

Development of the Plate Bending Element

Introduction

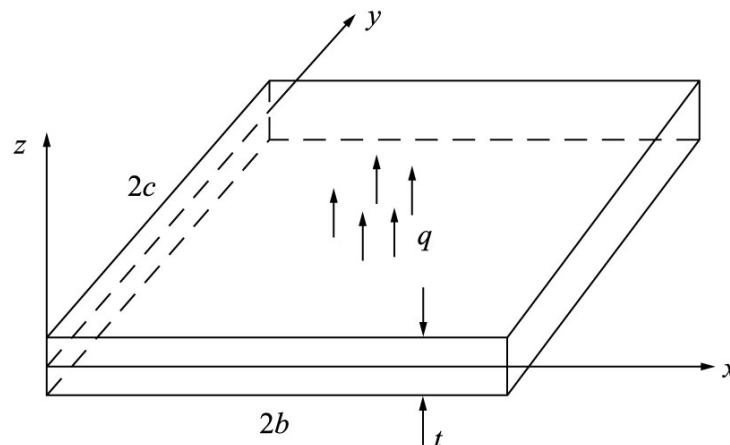
In this section we will begin by describing elementary concepts of plate bending behavior and theory. The plate element is one of the more important structural elements and is used to model and analyze such structures as pressure vessels, chimney stacks, and automobile parts. A large number of plate bending element formulations exist that would require lengthy chapter to cover. The purpose in this chapter is to present the derivation of the stiffness matrix for one of the most common plate bending finite elements and then to compare solutions to some classical problems for a variety of bending elements in the literature.

Basic Concepts of Plate Bending

A plate can be considered the two-dimensional extension of a beam in simple bending. Both plates and beams support loads transverse or perpendicular to their plane and through bending action. A plate is a flat (if it were curved, it would be a shell). A beam has a single bending moment resistance, while a plate resists bending about two axes and has a twisting moment. We will consider the classical thin-plate theory or Kirchhoff plate theory.

Basic Behavior of Geometry and Deformation

Consider the thin plate in the x - y plane of thickness t measured in the z direction shown in the figure below:

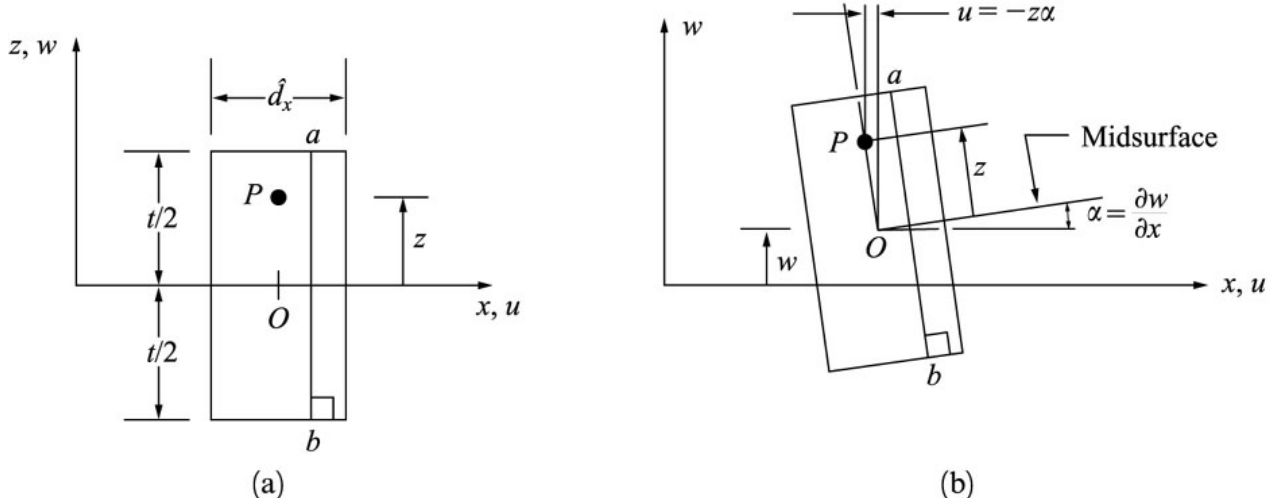


The plate surfaces are at $z = \pm t/2$, and its midsurface is at $z = 0$. The basic geometry of the plate is as follows:

1. The plate thickness is much smaller than its inplane dimensions b and c (that is, $t \ll b$ or c). If t is more than about one-tenth the span of the plate, then transverse shear deformation must be accounted for and the plate is then said to be thick.
2. The deflection w is much less than the thickness t (that is, $w/t \ll 1$).

Kirchhoff Assumptions

Consider the differential slice cut from the plate by planes perpendicular to the x axis as shown in the figure below:



Loading q causes the plate to deform laterally or upward in the z direction and, the deflection w of point P is assumed to be a function of x and y only; that is $w=w(x, y)$ and the plate does not stretch in the z direction. The line $a-b$ drawn perpendicular to the plate surface before loading remains perpendicular to the surface after loading. These conditions are consistent with the Kirchhoff assumptions:

1. Normals remain normal. This implies that transverse shears strains $\gamma_{yz} = 0$ and $\gamma_{xz} = 0$. However γ_{xy} does not equal to zero. Right angles in the plane of the plate may not remain right angles after loading. The plate may twist in the plane.
2. Thickness changes can be neglected and normals undergo no extension. This means that $\epsilon_z = 0$.

3. Normal stress σ_z has no effect on in-plane strains ε_x and ε_y in the stress-strain equations and is considered negligible.
4. Membrane or in-plane forces are neglected here, and the plane stress resistance can be superimposed later (that is, the constant-strain triangle behavior of Chapter 6 can be superimposed with the basic plate bending element resistance). Therefore, the in-plane deflections in the x and y directions at the midsurface, $t = 0$, are assumed to be zero; $u(x, y, 0) = 0$ and $v(x, y, 0) = 0$.

