$$\frac{1}{2T} E \left[|G_T(\omega)|^2 \right] = \frac{1}{2T} E \left[|G_T(\omega)|^2 \right] = \frac{1}{2T} E \left[G_T(\omega) \cdot \overline{G_T(\omega)} \right]$$

$$= \frac{1}{2T} E \left[\int_{-T}^T X(t) e^{-j\omega t} dt \cdot \overline{\int_{-T}^T X(s) e^{-j\omega s} ds} \right]$$

$$= \frac{1}{2T} E \left[\int_{-T}^T X(t) e^{-j\omega t} dt \cdot \int_{-T}^T X(s) e^{j\omega s} ds \right]$$

$$= \frac{1}{2T} E \left[\int_{-T}^T \int_{-T}^T X(t) X(s) e^{-j\omega(t-s)} dt ds \right]$$

$$= \frac{1}{2T} \left[\int_{-T}^T \int_{-T}^T E[X(t) X(s)] \cdot e^{-j\omega(t-s)} dt ds \right]$$

$$= \frac{1}{2T} \left[\int_{-T}^T \int_{-T}^T R_X(t-s) \cdot e^{-j\omega(t-s)} dt ds \right]$$

- Continuing...

$$\text{let } u = t + s, \quad \tau = t - s \Rightarrow t = \frac{u + \tau}{2}, s = \frac{u - \tau}{2}$$

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial \tau} \\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial \tau} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$S_X(\omega) = \frac{1}{2T} \left[\int_{-T}^T \int_{-T}^T R_X(t-s) \cdot e^{-j\omega(t-s)} dt ds \right]$$

$$= \frac{1}{2T} \left[\iint |J| \cdot R_X(\tau) \cdot e^{-j\omega\tau} du d\tau \right], \quad -T \leq \frac{u + \tau}{2} \leq T, \quad -T \leq \frac{u - \tau}{2} \leq T$$

$$= \frac{1}{2T} \left[\int_{-2T}^{2T} \left[\int_{-(2T-|\tau|)}^{2T-|\tau|} |J| \cdot R_X(\tau) \cdot e^{-j\omega\tau} du \right] d\tau \right]$$

$$= \int_{-2T}^{2T} \left[1 - \frac{|\tau|}{2T} \right] \cdot R_X(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\int_{-T}^T \int_{-T}^T R_X(t-s) \cdot e^{-j\omega(t-s)} dt ds \right] = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega\tau} d\tau$$

(3) showing: $S_X(\omega) = S_X(-\omega)$

$$R_X(\tau) = R_X(-\tau)$$

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$S_X(-\omega) = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{j\omega\tau} d\tau \quad \text{let } \tau_1 = -\tau$$

$$= \int_{+\infty}^{-\infty} R_X(-\tau_1) \cdot e^{-j\omega\tau_1} d(-\tau_1)$$

$$= \int_{-\infty}^{+\infty} R_X(\tau_1) \cdot e^{-j\omega\tau_1} d(\tau_1)$$

$$S_X(\omega) = S_X(-\omega)$$

Example: $X(t)$ is a WSS process, the mean $m_X(t) = 0$, the autocorrelation $R_X(\tau) = 4e^{-|\tau|} \cdot \cos(\pi\tau)$,
(1) Find the power spectral density $S_X(\omega)$;
(2) Find the total average power P_{Xav} .

Solution: (1)

$$R_X(\tau) \xleftrightarrow{\text{Fourier transform}} S_X(\omega)$$

$$\begin{aligned} S_X(\omega) &= \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j\omega\tau} d\tau \\ &= 4 \cdot \left[\frac{1}{1 + (\omega + \pi)^2} + \frac{1}{1 + (\omega - \pi)^2} \right] \end{aligned}$$

$$\begin{aligned}
 (2) \quad P_{X_{av}} &= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega \\
 &= R_X(\tau = 0) \\
 &= 4
 \end{aligned}$$

Example: $X(t)$ is a WSS process, the mean $m_X(t) = 0$, the power spectral density

$$S_X(\omega) = \frac{2\omega^2 + 1}{\omega^4 + 5\omega^2 + 4}$$

(1) Find the autocorrelation $R_X(\tau)$;

(2) Find the total average power P_{Xav} .

Solution: (1)

$$R_X(\tau) \xleftrightarrow{\text{Fourier transform}} S_X(\omega)$$

$$R_X(\tau) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) \cdot e^{j\omega\tau} d\omega = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \frac{2\omega^2 + 1}{\omega^4 + 5\omega^2 + 4} \cdot e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \left[\frac{2k_1}{\omega^2 + 1} + \frac{4k_2}{\omega^2 + 4} \right] \cdot e^{j\omega\tau} d\omega$$

$$= k_1 e^{-|\tau|} + k_2 e^{-2|\tau|}$$

$$k_1 = -\frac{1}{6}, \quad k_2 = \frac{7}{12}$$

$$(2) \quad P_{X_{av}} = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_X(\omega) d\omega$$

$$= R_X(\tau = 0)$$

$$= k_1 + k_2 = \frac{5}{12}$$