Based on Kirchhoff assumptions, at any point P the displacement in the x direction due to a small rotation α is:

$$u = -z\alpha = -z \left(\frac{\partial w}{\partial x} \right)$$

At the same point, the displacement in the y direction is:

$$v = -z\alpha = -z \left(\frac{\partial w}{\partial y} \right)$$

The curvatures of the plate are then given as the rate of change of the angular displacements of the normals and defined as:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} \quad \kappa_{xy} = -\frac{2\partial^2 w}{\partial x \partial y}$$

Using the definitions for in-plane strains, along with the curvature relationships, the in-plane strain/displacement equations are:

$$\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2} \quad \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2} \quad \gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}$$

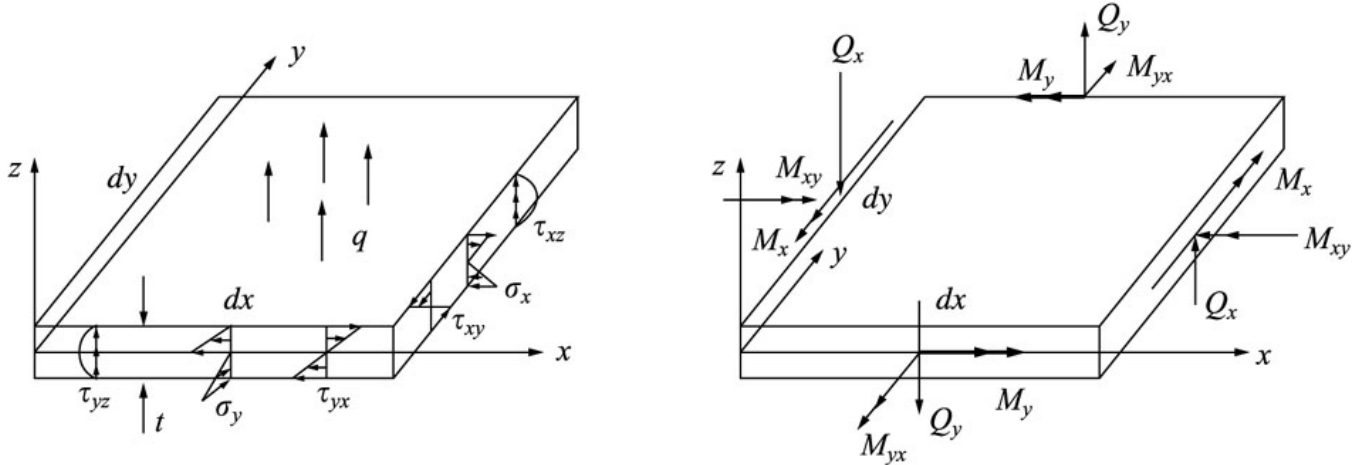
The first of the above equations is used in beam theory. The remaining two equations are new to plate theory.

Stress/Strain Relationship

Based on the third Kirchhoff assumption, the plane stress equations that relate in-plane stresses to in-plane strains for an isotropic material are:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \quad \tau_{xy} = G\gamma_{xy}$$

The in-plane normal stresses and shear stress are shown acting on the edges of the plate shown in figure below:



Similar to the stress variation in a beam, the stresses vary linearly in the z direction from the midsurface of the plate. The transverse shear stresses τ_{yz} and τ_{xz} are also present, even though transverse shear deformation is neglected. These stresses vary quadratically through the plate thickness.

The bending moments acting along the edge of the plate can be related to the stresses by:

$$M_x = \int_{-t/2}^{t/2} z \sigma_x dz \quad M_y = \int_{-t/2}^{t/2} z \sigma_y dz \quad M_{xy} = \int_{-t/2}^{t/2} z \tau_{xy} dz$$

Substituting strains for stresses gives:

$$M_x = \int_{-t/2}^{t/2} z \left(\frac{E}{1-\nu^2} (\varepsilon_x + \nu\varepsilon_y) \right) dz \quad M_y = \int_{-t/2}^{t/2} z \left(\frac{E}{1-\nu^2} (\varepsilon_y + \nu\varepsilon_x) \right) dz \quad M_{xy} = \int_{-t/2}^{t/2} z G \gamma_{xy} dz$$

Using the strain/curvature relationships, the moment expression become:

$$M_x = D(\kappa_x + \nu\kappa_y) \quad M_y = D(\kappa_y + \nu\kappa_x) \quad M_{xy} = \frac{D(1-\nu)}{2} \kappa_{xy}$$

where $D = Et^3/[12(1 - \nu^2)]$ is called the bending rigidity of the plate.

The maximum magnitude of the normal stress on each edge of the plate are located at the top or bottom at $z = t/2$.

$$\sigma_x = \frac{6M_x}{t^2}$$

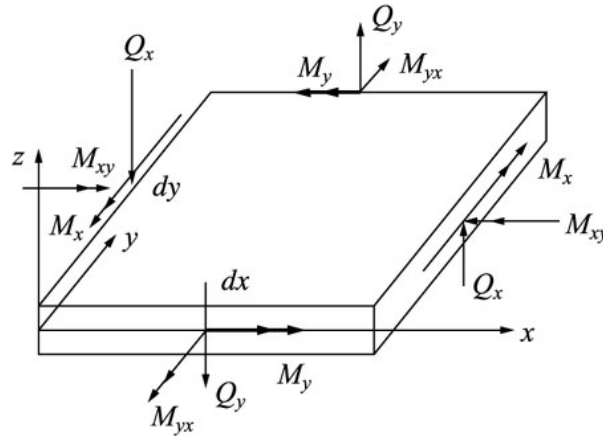
The equilibrium equations for plate bending are important in selecting the element displacement fields. The governing differential equations are:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

where q is the transverse distributed loading and Q_x and Q_y are the transverse shear line loads as shown below.



Substituting the moment/curvature expressions in the last two differential equations listed above, solving for Q_x and Q_y , and substituting the results into the first equation listed above, the governing partial differential equation for isotropic, thin-plate bending may be derived as:

$$D \left(\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q$$

where the solution to the thin-plate bending is a function of the transverse displacement w . If we neglect the differentiation with respect to the y direction, the

above equation simplifies to the equation for a beam and the flexural rigidity D of the plate reduces to the EI of the beam when the Poisson effect is set to zero.

Potential Energy of a Plate

The total potential energy of a plate is given as:

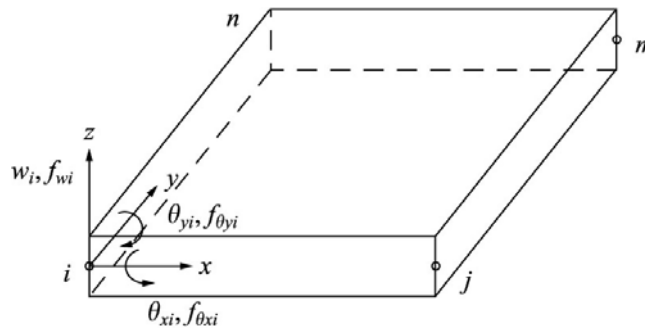
$$U = \frac{1}{2} \int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) dV$$

The potential energy can be expressed in terms of moments and curvatures as:

$$U = \frac{1}{2} \int_A (M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy}) dA$$

Derivation of a Plate Bending Element Stiffness Matrix and Equations

Numerous finite elements for plates bending have been developed over the years, references cite 88 different elements. In this section, we will introduce the basic 12-degree-of-freedom rectangular element shown below.



The formulation will be developed consistently with the stiffness matrix and equations for the bar, beam, plane stress/strain elements of previous chapters.

Step 1 - Discretize and Select Element Types

Consider the 12-degree-of-freedom plate element shown in the figure above. Each node has 3 degrees of freedom – a transverse displacement w in the z direction, a rotation θ_x about the x axis, and a rotation θ_y about the y axis.

The nodal displacements at node i are:

$$\{d\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix}$$

where the rotations are related to the transverse displacements by:

$$\theta_x = \frac{\partial w}{\partial y} \quad \theta_y = -\frac{\partial w}{\partial x}$$

The negative sign on θ_y is due to the fact that a negative displacement w is required to produce a positive rotation about the y axis.

The total element displacement matrix is:

$$\{d\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \\ d_n \end{Bmatrix}$$

Step 2 - Select Displacement Functions

Since the plate element has 12 degrees of freedom, we select a 12-term polynomial in x and y as:

$$w(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 \\ + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3$$

The function given above is an incomplete quartic polynomial; however, it is complete up to the third order (first ten terms), and the choice of the two more terms from the remaining five terms of the complete quartic must be made. The choice of x^3y and y^3x ensure that we will have continuity in the displacement among the interelement boundaries. The terms x^4 and y^4 would yield discontinuities along the interelement boundaries. The final term x^2y^2 cannot be paired with any other term so it is also rejected. The displacement function approximation also satisfies the basic differential equation over the unloaded part of the plate. In addition, the function accounts for rigid-body motion and constant strain in the

plate. However, interelement slope discontinuities along common boundaries of elements are not ensured.

To observe these discontinuities in slope, evaluate the polynomial and its slopes along a side or edge. For example, consider side $i-j$, the function gives:

$$w(x,y) = a_1 + a_2x + a_4x^2 + a_7x^3$$

$$\frac{\partial w}{\partial x} = a_2 + 2a_4x + 3a_7x^2$$

$$\frac{\partial w}{\partial y} = a_3 + a_5x + a_8x^2 + a_{11}x^3$$

The displacement w is cubic while the slope $\partial w / \partial x$ is the same as in beam bending. Based on the beam element, recall that the four constants a_1 , a_2 , a_4 , and a_7 can be defined by invoking the endpoint conditions of w_i , w_j , θ_i , and θ_j . Therefore, w and $\partial w / \partial x$ are completely defined along this edge. The normal slope $\partial w / \partial y$ is cubic in x : however, only two degrees of freedom remain for definition of this slope while four constants exist a_3 , a_5 , a_8 , and a_{11} . This slope is not uniquely defined and a slope discontinuity occurs. The solution obtained from the finite element analysis using this element will not be a minimum potential energy solution. However, this element has proven to give acceptable results.

The constant a_1 through a_{12} can be determined by expressing the 12 simultaneous equations linking the values of w and its slope at the nodes when the coordinates take their appropriate values.

$$\begin{Bmatrix} w \\ \frac{\partial w}{\partial y} \\ -\frac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & 2x^2 & -2xy & -y^2 & 0 & -2x^2y & -y^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{12} \end{Bmatrix}$$

or in matrix form as:

$$\{\psi\} = [P]\{a\}$$

where $[P]$ is the 3 x 12 first matrix on the right-hand side of the above equation.

Next, evaluate the matrix at each node point

$$\begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \\ w_j \\ \vdots \\ \theta_{yn} \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & x_i^2 & x_i y_i & y_i^2 & x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 & x_i^3 y_i & x_i y_i^3 \\ 0 & 0 & 1 & 0 & x_i & 2y_i & 0 & x_i^2 & 2x_i y_i & 3y_i^2 & x_i^3 & 3x_i y_i^2 \\ 0 & -1 & 0 & -2x_i & -y_i & 0 & -2x_i^2 & -2x_i y_i & -y_i^2 & 0 & -2x_i^2 y_i & -y_i^3 \\ 1 & x & y & x_j^2 & x_j y_j & y_j^2 & x_j^3 & x_j^2 y_j & x_j y_j^2 & y_j^3 & x_j^3 y_j & x_j y_j^3 \\ \vdots & & & & & & & & & & & \\ 0 & -1 & 0 & -2x_n & -y_n & 0 & -2x_n^2 & -2x_n y_n & -y_n^2 & 0 & -2x_n^2 y_n & -y_n^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{12} \end{Bmatrix}$$

In compact matrix form the above equations are:

$$\{d\} = [C]\{a\}$$

Therefore, the constants $\{a\}$ can be solved for by:

$$\{a\} = [C]^{-1}\{d\}$$

Substituting the above expression into the general form of the matrix gives:

$$\{\psi\} = [P][C]^{-1}\{d\} \quad \text{or} \quad \{\psi\} = [N]\{d\}$$

where $[N] = [P][C]^{-1}$ is the shape function matrix.

Step 3 - Define the Strain (Curvature)/Displacement and Stress (Moment)/Curvature Relationships

Recall the general form of the curvatures:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} \quad \kappa_{xy} = -\frac{2\partial^2 w}{\partial x \partial y}$$

The curvature matrix can be written as:

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -2a_4 - 6a_7x - 2a_8y - 6a_{11}xy \\ -2a_6 - 2a_9x - 6a_{10}y - 6a_{12}xy \\ -2a_5 - 4a_8x - 4a_9y - 6a_{11}x^2 - 6a_{12}y^2 \end{Bmatrix}$$

or in matrix form as:

$$\{\kappa\} = [Q]\{a\}$$

where [Q] is the coefficient matrix multiplied by the a 's in the curvature matrix equations.

$$\{Q\} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4x & -4y & 0 & -6x^2 & -6y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{12} \end{Bmatrix}$$

Therefore:

$$\{\kappa\} = [Q]\{a\} \quad \Rightarrow \quad \{\kappa\} = [Q][C]^{-1}\{d\} \quad \text{or} \quad \{\kappa\} = [B]\{d\}$$

where

$$[B] = [Q][C]^{-1}$$

The moment/curvature matrix for a plate is given by:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = [D][B]\{d\}$$

where the [D] matrix for isotropic materials is:

$$[D] = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix is given by the usually form of the stiffness matrix as:

$$[k] = \iint [B]^T [D][B] dxdy$$

The stiffness matrix for the four-node rectangular element is of a 12 x 12.

The surface force due to distributed loading q acting per unit area in the z direction is:

$$[F_s] = \iint [N_s]^T q dxdy$$

for a uniform load q acting over the surface of an element of dimensions $2b \times 2c$ the forces and moments at node i are:

$$\begin{Bmatrix} f_{wi} \\ f_{\theta xi} \\ f_{\theta yi} \end{Bmatrix} = \frac{qcb}{3} \begin{Bmatrix} 3 \\ -c \\ b \end{Bmatrix}$$

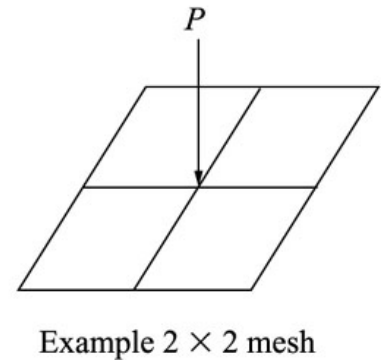
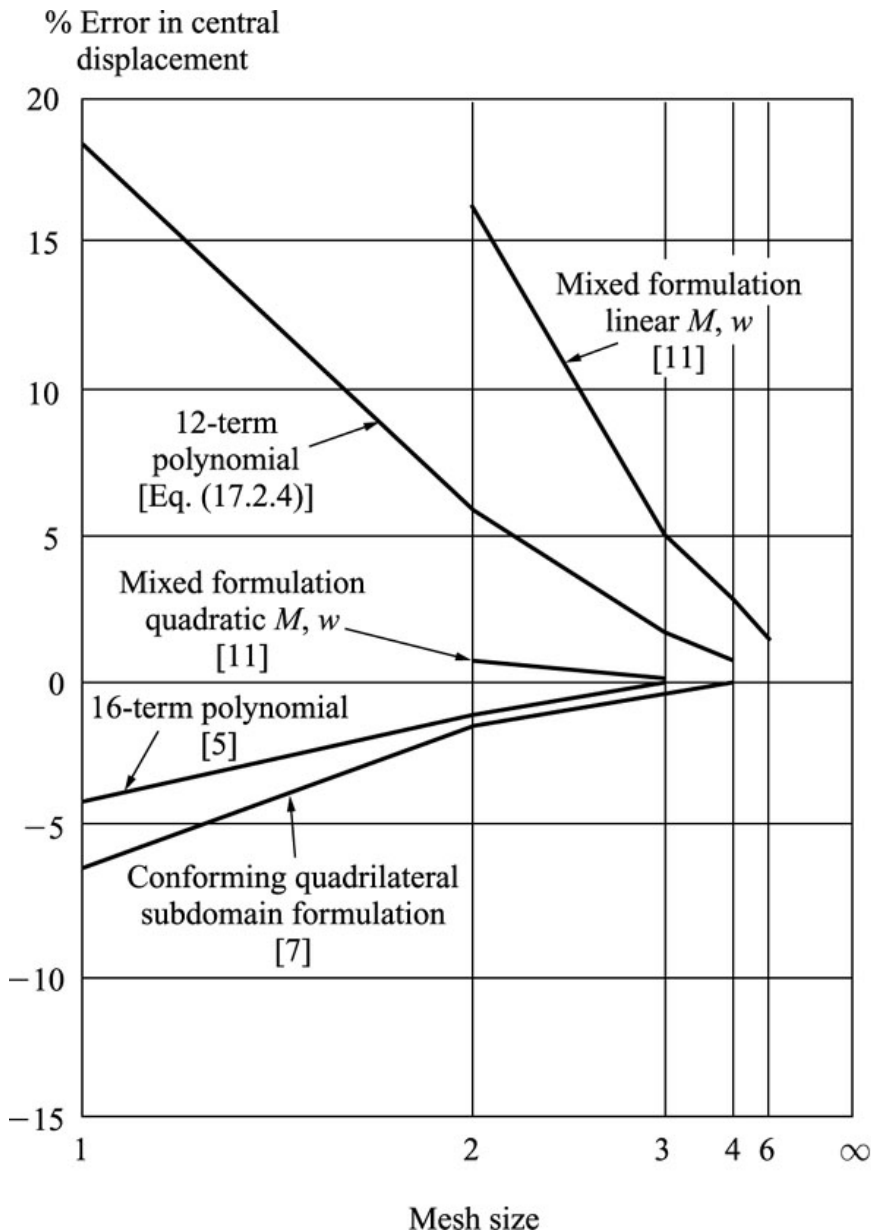
with similar expression at nodes j , m , and n .

The element equations are given by:

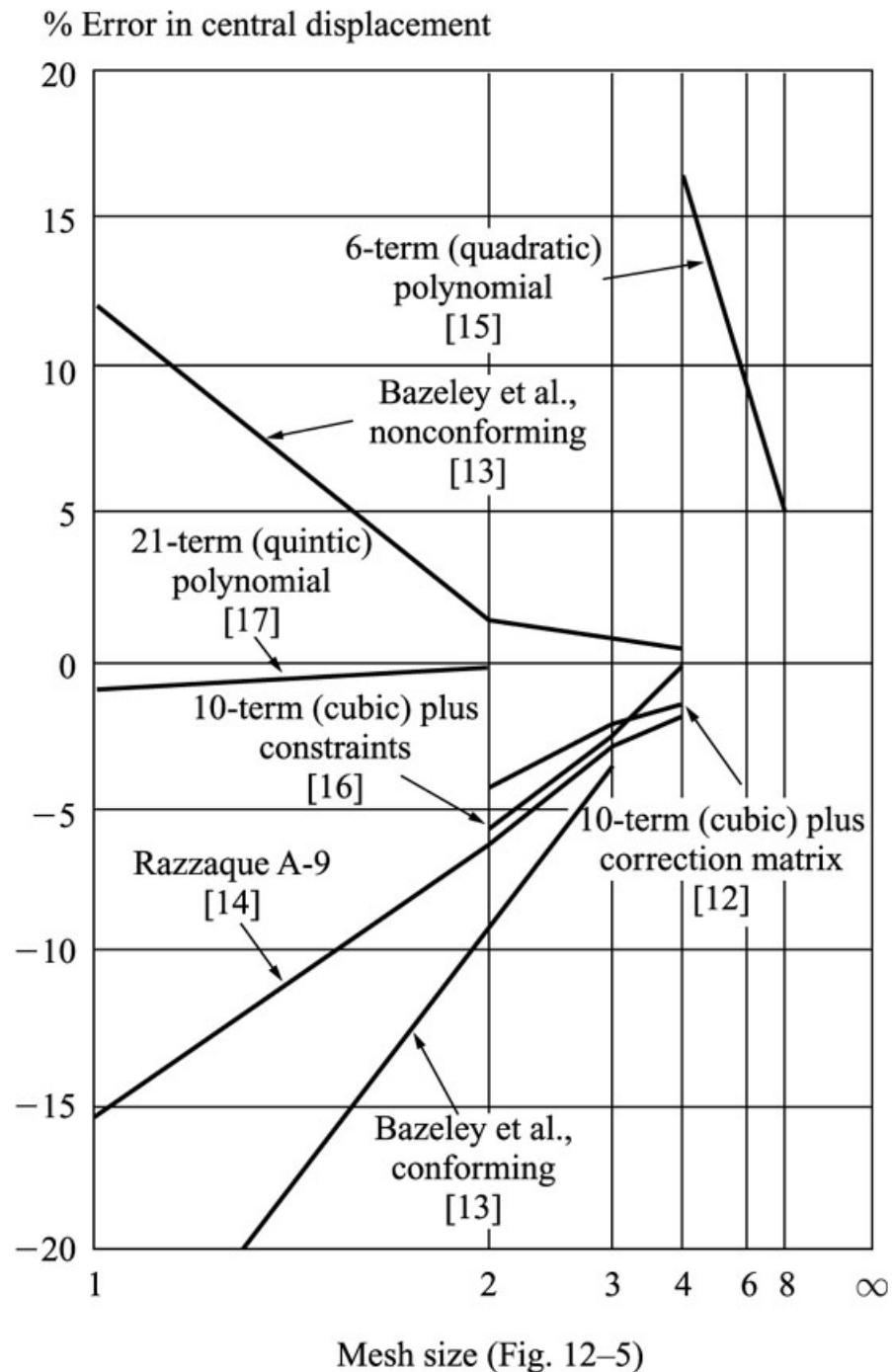
$$\begin{Bmatrix} f_{wi} \\ f_{\theta xi} \\ f_{\theta yi} \\ f_{wj} \\ \vdots \\ f_{\theta yn} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,12} \\ k_{21} & k_{22} & \cdots & k_{2,12} \\ k_{31} & k_{32} & \cdots & k_{3,12} \\ k_{41} & k_{42} & \cdots & k_{4,12} \\ \vdots & \vdots & \cdots & \vdots \\ k_{12,1} & k_{12,2} & \cdots & k_{12,12} \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \\ w_j \\ \vdots \\ \theta_{yn} \end{Bmatrix}$$

The remaining steps of assembling the global equations, applying boundary conditions, and solving the equations for nodal displacements and slopes follow the standard procedures introduced in previous chapters.

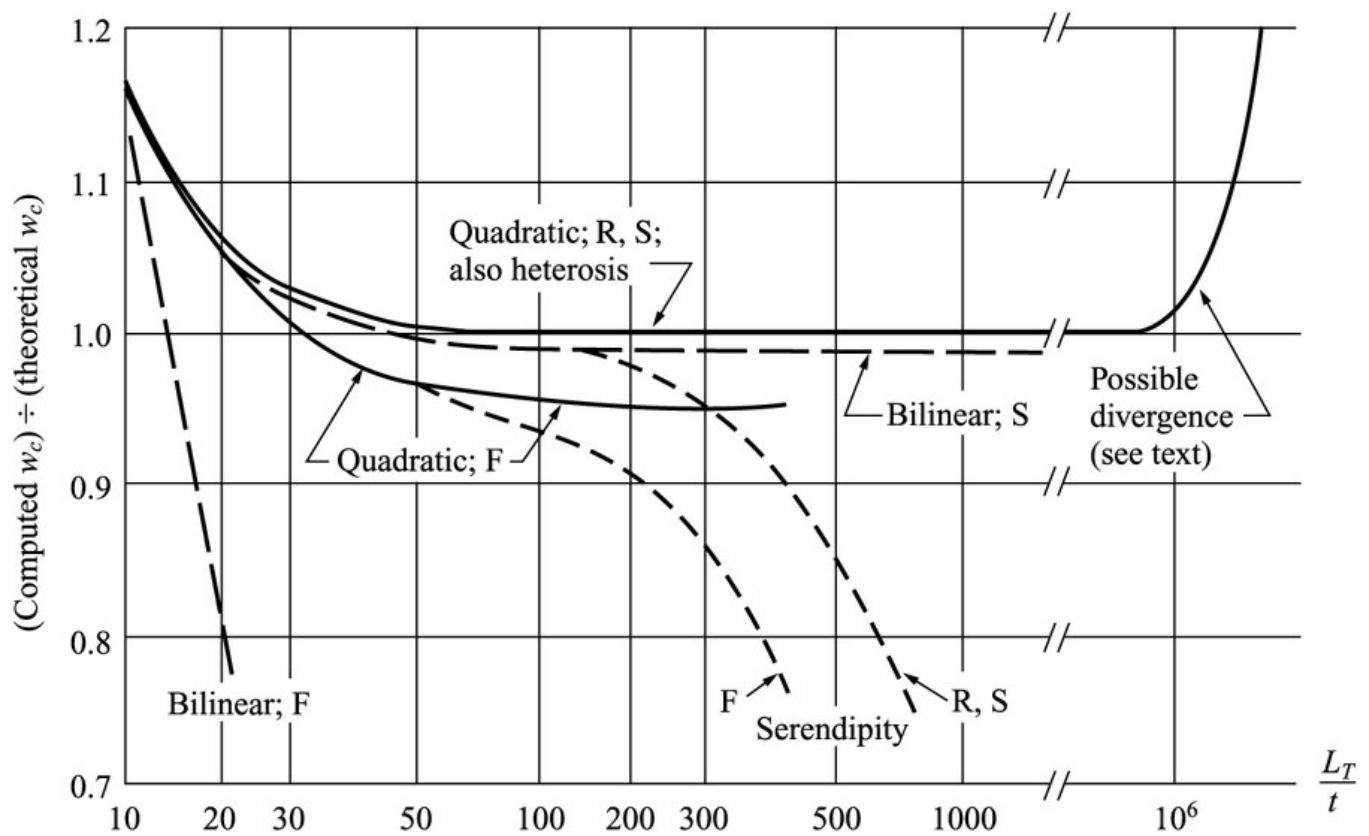
The figure below shows a number of plate element formulations results for a square plate simply supported all around and subjected to a concentrated vertical load applied at the center of the plate. The results show the upper and lower bound solutions behavior and demonstrate convergence of solution for various plate elements. Included in these results is the 12-term polynomial plate element introduced in this chapter.



The figure below shows comparisons of triangular plate formulations for the same centrally loaded simply supported plate. From both figures, we can observe a number of different formulations with results that converge for above and below. Some of these elements produce better results than others.

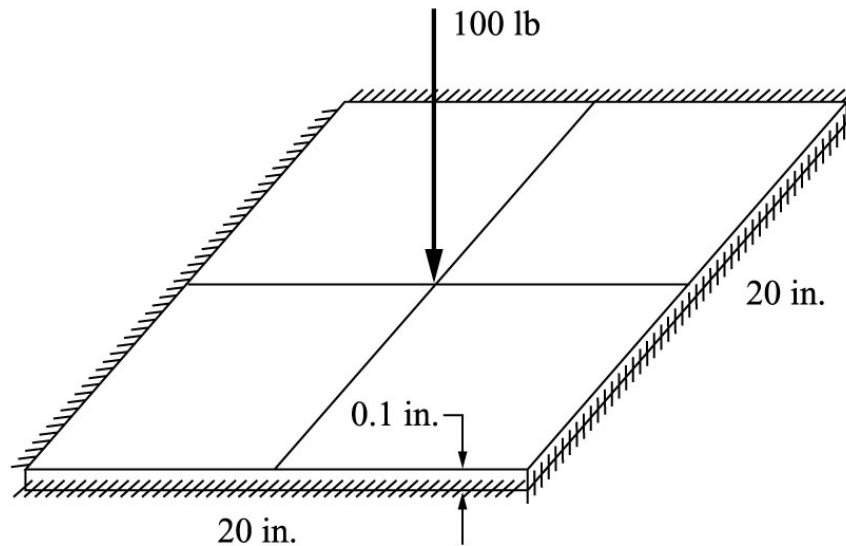


The figure below shows results for some selected Mindlin plate theory elements. Mindlin plate elements account for bending deformations and for transverse shear deformation.



Computer Solution for a Plate Bending Problem

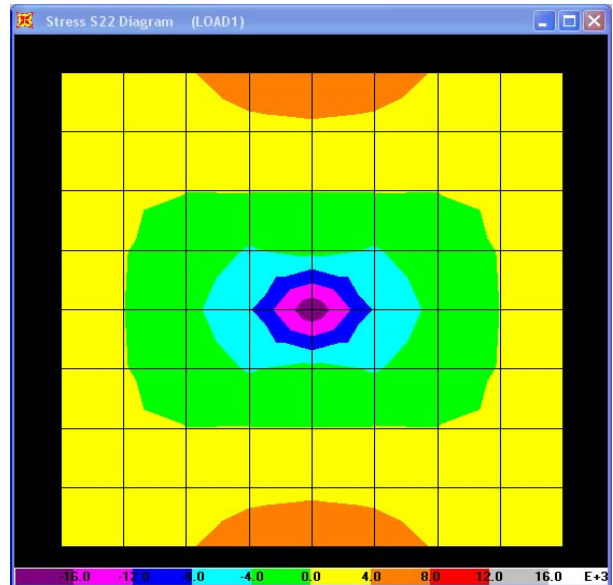
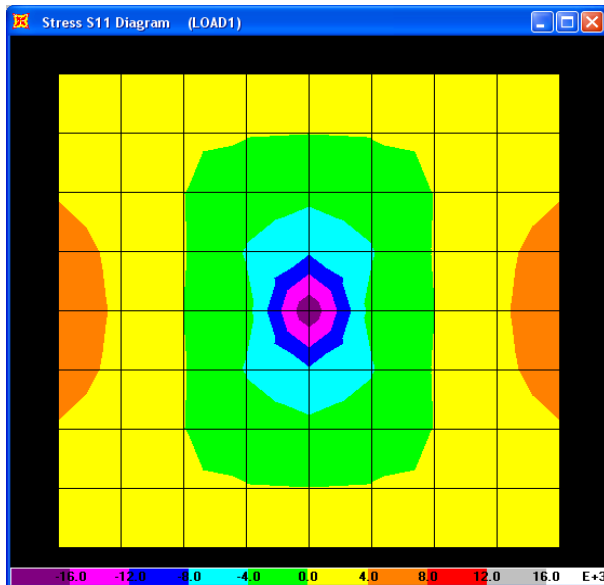
Consider the clamped plate shown below subjected to a 100 lb load applied at the center (let $E = 30 \times 10^6$ psi and $\nu = 0.3$).



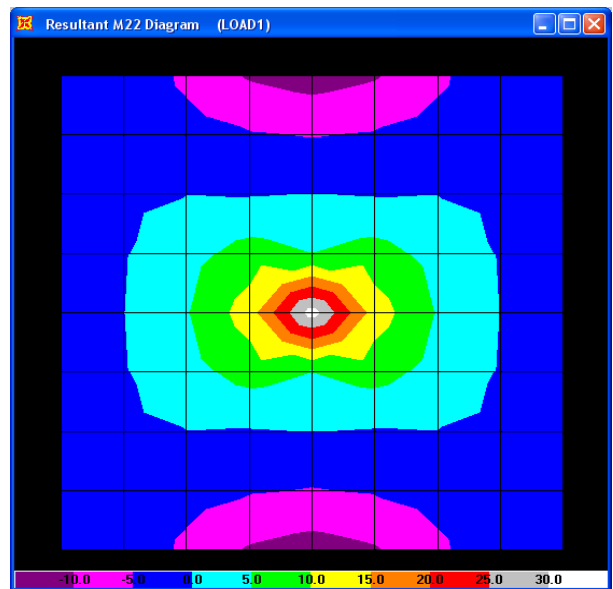
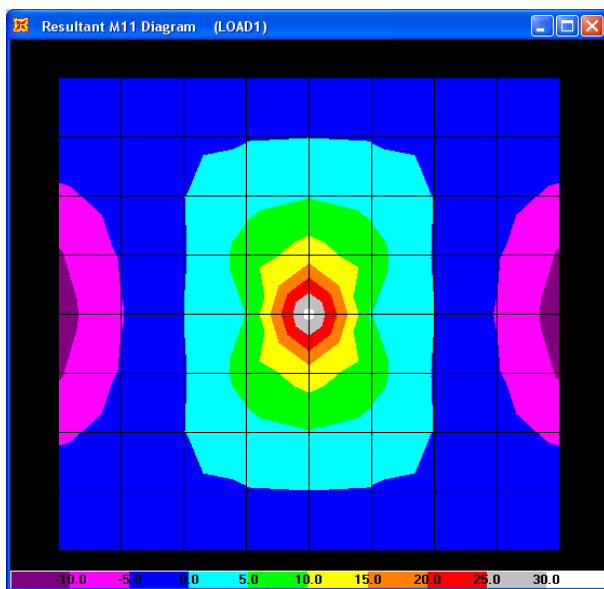
The exact solution for the displacement at the center of the plate is $w = 0.0056PL^2/D$. Substituting the values for the variables gives a numerical value of $w = 0.0815$ in. The table below shows the results of modeling this plate structure using SAP2000 (the educational version allows only 100 nodes) compares to the exact solution.

Number of square elements	Displacement at the center (in)	% error
4	0.09100	11.6
16	0.09334	14.5
36	0.08819	8.2
64	0.08584	5.3
256	0.08300	1.8
1,024	0.08209	0.7
4,096	0.08182	0.3
Exact Solution	0.08154	--

The figures below show non-node-averaged contour plot for the normal stress σ_x and σ_y .

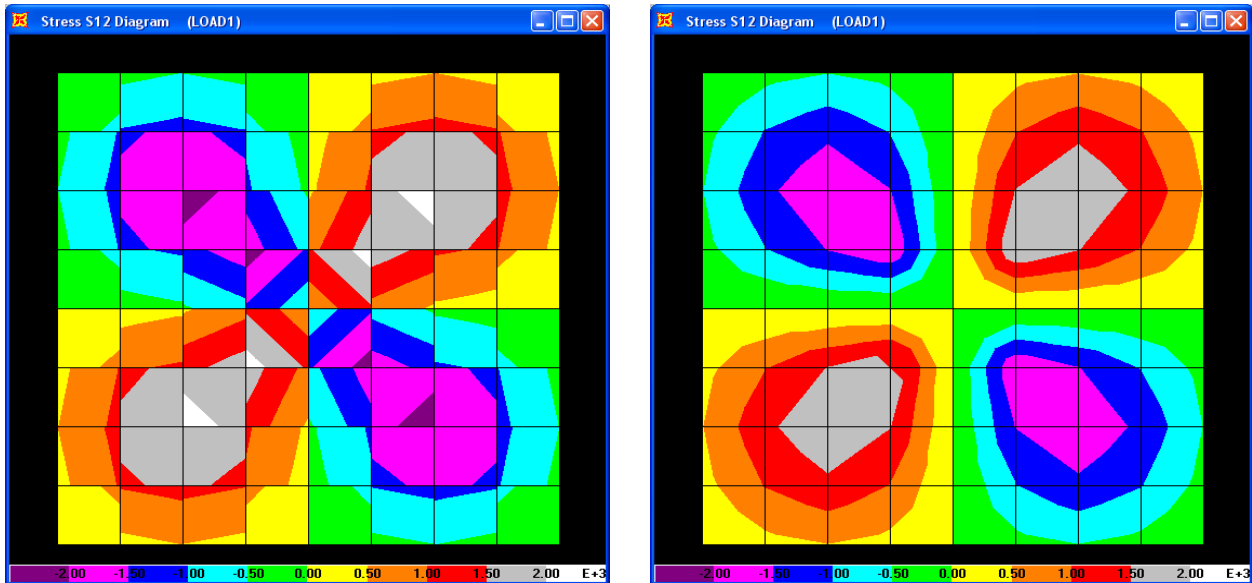


The next set of plots shows the non-node-averaged moments M_x and M_y .

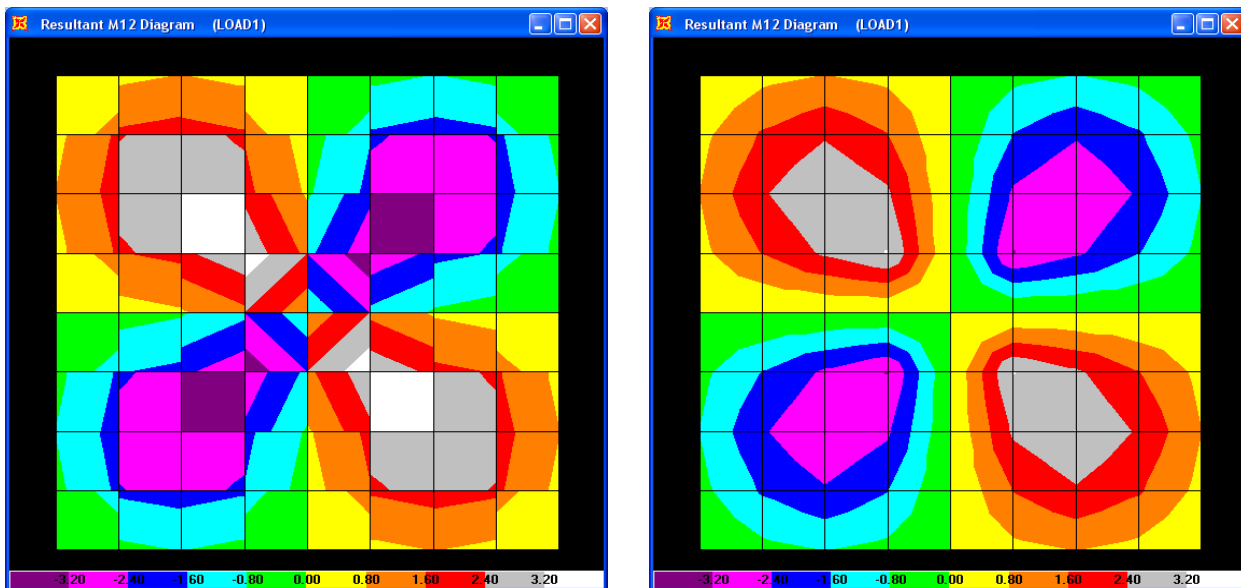


Both sets of plots seem to indicate good interelement continuity between normal stresses in the x and y direction and bending moments about the x and y axes.

The next set of plots shows the shear stress τ_{xy} and the node-average shear stress τ_{xy} .



The next set of plots shows the twisting moment M_{xy} and the node-average twisting moment M_{xy} .



Both sets of plots indicate interelement discontinuities for shear stress and twisting moment. However, if the node-average plots are viewed, the discontinuities are smoothed out and not visible.

Problems

21. Do problems **12.1** and **12.6** on pages 455 - 457 in your textbook "A First Course in the Finite Element Method" by D. Logan using SAP2000.